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Cohomology and Deformations of Relative Rota–Baxter Operators on Lie-Yamaguti Algebras

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Abstract: In this paper, we establish the cohomology of relative Rota–Baxter operators on Lie-Yamaguti algebras via the Yamaguti cohomology. Then, we use this type of cohomology to characterize deformations of relative Rota–Baxter operators on Lie-Yamaguti algebras. We show that if two linear or formal deformations of a relative Rota–Baxter operator are equivalent, then their infinitesimals are in the same cohomology class in the first cohomology group. Moreover, an order n deformation of a relative Rota–Baxter operator can be extended to an order $n+1$ deformation if and only if the obstruction class in the second cohomology group is trivial.

Keywords: Lie-Yamaguti algebra; relative Rota–Baxter operator; cohomology; deformation

MSC: 17B38; 17B60; 17A99

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1. Introduction

Lie-Yamaguti algebras are a generalization of Lie algebras and Lie triple systems, which can be traced back to Nomizu's work on the invariant affine connections on homogeneous spaces in the 1950s [1] and Yamaguti's work on general Lie triple systems and Lie triple algebras [2,3]. Kinyon and Weinstein first called this object a Lie-Yamaguti algebra when studying Courant algebroids in the earlier 21st century [4]. Since then, this system has been called a Lie-Yamaguti algebra, which has attracted much attention and has recently been widely investigated. For instance, Benito and his collaborators deeply explored irreducible Lie-Yamaguti algebras and their relations with orthogonal Lie algebras [5–8]. Deformations and extensions of Lie-Yamaguti algebras were examined in [9–11]. Sheng, the first author, and Zhou analyzed product structures and complex structures on Lie-Yamaguti algebras by means of Nijenhuis operators in [12]. Takahashi studied modules over quandles using representations of Lie-Yamaguti algebras in [13].

Another two topics of the present paper are deformation theory and Rota–Baxter operators, which play important roles in both mathematics and mathematical physics. In mathematics, informally speaking, a deformation of an object is another object that shares the same structure of the original object after a perturbation. Motivated by the foundational work of Kodaira and Spencer [14] for complex analytic structures, the generalization in the algebraic geometry of deformation theory was founded [15]. As an application in algebra, Gerstenhaber first studied the deformation theory on associative algebras [16]. Then, Nijenhuis and Richardson extended this idea and established similar results on Lie algebras [17,18]. Deformations of other algebraic structures such as pre-Lie algebras have also been developed [19]. In general, deformation theory was set up for binary quadratic operads by Balavoine [20]. Deformations are closely connected with cohomology in that the infinitesimal of a formal deformation is characterized by the cohomology class in the first cohomology group.

Rota–Baxter operators on associative algebras can be traced back to a study on fluctuation theory by G. Baxter [21]. In the context of Lie algebras, a Rota–Baxter of weight 0 was determined as the form of operators in the 1980s, which is the solution to the classical Yang–Baxter equation, named after Yang and Baxter [22,23]. Then, Kupershmidt introduced the notion of $\mathcal{O}$-operators (called relative Rota–Baxter operators in the present paper) on Lie algebras in [24]. For more details about the classical Yang–Baxter equation and Rota–Baxter operators, see [25,26].

Since deformation theory and Rota–Baxter operators have important applications in mathematics and mathematical physics, Sheng and his collaborators established cohomology and the deformation theory of relative Rota–Baxter operators on Lie algebras using Chevalley–Eilenberg cohomology [27]. See [28,29] for more details about cohomology and deformations of relative Rota–Baxter operators on 3-Lie algebras and Leibniz algebras, respectively. Furthermore, Sheng and the first author introduced the notion of relative Rota–Baxter operators on Lie-Yamaguti algebras and revealed the fact that a pre-Lie-Yamaguti algebra is the underlying algebraic structure of relative Rota–Baxter operators [30].

By the virtue of Lie-Yamaguti algebras and relative Rota–Baxter operators, it is natural to ask the following question: Does an appropriate cohomology theory of relative Rota–Baxter operators on Lie-Yamaguti algebras which can be used to classify certain types of deformations exist? We tackle this problem as follows.

The most important step is to construct a suitable cohomology theory for relative Rota–Baxter operators on Lie-Yamaguti algebras. Let $(\mathfrak{g}, [\cdot,\cdot][\![\cdot,\cdot,\cdot]\!])$ denote a Lie-Yamaguti algebra and $(V;\rho,\mu)$ a representation of $\mathfrak{g}$. It has been proven that in [30], if $T: V \to \mathfrak{g}$ is a relative Rota–Baxter operator on $\mathfrak{g}$ with respect to $(V;\rho,\mu)$, then there is a Lie-Yamaguti algebra structure $([\cdot,\cdot]_T, [\![\cdot,\cdot,\cdot]\!]_T)$ on V. The key role played in this step is to construct a representation of this Lie-Yamaguti algebra $(V, [\cdot,\cdot]_T, [\![\cdot,\cdot,\cdot]\!]_T)$ on $\mathfrak{g}$ (viewed as the representation space), that is, we shall present the explicit formulas of linear maps $\varrho: V \to \mathfrak{gl}(\mathfrak{g})$, $\varpi: \otimes^2 V \to \mathfrak{gl}(\mathfrak{g})$ and $D_{\varrho,\varpi}$, which are linked with the representation $(V;\rho,\mu)$ and the relative Rota–Baxter operator T, such that the triple $(\mathfrak{g};\varrho,\varpi)$ becomes a representation of Lie-Yamaguti algebra V (see Lemma 1 and Theorem 1). Consequently, we obtain the corresponding Yamaguti cohomology of $(V, [\cdot,\cdot]_T, [\![\cdot,\cdot,\cdot]\!]_T)$ with coefficients in the representation $(\mathfrak{g};\varrho,\varpi)$. However, note that the cochain complex of Yamaguti cohomology starts only from 1-cochain, *not* from 0-cochain. The main *difficulty* is to choose 0-cochain appropriately and build a proper coboundary map from the set of 0-cochains to that of 1-cochains. *Our strategy* is to define the set of 0-cochains to be $\wedge^2\mathfrak{g}$, then construct the coboundary map explicitly (see Proposition 4).

In this way, we obtain a cochain complex (associated to V) starting from 0-cochains, which gives rise to the cohomology of the relative Rota–Baxter operator T on Lie-Yamaguti algebras $(\mathfrak{g}, [\cdot,\cdot], [\![\cdot,\cdot,\cdot]\!])$ (see Definition 6). A Lie-Yamaguti algebra owns two algebraic operations, which makes its cochain complex much more complicated than others, while other algebras such as Lie algebras, pre-Lie algebras, Leibniz algebras or even 3-Lie algebras own only one structure map. As a result, the computation is technical in defining the cohomology of relative Rota–Baxter operators.

The next step is to make use of the cohomology theory to investigate deformations of relative Rota–Baxter operators on Lie-Yamaguti algebras. We consider three kinds of deformations: linear, formal and higher-order deformations. It turns out that our cohomology theory satisfies the rule that is mentioned above and works well (see Theorems 2, 4 and 5).

As was stated before, a Lie triple system is a spacial case of a Lie-Yamaguti algebra, so the conclusions in the present paper can also be adapted to the Lie triple system context. See [31] for more details about cohomology and deformations of relative Rota–Baxter operators on Lie triple systems. However, unlike other algebras such as Lie algebras or Leibniz algebras, a suitable graded Lie algebra whose Maurer–Cartan elements are only the Lie-Yamaguri algebra structure does not exist; thus, we did not find a suitable algebra that controls the deformations of relative Rota–Baxter operators. We will overcome this problem in the future and also expect new findings in this direction.

The paper is structured as follows. In Section 2, we recall some basic concepts, including those of Lie-Yamaguti algebras, representations and cohomology. In Section 3, the cohomology theory of relative Rota–Baxter operators on Lie-Yamaguti algebras is constructed by using that of Lie-Yamaguti algebras. Finally, we utilize our established cohomology theory to analyze three kinds of deformations of relative Rota–Baxter operators on Lie-Yamaguti algebras, namely linear, formal and higher-order deformations, in Section 4.

In this paper, all vector spaces are assumed to be over a field $\mathbb{K}$ of characteristic 0 and finite-dimensional.

2. Preliminaries: Lie-Yamaguti Algebras, Representations and Cohomology

In this section, we recall some basic notions such as Lie-Yamaguti algebras, representations and their cohomology theories. The notion of Lie-Yamaguti algebras was first defined by Yamaguti in [2,3].

Definition 1 ([4])**.** *A* **Lie-Yamaguti algebra** *is a vector space $\mathfrak{g}$ equipped with a bilinear bracket $[\cdot,\cdot] : \wedge^2\mathfrak{g} \to \mathfrak{g}$ and a trilinear bracket $[\![\cdot,\cdot,\cdot]\!] : \wedge^2\mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$, which meet the following conditions: for all $x, y, z, w, t \in \mathfrak{g}$,*

$$[[x,y],z] + [[y,z],x] + [[z,x],y] + [\![x,y,z]\!] + [\![y,z,x]\!] + [\![z,x,y]\!] = 0, \tag{1}$$

$$[\![[x,y],z,w]\!] + [\![[y,z],x,w]\!] + [\![[z,x],y,w]\!] = 0, \tag{2}$$

$$[\![x,y,[z,w]]\!] = [[\![x,y,z]\!],w] + [z,[\![x,y,w]\!]], \tag{3}$$

$$[\![x,y,[\![z,w,t]\!]]\!] = [\![[\![x,y,z]\!],w,t]\!] + [\![z,[\![x,y,w]\!],t]\!] + [\![z,w,[\![x,y,t]\!]]\!]. \tag{4}$$

In the sequel, we denote a Lie-Yamaguti algebra by $(\mathfrak{g}, [\cdot,\cdot][\![\cdot,\cdot,\cdot]\!])$.

Example 1. *Let $(\mathfrak{g}, [\cdot,\cdot])$ be a Lie algebra. Define a trilinear bracket*

$$[\![\cdot,\cdot,\cdot]\!] : \wedge^2\mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$$

by

$$[\![x,y,z]\!] := [[x,y],z], \quad \forall x,y,z \in \mathfrak{g}.$$

Then, via direct computation, we know that $(\mathfrak{g}, [\cdot,\cdot][\![\cdot,\cdot,\cdot]\!])$ forms a Lie-Yamaguti algebra.

The following example is even more interesting.

Example 2. *Let M be a closed manifold with an affine connection, and denote by $\mathfrak{X}(M)$ the set of vector fields on M. For all $x, y, z \in \mathfrak{X}(M)$, set*

$$\begin{aligned} [x,y] &:= -T(x,y), \\ [\![x,y,z]\!] &:= -R(x,y)z, \end{aligned}$$

where T and R are the torsion tensor and curvature tensor, respectively. It is found that the triple $(\mathfrak{X}(M), [\cdot,\cdot], [\![\cdot,\cdot,\cdot]\!])$ forms a Lie-Yamaguti algebra. See [1] for more details.

The following two notions are standard.

Definition 2 ([12,13])**.** *Suppose that $(\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g}, [\![\cdot,\cdot,\cdot]\!]_\mathfrak{g})$ and $(\mathfrak{h}, [\cdot,\cdot]_\mathfrak{h}, [\![\cdot,\cdot,\cdot]\!]_\mathfrak{h})$ are two Lie-Yamaguti algebras. A* **homomorphism** *from $(\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g}, [\![\cdot,\cdot,\cdot]\!]_\mathfrak{g})$ to $(\mathfrak{h}, [\cdot,\cdot]_\mathfrak{h}, [\![\cdot,\cdot,\cdot]\!]_\mathfrak{h})$ is a linear map $\phi : \mathfrak{g} \to \mathfrak{h}$ that preserves the Lie-Yamaguti algebra structures, that is, for all $x, y, z \in \mathfrak{g}$,*

$$\begin{aligned} \phi([x,y]_\mathfrak{g}) &= [\phi(x),\phi(y)]_\mathfrak{h}, \\ \phi([\![x,y,z]\!]_\mathfrak{g}) &= [\![\phi(x),\phi(y),\phi(z)]\!]_\mathfrak{h}. \end{aligned}$$

If, moreover, ϕ is a bijection, it is then called an **isomorphism***.*

Definition 3 ([3])**.** *Let $(\mathfrak{g}, [\cdot,\cdot], [\![\cdot,\cdot,\cdot]\!])$ be a Lie-Yamaguti algebra. A* **representation** *of $\mathfrak{g}$ is a vector space V equipped with a linear map $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ and a bilinear map $\mu : \otimes^2\mathfrak{g} \to \mathfrak{gl}(V)$, which meet the following conditions: for all $x, y, z, w \in \mathfrak{g}$,*

$$\mu([x,y],z) - \mu(x,z)\rho(y) + \mu(y,z)\rho(x) = 0, \tag{5}$$
$$\mu(x,[y,z]) - \rho(y)\mu(x,z) + \rho(z)\mu(x,y) = 0, \tag{6}$$
$$\rho([\![x,y,z]\!]) = [D_{\rho,\mu}(x,y), \rho(z)], \tag{7}$$
$$\mu(z,w)\mu(x,y) - \mu(y,w)\mu(x,z) - \mu(x,[\![y,z,w]\!]) + D_{\rho,\mu}(y,z)\mu(x,w) = 0, \tag{8}$$
$$\mu([\![x,y,z]\!],w) + \mu(z,[\![x,y,w]\!]) = [D_{\rho,\mu}(x,y), \mu(z,w)], \tag{9}$$

where the bilinear map $D_{\rho,\mu} : \otimes^2\mathfrak{g} \to \mathfrak{gl}(V)$ is given by

$$D_{\rho,\mu}(x,y) := \mu(y,x) - \mu(x,y) + [\rho(x),\rho(y)] - \rho([x,y]), \quad \forall x, y \in \mathfrak{g}. \tag{10}$$

It is obvious that $D_{\rho,\mu}$ is skew-symmetric. We denote a representation of $\mathfrak{g}$ by $(V;\rho,\mu)$.

Notice that the notion of representation on Lie-Yamaguti algebras can be reduced to that on Lie algebras or Lie triple systems when the original Lie-Yamaguti algebra is correspondingly reduced to a Lie algebra or a Lie triple system.

Example 3. *Let $(\mathfrak{g}, [\cdot,\cdot], [\![\cdot,\cdot,\cdot]\!])$ be a Lie-Yamaguti algebra. We define linear maps $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ and $\mathfrak{R} : \otimes^2\mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ by $x \mapsto \mathrm{ad}_x$ and $(x,y) \mapsto \mathfrak{R}_{x,y}$, respectively, where $\mathrm{ad}_x z = [x,z]$ and $\mathfrak{R}_{x,y}z = [\![z,x,y]\!]$ for all $z \in \mathfrak{g}$. Then, $(\mathrm{ad},\mathfrak{R})$ forms a representation of $\mathfrak{g}$ on itself, where $\mathfrak{L} := D_{\mathrm{ad},\mathfrak{R}}$ is given by*

$$\mathfrak{L}_{x,y} = \mathfrak{R}_{y,x} - \mathfrak{R}_{x,y} + [\mathrm{ad}_x, \mathrm{ad}_y] - \mathrm{ad}_{[x,y]}, \quad \forall x, y \in \mathfrak{g}.$$

By (1)*, we have*

$$\mathfrak{L}_{x,y}z = [\![x,y,z]\!], \quad \forall z \in \mathfrak{g}. \tag{11}$$

In this case, $(\mathfrak{g};\mathrm{ad},\mathfrak{R})$ is called the **adjoint representation** *of $\mathfrak{g}$.*

The representations of Lie-Yamaguti algebras can be characterized by the semidirect Lie-Yamaguti algebras. This fact is revealed via the following proposition.

Proposition 1 ([11])**.** *Let $(\mathfrak{g}, [\cdot,\cdot], [\![\cdot,\cdot,\cdot]\!])$ be a Lie-Yamaguti algebra and V a vector space. Let $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ and $\mu : \otimes^2\mathfrak{g} \to \mathfrak{gl}(V)$ be linear maps. Then, $(V;\rho,\mu)$ is a representation of $(\mathfrak{g}, [\cdot,\cdot], [\![\cdot,\cdot,\cdot]\!])$ if and only if there is a Lie-Yamaguti algebra structure $([\cdot,\cdot]_{\rho,\mu}, [\![\cdot,\cdot,\cdot]\!]_{\rho,\mu})$ on the direct sum $\mathfrak{g} \oplus V$, which is defined for all $x, y, z \in \mathfrak{g}$, $u, v, w \in V$ as*

$$[x+u, y+v]_{\rho,\mu} = [x,y] + \rho(x)v - \rho(y)u, \tag{12}$$
$$[\![x+u, y+v, z+w]\!]_{\rho,\mu} = [\![x,y,z]\!] + D_{\rho,\mu}(x,y)w + \mu(y,z)u - \mu(x,z)v, \tag{13}$$

where $D_{\rho,\mu}$ is given by (10)*. This Lie-Yamaguti algebra $(\mathfrak{g} \oplus V, [\cdot,\cdot]_{\rho,\mu}, [\![\cdot,\cdot,\cdot]\!]_{\rho,\mu})$ is called the* **semidirect product Lie-Yamaguti algebra***, and is denoted by $\mathfrak{g} \ltimes_{\rho,\mu} V$.*

Let us recall the cohomology theory on Lie-Yamaguti algebras given in [3]. Let $(\mathfrak{g}, [\cdot,\cdot], [\![\cdot,\cdot,\cdot]\!])$ be a Lie-Yamaguti algebra and $(V;\rho,\mu)$ a representation of $\mathfrak{g}$. We denote the set of p-cochains by $C^p_{\mathrm{LieY}}(\mathfrak{g}, V)$ $(p \geqslant 1)$, where

$$C^{n+1}_{\mathrm{LieY}}(\mathfrak{g}, V) \triangleq \begin{cases} \mathrm{Hom}(\underbrace{\wedge^2\mathfrak{g} \otimes \cdots \otimes \wedge^2\mathfrak{g}}_{n}, V) \times \mathrm{Hom}(\underbrace{\wedge^2\mathfrak{g} \otimes \cdots \otimes \wedge^2\mathfrak{g}}_{n} \otimes \mathfrak{g}, V), & \forall n \geqslant 1, \\ \mathrm{Hom}(\mathfrak{g}, V), & n = 0. \end{cases}$$

In the sequel, we recall the coboundary map of p-cochains:

- If $n \geqslant 1$, for any $(f,g) \in C^{n+1}_{\mathrm{LieY}}(\mathfrak{g},V)$, the coboundary map
$$\delta = (\delta_{\mathrm{I}}, \delta_{\mathrm{II}}) : C^{n+1}_{\mathrm{LieY}}(\mathfrak{g},V) \to C^{n+2}_{\mathrm{LieY}}(\mathfrak{g},V),$$
$$(f,g) \mapsto \Big(\delta_{\mathrm{I}}(f,g), \delta_{\mathrm{II}}(f,g)\Big),$$
is given as follows:
$$\begin{aligned}
&\Big(\delta_{\mathrm{I}}(f,g)\Big)(\mathfrak{X}_1,\cdots,\mathfrak{X}_{n+1}) \\
= \;& (-1)^n\Big(\rho(x_{n+1})g(\mathfrak{X}_1,\cdots,\mathfrak{X}_n,y_{n+1}) - \rho(y_{n+1})g(\mathfrak{X}_1,\cdots,\mathfrak{X}_n,x_{n+1}) \\
&\quad -g(\mathfrak{X}_1,\cdots,\mathfrak{X}_n,[x_{n+1},y_{n+1}])\Big) \\
&+\sum_{k=1}^{n}(-1)^{k+1}D_{\rho,\mu}(\mathfrak{X}_k)f(\mathfrak{X}_1,\cdots,\hat{\mathfrak{X}}_k,\cdots,\mathfrak{X}_{n+1}) \\
&+\sum_{1\leqslant k<l\leqslant n+1}(-1)^k f(\mathfrak{X}_1,\cdots,\hat{\mathfrak{X}}_k,\cdots,\mathfrak{X}_k\circ\mathfrak{X}_l,\cdots,\mathfrak{X}_{n+1}),
\end{aligned}$$
and
$$\begin{aligned}
&\Big(\delta_{\mathrm{II}}(f,g)\Big)(\mathfrak{X}_1,\cdots,\mathfrak{X}_{n+1},z) \\
= \;& (-1)^n\Big(\mu(y_{n+1},z)g(\mathfrak{X}_1,\cdots,\mathfrak{X}_n,x_{n+1}) - \mu(x_{n+1},z)g(\mathfrak{X}_1,\cdots,\mathfrak{X}_n,y_{n+1})\Big) \\
&+\sum_{k=1}^{n+1}(-1)^{k+1}D_{\rho,\mu}(\mathfrak{X}_k)g(\mathfrak{X}_1,\cdots,\hat{\mathfrak{X}}_k,\cdots,\mathfrak{X}_{n+1},z) \\
&+\sum_{1\leqslant k<l\leqslant n+1}(-1)^k g(\mathfrak{X}_1,\cdots,\hat{\mathfrak{X}}_k,\cdots,\mathfrak{X}_k\circ\mathfrak{X}_l,\cdots,\mathfrak{X}_{n+1},z) \\
&+\sum_{k=1}^{n+1}(-1)^k g(\mathfrak{X}_1,\cdots,\hat{\mathfrak{X}}_k,\cdots,\mathfrak{X}_{n+1},[\![x_k,y_k,z]\!]),
\end{aligned}$$
where $\mathfrak{X}_i = x_i \wedge y_i \in \wedge^2\mathfrak{g}$ $(i = 1,\cdots,n+1)$, $z \in \mathfrak{g}$ and the operation $\circ$ on $\wedge^2\mathfrak{g}$ means that
$$\mathfrak{X}_k\circ\mathfrak{X}_l := [\![x_k,y_k,x_l]\!]\wedge y_l + x_l\wedge[\![x_k,y_k,y_l]\!].$$
- if $n = 0$, for any $f \in C^1_{\mathrm{LieY}}(\mathfrak{g},V)$, the coboundary map
$$\delta : C^1_{\mathrm{LieY}}(\mathfrak{g},V) \to C^2_{\mathrm{LieY}}(\mathfrak{g},V),$$
$$f \mapsto \Big(\delta_{\mathrm{I}}(f), \delta_{\mathrm{II}}(f)\Big),$$
is defined by
$$\begin{aligned}
\Big(\delta_{\mathrm{I}}(f)\Big)(x,y) &= \rho(x)f(y) - \rho(y)f(x) - f([x,y]), \\
\Big(\delta_{\mathrm{II}}(f)\Big)(x,y,z) &= D_{\rho,\mu}(x,y)f(z) + \mu(y,z)f(x) - \mu(x,z)f(y) - f([\![x,y,z]\!]), \quad \forall x,y,z\in\mathfrak{g}.
\end{aligned}$$

Yamaguti showed the following fact.

Proposition 2 ([3]). *With the notations above, for any $f \in C^1_{\mathrm{LieY}}(\mathfrak{g},V)$, we have*
$$\delta_{\mathrm{I}}\Big(\delta_{\mathrm{I}}(f)),\delta_{\mathrm{II}}(f)\Big) = 0 \quad \text{and} \quad \delta_{\mathrm{II}}\Big(\delta_{\mathrm{I}}(f)),\delta_{\mathrm{II}}(f)\Big) = 0.$$

Moreover, for all $(f,g) \in C^p_{\mathrm{LieY}}(\mathfrak{g},V)$, $(p \geqslant 2)$, *we have*

$$\delta_{\mathrm{I}}\Big(\delta_{\mathrm{I}}(f,g)),\delta_{\mathrm{II}}(f,g)\Big) = 0 \quad \text{and} \quad \delta_{\mathrm{II}}\Big(\delta_{\mathrm{I}}(f,g)),\delta_{\mathrm{II}}(f,g)\Big) = 0.$$

Thus, the cochain complex $(C^\bullet_{\mathrm{LieY}}(\mathfrak{g},V) = \bigoplus_{p=1}^{\infty} C^p_{\mathrm{LieY}}(\mathfrak{g},V),\delta)$ *is well defined. For convenience, the cohomology of the cochain complex* $(C^\bullet_{\mathrm{LieY}}(\mathfrak{g},V),\delta)$ *is called the* **Yamaguti cohomology** *in this paper.*

Definition 4. *With the above notations, let* (f,g) *in* $C^p_{\mathrm{LieY}}(\mathfrak{g},V))$ *(resp.* $f \in C^1_{\mathrm{LieY}}(\mathfrak{g},V)$ *for* $p=1$*) be a p-cochain. If it satisfies* $\delta(f,g)=0$ *(resp.* $\delta(f)=0$*), then it is called a p-cocycle. If there exists* $(h,s) \in C^{p-1}_{\mathrm{LieY}}(\mathfrak{g},V)$, *(resp.* $t \in C^1(\mathfrak{g},V)$, *if* $p=2$*) such that* $(f,g)=\delta(h,s)$ *(resp.* $(f,g)=\delta(t)$*), then it is called a p-coboundary* $(p \geqslant 2)$*. The set of p-cocycles and that of p-coboundaries is denoted by* $Z^p_{\mathrm{LieY}}(\mathfrak{g},V)$ *and* $B^p_{\mathrm{LieY}}(\mathfrak{g},V)$*, respectively. The resulting p-cohomology group is defined to be the factor space:*

$$H^p_{\mathrm{LieY}}(\mathfrak{g},V) = Z^p_{\mathrm{LieY}}(\mathfrak{g},V)/B^p_{\mathrm{LieY}}(\mathfrak{g},V) \quad (p \geqslant 2).$$

3. Cohomology of Relative Rota–Baxter Operators on Lie-Yamaguti Algebras

In this section, we build the cohomology of relative Rota–Baxter operators on Lie-Yamaguti algebras. Once a relative Rota–Baxter operator on a Lie-Yamaguti algebra is given, we obtain a Lie-Yamaguti algebra structure on the representation space. Then, we construct a representation of the representation space (viewed as a Lie-Yamaguti algebra) on the Lie-Yamaguti algebra as a vector space. At the beginning, we recall some notions and conclusions in [30] about relative Rota–Baxter operators on Lie-Yamaguti algebras.

Definition 5 ([30])**.** *Let* $(\mathfrak{g},[\cdot,\cdot],[\![\cdot,\cdot,\cdot]\!])$ *be a Lie-Yamaguti algebra and* $(V;\rho,\mu)$ *a representation of* $\mathfrak{g}$*. A linear map* $T: V \to \mathfrak{g}$ *is called a* **relative Rota–Baxter operator** *on* $\mathfrak{g}$ *with respect to* $(V;\rho,\mu)$ *if T satisfies*

$$\begin{aligned} [Tu,Tv] &= T\Big(\rho(Tu)v - \rho(Tv)u\Big), &(14)\\ [\![Tu,Tv,Tw]\!] &= T\Big(D_{\rho,\mu}(Tu,Tv)w + \mu(Tv,Tw)u - \mu(Tu,Tw)v\Big), \quad \forall u,v,w \in V. &(15)\end{aligned}$$

Proposition 3 ([30])**.** *Let* $T: V \longrightarrow \mathfrak{g}$ *be a relative Rota–Baxter operator on a Lie-Yamaguti algebra* $(\mathfrak{g},[\cdot,\cdot],[\![\cdot,\cdot,\cdot]\!])$ *with respect to* $(V;\rho,\mu)$*. Define*

$$\begin{aligned} [u,v]_T &= \rho(Tu)v - \rho(Tv)u,\\ [\![u,v,w]\!]_T &= D_{\rho,\mu}(Tu,Tv)w + \mu(Tv,Tw)u - \mu(Tu,Tw)v, \quad \forall u,v,w \in V.\end{aligned}$$

Then, $(V,[\cdot,\cdot]_T,[\![\cdot,\cdot,\cdot]\!]_T)$ *is a Lie-Yamaguti algebra, which is the* **sub-adjacent Lie-Yamaguti algebra** *of T. Thus, T is a homomorphism from* $(V,[\cdot,\cdot]_T,[\![\cdot,\cdot,\cdot]\!]_T)$ *to* $(\mathfrak{g},[\cdot,\cdot],[\![\cdot,\cdot,\cdot]\!])$*.*

In the sequel, we present a representation of the sub-adjacent Lie-Yamaguti algebra V_T on $\mathfrak{g}$ (viewed as a vectors space). Define two linear maps $\varrho: V \to \mathfrak{gl}(\mathfrak{g})$ and $\varpi: \otimes^2 V \to \mathfrak{gl}(\mathfrak{g})$ by

$$\begin{aligned} \varrho(u)x &:= [Tu,x] + T\Big(\rho(x)u\Big), &(16)\\ \varpi(u,v)x &:= [\![x,Tu,Tv]\!] - T\Big(D_{\rho,\mu}(x,Tu)v - \mu(x,Tv)u\Big), \quad \forall x \in \mathfrak{g},\ u,v \in V. &(17)\end{aligned}$$

The following lemma gives the explicit formula of $D_{\varrho,\varpi}$.

Lemma 1. *Let T be a relative Rota–Baxter operator on a Lie-Yamaguti algebra $(\mathfrak{g}, [\cdot,\cdot], [\![\cdot,\cdot,\cdot]\!])$ with respect to $(V;\rho,\mu)$. Then, with the above notations, we have*

$$D_{\varrho,\varpi}(u,v)x = [\![Tu,Tv,x]\!] - T\Big(\mu(Tv,x)u - \mu(Tu,x)v\Big), \quad \forall u,v \in V,\ x \in \mathfrak{g}. \tag{18}$$

Proof. Since T is a relative Rota–Baxter operator, via direct computation, we have

$$\begin{aligned}
& D_{\varrho,\varpi}(u,v)x \\
\overset{(10)}{=}\ & \varpi(v,u)x - \varpi(u,v)x + [\varrho(u),\varrho(v)]x - \varrho([u,v]_T)x \\
\overset{(16),(17)}{=}\ & [\![x,Tv,Tu]\!] - T\Big(D_{\rho,\mu}(x,Tv)u - \mu(x,Tu)v\Big) - [\![x,Tu,Tv]\!] + T\Big(D_{\rho,\mu}(x,Tu)v - \mu(x,Tv)u\Big) \\
& +[Tu,[Tv,x]] + [Tu,T(\rho(x)v)] + T(\rho([Tv,x])u) + T(\rho(T(\rho(x)v)u)) \\
& -[Tv,[Tu,x]] - [Tv,T(\rho(x)u)] - T(\rho([Tu,x])v) - T(\rho(T(\rho(x)u)v)) \\
& -[T(\rho(Tu)v - \rho(Tv)u),x] - T(\rho(x)\rho(Tu)v) + T(\rho(x)\rho(Tv)u) \\
\overset{(14)}{=}\ & [\![x,Tv,Tu]\!] - [\![x,Tu,Tv]\!] + [Tu,[Tv,x]] - [Tv,[Tu,x]] - [[Tu,Tv],x] \\
& -T\Big(D_{\rho,\mu}(x,Tv)u - \mu(x,Tu)v\Big) + T\Big(D_{\rho,\mu}(x,Tu)v - \mu(x,Tv)u\Big) \\
& +T(\rho(Tu)\rho(x)v - \rho(T(\rho(x)v)u) - T(\rho(Tv)\rho(x)u - \rho(T(\rho(x)u)v) \\
& +T(\rho([Tv,x])u) + T(\rho(T(\rho(x)v)u)) - T(\rho([Tu,x])v) - T(\rho(T(\rho(x)u)v)) \\
& -T(\rho(x)\rho(Tu)v) + T(\rho(x)\rho(Tv)u) \\
\overset{(1),(10)}{=}\ & [\![Tu,Tv,x]\!] - T\Big(\mu(Tv,x)u - \mu(Tu,x)v\Big).
\end{aligned}$$

The conclusion thus follows. □

Theorem 1. *Let T be a relative Rota–Baxter operator on a Lie-Yamaguti algebra $(\mathfrak{g}, [\cdot,\cdot], [\![\cdot,\cdot,\cdot]\!])$ with respect to $(V;\rho,\mu)$. Then, $(\mathfrak{g};\varrho,\varpi)$ is a representation of the sub-adjacent Lie-Yamaguti algebra $(V, [\cdot,\cdot]_T, [\![\cdot,\cdot,\cdot]\!]_T)$, where ϱ, ϖ and $D_{\varrho,\varpi}$ are given by (16)–(18), respectively.*

Proof. It is evident that if $T : V \to \mathfrak{g}$ is a relative Rota–Baxter operator on a Lie-Yamaguti algebra $(\mathfrak{g}, [\cdot,\cdot], [\![\cdot,\cdot,\cdot]\!])$ with respect to a representation $(V;\rho,\mu)$, then $N_T = \begin{pmatrix} 0 & T \\ 0 & 0 \end{pmatrix}$ is a Nijenhuis operator. See [12] for more details about Nijenhuis operators on Lie-Yamaguti algebras on the semidirect product Lie-Yamaguti algebra $\mathfrak{g} \ltimes_{\rho,\mu} V$. Then, we deduce that there is a Lie-Yamaguti algebra structure on $V \oplus \mathfrak{g} \cong \mathfrak{g} \oplus V$ for all $x,y,z \in \mathfrak{g}$, $u,v,w \in V$, given by

$$\begin{aligned}
& [x+u,y+v]_{N_T} \\
=\ & [N_T(x+u),y+v]_{\rho,\mu} + [x+u,N_T(y+v)]_{\rho,\mu} - N_T[x+u,y+v]_{\rho,\mu} \\
=\ & [Tu,y+v]_{\rho,\mu} + [x+u,Tv]_{\rho,\mu} - N_T([x,y] + \rho(x)v - \rho(y)u) \\
=\ & [Tu,y] + \rho(Tu)v + [x,Tv] - \rho(Tv)u - T(\rho(x)v - \rho(y)u) \\
=\ & [u,v]_T + \varrho(u)y - \varrho(v)x,
\end{aligned}$$

and

$$
\begin{aligned}
& [\![x+u, y+v, z+w]\!]_{N_T} \\
= \; & [\![N_T(x+u), N_T(y+v), z+w]\!]_{\rho,\mu} + [\![N_T(x+u), y+v, N_T(z+w)]\!]_{\rho,\mu} \\
& + [\![x+u, N_T(y+v), N_T(z+w)]\!]_{\rho,\mu} \\
& - N_T\Big([\![N_T(x+u), y+v, z+w]\!]_{\rho,\mu} + [\![x+u, N_T(y+v), z+w]\!]_{\rho,\mu} \\
& + [\![x+u, y+v, N_T(z+w)]\!]_{\rho,\mu} \Big) \\
= \; & [\![Tu, Tv, z+w]\!]_{\rho,\mu} + [\![Tu, y+v, Tw]\!]_{\rho,\mu} + [\![x+u, Tv, Tw]\!]_{\rho,\mu} \\
& - N_T\Big([\![Tu, y+v, z+w]\!]_{\rho,\mu} + [\![x+u, Tv, z+w]\!]_{\rho,\mu} + [\![x+u, y+v, Tw]\!]_{\rho,\mu} \Big) \\
= \; & [\![Tu, Tv, z]\!] + D_{\rho,\mu}(Tu, Tv)w + [\![Tu, y, Tw]\!] - \mu(Tu, Tw)v + [\![x, Tv, Tw]\!] + \mu(Tv, Tw)u \\
& - T\Big(D_{\rho,\mu}(Tu, y)w - \mu(Tu, z)v + D_{\rho,\mu}(x, Tv)w + \mu(Tv, z)u + \mu(y, Tw)u - \mu(x, Tw)v \Big) \\
= \; & [\![u, v, w]\!]_T + D_{\varrho,\varpi}(u, v)z + \varpi(v, w)x - \varpi(u, w)y,
\end{aligned}
$$

which implies that $(\mathfrak{g}; \varrho, \varpi)$ is a representation of Lie-Yamaguti algebra $(V, [\cdot,\cdot]_T, [\![\cdot,\cdot,\cdot]\!]_T)$. This finishes the proof. $\square$

Having endowed the vector space V with a Lie-Yamaguti algebra structure $([\cdot,\cdot]_T, [\![\cdot,\cdot,\cdot]\!]_T)$ and established a representation $(\mathfrak{g}; \varrho, \varpi)$ of $(V, [\cdot,\cdot]_T, [\![\cdot,\cdot,\cdot]\!]_T)$, which gives rise to the corresponding Yamaguti cohomology of $(V, [\cdot,\cdot]_T, [\![\cdot,\cdot,\cdot]\!]_T)$, with coefficients in the representation $(\mathfrak{g}; \varrho, \varpi)$:

$$\Big(\oplus_{p=1}^{\infty} C^p_{\mathrm{LieY}}(V, \mathfrak{g}), \delta^T = (\delta^T_{\mathrm{I}}, \delta^T_{\mathrm{II}}) \Big).$$

More precisely, if $n \geqslant 1$, $\delta^T : C^{n+1}_{\mathrm{LieY}}(V, \mathfrak{g}) \to C^{n+2}_{\mathrm{LieY}}(V, \mathfrak{g})$ for any $(f, g) \in C^{n+1}_{\mathrm{LieY}}(V, \mathfrak{g})$ is given by

$$
\begin{aligned}
& \Big(\delta^T_{\mathrm{I}}(f, g)\Big)(\mathcal{V}_1, \cdots, \mathcal{V}_{n+1}) \\
= \; & (-1)^n \Big([Tu_{n+1}, g(\mathcal{V}_1, \cdots, \mathcal{V}_n, v_{n+1})] - [Tv_{n+1}, g(\mathcal{V}_1, \cdots, \mathcal{V}_n, u_{n+1})] \\
& \quad - g(\mathcal{V}_1, \cdots, \mathcal{V}_n, \rho(Tu_{n+1})v_{n+1} - \rho(Tv_{n+1})u_{n+1}) + T\big(\rho(g(\mathcal{V}_1, \cdots, \mathcal{V}_n, v_{n+1}))u_{n+1}\big) \\
& \quad - T\big(\rho(g(\mathcal{V}_1, \cdots, \mathcal{V}_n, u_{n+1}))v_{n+1}\big) \Big) \\
& + \sum_{k=1}^{n+1} (-1)^{k+1} \Big([\![Tu_k, Tv_k, f(\mathcal{V}_1, \cdots, \hat{\mathcal{V}}_k, \cdots, \mathcal{V}_{n+1})]\!] \\
& + T\Big(\mu(Tv_k, f(\mathcal{V}_1, \cdots, \hat{\mathcal{V}}_k, \cdots, \mathcal{V}_{n+1}))u_k \Big) - \mu(Tu_k, f(\mathcal{V}_1, \cdots, \hat{\mathcal{V}}_k, \cdots, \mathcal{V}_{n+1}))v_k \Big) \Big) \\
& + \sum_{1 \leqslant k < l \leqslant n+1} (-1)^k f(\mathcal{V}_1, \cdots, \hat{\mathcal{V}}_k, \cdots, \mathcal{V}_k \circ \mathcal{V}_l, \cdots, \mathcal{V}_{n+1}),
\end{aligned}
$$

and

$$
\begin{aligned}
& \Big(\delta^T_{\mathrm{II}}(f,g)\Big)(\mathcal{V}_1,\cdots,\mathcal{V}_{n+1},w) \\
= \; & (-1)^n\Big(\llbracket g(\mathcal{V}_1,\cdots,\mathcal{V}_n,u_{n+1}),Tv_{n+1},Tw\rrbracket - \llbracket g(\mathcal{V}_1,\cdots,\mathcal{V}_n,v_{n+1}),Tu_{n+1},Tw\rrbracket \\
& \quad +T\big(D_{\rho,\mu}(g(\mathcal{V}_1,\cdots,\mathcal{V}_n,u_{n+1}),Tv_{n+1})w - \mu(g(\mathcal{V}_1,\cdots,\mathcal{V}_n,u_{n+1}),Tw)v_{n+1} \\
& \quad +D_{\rho,\mu}(g(\mathcal{V}_1,\cdots,\mathcal{V}_n,v_{n+1}),Tu_{n+1})w - \mu(g(\mathcal{V}_1,\cdots,\mathcal{V}_n,v_{n+1}),Tw)u_{n+1}\big)\Big) \\
& +\sum_{k=1}^{n+1}(-1)^k\Big(\llbracket Tu_k,Tv_k,g(\mathcal{V}_1,\cdots,\hat{\mathcal{V}}_k,\cdots,\mathcal{V}_{n+1},w)\rrbracket \\
& +T\Big(\mu(Tv_k,g(\mathcal{V}_1,\cdots,\hat{\mathcal{V}}_k,\cdots,\mathcal{V}_{n+1},w))u_k - \mu(Tu_k,g(\mathcal{V}_1,\cdots,\hat{\mathcal{V}}_k,\cdots,\mathcal{V}_{n+1},w))v_k\Big)\Big) \\
& +\sum_{1\leqslant k<l\leqslant n+1}(-1)^k g(\mathcal{V}_1,\cdots,\hat{\mathcal{V}}_k,\cdots,\mathcal{V}_k\circ\mathcal{V}_l,\cdots,\mathcal{V}_{n+1},w) \\
& +\sum_{k=1}^{n+1}(-1)^k g(\mathcal{V}_1,\cdots,\hat{\mathcal{V}}_k,\cdots,\mathcal{V}_{n+1},\llbracket u_k,v_k,w\rrbracket_T),
\end{aligned}
$$

where $\mathcal{V}_i = u_i\wedge v_i \in \wedge^2 V$ $(1\leqslant i\leqslant n+1)$, $w\in V$ and $\mathcal{V}_k\circ\mathcal{V}_l = \llbracket u_k,v_k,u_l\rrbracket_T\wedge v_l + u_l\wedge \llbracket u_k,v_k,v_l\rrbracket_T$.

In particular, for any $f\in C^1_{\mathrm{LieY}}(V,\mathfrak{g}) = \mathrm{Hom}(V,\mathfrak{g})$,

$$\delta^T : C^1_{\mathrm{LieY}}(V,\mathfrak{g}) \to C^2_{\mathrm{LieY}}(V,\mathfrak{g}), \quad f\mapsto (\delta^T_{\mathrm{I}}(f),\delta^T_{\mathrm{II}}(f))$$

is given by

$$
\begin{aligned}
\Big(\delta^T_{\mathrm{I}}(f)\Big)(u,v) &= [Tu,f(v)] - [Tv,f(u)] + T\Big(\rho(f(v)u) - \rho(f(u)v)\Big) - f([u,v]_T), \\
\Big(\delta^T_{\mathrm{II}}(f)\Big)(u,v,w) &= \llbracket Tu,Tv,f(w)\rrbracket + \llbracket f(u),Tv,Tw\rrbracket - \llbracket f(v),Tu,Tw\rrbracket - f(\llbracket u,v,w\rrbracket_T) \\
& \quad -T\Big(D_{\rho,\mu}(f(u),Tv)w - D_{\rho,\mu}(f(v),Tu)w + \mu(Tv,f(w))u - \mu(Tu,f(w))v \\
& \quad -\mu(f(u),Tw)v + \mu(f(v),Tw)u\Big), \qquad \forall u,v,w\in V.
\end{aligned}
$$

In the following, we present the set of 0-cochains and the corresponding explicit coboundary map. For all $\mathfrak{X}\in\wedge^2\mathfrak{g}$, define $\delta(\mathfrak{X}) : V\to\mathfrak{g}$ by

$$\delta(\mathfrak{X})v := TD_{\rho,\mu}(\mathfrak{X})v - \llbracket\mathfrak{X},Tv\rrbracket, \quad \forall v\in V. \tag{19}$$

Proposition 4. *Let T be a relative Rota–Baxter operator on a Lie-Yamaguti algebra $(\mathfrak{g},[\cdot,\cdot]\llbracket\cdot,\cdot,\cdot\rrbracket)$ with respect to a representation $(V;\rho,\mu)$. Then, $\delta(\mathfrak{X})$ is a 1-cocycle of the Lie-Yamaguti algebra $(V,[\cdot,\cdot]_T,\llbracket\cdot,\cdot,\cdot\rrbracket_T)$ with the coefficients in the representation $(\mathfrak{g};\varrho,\varpi)$.*

Proof. It is sufficient to show that both $\delta_{\mathrm{I}}^T(\delta(\mathfrak{X}))$ and $\delta_{\mathrm{II}}^T(\delta(\mathfrak{X}))$ all vanish. Indeed, for any $u, v, w \in V$, we have

$$
\begin{aligned}
& \delta_{\mathrm{I}}^T\Big(\delta(\mathfrak{X})\Big)(u,v) \\
\overset{(14)}{=} \; & \varrho(u)\delta(\mathfrak{X})(v) - \varrho(v)\delta(\mathfrak{X})(u) - \delta(\mathfrak{X})([u,v]_T) \\
\overset{(16)}{=} \; & [Tu, \delta(\mathfrak{X})(v)] + T(\rho(\delta(\mathfrak{X})(v))u) - [Tv, \delta(\mathfrak{X})(u)] - T(\rho(\delta(\mathfrak{X})(u))v) \\
& -T(D_{\rho,\mu}(\mathfrak{X})[u,v]_T) + [\![\mathfrak{X}, T[u,v]_T]\!] \\
\overset{(19)}{=} \; & [Tu, TD_{\rho,\mu}(\mathfrak{X})v] - [Tu, [\![\mathfrak{X}, Tv]\!]] + T(\rho(T(D_{\rho,\mu}(\mathfrak{X})v))u) - T(\rho([\![\mathfrak{X}, Tv]\!])u) \\
& -[Tv, TD_{\rho,\mu}(\mathfrak{X})u] + [Tv, [\![\mathfrak{X}, Tu]\!]] - T(\rho(T(D_{\rho,\mu}(\mathfrak{X})u))v) + T(\rho([\![\mathfrak{X}, Tu]\!])v) \\
& -T(D_{\rho,\mu}(\mathfrak{X})[u,v]_T) + [\![\mathfrak{X}, T[u,v]_T]\!] \\
\overset{(14)}{=} \; & T(\rho(Tu)D_{\rho,\mu}(\mathfrak{X})v) - T(\rho(T(D_{\rho,\mu}(\mathfrak{X})v))u) - [Tu, [\![\mathfrak{X}, Tv]\!]] + T(\rho(T(D_{\rho,\mu}(\mathfrak{X})v))u) \\
& -T(\rho([\![\mathfrak{X}, Tv]\!])u) - T(\rho(Tv)D_{\rho,\mu}(\mathfrak{X})u) + T(\rho(T(D_{\rho,\mu}(\mathfrak{X})u))v) + [Tv, [\![\mathfrak{X}, Tu]\!]] \\
& -T(\rho(T(D_{\rho,\mu}(\mathfrak{X})u))v) + T(\rho([\![\mathfrak{X}, Tu]\!])v) \\
& -T(D_{\rho,\mu}(\mathfrak{X})(\rho(Tu)v - \rho(Tv)u)) + [\![\mathfrak{X}, [Tu, Tv]]\!] \\
\overset{(3),(7)}{=} \; & 0.
\end{aligned}
$$

Similarly, we also deduce that

$$\delta_{\mathrm{II}}^T(\delta(\mathfrak{X}))(u,v,w) = 0, \quad \forall u,v,w \in V.$$

This finishes the proof. $\square$

Thus far, we have constructed a new complex starting from 0-cochains, whose cohomology is defined to be that of relative Rota–Baxter operators.

Definition 6. *Let T be a relative Rota–Baxter operator on a Lie-Yamaguti algebra $(\mathfrak{g}, [\cdot,\cdot], [\![\cdot,\cdot,\cdot]\!])$ with respect to a representation $(V;\rho,\mu)$. Define the set of n-cochains by*

$$\mathcal{C}_T^n(V,\mathfrak{g}) := \begin{cases} C_{\mathrm{LieY}}^n(V,\mathfrak{g}), & n \geqslant 1, \\ \wedge^2\mathfrak{g}, & n = 0. \end{cases} \tag{20}$$

Define the coboundary map $\mathrm{d} : \mathcal{C}_T^n(V,\mathfrak{g}) \to \mathcal{C}_T^{n+1}(V,\mathfrak{g})$ by

$$\mathrm{d} := \begin{cases} \delta^T = (\delta_{\mathrm{I}}^T, \delta_{\mathrm{II}}^T), & n \geqslant 1, \\ \delta, & n = 0. \end{cases} \tag{21}$$

Thus, we obtain a well-defined cochain complex $(\mathcal{C}_T^\bullet(V,\mathfrak{g}) = \bigoplus_{n=0}^{\infty} \mathcal{C}_T^n(V,\mathfrak{g}), \mathrm{d})$, whose cohomology is called the **cohomology of relative Rota–Baxter operator** *T on the Lie-Yamaguti algebra $(\mathfrak{g}, [\cdot,\cdot], [\![\cdot,\cdot,\cdot]\!])$ with respect to the representation $(V;\rho,\mu)$. Denote the set of n-cocycles and n-coboundaries by $\mathcal{Z}^n(V,\mathfrak{g})$ and $\mathcal{B}^n(V,\mathfrak{g})$, respectively. The* **$n$-th cohomology group of relative Rota–Baxter operator** *T is taken to be*

$$\mathcal{H}_T^n(V,\mathfrak{g}) := \mathcal{Z}_T^n(V,\mathfrak{g}) / \mathcal{B}_T^n(V,\mathfrak{g}), \quad n \geqslant 1. \tag{22}$$

4. Deformatons of Relative Rota–Baxter Operators on Lie-Yamaguti Algebras

In this section, we use the cohomology theory constructed in the former section to characterize deformations of relative Rota–Baxter operators on Lie-Yamaguti algebras.

4.1. Linear Deformations of Relative Rota–Baxter Operators on Lie-Yamaguti Algebras

In this subsection, we aim to perform linear deformations of relative Rota–Baxter operators on Lie-Yamaguti algebras, and we show that the infinitesimals of two equivalent linear deformations of a relative Rota–Baxter operator on Lie-Yamaguti algebra are in the same cohomology class of the first cohomology group.

Definition 7. *Let T and T' be two relative Rota–Baxter operators on a Lie-Yamaguti algebra $(\mathfrak{g},[\cdot,\cdot],[\![\cdot,\cdot,\cdot]\!])$ with respect to a representation $(V;\rho,\mu)$. A* **homomorphism** *from T' to T is a pair $(\phi_{\mathfrak{g}},\phi_V)$, where $\phi_{\mathfrak{g}}:\mathfrak{g}\to\mathfrak{g}$ is a Lie-Yamaguti algebra homomorphism and $\phi_V:V\to V$ is a linear map satisfying*

$$
\begin{aligned}
T\circ\phi_V &= \phi_{\mathfrak{g}}\circ T' && (23)\\
\phi_V(\rho(x)v) &= \rho(\phi_{\mathfrak{g}}(x))\phi_V(v), && (24)\\
\phi_V\mu(x,y)(v) &= \mu(\phi_{\mathfrak{g}}(x),\phi_{\mathfrak{g}}(y))(\phi_V(v)), \quad \forall x,y\in\mathfrak{g},\ v\in V. && (25)
\end{aligned}
$$

In particular, if $\phi_{\mathfrak{g}}$ and ϕ_V are invertible, then $(\phi_{\mathfrak{g}},\phi_V)$ is called an **isomorphism** *from T' to T.*

Through direct computation, we have the following lemma.

Lemma 2. *Let T and T' be two relative Rota–Baxter operators on a Lie-Yamaguti algebra $(\mathfrak{g},[\cdot,\cdot],[\![\cdot,\cdot,\cdot]\!])$ with respect to a representation $(V;\rho,\mu)$, and $(\phi_{\mathfrak{g}},\phi_V)$ a homomorphism from T' to T; then, we have*

$$
\phi_V D_{\rho,\mu}(x,y)(v) = D_{\rho,\mu}(\phi_{\mathfrak{g}}(x),\phi_{\mathfrak{g}}(y))(\phi_V(v)), \quad \forall x,y\in\mathfrak{g},\ v\in V. \tag{26}
$$

Let $T:V\longrightarrow\mathfrak{g}$ be a relative Rota–Baxter operator on a Lie-Yamaguti algebra $(\mathfrak{g},[\cdot,\cdot],[\![\cdot,\cdot,\cdot]\!])$ with respect to a representation $(V;\rho,\mu)$; then a pre-Lie-Yamaguti algebra structure induces $(*_T,\{\cdot,\cdot,\cdot\}_T)$ on V, which is defined to be

$$
\begin{aligned}
u *_T v &= \rho(Tu)v,\\
\{u,v,w\}_T &= \mu(Tv,Tw)u, \quad \forall u,v,w\in V.
\end{aligned}
$$

For more details about pre-Lie-Yamaguti algebras, see [30]. In the sequel, we would write D for $D_{\rho,\mu}$ without ambiguity.

Proposition 5. *Let T and T' be two relative Rota–Baxter operators on a Lie-Yamaguti algebra $(\mathfrak{g},[\cdot,\cdot],[\![\cdot,\cdot,\cdot]\!])$ with respect to a representation $(V;\rho,\mu)$, and $(\phi_{\mathfrak{g}},\phi_V)$ a homomorphism from T' to T. Then, ϕ_V is a homomorphism from a pre-Lie-Yamaguti algebra from $(V,*_{T'},\{\cdot,\cdot,\cdot\}_{T'})$ to $(V,*_T,\{\cdot,\cdot,\cdot\}_T)$.*

Proof. For all $u,v,w\in V$, we have

$$
\begin{aligned}
\phi_V(u *_{T'} v) &= \phi_V(\rho(T'u)v) = \rho(\phi_{\mathfrak{g}}(T'u)\phi_V(v))\\
&= \rho(T(\phi_V(u))\phi_V(v)) = \phi_V(u) *_T \phi_V(v),\\
\phi_V(\{u,v,w\}_{T'}) &= \phi_V(\mu(T'v,T'w)u) = \mu(\phi_{\mathfrak{g}}(T'v),\phi_{\mathfrak{g}}(T'w))(\phi_V(u))\\
&= \mu(T(\phi_V(v)),T(\phi_V(w)))(\phi_V(u)) = \{\phi_V(u),\phi_V(v),\phi_V(w)\}_T.
\end{aligned}
$$

This finishes the proof. □

The notion of linear deformations of relative Rota–Baxter operators is given as follows.

Definition 8. *Let $(\mathfrak{g},[\cdot,\cdot],[\![\cdot,\cdot,\cdot]\!])$ be a Lie-Yamaguti algebra, and $(V;\rho,\mu)$ a representation of $\mathfrak{g}$. Suppose that T, $\mathfrak{T}:V\to\mathfrak{g}$ are two linear maps, where T is a relative Rota–Baxter operator on $\mathfrak{g}$ with respect to $(V;\rho,\mu)$. If $T_t=T+t\mathfrak{T}$ are still relative Rota–Baxter operators on $\mathfrak{g}$ with respect* *to $(V;\rho,\mu)$ for all t, we say that $\mathfrak{T}$ generates a* **linear deformation** *of the relative Rota–Baxter operator T.*

Remark 1. *It is easy to see that if $\mathfrak{T}$ generates a linear deformation of the relative Rota–Baxter operator T, then $\mathfrak{T}$ satisfies the following conditions:*

(i) *$\mathfrak{T}\in\mathcal{C}^1(V,\mathfrak{g})$ is a 1-cocycle of δ^T;*
(ii) *$\mathfrak{T}$ is a relative Rota–Baxter operator on the Lie-Yamaguti algebra $(\mathfrak{g},[\cdot,\cdot],[\![\cdot,\cdot,\cdot]\!])$ with respect to the representation $(V;\rho,\mu)$.*

Let $(A,*,\{\cdot,\cdot,\cdot\})$ be a pre-Lie-Yamaguti algebra, and let $\phi:\otimes^2A\to A$ and $\omega_1,\ \omega_2:\otimes^3A\to A$ be linear maps. If the linear operations $(*_t,\{\cdot,\cdot,\cdot\}_t)$ defined by

$$\begin{aligned} x*_t y &= x*y+t\phi(x,y), && (27)\\ \{x,y,z\}_t &= \{x,y,z\}+t\omega_1(x,y,z)+t^2\omega_2(x,y,z),\quad \forall x,y,z\in A, && (28)\end{aligned}$$

are still pre-Lie-Yamaguti algebra structures for all t, we say that (ϕ,ω_1,ω_2) generates a linear deformation of the pre-Lie-Yamaguti algebra A.

Thanks to the relationship between relative Rota–Baxter operators on Lie-Yamaguti algebras and pre-Lie-Yamaguti algebra, we have the following proposition.

Proposition 6. *If $\mathfrak{T}$ generates a linear deformation of the relative Rota–Baxter operator T on a Lie-Yamaguti algebra $(\mathfrak{g},[\cdot,\cdot],[\![\cdot,\cdot,\cdot]\!])$ with respect to a representation $(V;\rho,\mu)$, then the triple $(\phi_{\mathfrak{T}},\omega^1_{\mathfrak{T}},\omega^2_{\mathfrak{T}})$ generates a linear deformation of the underlying pre-Lie-Yamaguti algebra $(V,*,\{\cdot,\cdot,\cdot\}_T)$, where*

$$\begin{aligned} \phi_{\mathfrak{T}}(u,v) &= \rho(\mathfrak{T}(u))v, && (29)\\ \omega^1_{\mathfrak{T}}(u,v,w) &= \mu(Tv,\mathfrak{T}w)u+\mu(\mathfrak{T}v,Tw)u, && (30)\\ \omega^2_{\mathfrak{T}}(u,v,w) &= \mu(\mathfrak{T}v,\mathfrak{T}w)u,\quad \forall u,v,w\in V. && (31)\end{aligned}$$

Proof. Denote the corresponding pre-Lie-Yamaguti algebra structure induced by the relative Rota–Baxter operator $T_t:=T+t\mathfrak{T}$ by $(*_t,\{\cdot,\cdot,\cdot\}_t)$. Indeed, for all $u,v,w\in V$, we have that

$$\begin{aligned} u*_t v &= \rho((T+t\mathfrak{T})u)v=\rho(Tu)v+t\rho(\mathfrak{T}u)v=u*_T v+t\phi_{\mathfrak{T}}(u,v),\\ \{u,v,w\}_t &= \mu((T+t\mathfrak{T})v,(T+t\mathfrak{T})w)u\\ &= \mu(Tv,Tw)u+t\Big(\mu(Tv,\mathfrak{T}w)u+\mu(\mathfrak{T}v,Tw)u\Big)+t^2\mu(\mathfrak{T}v,\mathfrak{T}w)u\\ &= \{u,v,w\}_T+t\omega^1_{\mathfrak{T}}(u,v,w)+t^2\omega^2_{\mathfrak{T}}(u,v,w).\end{aligned}$$

This finishes the proof. □

Definition 9. *Let $T:V\to\mathfrak{g}$ be a relative Rota–Baxter operator on a Lie-Yamaguti algebra $(\mathfrak{g},[\cdot,\cdot],[\![\cdot,\cdot,\cdot]\!])$ with respect to a representation $(V;\rho,\mu)$:*

(i) *Two linear deformations $T^1_t=T+t\mathfrak{T}_1$ and $T^2_t=T+t\mathfrak{T}_2$ are said to be* **equivalent** *if there exists an element $\mathfrak{X}\in\wedge^2\mathfrak{g}$ such that $(\mathrm{Id}_{\mathfrak{g}}+t\mathfrak{L}_{\mathfrak{X}},\mathrm{Id}_V+tD(\mathfrak{X}))$ is a homomorphism from T^2_t to T^1_t.*
(ii) *A linear deformation $T_t=T+t\mathfrak{T}$ of a relative Rota–Baxter operator T is said to be* **trivial** *if there exists an element $\mathfrak{X}\in\wedge^2\mathfrak{g}$ such that $(\mathrm{Id}_{\mathfrak{g}}+t\mathfrak{L}_{\mathfrak{X}},\mathrm{Id}_V+tD(\mathfrak{X}))$ is a homomorphism from T_t to T.*

Let $(\mathrm{Id}_{\mathfrak{g}} + t\mathfrak{L}_{\mathfrak{X}}, \mathrm{Id}_V + tD(\mathfrak{X}))$ be a homomorphism from T_t^2 to T_t^1. Then, $\mathrm{Id}_{\mathfrak{g}} + t\mathfrak{L}_{\mathfrak{X}}$ is a Lie-Yamaguti algebra homomorphism of $\mathfrak{g}$, i.e., the following equalities hold: for all $x, y, z \in \mathfrak{g}$,

$$
\begin{aligned}
[\![\mathfrak{X}, x]\!], [\![\mathfrak{X}, y]\!]] &= 0, && (32)\\
[\![[\![\mathfrak{X}, x]\!], [\![\mathfrak{X}, y]\!], z]\!] + [\![[\![\mathfrak{X}, x]\!], y, [\![\mathfrak{X}, z]\!]]\!] + [\![x, [\![\mathfrak{X}, y]\!], [\![\mathfrak{X}, z]\!]]\!] &= 0, && (33)\\
[\![[\![\mathfrak{X}, x]\!], [\![\mathfrak{X}, y]\!], [\![\mathfrak{X}, z]\!]]\!] &= 0. && (34)
\end{aligned}
$$

By $T_t^1\big((\mathrm{Id}_V + tD(\mathfrak{X}))v\big) = \big(\mathrm{Id}_{\mathfrak{g}} + t\mathfrak{L}_{\mathfrak{X}}\big)T_t^2(v)$, we have

$$
\begin{aligned}
(\mathfrak{T}_2 - \mathfrak{T}_1)(v) &= T\Big(D(\mathfrak{X})v\Big) - [\![\mathfrak{X}, Tv]\!], && (35)\\
\mathfrak{T}_1\Big(D(\mathfrak{X})v\Big) &= [\![\mathfrak{X}, \mathfrak{T}_2(v)]\!]. && (36)
\end{aligned}
$$

By $\Big(\mathrm{Id}_V + tD(\mathfrak{X})\Big)(\rho(x)v) = \rho\Big((\mathrm{Id}_{\mathfrak{g}} + t\mathfrak{L}_{\mathfrak{X}})(x)\Big)(\mathrm{Id}_V + tD(\mathfrak{X}))(v)$, we have

$$
\rho([\![\mathfrak{X}, x]\!])D(\mathfrak{X}) = 0. \tag{37}
$$

Finally, by $\Big(\mathrm{Id}_V + tD(\mathfrak{X})\Big)\mu(z, w)v = \mu\Big((\mathrm{Id}_{\mathfrak{g}} + t\mathfrak{L}_{\mathfrak{X}})z, (\mathrm{Id}_{\mathfrak{g}} + t\mathfrak{L}_{\mathfrak{X}})w\Big)(\mathrm{Id}_V + tD(\mathfrak{X}))v$, we have

$$
\begin{aligned}
\mu(z, [\![\mathfrak{X}, w]\!])D(\mathfrak{X}) + \mu([\![\mathfrak{X}, z]\!], w)D(\mathfrak{X}) + \mu([\![\mathfrak{X}, z]\!], [\![\mathfrak{X}, w]\!]) &= 0, && (38)\\
\mu([\![\mathfrak{X}, z]\!], [\![\mathfrak{X}, w]\!])D(\mathfrak{X}) &= 0. && (39)
\end{aligned}
$$

Note that (35) means that there exists $\mathfrak{X} \in \wedge^2\mathfrak{g}$, such that $\mathfrak{T}_2 - \mathfrak{T}_1 = \delta(\mathfrak{X})$. Thus, we have the following key conclusion in this section.

Theorem 2. *Let $T : V \to \mathfrak{g}$ be a relative Rota–Baxter operator on a Lie-Yamaguti algebra $(\mathfrak{g}, [\cdot,\cdot], [\![\cdot,\cdot,\cdot]\!])$ with respect to a representation $(V; \rho, \mu)$. If two linear deformations $T_t^1 = T + t\mathfrak{T}_1$ and $T_t^2 = T + t\mathfrak{T}_2$ of T are equivalent, then $\mathfrak{T}_1$ and $\mathfrak{T}_2$ are in the same class of the cohomology group $\mathcal{H}_T^1(V, \mathfrak{g})$.*

Definition 10. *Let $T : V \to \mathfrak{g}$ be a relative Rota–Baxter operator on a Lie-Yamaguti algebra $(\mathfrak{g}, [\cdot,\cdot], [\![\cdot,\cdot,\cdot]\!])$ with respect to a representation $(V; \rho, \mu)$. An element $\mathfrak{X} \in \wedge^2\mathfrak{g}$ is called a* **Nijenhuis element** *with respect to T if $\mathfrak{X}$ satisfies (32)–(34), (38), (39) and the following equation*

$$
[\![\mathfrak{X}, T(D(\mathfrak{X})v) - [\![\mathfrak{X}, Tv]\!]]\!] = 0, \quad \forall v \in V. \tag{40}
$$

We denote the set of Nijenhuis elements with respect to T by $\mathsf{Nij}(T)$.

It is obvious that a trivial deformation of a relative Rota–Baxter operator on a Lie-Yamaguti algebra gives rise to a Nijenhuis element. Indeed, the converse is also true. Let us first present the following lemma.

Lemma 3. *Let $T : V \to \mathfrak{g}$ be a relative Rota–Baxter operator on a Lie-Yamaguti algebra $(\mathfrak{g}, [\cdot,\cdot], [\![\cdot,\cdot,\cdot]\!])$ with respect to a representation $(V; \rho, \mu)$. Let $\phi_{\mathfrak{g}} : \mathfrak{g} \to \mathfrak{g}$ be a Lie-Yamaguti algebra isomorphism and $\phi_V : V \to V$ an isomorphism between vector spaces such that Equations (24) and (25) hold. Then, $\phi_{\mathfrak{g}}^{-1} \circ T \circ \phi_V : V \to \mathfrak{g}$ is a relative Rota–Baxter operator on the Lie-Yamaguti algebra $(\mathfrak{g}, [\cdot,\cdot], [\![\cdot,\cdot,\cdot]\!])$ with respect to the representation $(V; \rho, \mu)$.*

Theorem 3. *Let $T : V \to \mathfrak{g}$ be a relative Rota–Baxter operator on a Lie-Yamaguti algebra $(\mathfrak{g}, [\cdot,\cdot], [\![\cdot,\cdot,\cdot]\!])$ with respect to a representation $(V; \rho, \mu)$. Then, for any Nijenhuis element $\mathfrak{X} \in \wedge^2\mathfrak{g}$, $T_t := T + t\mathfrak{T}$ with $\mathfrak{T} := \delta(\mathfrak{X})$ is a trivial linear deformation of the relative Rota–Baxter operator T.*

Proof. For any Nijenhuis element $\mathfrak{X} \in \mathsf{Nij}(T) \subset \wedge^2\mathfrak{g}$, we define

$$\mathfrak{T} = \delta\mathfrak{X}. \tag{41}$$

Since $\mathfrak{X}$ is a Nijenhuis element, for all t, $T_t = T + t\mathfrak{T}$ satisfies

$$\begin{aligned}
(\mathrm{Id}_{\mathfrak{g}} + t\mathfrak{L}_{\mathfrak{X}}) \circ T_t &= T \circ (\mathrm{Id}_V + tD(\mathfrak{X})), \\
(\mathrm{Id}_V + tD(\mathfrak{X}))\rho(x)v &= \rho((\mathrm{Id}_{\mathfrak{g}} + t\mathfrak{L}_{\mathfrak{X}})(x))(\mathrm{Id}_V + tD(\mathfrak{X}))(v), \\
(\mathrm{Id}_V + tD(\mathfrak{X}))\mu(x,y)v &= \mu(\mathrm{Id}_{\mathfrak{g}} + t\mathfrak{L}_{\mathfrak{X}})(x), \mathrm{Id}_{\mathfrak{g}} + t\mathfrak{L}_{\mathfrak{X}})(y))(\mathrm{Id}_V + tD(\mathfrak{X}))(v), \quad \forall x, y \in \mathfrak{g},\ v \in V.
\end{aligned}$$

For a sufficently small t, we see that $\mathrm{Id}_{\mathfrak{g}} + t\mathfrak{L}_{\mathfrak{X}}$ is a Lie-Yamaguti algebra isomorphism and that $\mathrm{Id}_V + tD(\mathfrak{X})$ is an isomorphism between vector spaces. Thus, we have

$$T_t = (\mathrm{Id}_{\mathfrak{g}} + t\mathfrak{L}_{\mathfrak{X}})^{-1} \circ T \circ (\mathrm{Id}_V + tD(\mathfrak{X})).$$

By Lemma 3, we see that T_t is a relative Rota–Baxter operator on the Lie-Yamaguti algebra $(\mathfrak{g}, [\cdot,\cdot], [\![\cdot,\cdot,\cdot]\!])$ with respect to $(V;\rho,\mu)$ for a sufficiently small t. Thus, $\mathfrak{T} = \delta\mathfrak{X}$ satisfies conditions (i) and (ii) in Remark 1. Therefore, T_t is a relative Rota–Baxter operator for all t, which implies that $\mathfrak{T}$ generates a liner deformation of T. It is easy to see that this deformation is trivial. □

At the end of this subsection, we present two examples of Nijenhuis elements associated to Rota–Baxter operators.

Example 4. *Let $(\mathfrak{g}, [\cdot,\cdot], [\![\cdot,\cdot,\cdot]\!])$ be a two-dimensional Lie-Yamaguti algebra, whose nontrivial brackets are given by, with respect to a basis $\{e_1, e_2\}$:*

$$[e_1, e_2] = e_1, \quad [\![e_1, e_2, e_2]\!] = e_1.$$

Moreover,

$$R = \begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix}$$

is a Rota–Baxter operator on $\mathfrak{g}$. Then, via direct computation, any element in $\wedge^2\mathfrak{g}$ is a Nijenhuis element of R.

Example 5. *Let $\mathfrak{g}$ be a four-dimensional Lie-Yamaguti algebra with a basis $\{e_1, e_2, e_3, e_4\}$ defined by*

$$[e_1, e_2] = 2e_4, \quad [\![e_1, e_2, e_1]\!] = e_4$$

and

$$R = \begin{pmatrix} 0 & a_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$$

is a Rota–Baxter operator on $\mathfrak{g}$. Then any element in $\wedge^2\mathfrak{g}$ is a Nijenhuis element of R. In particular,

$$\begin{aligned}
&\mathfrak{X}_1 = e_1 \wedge e_2, \quad \mathfrak{X}_2 = e_1 \wedge e_3, \quad \mathfrak{X}_3 = e_1 \wedge e_4, \\
&\mathfrak{X}_4 = e_2 \wedge e_3, \quad \mathfrak{X}_5 = e_2 \wedge e_4, \quad \mathfrak{X}_6 = e_3 \wedge e_4,
\end{aligned}$$

are all Nijenhuis elements of R.

4.2. Formal Deformations of Relative Rota–Baxter Operators on Lie-Yamaguti Algebras

In this subsection, we study formal deformations of relative Rota–Baxter operators on Lie-Yamaguti algebras. Let $\mathbb{K}[[t]]$ be a ring of power series of one variable t. For any linear vector space V, $V[[t]]$ denotes the vector space of a formal power series of t with the coefficients in V. If $(\mathfrak{g}, [\cdot,\cdot], [\![\cdot,\cdot,\cdot]\!])$ is a Lie-Yamaguti algebra, then there is a Lie-Yamaguti algebra structure over the ring $\mathbb{K}[[t]]$ on $\mathfrak{g}[[t]]$ given by

$$\Big[\sum_{i=0}^{\infty} x_i t^i, \sum_{j=0}^{\infty} y_j t^j\Big] = \sum_{s=0}^{\infty} \sum_{i+j=s} \Big[x_i, y_j\Big] t^s, \tag{42}$$

$$\Big[\!\!\Big[\sum_{i=0}^{\infty} x_i t^i, \sum_{j=0}^{\infty} y_j t^j, \sum_{k=0}^{\infty} z_k t^k\Big]\!\!\Big] = \sum_{s=0}^{\infty} \sum_{i+j+k=s} [\![x_i, y_j, z_k]\!] t^s, \quad \forall x_i, y_j, z_k \in \mathfrak{g}. \tag{43}$$

For any representation $(V;\rho,\mu)$ of a Lie-Yamaguti algebra $(\mathfrak{g}, [\cdot,\cdot], [\![\cdot,\cdot,\cdot]\!])$, there is a natural representation of the Lie-Yamaguti algebra $\mathfrak{g}[[t]]$ on the $\mathbb{K}[[t]]$-module $V[[t]]$, given by

$$\rho\Big(\sum_{i=0}^{\infty} x_i t^i\Big)\Big(\sum_{k=0}^{\infty} v_k t^k\Big) = \sum_{s=0}^{\infty} \sum_{i+k=s} \rho(x_i) v_k t^s, \tag{44}$$

$$\mu\Big(\sum_{i=0}^{\infty} x_i t^i, \sum_{j=0}^{\infty} y_j t^j\Big)\Big(\sum_{k=0}^{\infty} v_k t^k\Big) = \sum_{s=0}^{\infty} \sum_{i+j+k=s} \mu(x_i, x_j) v_k t^s, \quad \forall x_i, y_j \in \mathfrak{g},\ v_k \in V. \tag{45}$$

Let T be a relative Rota–Baxter operator on a Lie-Yamaguti algebra $(\mathfrak{g}, [\cdot,\cdot], [\![\cdot,\cdot,\cdot]\!])$ with respect to a representation $(V;\rho,\mu)$. Consider the power series

$$T_t = \sum_{i=0}^{\infty} \mathfrak{T}_i t^i, \quad \mathfrak{T}_i \in \mathrm{Hom}(V, \mathfrak{g}), \tag{46}$$

that is, $T_t \in \mathrm{Hom}_{\mathbb{K}}(V, \mathfrak{g})[[t]] = \mathrm{Hom}_{\mathbb{K}}(V, \mathfrak{g}[[t]])$.

Definition 11. *Let T be a relative Rota–Baxter operator on a Lie-Yamaguti algebra $(\mathfrak{g}, [\cdot,\cdot], [\![\cdot,\cdot,\cdot]\!])$ with respect to a representation $(V;\rho,\mu)$. Suppose that T_t is given by (46), where $\mathfrak{T}_0 = T$, and T_t also satisfies*

$$[T_t u, T_t v] = T_t\Big(\rho(T_t u)v - \rho(T_t v)u\Big), \tag{47}$$

$$[\![T_t u, T_t v, T_t w]\!] = T_t\Big(D_{\rho,\mu}(T_t u, T_t v)w + \mu(T_t v, T_t w)u - \mu(T_t u, T_t w)v\Big), \quad \forall u, v, w \in V. \tag{48}$$

We say that T_t is a **formal deformation** *of T.*

Recall that a formal deformation of a Lie-Yamaguti algebra $(\mathfrak{g}, [\cdot,\cdot][\![\cdot,\cdot,\cdot]\!])$ is a pair of power series $f_t = \sum_{i=0}^{\infty} f_i t^i$ and $g_t = \sum_{j=0}^{\infty} g_j t^j$, where $f_0 = [\cdot,\cdot]$ and $g_0 = [\![\cdot,\cdot,\cdot]\!]$, and (f_t, g_t) defines a Lie-Yamaguti algebra structure on $\mathfrak{g}[[t]]$ ([9]). Based on the relationship between the relative Rota–Baxter operators and the pre-Lie-Yamaguti algebras, we have the following proposition.

Proposition 7. *If $T_t = \sum_{i=0}^{\infty} \mathfrak{T}_i t^i$ is a formal deformation of a relative Rota–Baxter operator T on a Lie-Yamaguti algebra $(\mathfrak{g}, [\cdot,\cdot], [\![\cdot,\cdot,\cdot]\!])$ with respect to $(V;\rho,\mu)$, then $([\cdot,\cdot]_{T_t}, [\![\cdot,\cdot,\cdot]\!]_{T_t})$ is a formal deformation of the Lie-Yamaguti algebra $(V, [\cdot,\cdot]_T, [\![\cdot,\cdot,\cdot]\!]_T)$, where*

$$[u, v]_{T_t} = \sum_{i=0}^{\infty} \Big(\rho(\mathfrak{T}_i u)v - \rho(\mathfrak{T}_i v)u\Big) t^i, \tag{49}$$

$$[\![u, v, w]\!]_{T_t} = \sum_{k=0}^{\infty} \sum_{i+j=k} \Big(D_{\rho,\mu}(\mathfrak{T}_i u, \mathfrak{T}_j v)w + \mu(\mathfrak{T}_i v, \mathfrak{T}_j w)u - \mu(\mathfrak{T}_i u, \mathfrak{T}_j w)v\Big) t^k, \quad u, v, w \in V. \tag{50}$$

Substituting Equation (46) into Equations (47) and (48) and comparing the coefficients of t^s $(s \geqslant 0)$ yields that, for all $u, v, w \in V$,

$$\sum_{\substack{i+j=s,\\ i,j\geqslant 0}}\Big([\mathfrak{T}_i u,\mathfrak{T}_j v]-\mathfrak{T}_i\Big(\rho(\mathfrak{T}_j u)v-\rho(\mathfrak{T}_j v)u\Big)\Big)t^s=0, \tag{51}$$

$$\sum_{\substack{i+j+k=s,\\ i,j,k\geqslant 0}}\Big([\![\mathfrak{T}_i u,\mathfrak{T}_j v,\mathfrak{T}_k w]\!]-\mathfrak{T}_i\Big(D_{\rho,\mu}(\mathfrak{T}_j u,\mathfrak{T}_k v)w+\mu(\mathfrak{T}_j v,\mathfrak{T}_k w)u-\mu(\mathfrak{T}_j u,\mathfrak{T}_k w)v\Big)\Big)t^s=0. \tag{52}$$

Proposition 8. *If $T_t=\sum_{i=0}^{\infty}\mathfrak{T}_i t^i$ is a formal deformation of a relative Rota–Baxter operator T on a Lie-Yamaguti algebra $(\mathfrak{g},[\cdot,\cdot],[\![\cdot,\cdot,\cdot]\!])$ with respect to $(V;\rho,\mu)$. Then $\delta^T\mathfrak{T}_1=0$, i.e., $\mathfrak{T}_1\in\mathcal{C}^1_T(V,\mathfrak{g})$ is a 1-cocycle of the relative Rota–Baxter operator T.*

Proof. When $s=1$, Equations (51) and (52) are equivalent to

$$\begin{aligned}&[Tu,\mathfrak{T}_1 v]-[\mathfrak{T}_1 u,Tv]\\ =\;&T\big(\rho(\mathfrak{T}_1 u)v-\rho(\mathfrak{T}_1 v)u\big)+\mathfrak{T}_1\big(\rho(Tu)v-\rho(Tv)u\big),\end{aligned}$$

and

$$\begin{aligned}&[\![\mathfrak{T}_1 u,Tv,Tw]\!]+[\![Tu,\mathfrak{T}_1 v,Tw]\!]+[\![Tu,Tv,\mathfrak{T}_1 w]\!]\\ =\;&\mathfrak{T}_1\Big(D_{\rho,\mu}(Tu,Tv)w+\mu(Tv,Tw)u-\mu(Tu,Tw)v\Big)\\ &+T\Big(D_{\rho,\mu}(\mathfrak{T}_1 u,Tv)w+\mu(\mathfrak{T}_1 v,Tw)u-\mu(\mathfrak{T}_1 u,Tw)v\Big)\\ &+T\Big(D_{\rho,\mu}(Tu,\mathfrak{T}_1 v)w+\mu(Tv,\mathfrak{T}_1 w)u-\mu(Tu,\mathfrak{T}_1 w)v\Big),\quad \forall u,v,w\in V,\end{aligned}$$

respectively, which implies that $\delta^T(\mathfrak{T}_1)=0$, i.e., $\mathfrak{T}_1$ is a 1-cocycle of δ^T. □

Definition 12. *Let T be a relative Rota–Baxter operator on a Lie-Yamaguti algebra $(\mathfrak{g},[\cdot,\cdot],[\![\cdot,\cdot,\cdot]\!])$ with respect to a representation $(V;\rho,\mu)$. Then the 1-cocycle $\mathfrak{T}_1$ is called the* **infinitesimal** *of the formal deformation $T_t=\sum_{i=0}^{\infty}\mathfrak{T}_i t^i$ of T.*

In the sequel, let us present the notion of equivalent formal deformations of relative Rota–Baxter operators on Lie-Yamaguti algebras.

Definition 13. *Let T be a relative Rota–Baxter operator on a Lie-Yamaguti algebra $(\mathfrak{g},[\cdot,\cdot],[\![\cdot,\cdot,\cdot]\!])$ with respect to a representation $(V;\rho,\mu)$. Two formal deformations $\overline{T}_t=\sum_{i=0}^{\infty}\overline{\mathfrak{T}}_i t^i$ and $T_t=\sum_{i=0}^{\infty}\mathfrak{T}_i t^i$, where $\overline{\mathfrak{T}}_0=\mathfrak{T}_0=T$ are said to be* **equivalent** *if there exists $\mathfrak{X}\in\wedge^2\mathfrak{g}$, $\phi_i\in\mathfrak{gl}(\mathfrak{g})$ and $\varphi_i\in\mathfrak{gl}(V)$, $i\geqslant 2$, such that for*

$$\phi_t=\mathrm{Id}_{\mathfrak{g}}+t\mathfrak{L}_{\mathfrak{X}}+\sum_{i=2}^{\infty}\phi_i t^i,\qquad \varphi_t=\mathrm{Id}_V+tD(\mathfrak{X})+\sum_{i=2}^{\infty}\varphi_i t^i, \tag{53}$$

the following holds:

$$[\phi_t(x),\phi_t(y)]=\phi_t[x,y],\quad [\![\phi_t(x),\phi_t(y),\phi_t(z)]\!]=\phi_t[\![x,y,z]\!],\quad \forall x,y,z\in\mathfrak{g}, \tag{54}$$

$$\varphi_t\rho(x)v=\rho(\phi_t(x))(\varphi_t(v)),\quad \varphi_t\mu(x,y)v=\mu(\phi_t(x),\phi_t(y))(\varphi_t(v)),\quad \forall x,y\in\mathfrak{g},\ v\in V, \tag{55}$$

and

$$T_t\circ\varphi_t=\phi_t\circ\overline{T}_t \tag{56}$$

as $\mathbb{K}[[t]]$-module maps.

The following theorem is the second key conclusion in this section.

Theorem 4. *Let T be a relative Rota–Baxter operator on a Lie-Yamaguti algebra $(\mathfrak{g},[\cdot,\cdot],[\![\cdot,\cdot,\cdot]\!])$ with respect to a representation $(V;\rho,\mu)$. If two formal deformations $\overline{T}_t=\sum_{i=0}^{\infty}\overline{\mathfrak{T}}_i t^i$ and $T_t=\sum_{i=0}^{\infty}\mathfrak{T}_i t^i$ are equivalent, then their infinitesimals are in the same cohomology classes.*

Proof. Let (ϕ_t,φ_t) be the maps defined by (53), which makes two deformations $\overline{T}_t=\sum_{i=0}^{\infty}\overline{\mathfrak{T}}_i t^i$ and $T_t=\sum_{i=0}^{\infty}\mathfrak{T}_i t^i$ equivalent. By (56), we have

$$\overline{\mathfrak{T}}_1 v=\mathfrak{T}_1 v+TD(\mathfrak{X})v-[\![\mathfrak{X},Tv]\!]=\mathfrak{T}_1 v+\delta(\mathfrak{X})(v),\quad \forall v\in V,$$

which implies that $\overline{\mathfrak{T}}_1$ and $\mathfrak{T}_1$ are in the same cohomology classes. □

Definition 14. *A relative Rota–Baxter operator T is* **rigid** *if all formal deformations of T are trivial.*

Proposition 9. *Let T be a relative Rota–Baxter operator on a Lie-Yamaguti algebra $(\mathfrak{g},[\cdot,\cdot],[\![\cdot,\cdot,\cdot]\!])$ with respect to a representation $(V;\rho,\mu)$. If $\mathcal{Z}^1(V,\mathfrak{g})=\delta(\mathsf{Nij}(T))$, then T is rigid.*

Proof. Let $T_t=\sum_{i=0}^{\infty}\mathfrak{T}_i t^i$ be a formal deformation of T, then Proposition 8 gives $\mathfrak{T}_1\in\mathcal{Z}^1(V,\mathfrak{g})$. By the assumption, $\mathfrak{T}_1=\delta(\mathfrak{X})$ for some $\mathfrak{X}\in\wedge^2\mathfrak{g}$. Then setting $\phi_t=\mathrm{Id}_{\mathfrak{g}}+t\mathfrak{L}_{\mathfrak{X}}$, $\varphi_t=\mathrm{Id}_V+tD(\mathfrak{X})$, we obtain a formal deformation:

$$\overline{T}_t:=\phi_t^{-1}\circ T_t\circ\varphi_t.$$

Thus, $\overline{T}_t$ is equivalent to T_t. Moreover, we have

$$\begin{aligned}\overline{T}_t&=(\mathrm{Id}-\mathfrak{L}_{\mathfrak{X}}t+(\mathfrak{L}_{\mathfrak{X}})^2t^2+\cdots+(-1)^i(\mathfrak{L}_{\mathfrak{X}})^it^i+\cdots)(T_t(v+tD(\mathfrak{X})v))\\&=Tv+(\mathfrak{T}_1v+T(D(\mathfrak{X})v)-[\![\mathfrak{X},Tv]\!])t+\bar{\mathfrak{T}}_2vt^2+\cdots\\&=Tv+\bar{\mathfrak{T}}_2(v)t^2+\cdots.\end{aligned}$$

By repeating this procedure, we can determine that T_t is equivalent to T. □

4.3. Higher-Order Deformations of Relative Rota–Baxter Operators on Lie-Yamaguti Algebras

In this subsection, we introduce a special cohomology class associated with an order n deformation of a relative Rota–Baxter operator, and show that a deformation of order n is extendable if and only if this cohomology class in the second cohomology group is trivial. Thus, we call this cohomology class: the obstruction class of a deformation of an extendable order n.

Definition 15. *Let T be a relative Rota–Baxter operator on a Lie-Yamaguti algebra $(\mathfrak{g},[\cdot,\cdot],[\![\cdot,\cdot,\cdot]\!])$ with respect to a representation $(V;\rho,\mu)$. If $T_t=\sum_{i=0}^{n}\mathfrak{T}_i t^i$ with $\mathfrak{T}_0=T$, $\mathfrak{T}_i\in\mathrm{Hom}_{\mathbb{K}}(V,\mathfrak{g})$, $i=1,2,\cdots,n$, a $\mathbb{K}[t]/(t^{n+1})$-module from $V[t]/(t^{n+1})$ to the Lie-Yamaguti algebra $\mathfrak{g}[t]/(t^{n+1})$ is defined, satisfying*

$$[T_tu,T_tv]=T_t\Big(\rho(T_t)u-\rho(T_tv)u\Big),\tag{57}$$

$$[\![T_tu,T_tv,T_tw]\!]=T_t\Big(D_{\rho,\mu}(T_tu,T_tv)w+\mu(T_tv,T_tw)u-\mu(T_tu,T_tw)v\Big),\quad\forall u,v,w\in V,\tag{58}$$

hence, we say that T_t is an **order** n **deformation** *of T.*

Remark 2. *The left-hand side of Equations (57) and (58) hold in the Lie-Yamaguti algebra $\mathfrak{g}[t]/(t^{n+1})$ and the right-hand side of Equations (57) and (58) make sense since T_t is a $\mathbb{K}[t]/(t^{n+1})$-module map.*

Definition 16. *Let $T_t = \sum_{i=0}^{n} \mathfrak{T}_i t^i$ be an order n deformation of a relative Rota–Baxter operator T on a Lie-Yamaguti algebra $(\mathfrak{g}, [\cdot,\cdot], [\![\cdot,\cdot,\cdot]\!])$ with respect to a representation $(V;\rho,\mu)$. If there exists a 1-cochain $\mathfrak{T}_{n+1} \in \mathrm{Hom}_{\mathbb{K}}(V,\mathfrak{g})$ such that $\widetilde{T}_t = T_t + \mathfrak{T}_{n+1} t^{n+1}$ is an order $n+1$ deformation of T, then we say that T_t is* **extendable**.

The following theorem is the third key conclusion in this section.

Theorem 5. *Let T be a relative Rota–Baxter operator on a Lie-Yamaguti algebra $(\mathfrak{g}, [\cdot,\cdot], [\![\cdot,\cdot,\cdot]\!])$ with respect to a representation $(V;\rho,\mu)$, and $T_t = \sum_{i=0}^{n} \mathfrak{T}_i t^i$ be an order n deformation of T. Then T_t is extendable if and only if the cohomology class $[\mathsf{Ob}^{\mathsf{T}}] \in \mathcal{H}_T^2(V,\mathfrak{g})$ is trivial, where $\mathsf{Ob}^{\mathsf{T}} = (\mathsf{Ob}_{\mathrm{I}}^{\mathsf{T}}, \mathsf{Ob}_{\mathrm{II}}^{\mathsf{T}}) \in \mathcal{C}_T^2(V,\mathfrak{g})$ is defined by*

$$\mathsf{Ob}_{\mathrm{I}}^{\mathsf{T}}(v_1,v_2) = \sum_{\substack{i+j=n+1,\\ i,j\geqslant 0}} \Big([\mathfrak{T}_i v_1, \mathfrak{T}_j v_2] - \mathfrak{T}_i\big(\rho(\mathfrak{T}_j v_1)v_2 - \rho(\mathfrak{T}_j v_2)v_1\big)\Big), \tag{59}$$

$$\begin{aligned}\mathsf{Ob}_{\mathrm{II}}^{\mathsf{T}}(v_1,v_2,v_3) = & \sum_{\substack{i+j+k=n+1,\\ i,j,k\geqslant 0}} \Big([\![\mathfrak{T}_i v_1, \mathfrak{T}_j v_2, \mathfrak{T}_k v_3]\!] - \mathfrak{T}_i\big(D(\mathfrak{T}_j v_1, \mathfrak{T}_k v_2)v_3 + \mu(\mathfrak{T}_j v_2, \mathfrak{T}_k v_3)v_1 \\ & -\mu(\mathfrak{T}_j v_1, \mathfrak{T}_k v_3)v_2\big)\Big), \quad \forall v_1,v_2,v_3 \in V.\end{aligned} \tag{60}$$

Proof. Let $\widetilde{T}_t = \sum_{i=0}^{n+1} \mathfrak{T}_i t^i$ be the extension of T_t, then for all $u,v,w \in V$,

$$[\widetilde{T}_t u, \widetilde{T}_t v] = \widetilde{T}_t\big(\rho(\widetilde{T}_t u)v - \rho(\widetilde{T}_t v)u\big), \tag{61}$$

$$[\![\widetilde{T}_t u, \widetilde{T}_t v, \widetilde{T}_t w]\!] = \widetilde{T}_t\big(D(\widetilde{T}_t u, \widetilde{T}_t v)w + \mu(\widetilde{T}_t v, \widetilde{T}_t w)u - \mu(\widetilde{T}_t u, \widetilde{T}_t w)v\big). \tag{62}$$

Expanding Equation (61) and comparing the coefficients of t^n yields that

$$\sum_{\substack{i+j=n+1,\\ i,j\geqslant 0}} \Big([\mathfrak{T}_i u, \mathfrak{T}_j v] - \mathfrak{T}_i\big(\rho(\mathfrak{T}_j u)v - \rho(\mathfrak{T}_j v)u\big)\Big) = 0,$$

which is equivalent to

$$\begin{aligned} & \sum_{\substack{i+j=n+1,\\ i,j\geqslant 1}} \Big([\mathfrak{T}_i u, \mathfrak{T}_j v] - \mathfrak{T}_i\big(\rho(\mathfrak{T}_j u)v - \rho(\mathfrak{T}_j v)u\big)\Big) + [\mathfrak{T}_{n+1}u, Tv] + [Tu, \mathfrak{T}_{n+1}v] \\ & -\Big(T\big(\rho(\mathfrak{T}_{n+1}u)v - \rho(\mathfrak{T}_{n+1}v)u\big) + \mathfrak{T}_{n+1}\big(\rho(Tu)v - \rho(Tv)u\big)\Big) = 0,\end{aligned}$$

i.e.,

$$\mathsf{Ob}_{\mathrm{I}}^{\mathsf{T}} + \delta_{\mathrm{I}}^{T}(\mathfrak{T}_{n+1}) = 0. \tag{63}$$

Similarly, expanding Equation (62) and comparing the coefficients of t^n yields that

$$\mathsf{Ob}_{\mathrm{II}}^{\mathsf{T}} + \delta_{\mathrm{II}}^{T}(\mathfrak{T}_{n+1}) = 0. \tag{64}$$

From (63) and (64), we obtain

$$\mathsf{Ob}^{\mathsf{T}} = -\delta^{T}(\mathfrak{T}_{n+1}).$$

Thus, the cohomology class $[\mathsf{Ob}^{\mathsf{T}}]$ is trivial.

Conversely, suppose that the cohomology class $[\mathsf{Ob}^{\mathsf{T}}]$ is trivial, then there exists $\mathfrak{T}_{n+1} \in \mathcal{C}_T^1(V,\mathfrak{g})$, such that $\mathsf{Ob}^{\mathsf{T}} = -\delta^T(\mathfrak{T}_{n+1})$. Set $\widetilde{T}_t = T_t + \mathfrak{T}_{n+1}t^{n+1}$. Then, for all $0 \leqslant s \leqslant n+1$, $\widetilde{T}_t$ satisfies

$$\sum_{i+j=s}\left([\mathfrak{T}_i u, \mathfrak{T}_j v] - \mathfrak{T}_i\Big(\rho(\mathfrak{T}_j u)v - \rho(\mathfrak{T}_j v)u\Big)\right) = 0,$$

$$\sum_{i+j+k=s}\left([\![\mathfrak{T}_i u, \mathfrak{T}_j v, \mathfrak{T}_k w]\!] - \mathfrak{T}_i\Big(D(\mathfrak{T}_j u, \mathfrak{T}_k v)w + \mu(\mathfrak{T}_j v, \mathfrak{T}_k w)u - \mu(\mathfrak{T}_j u, \mathfrak{T}_k w)v\Big)\right) = 0.$$

which implies that $\widetilde{T}_t$ is an order $n+1$ deformation of T. Hence, $\widetilde{T}_t$ is an extension of T_t. □

Definition 17. *Let T be a relative Rota–Baxter operator on a Lie-Yamaguti algebra $(\mathfrak{g}, [\cdot,\cdot], [\![\cdot,\cdot,\cdot]\!])$ with respect to a representation $(V;\rho,\mu)$, and $T_t = \sum_{i=0}^{n} \mathfrak{T}_i t^i$ be an order n deformation of T. Then the cohomology class $[\mathsf{Ob}^{\mathsf{T}}] \in \mathcal{H}_T^2(V,\mathfrak{g})$ defined in Theorem 5 is called the* **obstruction class** *of T_t being extendable.*

Corollary 1. *Let T be a relative Rota–Baxter operator on a Lie-Yamaguti algebra $(\mathfrak{g}, [\cdot,\cdot], [\![\cdot,\cdot,\cdot]\!])$ with respect to a representation $(V;\rho,\mu)$. If $\mathcal{H}_T^2(V,\mathfrak{g}) = 0$, then every 1-cocycle in $\mathcal{Z}_T^1(V,\mathfrak{g})$ is the infinitesimal of some formal deformation of the relative Rota–Baxter operator T.*

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