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Generalized Bayes Estimation Based on a Joint Type-II Censored Sample from K-Exponential Populations

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Abstract: Generalized Bayes is a Bayesian study based on a learning rate parameter. This paper considers a generalized Bayes estimation to study the effect of the learning rate parameter on the estimation results based on a joint censored sample of type-II exponential populations. Squared error, Linex, and general entropy loss functions are used in the Bayesian approach. Monte Carlo simulations were performed to assess how well the different approaches perform. The simulation study compares the Bayesian estimators for different values of the learning rate parameter and different losses.

Keywords: generalized bayes; learning rate parameter; exponential distribution; joint type-II censoring; squared-error loss; Linex loss; general entropy loss

MSC: 62F10; 62F15

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1. Introduction

Generalized Bayes is a Bayesian study based on a learning rate parameter $(0 < \eta < 1)$. As a fractional power on the likelihood function $L \equiv L(\theta; data)$ for the parameter $\theta \in \Theta$, the traditional Bayesian framework is obtained for $\eta = 1$. In this paper, we will show the effect of the learning rate parameter on the estimation results. That is, if the prior distribution of the parameter θ is $\pi(\theta)$, then the generalized Bayes posterior distribution for θ is:

$$\pi^*(\theta|data) \propto L^{\eta}\pi(\theta), \theta \in \Theta, 0 < \eta < 1. \tag{1}$$

For more details on the generalized Bayes method and the choice of the value of the rate parameter, see, for example [1–10]. In addition, we refer readers to [11,12] for recent work on Bayesian inversion.

An exact inference method based on maximum likelihood estimates (MLEs), and compared its performance with approximate, Bayesian, and bootstrap methods developed by [13]. A joint progressive censoring of type-II and the expected values of the number of failures for two populations under joint progressive censoring of type-II introduced and studied by [14]. Exact likelihood inference for two exponential populations under joint progressive censoring of type II was studied by [15]. A precise result based on maximum likelihood estimates developed by [16]. A study of Bayesian estimation and prediction based on a joint type- II censored sample from two exponential populations was presented by [17]. Exact likelihood inference for two populations of two-parameter exponential distributions under joint censoring of type II was studied by [18].

Suppose that products from k different lines are manufactured in the same factory and that k independent samples of size $n_j, 1 \leq j \leq k$ are selected from these k lines and simultaneously subjected to a lifetime test. To reduce the cost of the experiment and shorten the duration of the experiment, the experimenter can terminate the lifetime test experiment once a certain number (say r) of failures occur. In this situation, one is interested in either a point or interval estimate of the mean lifetime of the units produced by these k lines.

Suppose $\{\mathbf{X}_j^{n_j}, j = 1, \ldots, k\}$ are k-samples where, $\mathbf{X}_j^{n_j} = \{X_{j1}, X_{j2}, \ldots, X_{jn_j}\}$ are the lifetimes of n_j copies of product line A_j and assumed to be independent and identically distributed (iid) random variables from a population with cumulative distribution function (cdf) $F_j(x)$ and probability density function (pdf) $f_j(x)$.

Furthermore, let $N = \sum_{j=1}^{k} n_j$ be the total sample size and r be the total number of observed failures. Let $W_1 \leq \ldots \leq W_N$ denote the order statistics of the N random variables $\{\mathbf{X}_j^{n_j}, j = 1, \ldots, k\}$. Under the joint Type-II censoring scheme for the k-samples, the observable data consist of $(\boldsymbol{\delta}, \mathbf{W})$, where $\mathbf{W} = (W_1, \ldots, W_r), W_i \in \{\mathbf{X}_{j_i}^{n_{j_i}}, j_i = 1, \ldots, k\}$, and r is a predefined integerand $\boldsymbol{\delta} = (\delta_{1j}, \ldots, \delta_{rj})$ associated with $(j_1, \ldots, j_r)$ is defined by:

$$\delta_{ij} = \begin{cases} 1, & \text{if } j = j_i \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

If $r_j = \sum_{i=1}^{r} \delta_{ij}$ denotes the number of X_j-failures in $\mathbf{W}$ and $r = \sum_{j=1}^{k} r_j$, then, the joint density function of $(\boldsymbol{\delta}, \mathbf{W})$ is given by:

$$f(\boldsymbol{\delta}, \mathbf{w}) = c_r \prod_{i=1}^{r} \prod_{j=1}^{k} (f_j(w_i))^{\delta_{ij}} . \prod_{j=1}^{k} (\bar{F}_j(w_r))^{n_j - r_j} \tag{3}$$

where $\bar{F}_j = 1 - F_j$ is the survival functions of the jth population and $c_r = \prod_{j=1}^{k} \frac{n_j!}{(n_j - r_j)!}$.

The main goal of this paper is to consider the Bayesian estimation of the parameter based on the learning rate parameter under a joint censoring scheme of type-II for exponential populations when censoring is applied to the samples in a combined manner. Section 2 presents the maximum likelihood and generalized Bayes estimators, using squared error, Linex, and general entropy loss functions in the Bayesian approach to estimate the population parameters. A numerical investigation of the results from Section 2 is presented in Section 3. Finally, we conclude the paper in Section 4.

2. Estimation of the Parameters

Suppose that for $1 \leq j \leq k$, the k populations are exponential with the following pdf and cdf:

$$f_h = \theta_j \exp(-\theta_j x),\ F_j = 1 - \exp(-\theta_j x),\ x > 0, \theta_j > 0. \tag{4}$$

Then, the likelihood function in (3) becomes:

$$\begin{aligned} L(\boldsymbol{\Theta}, \boldsymbol{\delta}, \mathbf{w}) &= c_r \prod_{i=1}^{r} \prod_{j=1}^{k} \{\theta_j \exp(-\theta_j w_i)\}^{\delta_{ij}} \prod_{j=1}^{k} \{\exp(-\theta_j w_r)\}^{n_j - r_j} \\ &= c_r \prod_{j=1}^{k} \theta_j^{r_j} \exp\{-\theta_j u_j\} \end{aligned} \tag{5}$$

where $\Theta = (\theta_1, \ldots, \theta_k)$ and $u_j = \sum_{i=1}^{r} w_i \delta_{ij} + w_r(n_j - r_j)$.

2.1. Maximum Likelihood Estimation

From (5), the MLE of θ_j, for $1 \leq j \leq k$, is given by:

$$\hat{\theta}_{jM} = \frac{r_j}{u_j}. \tag{6}$$

Remark 1. *MLEs of θ_j exist if we have at least k failures $(r \geq k)$, which means at least one failure from each sample, i.e., $1 \leq r_j \leq r - k + 1$ and $r_j \leq n_j$.*

We determined the MLEs to compare their results with those of Bayesian estimation, which uses the three types of loss functions for different values of the rate parameters, as described in Section 3.

2.2. Generalized Bayes Estimation

Since the parameters Θ are assumed to be unknown, we can consider the conjugate prior distributions of Θ as independent gamma prior distributions, i.e., $\theta_j \sim Gam\left(a_j, b_j\right)$. Therefore, the joint prior distribution of Θ is given by:

$$\pi(\Theta) = \prod_{j=1}^{k} \pi_j(\theta_j), \tag{7}$$

where

$$\pi_j(\theta_j) = \frac{b_j^{a_j}}{\Gamma\left(a_j\right)} \theta_j^{a_j-1} e^{-b_j\theta_j}, \tag{8}$$

and $\Gamma(\cdot)$ denotes the complete gamma function.

Combining (5) and (7), after raising (5) to the fractional power η, the posterior joint density function of Θ is then:

$$\pi^{*}(\Theta|data) = \prod_{j=1}^{k} \frac{(u_j\eta + b_j)^{r_j\eta + a_j} \theta_j^{r_j\eta + a_j - 1}}{\Gamma\left(r_j\eta + a_j\right)} \exp[-\{\theta_j(u_j\eta + b_j)\}]. \tag{9}$$

Since π_j is a conjugate prior, we see that if $\theta_j \sim Gam\left(a_j, b_j\right)$, then it has the posterior density function as $\left(\theta_j|data\right) \sim Gam\left(r_j\eta + a_j, u_j\eta + b_j\right)$.

In generalized Bayes estimation, we consider three types of loss functions:

(i) The squared error loss function (SE), which is classified as a symmetric function and gives equal importance to losses for overestimates and underestimates of the same magnitude;
(ii) The Linex loss function, which is asymmetric;
(iii) The generalization of the entropy (GE) loss function.

Using (9), the Bayesian estimators of θ_j under the squared error (SE) loss function are:

$$\hat{\theta}_{jS} = E(\theta_j) = \frac{r_j\eta + a_j}{u_j\eta + b_j}, \quad 1 \le j \le k. \tag{10}$$

Under the Linex loss function, the Bayesian estimators of θ_j are given by:

$$\hat{\theta}_{jL} = -\frac{1}{\nu} E\left(e^{-\nu\theta_j}\right) = \frac{r_j\eta + a_j}{\nu} \log\left(1 + \frac{\nu}{u_j\eta + b_j}\right), \quad \nu \neq 0, \quad 1 \le j \le k, \tag{11}$$

and under the GE loss function, the Bayesian estimators of θ_j are given by

$$\hat{\theta}_{jE} = \{E(\theta_j^{-c})\}^{-\frac{1}{c}} = \left(\frac{\Gamma\left(r_j\eta + a_j - c\right)}{\Gamma\left(r_j\eta + a_j\right)}\right)^{-\frac{1}{c}} \frac{1}{u_j\eta + b_j}, 1 \le j \le k. \tag{12}$$

Remark 2. *Obviously, $\hat{\theta}_j$ for $1 \le j \le k$ in the above three cases are the unique Bayes estimators of θ_j and thus admissible. The estimators $\hat{\theta}_{jJ}$, are Bayes estimators of θ_j using the noninformative Jeffreys priors $\pi_J \propto \prod_{j=1}^{k} \frac{1}{\theta_j}$, obtained directly by substituting $a_j = b_j = 0$ into (9), so that (10) leads to MLEs $\hat{\theta}_{jM}$.*

Remark 3. *For $c = 1, -1, -2$, the Bayes estimates $\hat{\theta}_{jE}$ agree with the Bayes estimates under the following losses: weighted squared error loss function, squared error loss function, and precautionary loss function.*

3. Numerical Study

This section examines the results of a Monte Carlo simulation study to evaluate the performance of the inference procedures derived in the previous section. An example is then presented to illustrate the inference methods discussed here.

3.1. Simulation Study

We have considered the different choices for the three populations sample sizes (n_1, n_2, n_3) and also for r. We choose the exponential parameters $(\theta_1, \theta_2, \theta_3)$ as (1, 2, 3) and for the Monte Carlo simulations, we use 10,000 replicates. Using (6), we obtain the MLEs of $\theta_1, \theta_2, \theta_3$ and their estimated risk, which are shown in Table 1.

Table 1. Average value of (r_1, r_2, r_3) and the average value and estimated risk (ER) of the MLEs $\hat{\theta}_{1M}, \hat{\theta}_{2M}, \hat{\theta}_{3M}$ for different choices of n_1, n_2, n_3, r.

(n_1, n_2, n_3)	r	(r_1, r_2, r_3)	$\hat{\theta}_{1M}$	ER$(\hat{\theta}_{1M})$	$\hat{\theta}_{2M}$	ER$(\hat{\theta}_{2M})$	$\hat{\theta}_{3M}$	ER$(\hat{\theta}_{3M})$
(10, 10, 10)	15	(3.2, 5.2, 6.6)	1.6578	2.4878	2.5768	1.4083	3.6371	1.7036
	20	(4.6, 7, 8.4)	1.3293	0.8232	2.3981	1.0986	3.5154	1.4353
	25	(6.6, 8.8, 9.6)	1.2090	0.5635	2.3328	0.9205	3.4128	1.2615
(7, 10, 13)	15	(2.1, 4.8, 8.1)	2.4920	8.3330	2.6255	1.5384	3.4742	1.4147
	20	(3, 6.6, 10.4)	1.6938	2.2329	2.4420	1.1270	3.4024	1.2115
	25	(4.3, 8.5, 12.2)	1.3595	0.8856	2.3450	0.9430	3.2879	1.0874
(13, 10, 7)	15	(4.4, 5.6, 5)	1.3523	0.8575	2.5333	1.3429	3.9389	2.2551
	20	(6.4, 7.5, 6.1)	1.2105	0.5738	2.4012	1.0682	3.7394	1.9000
	25	(9.1, 9.1, 6.8)	1.1381	0.4423	2.3021	0.8948	3.5907	1.7008
(20, 20, 20)	30	(6.1, 10.4, 13.4)	1.2188	0.5791	2.2398	0.7672	3.3125	1.0128
	40	(9.1, 14.1, 16.8)	1.1385	0.4186	2.1901	0.6409	3.2480	0.8648
	50	(13.2, 17.6, 19.2)	1.0948	0.3280	2.1634	0.5686	3.2370	0.8013
(14, 20, 26)	30	(3.9, 9.7, 16.4)	1.4297	1.5887	2.2755	0.8272	3.2351	0.8782
	40	(5.8, 13.3, 20.9)	1.2491	0.6099	2.2036	0.6618	3.1838	0.7458
	50	(8.6, 17, 24.4)	1.1622	0.4534	2.1639	0.5724	3.1772	0.6906
(26, 20, 14)	30	(8.8, 11.2, 10)	1.1386	0.4196	2.2234	0.7277	3.4242	1.2191
	40	(12.9, 14.9, 12.2)	1.0949	0.3263	2.1899	0.6219	3.3609	1.0732
	50	(18.2, 18.2, 13.6)	1.0667	0.2733	2.1621	0.5621	3.3013	0.9981
(30, 30, 30)	45	(9.3, 15.6, 20.1)	1.1421	0.4090	2.1576	0.5835	3.1945	0.7693
	60	(13.7, 21.1, 25.2)	1.089	0.3150	2.1253	0.4954	3.1738	0.6840
	75	(19.7, 26.5, 28.8)	1.0629	0.2525	2.1112	0.4364	3.1598	0.6248
(21, 30, 39)	45	(5.9, 14.5, 24.6)	1.2336	0.61024	2.1745	0.6116	3.1535	0.6711
	60	(8.8, 19.9, 31.3)	1.1510	0.4301	2.1295	0.5067	3.1324	0.5986
	75	(12.9, 25.4, 36.7)	1.1138	0.3452	2.1040	0.4508	3.1181	0.5397
(39, 30, 21)	45	(13.2, 16.9, 14.9)	1.0910	0.3179	2.1477	0.55604	3.2719	0.9112
	60	(19.3, 22.4, 18.3)	1.0633	0.2549	2.1212	0.4872	3.2461	0.8316
	75	(27.3, 27.3, 20.4)	1.0424	0.2127	2.11531	0.4348	3.2057	0.7510

In the simulation study, it should be noted that some of the simulated samples do not meet the condition in Remark 1 and, therefore, must be discarded. Thus, the average values of the observed failures (r_1, r_2, r_3) are calculated and reported in Table 1. For the Bayesian study, the hyperparameters are represented by $\Delta = (a_1, b_1, a_2, b_2, a_3, b_3)$, where $\Delta = \Delta_1 = (1, 1, 2, 1, 3, 1)$.

The results of Bayesian estimators of $\theta_1, \theta_2, \theta_3$ at $\Delta_1; c = -1, -0.5, -0.75; \nu = 0.1, 0.5;$ $\eta = 0.1, 0.5, 0.9$ are shown in Tables 2–7, where the loss at $c = -1$ is SE. Tables 2–4 studied generalized Bayes under general entropy loss function for the chosen values $c = -1, -0.75, -0.5$.

Table 2. Average value of Bayesian estimators $\hat{\theta}_{1E}, \hat{\theta}_{2E}, \hat{\theta}_{3E}$ and estimated risk for different choices of n_1, n_2, n_3, r and $\Delta = \Delta_1$, c $= -1, -0.75, -0.5$, η = 0.1.

n_1, n_2, n_3	r	η = 0.1, c = -1					
		$\hat{\theta}_{1E}$	ER $(\hat{\theta}_{1E})$	$\hat{\theta}_{2M}$	ER $(\hat{\theta}_{2E})$	$\hat{\theta}_{3E}$	ER($\hat{\theta}_{3E}$)
(10, 10, 10)	15	1.0639	0.1289	2.0514	0.1792	3.0520	0.2142
	20	1.0578	0.1496	2.0468	0.2033	3.0579	0.2191
	25	1.0487	0.1610	2.0341	0.2164	3.0451	0.2218
(20, 20, 20)	30	1.0547	0.1607	2.0533	0.2211	3.0606	0.2535
	40	1.0454	0.1700	2.0483	0.2282	3.0492	0.2677
	50	1.0446	0.1655	2.0537	0.2323	3.0512	0.2691
(30, 30, 30)	45	1.0477	0.1676	2.0383	0.2294	3.0393	0.2778
	60	1.0439	0.16566	2.0537	0.2305	3.0545	0.2792
	75	1.0351	0.1604	2.0213	0.2299	3.0516	0.2775
n_1, n_2, n_3	r	η = 0.1, c = -0.75					
		$\hat{\theta}_{1E}$	ER$(\hat{\theta}_{1E})$	$\hat{\theta}_{2M}$	ER$(\hat{\theta}_{2E})$	$\hat{\theta}_{3E}$	ER$(\hat{\theta}_{3E})$
(10, 10, 10)	15	0.9690	0.1198	1.9528	0.1692	2.9585	0.2070
	20	0.9689	0.1396	1.9652	0.1899	2.9515	0.2180
	25	0.9680	0.1524	1.9530	0.1987	2.9515	0.2228
(20, 20, 20)	30	0.9837	0.1519	1.9816	0.2124	2.9822	0.2563
	40	0.9911	0.1579	1.9748	0.2184	2.9777	0.2579
	50	0.9884	0.1574	1.9756	0.2178	2.9714	0.2590
(30, 30, 30)	45	0.9730	0.1538	1.9882	0.2189	2.9654	0.2641
	60	0.9939	0.1562	1.9855	0.2222	2.9854	0.2720
	75	0.9992	0.1526	1.9837	0.2204	3.0073	0.2705
n_1, n_2, n_3	r	η = 0.1, c = -0.5					
		$\hat{\theta}_{1E}$	ER$(\hat{\theta}_{1E})$	$\hat{\theta}_{2M}$	ER$(\hat{\theta}_{2E})$	$\hat{\theta}_{3E}$	ER$(\hat{\theta}_{3E})$
	15	0.8779	0.1573	1.8597	0.2176	2.8503	0.2425
(10, 10, 10)	20	0.8906	0.1771	1.8768	0.2376	2.8590	0.2488
	25	0.8894	0.1645	1.8830	0.2310	2.8545	0.2594
(20, 20, 20)	30	0.9112	0.1775	1.8823	0.2258	2.8996	0.2945
	40	0.9194	0.1726	1.9015	0.2273	2.8834	0.2640
	50	0.9343	0.1666	1.9111	0.2256	2.9009	0.2709
(30, 30, 30)	45	0.9281	0.1771	1.9025	0.2301	2.8995	0.2723
	60	0.9300	0.1638	1.9281	0.2316	2.9065	0.2813
	75	0.9460	0.1518	1.9388	0.2241	2.8961	0.2675

Table 3. Average value of Bayesian estimators $\hat{\theta}_{1E}, \hat{\theta}_{2E}, \hat{\theta}_{3E}$ and estimated risk for different choices of n_1, n_2, n_3, r and $\Delta = \Delta_1$, c $= -1, -0.75, -0.5$, η = 0.5.

n_1, n_2, n_3	r	η = 0.5, c = -1					
		$\hat{\theta}_{1E}$	ER $(\hat{\theta}_{1E})$	$\hat{\theta}_{2E}$	ER $(\hat{\theta}_{2E})$	$\hat{\theta}_{3E}$	ER$(\hat{\theta}_{3E})$
	15	1.1972	0.4274	2.2105	0.5615	3.1933	0.6662
(10, 10, 10)	20	1.1473	0.4071	2.1902	0.5528	3.1851	0.6597
	25	1.1119	0.3704	2.1729	0.5229	3.1584	0.6367
(20, 20, 20)	30	1.1510	0.3735	2.1389	0.5018	3.1645	0.6193
	40	1.1177	0.3201	2.0983	0.4543	3.1587	0.5890
	50	1.0396	0.2742	2.0935	0.4334	3.1136	0.5620
(30, 30, 30)	45	1.0881	0.3076	2.1095	0.4423	3.0956	0.5516
	60	1.0669	0.2672	2.0774	0.4059	3.1780	0.5307
	75	1.0524	0.2298	2.0732	0.3664	3.1116	0.5005

Table 3. *Cont.*

n_1, n_2, n_3	r	η = 0.5, c = −0.75					
		$\hat{\theta}_{1E}$	ER$(\hat{\theta}_{1E})$	$\hat{\theta}_{2E}$	ER$(\hat{\theta}_{2E})$	$\hat{\theta}_{3E}$	ER$(\hat{\theta}_{3E})$
	15	1.1439	0.3947	2.1345	0.5446	3.1490	0.6481
(10, 10, 10)	20	1.1412	0.3874	2.1343	0.5279	3.1305	0.6435
	25	1.1130	0.3539	2.1098	0.5055	3.1251	0.6169
(20, 20, 20)	30	1.0886	0.3475	2.1274	0.4826	3.1118	0.6067
	40	1.0645	0.3073	2.0875	0.4494	3.0735	0.5728
	50	1.0652	0.2710	2.1057	0.4221	3.1000	0.5494
(30, 30, 30)	45	1.0565	0.2997	2.1020	0.4353	3.1070	0.5508
	60	1.0676	0.2587	2.0958	0.4019	3.0733	0.5188
	75	1.0402	0.2230	2.0471	0.3625	3.0563	0.4894

n_1, n_2, n_3	r	η = 0.5, c = −0.5					
		$\hat{\theta}_{1E}$	ER$(\hat{\theta}_{1E})$	$\hat{\theta}_{2E}$	ER$(\hat{\theta}_{2E})$	$\hat{\theta}_{3E}$	ER$(\hat{\theta}_{3E})$
	15	1.0931	0.3704	2.0916	0.5080	3.0920	0.6268
(10, 10, 10)	20	1.0857	0.3639	2.0800	0.5061	3.0810	0.6153
	25	1.0680	0.3335	2.0566	0.4885	3.0687	0.6097
(20, 20, 20)	30	1.0768	0.3401	2.0929	0.4806	3.0683	0.5892
	40	1.0515	0.2975	2.0572	0.4450	3.1014	0.5688
	50	1.0339	0.2639	2.0787	0.4191	3.0639	0.5405
(30, 30, 30)	45	1.0724 0	.2963	2.0583	0.4274	3.0872	0.5358
	60	1.0252	0.2540	2.0957	0.3907	3.0558	0.5065
	75	1.0153	0.2169	2.0354	0.3572	3.0766	0.4857

Table 4. Average value of Bayesian estimators $\hat{\theta}_{1E}, \hat{\theta}_{2E}, \hat{\theta}_{3E}$ and estimated risk for different choices of n_1, n_2, n_3, r and $\Delta = \Delta_1$, c $= -1, -0.75, -0.5$, η = 0.9.

n_1, n_2, n_3	r	η = 0.9, c = −1					
		$\hat{\theta}_{1E}$	ER $(\hat{\theta}_{1E})$	$\hat{\theta}_{2E}$	ER $(\hat{\theta}_{2E})$	$\hat{\theta}_{3E}$	ER($\hat{\theta}_{3E}$)
(10, 10, 10)	15	1.2924	0.5662	2.2548	0.7427	3.3118	0.9021
	20	1.1990	0.5045	2.2362	0.6836	3.2827	0.8566
	25	1.1805	0.4457	2.2098	0.6479	3.2353	0.8091
(20, 20, 20)	30	1.1664	0.4280	2.1739	0.5844	3.2105	0.7427
	40	1.1012	0.3541	2.1414	0.5186	3.1937	0.7014
	50	1.0763	0.2996	2.1398	0.4826	3.1701	0.6397
(30, 30, 30)	45	1.1127	0.3468	2.1500	0.5000	3.2210	0.6480
	60	1.0911	0.2916	2.0927	0.4305	3.1516	0.5857
	75	1.0764	0.2452	2.1118	0.4011	3.1384	0.5517

n_1, n_2, n_3	r	η = 0.9, c = −0.75					
		$\hat{\theta}_{1E}$	ER$(\hat{\theta}_{1E})$	$\hat{\theta}_{2E}$	ER$(\hat{\theta}_{2E})$	$\hat{\theta}_{3E}$	ER$(\hat{\theta}_{3E})$
(10, 10, 10)	15	1.2242	0.5390	2.2262	0.7190	3.2448	0.8698
	20	1.1704	0.4868	2.1929	0.6805	3.2537	0.8441
	25	1.1217	0.4186	2.1849	0.6323	3.2040	0.7893
(20, 20, 20)	30	1.1633	0.4295	2.1419	0.5709	3.1816	0.7413
	40	1.1009	0.3548	2.0763	0.5131	3.1570	0.6703
	50	1.0710	0.2932	2.1185	0.4773	3.1497	0.6551
(30, 30, 30)	45	1.0610	0.3314	2.1260	0.4933	3.1146	0.6249
	60	1.0666	0.2838	2.1065	0.4281	3.1417	0.5752
	75	1.0353	0.2327	2.1163	0.3986	3.0907	0.5386

Table 4. *Cont.*

n_1, n_2, n_3	r	η = 0.9, c = −0.5					
		$\hat{\theta}_{1E}$	ER$(\hat{\theta}_{1E})$	$\hat{\theta}_{2E}$	ER$(\hat{\theta}_{2E})$	$\hat{\theta}_{3E}$	ER$(\hat{\theta}_{3E})$
(10, 10, 10)	15	1.1966	0.5169	2.2178	0.7047	3.1951	0.8582
	20	1.1408	0.4747	2.1926	0.6692	3.1770	0.8263
	25	1.1171	0.4128	2.1510	0.6177	3.1755	0.7808
(20, 20, 20)	30	1.1210	0.4090	2.1776	0.5749	3.1854	0.7334
	40	1.0767	0.3399	2.0959	0.5071	3.1243	0.6774
	50	1.0738	0.2905	2.0930	0.4726	3.1130	0.6230
(30, 30, 30)	45	1.0529	0.3257	2.1211	0.4892	3.0968	0.6160
	60	1.0611	0.2799	2.1081	0.4329	3.0707	0.5613
	75	1.0411	0.2381	2.0810	0.3891	3.0885	0.5405

Table 5. Average value of Bayesian estimators $\hat{\theta}_{1L}, \hat{\theta}_{2L}, \hat{\theta}_{3L}$ and estimated risk for different choices of n_1, n_2, n_3, r and $\Delta = \Delta_1$, $\upsilon = 0.1, 0.5$, $\eta = 0.1$.

n_1, n_2, n_3	r	η = 0.1, υ = 0.1					
		$\hat{\theta}_{1L}$	ER $(\hat{\theta}_{1L})$	$\hat{\theta}_{2L}$	ER $(\hat{\theta}_{2L})$	$\hat{\theta}_{3L}$	ER$(\hat{\theta}_{3L})$
(10, 10, 10)	15	1.0210	0.1170	1.9743	0.1653	2.9286	0.2008
	20	1.0142	0.1380	1.9768	0.1841	2.9291	0.2122
	25	1.0101	0.1492	1.9707	0.1888	2.9283	0.2173
(20, 20, 20)	30	1.0051	0.1531	1.9962	0.2068	2.9430	0.2423
	40	1.0176	0.1589	1.9997	0.2128	2.9496	0.2489
	50	1.0070	0.1599	2.0026	0.2152	2.9737	0.2562
(30, 30, 30)	45	1.0132	0.1556	1.9823	0.2144	2.9669	0.2624
	60	1.0328	0.1571	2.0014	0.2161	2.9830	0.2670
	75	1.0286	0.1525	1.9916	0.2147	2.9812	0.2658
n_1, n_2, n_3	r	η = 0.1, υ = 0.5					
		$\hat{\theta}_{1L}$	ER$(\hat{\theta}_{1L})$	$\hat{\theta}_{2L}$	ER$(\hat{\theta}_{2L})$	$\hat{\theta}_{3L}$	ER$(\hat{\theta}_{3L})$
(10, 10, 10)	15	0.8878	0.1371	1.7221	0.3093	2.5511	0.4943
	20	0.8993	0.1604	1.7433	0.3208	2.5696	0.4673
	25	0.9016	0.1529	1.7496	0.2950	2.5751	0.4683
(20, 20, 20)	30	0.9071	0.1482	1.7606	0.2691	2.6117	0.4220
	40	0.9295	0.1565	1.7884	0.2818	2.6441	0.4162
	50	0.9393	0.1527	1.8166	0.2720	2.6610	0.4167
(30, 30, 30)	45	0.9200	0.1488	1.8062	0.2794	2.6713	0.4136
	60	0.9372	0.1515	1.8286	0.2558	2.6870	0.3724
	75	0.9495	0.1436	1.8550	0.2523	2.7153	0.3719

Table 6. Average value of Bayesian estimators $\hat{\theta}_{1L}, \hat{\theta}_{2L}, \hat{\theta}_{3L}$ and estimated risk for different choices of n_1, n_2, n_3, r and $\Delta = \Delta_1$, $\upsilon = 0.1, 0.5$, $\eta = 0.5$.

n_1, n_2, n_3	r	η = 0.5, υ = 0.1					
		$\hat{\theta}_{1L}$	ER $(\hat{\theta}_{1L})$	$\hat{\theta}_{2L}$	ER $(\hat{\theta}_{2L})$	$\hat{\theta}_{3L}$	ER$(\hat{\theta}_{3L})$
(10, 10, 10)	15	1.1664	0.3981	2.1587	0.5353	3.1247	0.6201
	20	1.1304	0.3859	2.1463	0.5218	3.1495	0.6164
	25	1.1077	0.3516	2.1239	0.4970	3.0751	0.6016
(20, 20, 20)	30	1.1254	0.3542	2.1249	0.4814	3.1296	0.5894
	40	1.0891	0.3111	2.1256	0.4537	3.1219	0.5709
	50	1.0726	0.2733	2.0812	0.4193	3.1096	0.5423
(30, 30, 30)	45	1.0923	0.3031	2.0966	0.4231	3.0681	0.5447
	60	1.0882	0.2650	2.0497	0.3894	3.0733	0.5108
	75	1.0281	0.2246	2.0805	0.3624	3.1313	0.4823

Table 6. *Cont.*

n_1, n_2, n_3	r	η = 0.5, υ = 0.5					
		$\hat{\theta}_{1L}$	ER$(\hat{\theta}_{1L})$	$\hat{\theta}_{2L}$	ER$(\hat{\theta}_{2L})$	$\hat{\theta}_{3L}$	ER$(\hat{\theta}_{3L})$
(10, 10, 10)	15	1.0765	0.3300	1.9605	0.4364	2.8693	0.5419
	20	1.0658	0.3294	1.9662	0.4388	2.8899	0.5379
	25	1.0588	0.3133	1.9698	0.4253	2.8856	0.5215
(20, 20, 20)	30	1.0418	0.3126	2.0013	0.4207	2.9282	0.5278
	40	1.0619	0.2852	1.9951	0.4012	2.9664	0.5122
	50	1.0700	0.2616	2.0169	0.3881	2.9589	0.4893
(30, 30, 30)	45	1.0319	0.2783	2.0101	0.3914	2.9623	0.5000
	60	1.0443	0.2474	2.0633	0.3658	2.9677	0.4717
	75	1.0523	0.2186	2.0166	0.3378	2.9503	0.4375

Table 7. Average value of Bayesian estimators $\hat{\theta}_{1L}, \hat{\theta}_{2L}, \hat{\theta}_{3L}$ and estimated risk for different choices of n_1, n_2, n_3, r and $\Delta = \Delta_1, \upsilon = 0.1, 0.5$, η = 0.9.

n_1, n_2, n_3	r	η = 0.9, υ = 0.1					
		$\hat{\theta}_{1L}$	ER $(\hat{\theta}_{1L})$	$\hat{\theta}_{2L}$	ER $(\hat{\theta}_{2L})$	$\hat{\theta}_{3L}$	ER($\hat{\theta}_{3L}$)
(10, 10, 10)	15	1.2562	0.5469	2.2520	0.7191	3.2732	0.8542
	20	1.1807	0.4765	2.1859	0.6733	3.2325	0.8172
	25	1.1446	0.4205	2.1617	0.6125	3.1943	0.7835
(20, 20, 20)	30	1.1599	0.4211	2.1708	0.5825	3.1887	0.7187
	40	1.0989	0.3501	2.1293	0.5098	3.1658	0.6590
	50	1.0761	0.2956	2.1474	0.4839	3.1444	0.6305
(30, 30, 30)	45	1.0794	0.3410	2.1228	0.4857	3.1381	0.6270
	60	1.0735	0.2820	2.0872	0.4301	3.0966	0.5732
	75	1.0380	0.2370	2.0931	0.3952	3.1160	0.5311
n_1, n_2, n_3	r	η = 0.9, υ = 0.5					
		$\hat{\theta}_{1L}$	ER$(\hat{\theta}_{1L})$	$\hat{\theta}_{2L}$	ER$(\hat{\theta}_{2L})$	$\hat{\theta}_{3L}$	ER$(\hat{\theta}_{3L})$
(10, 10, 10)	15	1.1740	0.4682	2.1289	0.6062	3.0448	0.7299
	20	1.1486	0.4443	2.1051	0.5959	3.0577	0.7124
	25	1.0937	0.3885	2.0651	0.5546	3.0121	0.6752
(20, 20, 20)	30	1.1146	0.3892	2.0960	0.5320	3.0475	0.6468
	40	1.0707	0.3259	2.1091	0.4891	3.0329	0.6075
	50	1.0769	0.2872	2.0808	0.4544	3.0330	0.5710
(30, 30, 30)	45	1.0966	0.3287	2.0929	0.4571	3.0283	0.5798
	60	1.0432	0.2743	2.0772	0.4116	3.0108	0.5305
	75	1.0484	0.2314	2.0513	0.3707	3.0292	0.5052

Tables 5–7 studied generalized Bayes under Linex loss function for $\upsilon = 0.1, 0.5$, $\eta = 0.1, 0.5, 0.9$.

3.2. Illustrative Example

To illustrate the usefulness of the results developed in the previous sections, we consider three samples, each of size $n_1 = n_2 = n_3 = 10$, from Nelson's data (groups 1, 4, and 5) (p. 462, [19]), corresponding to the breakdown in minutes of an insulating fluid subjected to a high load. These failure times as samples $X_i, i = 1, 2, 3$, and their order statistics for (W, j_i) are shown in Table 8.

Table 8. The failure time data for X_1, X_2, and X_3, and their order (w, ji), where δji = 1.

Sample	Data
X_1	1.89, 4.03, 1.54, 0.31, 0.66, 1.7, 2.17, 1.82, 9.99, 2.24
X_2	1.17, 3.87, 2.8, 0.7, 3.82, 0.02, 0.5, 3.72, 0.06, 3.57
X_3	8.11, 3.17, 5.55, 0.80, 0.20, 1.13, 6.63, 1.08, 2.44, 0.78
Ordered data (w, j_i)	
(0.02,2), (0.06,2), (0.20,3), (0.31,1), (0.50,2), (0.66,1), (0.70,2), (0.78,3), (0.80,3), (1.083) (1.13,3), (1.17,2), (1.54,1), (1.70,1), (1.82,1), (1.89,1), (2.17,1), (2.24,1), (2.44,3), (2.80,2) (3.17,3), (3.57,2), (3.72,2), (3.82,2), (3.87,2), (4.03,1), (5.55,3), (6.63,3), (8.11,3), (9.99,1)	

For $r = 20, 25, 30$, the MLE and Bayesian estimates of the parameters are shown in Tables 9 and 10 for $\eta = 0.1, 0.5$, and using $\Delta = \Delta_2 = (1, 2.6, 1, 2, 1, 3)$; $c = -1, -0.75, -0.5$ and ν = 0.1, 0.5, respectively.

Table 9. ML and Bayesian estimates of the parameters θ_1, θ_2, θ_3 for different choices of r, η = 0.1 and $\Delta = \Delta_2$.

r	r_1, r_2, r_3		$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\theta}_3$
20	(8, 6, 6)	ML	0.4759	0.3647	0.3706
		Bayesian	η = 0.1	Δ_2	
		SE $c = -1$	0.4205	0.4390	0.3464
		GE $c = -0.75$	0.3937	0.4079	0.3219
		GE $c = -0.5$	0.4098	0.4266	0.3366
		Linex ν = 0.1	0.4156	0.4330	0.3427
		Linex ν = 0.5	0.3977	0.4113	0.3289
25	(8, 10, 7)	ML	0.4759	0.4943	0.3663
		Bayesian	η = 0.1	Δ_2	
		SE $c = -1$	0.4205	0.4971	0.3462
		GE $c = -0.75$	0.3937	0.4684	0.3230
		GE $c = -0.5$	0.3665	0.4393	0.2994
		Linex ν = 0.1	0.4156	0.4911	0.3427
		Linex ν = 0.5	0.3977	0.4686	0.3297
30	(10, 10, 10)	ML	0.3795	0.4943	0.3346
		Bayesian	η = 0.1	Δ_2	
		SE $c = -1$	0.3820	0.4971	0.3339
		GE $c = -0.75$	0.3600	0.4684	0.3146
		GE $c = -0.5$	0.3376	0.4393	0.2951
		Linex ν = 0.1	0.3784	0.4911	0.3312
		Linex ν = 0.5	0.3649	0.4686	0.3207

Table 10. Bayesian estimates of the parameters θ_1, θ_2, θ_3 for different choices of r; η = 0.5 and $\Delta = \Delta_2$.

r	r_1, r_2, r_3		$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\theta}_3$
20	(8, 6, 6)	SE $c = -1$	0.4543	0.3912	0.3605
		GE $c = -0.75$	0.4433	0.3794	0.3497
		GE $c = -0.5$	0.4322	0.3676	0.3387
		Linex ν = 0.1	0.4523	0.3893	0.3589
		Linex ν = 0.5	0.4443	0.3819	0.3526
25	(8, 10, 7)	SE $c = -1$	0.4543	0.4953	0.3584
		GE $c = -0.75$	0.4433	0.4852	0.3488
		GE $c = -0.5$	0.4322	0.4751	0.3391
		Linex ν = 0.1	0.4523	0.4932	0.3570
		Linex ν = 0.5	0.4443	0.4853	0.3515
30	(10, 10, 10)	SE $c = -1$	0.3803	0.4953	0.3344
		GE $c = -0.75$	0.3726	0.4852	0.3276
		GE $c = -0.5$	0.3648	0.4751	0.3207
		Linex ν = 0.1	0.3791	0.4932	0.3334
		Linex ν = 0.5	0.3744	0.4853	0.3298

4. Conclusions

In this work, we considered a joint type-II censoring scheme when the lifetimes of three populations have exponential distributions. We obtained the MLEs and the Bayesian estimates of the parameters using different values for the learning rate parameter η and the loss functions SE, GE, and Linex in a simulation study and an illustrative example. In both methods, the MLEs and the generalized Bayes estimates $\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3$ become better with more significant $n_{\rm i}$; i = 1, 2, 3 for different values of r; of course, the estimates become better, but the Bayes estimators are better than the MLEs. From Table 2 to Table 7, it can be seen that the results improve as c increases. Generally, the best result is obtained for generalized Bayes estimators for η = 0.1, i.e., the result improve better when η becomes small. Studying this work with a different type of censoring might be interesting.

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