mathematics

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Ricci Vector Fields Revisited

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Abstract: We continue studying the σ-Ricci vector field $\mathbf{u}$ on a Riemannian manifold (N^m, g), which is not necessarily closed. A Riemannian manifold with Ricci operator T, a Coddazi-type tensor, is called a *T-manifold*. In the first result of this paper, we show that a complete and simply connected *T-manifold* (N^m, g), $m > 1$, of positive scalar curvature τ, admits a closed σ-Ricci vector field $\mathbf{u}$ such that the vector $\mathbf{u} - \nabla\sigma$ is an eigenvector of T with eigenvalue τm^{-1}, if and only if it is isometric to the m-sphere S^m_α. In the second result, we show that if a compact and connected *T-manifold* (N^m, g), $m > 2$, admits a σ-Ricci vector field $\mathbf{u}$ with $\sigma \neq 0$ and is an eigenvector of a rough Laplace operator with the integral of the Ricci curvature $Ric(\mathbf{u}, \mathbf{u})$ that has a suitable lower bound, then (N^m, g) is isometric to the m-sphere S^m_α, and the converse also holds. Finally, we show that a compact and connected Riemannian manifold (N^m, g) admits a σ-Ricci vector field $\mathbf{u}$ with σ as a nontrivial solution of the static perfect fluid equation, and the integral of the Ricci curvature $Ric(\mathbf{u}, \mathbf{u})$ has a lower bound depending on a positive constant α, if and only if (N^m, g) is isometric to the m-sphere S^m_α.

Keywords: Ricci vector field; m-sphere; Riemannian manifold; static perfect fluid equation

MSC: 53C20, 53C21, 53B50

Citation: Alohali, H.; Deshmukh, S.; Vîlcu, G.-E. Ricci Vector Fields Revisited. *Mathematics* **2024**, *12*, 144. https://doi.org/10.3390/math12010144

Academic Editor: Jonathan Blackledge

Received: 25 November 2023
Accepted: 29 December 2023
Published: 1 January 2024
Corrected: 7 February 2024

1. Introduction

In a recent paper, (cf. [1]), a σ-Ricci vector field (abbreviated as σ-*RVF*) $\mathbf{u}$ on a m-Riemannian manifold (N^m, g) is introduced, being defined by

$$\frac{1}{2}\pounds_{\mathbf{u}} g = \sigma Ric, \tag{1}$$

where $\pounds_{\mathbf{u}} g$ is the Lie derivative of the metric g with respect to $\mathbf{u}$, σ is a smooth function and Ric is the Ricci tensor of (N^m, g). A σ-*RVF* is a generalization of conformal vector fields (known for their utility in studying geometry and relativity), on Einstein manifolds (see [1–11]). Moreover, it represents a Killing vector field, which is known to have a great influence on the geometry as well as topology on which it lives (see [12–15]). Apart from these generalizations, a σ-*RVF* is a particular form of potential field of generalized solitons considered in [16–18]. Note that a 1-*RVF* $\mathbf{u}$ on a m-Riemannian manifold (N^m, g) is a stable Ricci soliton $(N^m, g, \mathbf{u}, 0)$ (see [19]). Indeed, in [1], it has been observed that a σ-*RVF* on (N^m, g) is a stable solution of the generalized Ricci flow (or a σ-Ricci flow),

$$\partial_t g = 2\sigma Ric, \quad g(0) = g, \tag{2}$$

of the form $g(t) = \rho(t)\varphi_t^*(g)$, where $\varphi_t : N^m \to N^m$ is a 1-parameter family of diffeomorphisms generated by the vector fields $\mathbf{U}(t)$ and $\rho(t)$ is a scale factor, under the initial conditions $\rho(0) = 1$, $\dot{\rho}(0) = 0$, $\mathbf{U}(0) = \mathbf{u}$ and $\varphi_0 = id$.

In [1], a closed σ-*RVF* $\mathbf{u}$, with $\sigma \neq 0$, on a compact and connected m-Riemannian manifold (N^m, g), $m > 2$, of nonzero scalar curvature is used with an appropriate lower bound on the integral of the Ricci curvature $Ric(\mathbf{u}, \mathbf{u})$ to find a characterization of the m-sphere $S^m(c)$. Moreover, in [1], a closed σ-*RVF* $\mathbf{u}$ on a complete and simply connected m-Riemannian manifold (N^m, g), $m > 2$, of positive scalar curvature, is used, where the function σ is a nontrivial solution of the Fischer–Marsden equation (cf. [20]) with an appropriate upper bound on the length $\|\nabla \mathbf{u}\|$ of the covariant derivative of $\mathbf{u}$, to find another characterization of the sphere $S^m(c)$.

The Ricci operator T of a Riemannian manifold (N^m, g) is a symmetric operator defined by

$$Ric(E, F) = g(T(E), F), \quad E, F \in \Gamma(N^m),$$

where $\Gamma(N^m)$ is a space of vector fields on N^m. A Riemannian manifold (N^m, g) is said to be a *T-manifold*, if the Ricci operator T is a Codazzi tensor, i.e., it satisfies

$$(D_E T)(F) = (D_F T)(E), \quad E, F \in \Gamma(N^m), \tag{3}$$

where D is the Riemannian connection on (N^m, g). It is worth noting that a *T-manifold* (N^m, g) has a constant scalar curvature.

In this article, we are interested in studying the geometry of (N^m, g) equipped with a σ-*RVF* $\mathbf{u}$. In the first result, we consider a *T-manifold* (N^m, g) that possesses a closed σ-*RVF* $\mathbf{u}$ and we observe that, in this case, the vector field $\mathbf{u} - \nabla\sigma$ has a special role to play in shaping the geometry of the *T-manifold* (N^m, g). It is shown that if the scalar curvature τ of a compact *T-manifold* (N^m, g) is positive (note that τ is a constant for a *T-manifold*) and the vector field $\mathbf{u} - \nabla\sigma$ satisfies

$$T(\mathbf{u} - \nabla\sigma) = \frac{\tau}{m}(\mathbf{u} - \nabla\sigma),$$

then (N^m, g) is isometric to the m-sphere S^m_c of constant curvature c, where $\tau = m(m-1)c$, and the converse also holds (cf. Theorem 1).

Then, we concentrate on a σ-*RVF* $\mathbf{u}$ on (N^m, g) that is not necessarily closed. In this case, the 1-form β dual to $\mathbf{u}$ gives rise to a skew symmetric operator $\Psi : \Gamma(N^m) \to \Gamma(N^m)$ defined by

$$g(\Psi(E), F) = \frac{1}{2} d\beta(E, F), \quad E, F \in \Gamma(N^m),$$

and we call the operator Ψ the associated operator of the σ-*RVF* $\mathbf{u}$. In the second result of this paper, we consider a compact and connected *T-manifold* (N^m, g) with scalar curvature $\tau = m(m-1)c$ that possesses a σ-*RVF* $\mathbf{u}$, $\sigma \neq 0$, with associated operator Ψ satisfying

$$\Delta\mathbf{u} = -c\mathbf{u}, \quad \int_{N^m} Ric(\mathbf{u}, \mathbf{u}) \geq \int_{N^m} \left[\frac{m-1}{m}\tau^2\sigma^2 + \|\Psi\|^2\right],$$

which necessarily implies that (N^m, g) is isometric to the m-sphere S^m_c of constant curvature c, and the converse is also true (cf. Theorem 2), where Δ is the rough Laplace operator acting on vector fields on (N^m, g).

Recall the differential equation on a Riemannian manifold (N^m, g) considered by Obata (cf. [18,21]), namely

$$Hess(\sigma) = -c\sigma g, \tag{4}$$

where σ is a non-constant smooth function, c is a positive constant and $Hess(\sigma)$ is the Hessian of σ defined by

$$Hess(\sigma)(E, F) = g(D_E \nabla\sigma, F), \quad E, F \in \Gamma(N^m).$$

It is known that a complete, simply connected (N^m, g) admits a nontrivial solution of (4) if and only if (N^m, g) is isometric to the sphere S^m_c (cf. [18,21]).

There is yet another important differential equation on a Riemannian manifold (N^m, g) (cf. [7] and references therein), given by

$$\sigma Ric - Hess(\sigma) = \frac{1}{m}(\tau\sigma - \Delta\sigma)g, \tag{5}$$

known as the static fluid equation, where $\Delta\sigma$ is the Laplacian of σ with respect to the metric g. A Riemannian manifold (N^m, g) that admits a nontrivial solution of the static fluid equation is called a *static space*. Note that under the additional assumption

$$\Delta\sigma = -\frac{\tau}{m-1}\sigma,$$

the static fluid equation reduces to the Fischer–Marsden equation (cf. [20])

$$(\Delta\sigma)g + \sigma Ric = Hess(\sigma). \tag{6}$$

In the last result of this paper, we show that a compact and connected Riemannian manifold (N^m, g) with scalar curvature τ possessing a σ-*RVF* $\mathbf{u}$ with associated operator Ψ and the function σ is a nontrivial solution of the static perfect fluid Equation (5); furthermore, for a positive constant c, the following inequality holds:

$$\int_{N^m} Ric(\mathbf{u}, \mathbf{u}) \geq \int_{N^m} \left[\frac{m-1}{m}\tau^2\sigma^2 + \frac{1}{m}(\Delta\sigma + nc\sigma)^2 + \|\Psi\|^2\right],$$

which necessarily implies that (N^m, g) is isometric to the sphere S^m_c, and the converse is also true (cf. Theorem 3).

2. Preliminaries

For a σ-*RVF* $\mathbf{u}$ on an m-dimensional Riemannian manifold (N^m, g), we let β be the 1-form dual to $\mathbf{u}$, i.e.,

$$\beta(E) = g(\mathbf{u}, E), \quad E \in \Gamma(N^m). \tag{7}$$

Then, we have the associated operator Ψ satisfying

$$d\beta(E, F) = \frac{1}{2}g(\Psi(E), F), \quad E, F \in \Gamma(N^m), \tag{8}$$

which shows that Ψ is a skew symmetric operator. Using Equations (1) and (8), we obtain the following expression for the covariant derivative $\nabla_E \mathbf{u}$

$$D_E\mathbf{u} = \sigma T(E) + \Psi(E), \quad E \in \Gamma(N^m). \tag{9}$$

where T is the Ricci operator defined by

$$Ric(E, F) = g(T(E), F), \quad E, F \in \Gamma(N^m).$$

On employing the following expression for the curvature tensor field R of (N^m, g),

$$R(E, F)G = [D_E, D_F]G - D_{[E,F]}G, \quad E, F, G \in \Gamma(N^m),$$

with Equation (9), we obtain

$$\begin{aligned} R(E, F)\mathbf{u} = {} & E(\sigma)T(F) - F(\sigma)T(E) + \rho((D_E T)(F) - (D_F T)(E)) \\ & + (D_E\Psi)(F) - (D_F\Psi)(E), \end{aligned} \tag{10}$$

for any $E, F \in \Gamma(N^m)$, where

$$(D_E T)(F) = D_E T(F) - T(D_E F).$$

The scalar curvature τ of (N^m, g) is given by

$$\tau = \sum_{\alpha=1}^{m} g(T(E_\alpha), E_\alpha),$$

where $\{E_1, \ldots, E_m\}$ is a local frame on N^m. The Ricci tensor is given by

$$Ric(E, F) = \sum_{\alpha=1}^{m} g(R(E_\alpha, E)F, E_\alpha),$$

and employing it in Equation (10), we conclude

$$\begin{aligned} Ric(F, \mathbf{u}) = Ric(F, \nabla\sigma) - \tau F(\sigma) + \sigma g\left(F, \sum_{\alpha=1}^{m} (\nabla_{F_\alpha} T)(F_\alpha)\right) \\ - \rho Y(\tau) - g\left(F, \sum_{\alpha=1}^{m} (\nabla_{F_\alpha} \Psi)(F_\alpha)\right), \end{aligned} \tag{11}$$

where $\nabla\sigma$ is the gradient of σ and we have used the symmetry of the Ricci operator T and the skew symmetry of the associated operator Ψ. It is known that the gradient of scalar curvature τ satisfies (cf. [22])

$$\frac{1}{2}\nabla\tau = \sum_{\alpha=1}^{m} (D_{F_\alpha} T)(F_\alpha). \tag{12}$$

Thus, on using Equation (12) in (11), we arrive at

$$Ric(F, \mathbf{u}) = Ric(F, \nabla\sigma) - \tau F(\sigma) - \frac{1}{2}\sigma F(\tau) - g\left(F, \sum_{\alpha=1}^{m} (\nabla_{F_\alpha} \Psi)(F_\alpha)\right) \tag{13}$$

and, therefore,

$$T(\mathbf{u}) = T(\nabla\sigma) - \tau\nabla\sigma - \frac{1}{2}\rho\nabla\tau - \sum_{\alpha=1}^{m} (\nabla_{F_\alpha} \Psi)(F_\alpha). \tag{14}$$

Lemma 1. *For a σ-RVF $\mathbf{u}$ on a T-manifold (N^m, g), the associated operator Ψ satisfies*

$$(D_E\Psi)(F) = R(E, \mathbf{u})F - Ric(E, F)\nabla\sigma + F(\sigma)T(E), \quad E, F \in \Gamma(N^m).$$

Proof. Suppose that $\mathbf{u}$ is a σ-*RVF* on a T-manifold (N^m, g). Then, Equation (10) changes to

$$(D_E\Psi)(F) - (D_F\Psi)(E) = R(E, F)\mathbf{u} - E(\sigma)T(F) + F(\sigma)T(E). \tag{15}$$

Now, using the fact that the 2-form $d\beta$ in Equation (8) is closed and the associated operator Ψ is skew symmetric, we have

$$g((D_E\Psi)(F) - (D_F\Psi)(E), G) + g((D_G\Psi)(E), F) = 0$$

and employing Equation (15) in the above equation yields

$$g(R(E, F)\mathbf{u} - E(\sigma)T(F) + F(\sigma)T(E), G) + g((D_G\Psi)(E), F) = 0.$$

Thus, we have

$$g((D_G\Psi)(E), F) = g(R(G, \mathbf{u})E, F) + E(\sigma)g(T(G), F) - Ric(E, G)g(\nabla\sigma, F)$$

and this proves the lemma. $\square$

On a Riemannian manifold (N^m, g) possessing a σ-*RVF* $\mathbf{u}$, we have the second-order differential operator $\nabla^2\mathbf{u}$ defined by

$$\left(\nabla^2\mathbf{u}\right)(E,F) = D_E D_F \mathbf{u} - D_{D_E F}\mathbf{u}, \quad E,F \in \Gamma(N^m)$$

and its trace

$$\Delta\mathbf{u} = \sum_{\alpha=1}^{m} \left(\nabla^2\mathbf{u}\right)(E_\alpha, E_\alpha) \tag{16}$$

is the rough Laplacian of the σ-*RVF* $\mathbf{u}$.

Lemma 2. *On a connected T-manifold (N^m, g), the scalar curvature τ is a constant, and for a σ-RVF $\mathbf{u}$ on a connected T-manifold (N^m, g) with associated operator Ψ, the rough Laplacian satisfies*

$$\Delta\mathbf{u} = T(\nabla\sigma) + \sum_{\alpha=1}^{m} (D_{E_\alpha}\Psi)(E_\alpha),$$

where $\{E_1, \ldots, E_m\}$ is a local frame on N^m.

Proof. First, note that for a T-manifold (N^m, g), using Equation (3), we have

$$\begin{aligned}
E(\tau) &= E\sum_{\alpha=1}^{m} g(T(E_\alpha), E_\alpha) \\
&= \sum_{\alpha=1}^{m} g((D_E T)(E_\alpha) + T(D_E E_\alpha), E_\alpha) + \sum_{\alpha=1}^{m} g(T(E_\alpha), D_E E_\alpha) \\
&= \sum_{\alpha=1}^{m} g((D_{E_\alpha} T)(E), E_\alpha) + 2\sum_{\alpha=1}^{m} g(T(E_\alpha), D_E E_\alpha) \\
&= \sum_{\alpha=1}^{m} g(E, (D_{E_\alpha} T)(E_\alpha)) + 2\sum_{\alpha=1}^{m} g(T(E_\alpha), D_E E_\alpha).
\end{aligned} \tag{17}$$

Note that

$$D_E E_\alpha = \sum_k \wedge_\alpha^k(E)E_k, \quad T(E_\alpha) = \sum_j \mu_\alpha^j E_j,$$

where the connection forms $\wedge_\alpha^k$ are skew symmetric and coefficients μ_α^j are symmetric and, as such, we have

$$\sum_{\alpha=1}^{m} g(T(E_\alpha), D_E E_\alpha) = 0.$$

Consequently, Equation (17) yields

$$\nabla\tau = \sum_{\alpha=1}^{m} (D_{E_\alpha} T)(E_\alpha).$$

Combining it with Equation (11), we obtain $\nabla\tau = 0$, i.e., the scalar curvature τ of a *T-manifold* is a constant.

Employing Equation (9), we have

$$\left(\nabla^2\mathbf{u}\right)(E,F) = E(\sigma)T(F) + \sigma(D_E T)(F) + (D_E\Psi)(F)$$

and taking the trace in the above equation, while using Equation (11) with $\nabla\tau = 0$, we obtain

$$\Delta\mathbf{u} = T(\nabla\sigma) + \sum_{\alpha=1}^{m} (D_{E_\alpha}\Psi)(E_\alpha).$$

□

Next, the sphere S_α^m of constant curvature α possesses a σ-*RVF* induced by a coordinate unit vector field $\frac{\partial}{\partial u}$ on the Euclidean space R^{m+1}. Indeed, on treating S_α^m as an embedded surface in R^{m+1} with unit normal ζ and Weingarten operator$-\sqrt{\alpha}I$, we express $\frac{\partial}{\partial u}$ as

$$\frac{\partial}{\partial u} = \mathbf{u} + f\zeta, f = \left\langle \frac{\partial}{\partial u}, \zeta \right\rangle, \tag{18}$$

where $\langle , \rangle$ is a Euclidean inner product and $\mathbf{u} \in \Gamma(S_\alpha^m)$. On taking g as the induced metric on S_α^m and D as the Riemannian connection with respect to g and differentiating the above equation with respect to the vector field $E \in \Gamma(S_\alpha^m)$, we have

$$D_E\mathbf{u} = -\sqrt{\alpha} f E, \quad \nabla f = \sqrt{\alpha}\mathbf{u}. \tag{19}$$

Using the first equation in (19), it follows that

$$\pounds_{\mathbf{u}} g = -2\sqrt{\alpha} f g$$

and for the Ricci tensor of S_α^m, we have

$$Ric = (m-1)\alpha g, \quad \tau = m(m-1)\alpha. \tag{20}$$

Hence, the vector field $\mathbf{u}$ on S_α^m obeys

$$\frac{1}{2}\pounds_{\mathbf{u}} g = \sigma Ric, \quad \sigma = -\frac{1}{(m-1)\sqrt{\alpha}} f, \tag{21}$$

i.e., $\mathbf{u}$ is a σ-*RVF* on S_α^m.

Moreover, note that Equation (21) in view of Equation (19) confirms

$$\begin{aligned} Hess(\sigma)(E,F) &= g(D_E\nabla\sigma, F) \\ &= -\frac{1}{(m-1)\sqrt{\alpha}} g(D_E\nabla f, F) \\ &= -\frac{1}{m-1} g(D_E\mathbf{u}, F) \\ &= \frac{\sqrt{\alpha} f}{m=1} g(E,F), \end{aligned}$$

i.e.,

$$Hess(\sigma) = -\alpha\sigma g, \quad \Delta\sigma = -m\alpha\sigma. \tag{22}$$

Combining Equations (20) and (22), we see that the function σ of the σ-*RVF* $\mathbf{u}$ on S_α^m satisfies the static fluid equation

$$\sigma Ric - Hess(\sigma) = \frac{1}{m}(\tau\sigma - \Delta\sigma) g. \tag{23}$$

We investigate now whether σ is a nontrivial solution. If σ was a constant, by virtue of Equation (21), it would mean that f was a constant, and, in turn, by (19), it would mean that $\mathbf{u} = 0$ and, by the same equation, would imply $f = 0$. Inserting this information in (18), we have $\frac{\partial}{\partial u} = 0$, a contradiction. Hence, σ is a nontrivial solution of the static fluid equation on S_α^m.

3. σ-Ricci Vector Fields on T-Manifolds

In this section, we consider an m-dimensional T-*manifold* (N^m, g) that possesses a closed σ-*RVF* $\mathbf{u}$. It is interesting to observe that, in this situation, the vector field $\mathbf{u} - \nabla\sigma$ plays an interesting role while treating the Ricci operator T of (N^m, g). Note that, by

Lemma 2, the scalar curvature τ of a *T-manifold* (N^m, g) is a constant and we put $\tau = m(m-1)\alpha$, for a constant α. Here, we prove the following result.

Theorem 1. *An m-dimensional, $m > 1$, complete, and simply connected T-manifold (N^m, g) with positive scalar curvature τ admits a nonzero closed σ-RVF $\mathbf{u}$, $\sigma \neq 0$ satisfying*

$$T(\mathbf{u} - \nabla\sigma) = \frac{\tau}{m}(\mathbf{u} - \nabla\sigma),$$

if and only if (N^m, g) is isometric to S^m_α, where $\tau = m(m-1)\alpha$.

Proof. Suppose that the complete and simply connected *T-manifold* (N^m, g), $m > 1$, of scalar curvature $\tau > 0$, admits a nonzero closed σ*-RVF* $\mathbf{u}$, $\sigma \neq 0$, which satisfies

$$T(\mathbf{u} - \nabla\sigma) = \frac{\tau}{m}(\mathbf{u} - \nabla\sigma). \tag{24}$$

As the σ*-RVF* $\mathbf{u}$ is closed, its associated operator $\Psi = 0$, and by Lemma 2, the scalar curvature τ is a constant, and Equation (14) becomes

$$T(\mathbf{u}) = T(\nabla\sigma) - \tau\nabla\sigma. \tag{25}$$

Treating it with Equation (24) yields

$$\frac{\tau}{m}(\mathbf{u} - \nabla\sigma) = -\tau\nabla\sigma$$

and, as $\tau > 0$, it transforms into

$$\mathbf{u} = -(m-1)\nabla\sigma. \tag{26}$$

Note that, by Equation (9), we have $\operatorname{div}\mathbf{u} = \sigma\tau$, and taking the divergence in Equation (26) gives

$$\sigma\tau = -(m-1)\Delta\sigma. \tag{27}$$

Now, inserting the value of $\nabla\sigma$ from Equation (26) into Equation (25), we arrive at

$$T(\mathbf{u}) = -\frac{m-1}{m}\tau\nabla\sigma. \tag{28}$$

Note that as $\mathbf{u}$ is closed, Equation (9) has the form

$$D_E\mathbf{u} = \sigma T(E), \quad E \in \Gamma(N^m). \tag{29}$$

Next, we intend to compute the divergence $\operatorname{div}(T\mathbf{u})$ and we proceed by choosing a local frame $\{E_1, \ldots, E_m\}$ and using Equation (29)

$$\begin{aligned}
\operatorname{div}(T\mathbf{u}) &= \sum_{\alpha=1}^{m} g(\nabla_{E_\alpha} T\mathbf{u}, E_\alpha) \\
&= \sum_{\alpha=1}^{m} g((\nabla_{E_\alpha} T)(\mathbf{u}) + T(\nabla_{E_\alpha}\mathbf{u}), E_\alpha) \\
&= \sum_{\alpha=1}^{m} g(\mathbf{u}, (\nabla_{E_\alpha} T)(E_\alpha)) + \sum_{\alpha=1}^{m} g(\nabla_{E_\alpha}\mathbf{u}, T(E_\alpha)).
\end{aligned}$$

Note that on *T-manifold* (N^m, g), by Lemma 2, τ is a constant and, thus, employing Equations (12) and (29), we arrive at

$$\operatorname{div}(T\mathbf{u}) = \sigma\|T\|^2.$$

Now, utilizing this equation in Equation (28) yields

$$\sigma\|T\|^2 = -\frac{m-1}{m}\tau\Delta\sigma. \tag{30}$$

Inserting Equation (27) in the above equation gives

$$\sigma\|T\|^2 = \frac{1}{m}\sigma\tau^2,$$

i.e.,

$$\sigma\left(\|T\|^2 - \frac{1}{m}\tau^2\right) = 0.$$

As N^k is connected (being simply connected) and $\sigma \neq 0$, in this situation, the above equation yields

$$\|T\|^2 = \frac{1}{m}\tau^2. \tag{31}$$

However, Equation (31) is the equality in Schwartz's inequality

$$\|T\|^2 \geq \frac{1}{m}\tau^2.$$

Hence, equality (31) holds if and only if

$$T = \frac{\tau}{m}I$$

and Equation (29) changes to

$$D_E\mathbf{u} = \frac{\tau}{m}\rho E, \quad E \in \Gamma(N^m).$$

Thus, on employing Equation (26) in the above equation, we confirm

$$D_E\nabla\sigma = -\frac{\tau}{m(m-1)}\sigma E, \quad E \in \Gamma(N^m). \tag{32}$$

Note that as $\mathbf{u} \neq 0$ by Equation (26), the function σ is a non-constant function and, also, τ being a positive constant, letting $\tau = m(m-1)\alpha$, we obtain a positive constant α and Equation (32) is Obata's equation

$$Hess(\sigma) = -\alpha\rho g,$$

proving that (N^m, g) is isometric to the sphere S^m_α (cf. [18,21]).

Conversely, suppose that (N^m, g) is isometric to the sphere S^m_α. Then, by Equations (19)–(21), there is a nonzero *σ-RVF* $\mathbf{u}$ on S^m_α and, as seen earlier, the function $\sigma \neq 0$ and is a non-constant function. Moreover, the Ricci operator of S^m_α being

$$T = \frac{\tau}{m}I,$$

the condition

$$T(\mathbf{u} - \nabla\sigma) = \frac{\tau}{m}(\mathbf{u} - \nabla\sigma)$$

holds, and this finishes the proof. □

In an earlier result, we considered a closed *σ-RVF* $\mathbf{u}$ on an m-dimensional *T-manifold* (N^m, g) to find a characterization of the sphere S^m_α. Next, we consider a *σ-RVF* $\mathbf{u}$ on an m-dimensional *T-manifold* (N^m, g) not necessarily closed and prove the following.

Theorem 2. *An m-dimensional compact and connected T-manifold (N^m, g), $m > 2$ of positive scalar curvature τ admits a σ-RVF $\mathbf{u}$ with associated operator Ψ, $\sigma \neq 0$, $\Delta\mathbf{u} = -\frac{\tau}{m(m-1)}\mathbf{u}$ and the Ricci curvature $Ric(\mathbf{u}, \mathbf{u})$ satisfies*

$$\int\limits_{N^m} Ric(\mathbf{u}, \mathbf{u}) \geq \int\limits_{N^m} \left[\frac{m-1}{m}\tau^2\sigma^2 + \|\Psi\|^2\right]$$

if and only if (N^m, g) is isometric to S^m_α, where $\tau = m(m-1)\alpha$.

Proof. Let an m-dimensional *T-manifold* (N^m, g), $m > 2$, with scalar curvature $\tau > 0$ be equipped with a σ-*RVF* $\mathbf{u}$ with $\sigma \neq 0$ and associated operator Ψ such that

$$\Delta\mathbf{u} = -\frac{\tau}{m(m-1)}\mathbf{u} \tag{33}$$

and

$$\int_{N^m} Ric(\mathbf{u}, \mathbf{u}) \geq \int_{N^m} \left[\frac{m-1}{m}\tau^2\sigma^2 + \|\Psi\|^2\right]. \tag{34}$$

Using Lemma 1, we have

$$R(E, \mathbf{u})F = (D_E\Psi)(F) + Ric(E, F)\nabla\sigma - F(\sigma)T(E), \quad E, F \in \Gamma(N^m)$$

Employing a local frame $\{E_1, \ldots, E_m\}$ in the above equation, we conclude

$$Ric(\mathbf{u}, F) = Ric(\nabla\sigma, F) - \tau F(\sigma) - \sum_{\alpha=1}^{m} g(F, (D_{E_\alpha}\Psi)(E_\alpha)), \quad F \in \Gamma(N^m)$$

and the above equation implies

$$Ric(\mathbf{u}, \mathbf{u}) = Ric(\nabla\sigma, \mathbf{u}) - \tau\mathbf{u}(\sigma) - \sum_{\alpha=1}^{m} g(\mathbf{u}, (D_{E_\alpha}\Psi)(E_\alpha)).$$

Note that, by Equation (9), we have

$$\mathrm{div}\mathbf{u} = \tau\sigma$$

and using

$$\mathrm{div}(\sigma\mathbf{u}) = \mathbf{u}(\sigma) + \tau\sigma^2,$$

in the above equation containing the expression of $Ric(\mathbf{u}, \mathbf{u})$, we derive

$$Ric(\mathbf{u}, \mathbf{u}) = Ric(\nabla\sigma, \mathbf{u}) + \tau^2\sigma^2 - \tau\mathrm{div}(\sigma\mathbf{u}) - \sum_{\alpha=1}^{m} g(\mathbf{u}, (D_{E_\alpha}\Psi)(E_\alpha)). \tag{35}$$

Next, using a local frame $\{E_1, \ldots, E_m\}$ on (N^m, g), to compute the $\mathrm{div}(\Psi\mathbf{u})$, we have, on using the skew symmetry of the associated operator Ψ and Equation (9),

$$\begin{aligned}
\mathrm{div}(\Psi\mathbf{u}) &= \sum_{\alpha=1}^{m} g(D_{E_\alpha}\Psi\mathbf{u}, E_\alpha) \\
&= \sum_{\alpha=1}^{m} g((D_{E_\alpha}\Psi)(\mathbf{u}) + \Psi(\sigma T E_\alpha + \Psi E_\alpha), E_\alpha) \\
&= -\sum_{\alpha=1}^{m} g(\mathbf{u}, (D_{E_\alpha}\Psi)(E_\alpha)) - \sigma\sum_{\alpha=1}^{m} g(TE_\alpha, \Psi E_\alpha) - \|\Psi\|^2
\end{aligned} \tag{36}$$

Since T is symmetric and the associated operator Ψ is skew symmetric, it follows that

$$\sum_{\alpha=1}^{m} g(TE_\alpha, \Psi E_\alpha) = 0 \tag{37}$$

and Equation (36) now becomes

$$\mathrm{div}(\Psi\mathbf{u}) = -\sum_{\alpha=1}^{m} g(\mathbf{u}, (D_{E_\alpha}\Psi)(E_\alpha)) - \|\Psi\|^2$$

and, inserting this equation into Equation (35), we arrive at

$$Ric(\mathbf{u},\mathbf{u}) = Ric(\nabla\sigma,\mathbf{u}) + \tau^2\sigma^2 - \tau\mathrm{div}(\sigma\mathbf{u}) + \|\Psi\|^2 + \mathrm{div}(\Psi\mathbf{u}).$$

Note that on a *T-manifold* (N^m, g), τ is a constant and keeping this in mind and integrating the above equation brings us to

$$\int_{N^m} \left(Ric(\mathbf{u},\mathbf{u}) - Ric(\nabla\sigma,\mathbf{u}) - \tau^2\sigma^2 - \|\Psi\|^2 \right) = 0. \tag{38}$$

Observe that, by virtue of the symmetry of the operator T and Equations (9), (12) and (37), and the fact that τ is a constant, we have

$$\begin{aligned}
\mathrm{div}(T\mathbf{u}) &= \sum_{\alpha=1}^{m} g(D_{E_\alpha} T\mathbf{u}, E_\alpha) \\
&= \sum_{\alpha=1}^{m} g((D_{E_\alpha}T)(\mathbf{u}) + T(\sigma TE_\alpha + \Psi E_\alpha), E_\alpha) \\
&= \sigma\|T\|^2.
\end{aligned} \tag{39}$$

Now, using the fact that

$$\mathrm{div}(\sigma T\mathbf{u}) = Ric(\nabla\sigma,\mathbf{u}) + \sigma\mathrm{div}(T\mathbf{u})$$

in Equation (39), we arrive at

$$Ric(\nabla\sigma,\mathbf{u}) = \mathrm{div}(\sigma T\mathbf{u}) - \sigma^2\|T\|^2.$$

Inserting the above equation in Equation (38), we confirm

$$\int_{N^m} \left(Ric(\mathbf{u},\mathbf{u}) + \sigma^2\|T\|^2 - \tau^2\sigma^2 - \|\Psi\|^2 \right) = 0$$

and the above integral could be rearranged as

$$\int_{N^m} \sigma^2 \left(\|T\|^2 - \frac{1}{m}\tau^2 \right) = \int_{N^m} \left(\frac{m-1}{m}\tau^2\sigma^2 + \|\Psi\|^2 \right) - \int_{N^m} Ric(\mathbf{u},\mathbf{u}). \tag{40}$$

Treating the above equation with the inequality (34), we arrive at

$$\int_{N^m} \sigma^2 \left(\|T\|^2 - \frac{1}{m}\tau^2 \right) \leq 0.$$

The integrand in the above inequality by virtue of Schwartz's inequality is non-negative, and, therefore, we conclude

$$\sigma^2 \left(\|T\|^2 - \frac{1}{m}\tau^2 \right) = 0.$$

As $\sigma \neq 0$ and N^m is connected, we conclude that

$$\|T\|^2 = \frac{1}{m}\tau^2,$$

which, being the equality in Schwartz's inequality, it holds if and only if

$$T = \frac{\tau}{m} I. \tag{41}$$

Consequently, as τ is a constant, Equations (14) and (41) combine to arrive at

$$\frac{\tau}{m}\mathbf{u} = \frac{\tau}{m}(\nabla\sigma) - \tau\nabla\sigma - \sum_{\alpha=1}^{m}(\nabla_{F_\alpha}\Psi)(F_\alpha),$$

for a local frame $\{E_1, \dots, E_m\}$ on (N^m, g), i.e., we have

$$\frac{\tau}{m}\mathbf{u} = -\frac{m-1}{m}\tau\nabla\sigma - \sum_{\alpha=1}^{m}(\nabla_{F_\alpha}\Psi)(F_\alpha). \tag{42}$$

Moreover, using Equations (33) and (41) with Lemma 2, we obtain the following:

$$-\frac{\tau}{m(m-1)}\mathbf{u} = \frac{\tau}{m}(\nabla\sigma) + \sum_{\alpha=1}^{m}(\nabla_{F_\alpha}\Psi)(F_\alpha). \tag{43}$$

Adding Equations (42) and (43), we find

$$\frac{m-2}{m(m-1)}\tau\mathbf{u} = -\frac{m-2}{m}\tau\nabla\sigma$$

and, as $m > 2$, $\tau > 0$, it confirms

$$\mathbf{u} = -(m-1)\nabla\sigma.$$

Differentiating the above equation and using Equations (9) and (41), we have

$$D_E\nabla\sigma = -\frac{1}{m-1}\left(\frac{\tau}{m}\sigma E + \Psi(E)\right), \quad E \in \Gamma(N^m),$$

which, on taking the inner product with E and noticing that Ψ is a skew symmetric operator, leads to

$$Hess(\sigma)(E,E) = -\alpha\sigma g(E,E), \quad E \in \Gamma(N^m),$$

where $\tau = m(m-1)\alpha$, i.e., α is a positive constant. Now, polarizing the above equation confirms

$$Hess(\sigma) = -\alpha\sigma g.$$

Hence, (N^m, g) is isometric to S^m_α (cf. [18,21]).

Conversely, suppose that (N^m, g) is isometric to S^m_α. Then, by Equation (21), there is a nonzero σ-*RVF* $\mathbf{u}$ on S^m_α with $\sigma \neq 0$ and, as $\mathbf{u}$ is is closed, the associated operator $\Psi = 0$. Moreover, it is obvious that S^m_α is a *T-manifold*. Thus, using Equation (19), we have

$$\begin{aligned} \left(\nabla^2\mathbf{u}\right)(E,F) &= D_E D_F\mathbf{u} - D_{D_E F}\mathbf{u} \\ &= -\sqrt{\alpha}E(f)F \end{aligned}$$

and, therefore, by treating the above equation with (16), we have

$$\begin{aligned} \Delta \mathbf{u} &= \sum_{\alpha=1}^{m} \left(\nabla^2 \mathbf{u}\right)(E_\alpha, E_\alpha) \\ &= -\sqrt{\alpha}\nabla f, \end{aligned}$$

which, by virtue of Equation (19), implies

$$\begin{aligned} \Delta \mathbf{u} &= -\alpha \mathbf{u} \\ &= -\frac{\tau}{m(m-1)}\mathbf{u}, \end{aligned}$$

where $\tau = m(m-1)\alpha$. Finally, using Equations (19) and (21), we have

$$\begin{aligned} \nabla \sigma &= -\frac{1}{(m-1)\sqrt{\alpha}}\nabla f \\ &= -\frac{1}{m-1}\mathbf{u}, \end{aligned}$$

i.e.,

$$\|\mathbf{u}\|^2 = (m-1)^2\|\nabla\sigma\|^2. \tag{44}$$

Now, Equation (22) implies

$$\sigma\Delta\sigma = -m\alpha\sigma^2,$$

which, on integrating by parts, confirms

$$\begin{aligned} \int_{S_\alpha^m} \|\nabla\sigma\|^2 &= m\alpha \int_{S_\alpha^m} \sigma^2 \\ &= \frac{1}{m(m-1)^2\alpha}\int_{S_\alpha^m} \tau^2\sigma^2. \end{aligned} \tag{45}$$

The Ricci curvature of S_α^m is

$$Ric(\mathbf{u},\mathbf{u}) = (m-1)\alpha\|\mathbf{u}\|^2$$

and, thus, using $\Psi = 0$ and Equations (44) and (45), we conclude

$$\begin{aligned} \int_{S_\alpha^m} Ric(\mathbf{u},\mathbf{u}) &= \int_{S_\alpha^m} (m-1)\alpha\|\mathbf{u}\|^2 \\ &= \int_{S_\alpha^m} (m-1)^3\alpha\|\nabla\sigma\|^2 \\ &= \int_{S_\alpha^m} \left[\frac{m-1}{m}\tau^2\sigma^2 + \|\Psi\|^2\right] \end{aligned} \tag{46}$$

and this completes the proof. □

4. σ-Ricci Vector Fields on Static Spaces

Now, we are interested in a σ-*RVF* $\mathbf{u}$, not necessarily closed, on a Riemannian manifold (N^m, g) with function σ as a nontrivial solution of the static fluid Equation (5). Indeed, we prove the following.

Theorem 3. *If an m-dimensional compact and connected Riemannian manifold (N^m, g) admits a σ-RVF $\mathbf{u}$ with associated operator Ψ, such that σ is a nontrivial solution of the static perfect fluid equation, for a positive constant α and the Ricci curvature $Ric(\mathbf{u},\mathbf{u})$, it satisfies*

$$\int_{N^m} Ric(\mathbf{u},\mathbf{u}) \geq \int_{N^m} \left[\frac{m-1}{m}\tau^2\sigma^2 + \frac{1}{m}(\Delta\sigma + m\alpha\sigma)^2 + \|\Psi\|^2\right],$$

and (N^m, g) is isometric to S^m_α, and the converse also holds.

Proof. Assume that (N^m, g) admits a σ-*RVF* $\mathbf{u}$ with associated operator Ψ, such that σ is a nontrivial solution of the static perfect fluid Equation (5) and the Ricci curvature $Ric(\mathbf{u}, \mathbf{u})$ satisfies

$$\int_{N^m} Ric(\mathbf{u}, \mathbf{u}) \geq \int_{N^m} \left[\frac{m-1}{m} \tau^2 \sigma^2 + \frac{1}{m} (\Delta\sigma + m\alpha\sigma)^2 + \|\Psi\|^2 \right]. \tag{47}$$

Then, the Hessian operator H_σ of the function σ defined by

$$g(H_\sigma(E), F) = Hess(\sigma)(E, F),$$

by virtue of Equation (5) satisfies

$$H_\sigma(E) = \sigma T(E) + \frac{1}{m}(\Delta\sigma - \tau\sigma)E, \quad E \in \Gamma(N^m)$$

Utilizing Equation (9) in the above equation, we arrive at

$$H_\sigma(E) = D_E \mathbf{u} - \Psi(E) + \frac{1}{m}(\Delta\sigma - \tau\sigma)E$$

and, for a positive constant α, the above equation could be rearranged as

$$(H_\sigma + \alpha\sigma I)(E) = D_E \mathbf{u} - \Psi(E) + \frac{1}{m}(\Delta\sigma + m\alpha\sigma - \tau\sigma)E, \quad E \in \Gamma(N^m).$$

Choosing a local frame $\{E_1, \ldots, E_m\}$, and using the above equation, we compute

$$\begin{aligned}
\|H_\sigma + \alpha\sigma I\|^2 &= \sum_{j=1}^m g\big((H_\sigma + \alpha\sigma I)(E_j), (H_\sigma + \alpha\sigma I)(E_j)\big) \\
&= \sum_{j=1}^m g\Big(D_{E_j}\mathbf{u} - \Psi(E_j) + \frac{1}{m}(\Delta\sigma + m\alpha\sigma - \tau\sigma)E_j, \\
&\qquad D_{E_j}\mathbf{u} - \Psi(E_j) + \frac{1}{m}(\Delta\sigma + m\alpha\sigma - \tau\sigma)E_j\Big) \\
&= \|D\mathbf{u}\|^2 + \|\Psi\|^2 + \frac{1}{m}(\Delta\sigma + m\alpha\sigma - \tau\sigma)^2 \\
&\quad -2\sum_{j=1}^m g\Big(D_{E_j}\mathbf{u}, \Psi(E_j)\Big) \\
&\quad + \frac{2}{m}(\Delta\sigma + m\alpha\sigma - \tau\sigma)\mathrm{div}(\mathbf{u}).
\end{aligned}$$

Now, using Equation (9) and $\mathrm{div}(\mathbf{u}) = \tau\sigma$ in the above equation, we arrive at

$$\begin{aligned}
\|H_\sigma + \alpha\sigma I\|^2 &= \|D\mathbf{u}\|^2 - \|\Psi\|^2 + \frac{1}{m}(\Delta\sigma + m\alpha\sigma - \tau\sigma)^2 \\
&\quad + \frac{2\tau\sigma}{m}(\Delta\sigma + m\alpha\sigma - \tau\sigma),
\end{aligned}$$

i.e.,

$$\|H_\sigma + \alpha\sigma I\|^2 = \|D\mathbf{u}\|^2 - \|\Psi\|^2 + \frac{1}{m}(\Delta\sigma + m\alpha\sigma - \tau\sigma)(\Delta\sigma + m\alpha\sigma + \tau\sigma)$$

or

$$\|H_\sigma + \alpha\sigma I\|^2 = \|D\mathbf{u}\|^2 - \|\Psi\|^2 + \frac{1}{m}(\Delta\sigma + m\alpha\sigma)^2 - \frac{1}{m}(\tau\sigma)^2. \tag{48}$$

We recall the integral formula (cf. [23])

$$\int_{N^m} \|D\mathbf{u}\|^2 = \int_{N^m} \left(Ric(\mathbf{u}, \mathbf{u}) + \frac{1}{2} |\pounds_{\mathbf{u}} g|^2 - (\mathrm{div}\mathbf{u})^2 \right).$$

Using $\mathrm{div}(\mathbf{u}) = \tau\sigma$ and the outcome of Equation (1) in the form

$$\frac{1}{2} |\pounds_{\mathbf{u}} g|^2 = 2\sigma^2 \|T\|^2$$

in the above integral equation, we have

$$\int_{N^m} \|D\mathbf{u}\|^2 = \int_{N^m} \left(Ric(\mathbf{u}, \mathbf{u}) + 2\sigma^2 \|T\|^2 - (\tau\sigma)^2 \right).$$

Now, integrating Equation (48) and using the above equation, we arrive at

$$\begin{aligned} \int_{N^m} \|H_\sigma + \alpha\sigma I\|^2 = \int_{N^m} \Bigg(& Ric(\mathbf{u}, \mathbf{u}) + 2\sigma^2 \|T\|^2 - \frac{m+1}{m} (\tau\sigma)^2 \\ & - \|\Psi\|^2 + \frac{1}{m} (\Delta\sigma + m\alpha\sigma)^2 \Bigg) \end{aligned} \tag{49}$$

Notice that

$$2\sigma^2 \|T\|^2 - \frac{m+1}{m} (\tau\sigma)^2 = 2\sigma^2 \left(\|T\|^2 - \frac{1}{m} \tau^2 \right) - \frac{m-1}{m} (\tau\sigma)^2 \tag{50}$$

and, by Equation (5), we have that

$$\sigma T(E) - \frac{1}{m} \tau\sigma E = H_\sigma(E) - \frac{1}{m} (\Delta\sigma) E,$$

which implies

$$\sigma^2 \left\| T - \frac{\tau}{m} I \right\|^2 = \left\| H_\sigma - \frac{1}{m} (\Delta\sigma) I \right\|^2.$$

Combining it with Equation (50), we arrive at

$$2\sigma^2 \|T\|^2 - \frac{m+1}{m} (\tau\sigma)^2 = 2 \left\| H_\sigma - \frac{1}{m} (\Delta\sigma) I \right\|^2 - \frac{m-1}{m} (\tau\sigma)^2. \tag{51}$$

Moreover, we have

$$\begin{aligned} \left\| H_\sigma - \frac{1}{m} (\Delta\sigma) I \right\|^2 &= \|H_\sigma\|^2 + \frac{1}{m} (\Delta\sigma)^2 - \frac{2}{m} (\Delta\sigma) \sum_{j=1}^{m} g\left(H_\sigma\left(E_j\right), E_j\right) \\ &= \|H_\sigma\|^2 - \frac{1}{m} (\Delta\sigma)^2 \end{aligned} \tag{52}$$

Similarly, we have

$$\|H_\sigma + \alpha\sigma I\|^2 = \|H_\sigma\|^2 + 2\alpha\sigma\Delta\sigma + m\alpha^2\sigma^2$$

and utilizing it in Equation (52), we obtain

$$\left\| H_\sigma - \frac{1}{m} (\Delta\sigma) I \right\|^2 = \|H_\sigma + \alpha\sigma I\|^2 - 2\alpha\sigma\Delta\sigma - m\alpha^2\sigma^2 - \frac{1}{m} (\Delta\sigma)^2,$$

i.e.,

$$\left\| H_\sigma - \frac{1}{m}(\Delta\sigma)I \right\|^2 = \|H_\sigma + \alpha\sigma I\|^2 - \frac{1}{m}(\Delta\sigma + m\alpha\sigma)^2.$$

Thus, in view of the above equation, (51) assumes the form

$$2\sigma^2\|T\|^2 - \frac{m+1}{m}(\tau\sigma)^2 = 2\|H_\sigma + \alpha\sigma I\|^2 - \frac{2}{m}(\Delta\sigma + m\alpha\sigma)^2 - \frac{m-1}{m}(\tau\sigma)^2.$$

Now, inserting this value in Equation (49), we arrive at

$$\int_{N^m} \|H_\sigma + \alpha\sigma I\|^2 = \int_{N^m} \Big(Ric(\mathbf{u}, \mathbf{u}) + 2\|H_\sigma + \alpha\sigma I\|^2 - \frac{m-1}{m}(\tau\sigma)^2 \\ - \|\Psi\|^2 - \frac{1}{m}(\Delta\sigma + m\alpha\sigma)^2 \Big),$$

i.e.,

$$\int_{N^m} \|H_\sigma + \alpha\sigma I\|^2 = \int_{N^m} \left(\frac{m-1}{m}(\tau\sigma)^2 + \frac{1}{m}(\Delta\sigma + m\alpha\sigma)^2 + \|\Psi\|^2 \right) - \int_{N^m} Ric(\mathbf{u}, \mathbf{u}).$$

Using inequality (47) in the above equation, we conclude

$$\int_{N^m} \|H_\sigma + \alpha\sigma I\|^2 \leq 0,$$

which proves

$$Hess(\sigma) = -\alpha\sigma g,$$

where $\alpha > 0$ is a constant and σ, being a nontrivial solution of a static perfect fluid, is a non-constant function. Hence, (N^m, g) is isometric to S^m_α (cf. [18,21]).

The converse is trivial, because, by Equation (21), S^m_α admits a σ-*RVF* $\mathbf{u}$, and by Equation (23) and the paragraph that follows (23), σ is a nontrivial solution of the static perfect fluid equation. Moreover, we have, by Equation (22), that

$$\Delta\sigma + m\alpha\sigma = 0$$

and, by Equation (46), we have

$$\int_{N^m} Ric(\mathbf{u}, \mathbf{u}) = \int_{N^m} \left[\frac{m-1}{m}\tau^2\sigma^2 + \frac{1}{m}(\Delta\sigma + m\alpha\sigma)^2 + \|\Psi\|^2 \right]$$

This finishes the proof. □

Author Contributions: Conceptualization, H.A., S.D. and G.-E.V.; methodology, H.A., S.D. and G.-E.V.; software, H.A., S.D. and G.-E.V.; validation, H.A., S.D. and G.-E.V.; formal analysis, H.A., S.D. and G.-E.V.; investigation, H.A., S.D. and G.-E.V.; resources, H.A., S.D. and G.-E.V.; data curation, H.A., S.D. and G.-E.V.; writing—original draft preparation, H.A., S.D. and G.-E.V.; writing—review and editing, H.A., S.D. and G.-E.V.; visualization, H.A., S.D. and G.-E.V.; supervision, H.A., S.D. and G.-E.V.; project administration, H.A., S.D. and G.-E.V.; funding acquisition, H.A. All authors have read and agreed to the published version of the manuscript.

Funding: The authors would like to extend their sincere appreciation to Supporting Project Number (RSPD2023R860), King Saud University, Riyadh, Saudi Arabia.

Data Availability Statement: Data are contained within the article.

Acknowledgments: The authors would like to extend their sincere appreciations to Supporting project number (RSPD2024R860) King Saud University, Riyadh, Saudi Arabia.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Alohali, H.; Deshmukh, S. Ricci vector fields. *Mathematics* **2023**, *11*, 4622. [CrossRef]
2. Chen, B.-Y. Some results on concircular vector fields and their applications to Ricci solitons. *Bull. Korean Math. Soc.* **2015**, *52*, 1535–1547. [CrossRef]
3. Deshmukh, S.; Turki, N.B.; Vîlcu, G.-E. A note on static spaces. *Results Phys.* **2021**, *27*, 104519. [CrossRef]
4. Falcitelli, M.; Sarkar, A.; Halder, S. Conformal vector fields and conformal Ricci solitons on α-Kenmotsu manifolds. *Mediterr. J. Math.* **2023**, *20*, 127. [CrossRef]
5. Hwang, S.; Yun, G. Conformal vector fields and their applications to Einstein-type manifolds. *Results Math.* **2024**, *79*, 45. [CrossRef]
6. Khan, S.; Bukhari, M.; Alkhaldi, A.; Ali, A. Conformal vector fields of Bianchi type-I spacetimes. *Mod. Phys. Lett. A* **2021**, *36*, 2150254. [CrossRef]
7. Narmanov, A.; Rajabov, E. On the geometry of orbits of conformal vector fields. *J. Geom. Symmetry Phys.* **2019**, *51*, 29–39. [CrossRef]
8. Poddar, R.; Balasubramanian, S.; Sharma, R. Quasi-Einstein manifolds admitting conformal vector fields. *Colloq. Math.* **2023**, *174*, 81–87. [CrossRef]
9. Sharma, R. Gradient Ricci solitons with a conformal vector field. *J. Geom.* **2018**, *109*, 33. [CrossRef]
10. Sharma, R. Conformal flatness and conformal vector fields on umbilically synchronized space-times. *Acta Phys. Pol. B* **2023**, *54*, A3. [CrossRef]
11. Filho, J.F.S. Quasi-Einstein manifolds admitting a closed conformal vector field. *Differ. Geom. Appl.* **2024**, *92*, 102083. [CrossRef]
12. Nikonorov, Y. Spectral properties of Killing vector fields of constant length. *J. Geom. Phys.* **2019**, *145*, 103485. [CrossRef]
13. Lynge, W.C. Sufficient conditions for periodicity of a Killing vector field. *Proc. Am. Math. Soc.* **1973**, *38*, 614–616. [CrossRef]
14. Rong, X. Positive curvature, local and global symmetry, and fundamental groups. *Am. J. Math.* **1999**, *121*, 931–943. [CrossRef]
15. Yorozu, S. Killing vector fields on complete Riemannian manifolds. *Proc. Am. Math. Soc.* **1982**, *84*, 115–120. [CrossRef]
16. Blaga, A.M.; Chen, B.-Y. Harmonic forms and generalized solitons. *Results Math.* **2024**, *79*, 16. [CrossRef]
17. Lee, K.-H. Stability and moduli space of generalized Ricci solitons. *Nonlinear Anal.* **2024**, *240*, 113458. [CrossRef]
18. Obata, M. The conjectures about conformal transformations. *J. Differ. Geom.* **1971**, *6*, 247–258. [CrossRef]
19. Chow, B.; Lu, P.; Ni, L. *Hamilton's Ricci Flow*; Graduate Studies in Mathematics; American Mathematical Society: Providence, RI, USA, 2006; Volume 77.
20. Fischer, A.E.; Marsden, J.E. Manifolds of Riemannian metrics with prescribed scalar curvature. *Bull. Am. Math. Soc.* **1974**, *80*, 479–484. [CrossRef]
21. Obata, M. Conformal transformations of Riemannian manifolds. *J. Differ. Geom.* **1970**, *4*, 311–333. [CrossRef]
22. Besse, A.L. *Einstein Manifolds*; Springer: Berlin/Heidelberg, Germany, 1987.
23. Yano, K. *Integral Formulas in Riemannian Geometry*; Marcel Dekker Inc.: New York, NY, USA, 1970.