SUPPLEMENTARY MATERIALS

Quantum theory of scattering of nonclassical fields by free electrons

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Searching for a solution to the equation (1) articles in the form of operators $\hat{a}_i$ and $\hat{a}_i^\dagger$ is inconvenient, since there are no methods for solving such equations based on the properties of these operators. It is much easier to pass to writing in the form of a differential equation in canonical field variables, where $\hat{a}_i = \frac{1}{\sqrt{2}}(Q_i + \frac{\partial}{\partial Q_i})$ and $\hat{a}_i^\dagger = \frac{1}{\sqrt{2}}(Q_i - \frac{\partial}{\partial Q_i})$. It is then convenient to represent the vector potential in the form of canonical variables [1–4] $\hat{\mathbf{A}}_{i,a} = \beta_i c \mathbf{u}_i \left(Q_i \cos(\mathbf{k}_i \mathbf{r}_a) + i \frac{\partial}{\partial Q_i} \sin(\mathbf{k}_i \mathbf{r}_a) \right)$, where $\beta_i = \sqrt{\frac{4\pi}{\omega_i V}}$. The quantity β_i^2 has the dimension of energy and, relative to the photon energy ω_i, is a very small quantity even in very strong electromagnetic fields [5]. It is easy to see this at a qualitative level if we make estimates of the dimensionless parameter β_i^2/ω_i, which is always $\beta_i^2/\omega_i \ll 1$ in reasonable cases (in the CGS system of units, the value $\frac{4\pi e^2}{m\omega_i^2 V}$). As a result, it is necessary to solve the differential equation with the Hamiltonian

$$\hat{H} = \frac{1}{2}\sum_{i=1}^{2} \omega_i^{'} \left(Q_i^2 - \frac{\partial^2}{\partial Q_i^2} \right) + \frac{1}{2}\sum_a \hat{\mathbf{p}}_a^2 + \sum_{i=1}^{2} \beta_i \mathbf{u}_i \sum_a \left(Q_i \cos(\mathbf{k}_i \mathbf{r}_a) + i \frac{\partial}{\partial Q_i} \sin(\mathbf{k}_i \mathbf{r}_a) \right) \hat{\mathbf{p}}_a + \sum_{i,j=1}^{2} A_{i,j} Q_i \frac{\partial}{\partial Q_j} + B Q_1 Q_2 + C \frac{\partial}{\partial Q_1}\frac{\partial}{\partial Q_2}. \quad \text{(S1)}$$

In the Eq.(S1) the coefficients will be defined as

$$\omega_i^{'} = \omega_i \sqrt{1 + \beta_i^2/\omega_i}, \quad A_{i,j} = i\beta_i\beta_j \mathbf{u}_i \mathbf{u}_j \sum_a \sin(\mathbf{k}_j \mathbf{r}_a)\cos(\mathbf{k}_i \mathbf{r}_a),$$
$$B = \beta_1\beta_2 \mathbf{u}_1\mathbf{u}_2 \sum_a \cos(\mathbf{k}_1 \mathbf{r}_a)\cos(\mathbf{k}_2 \mathbf{r}_a), \; C = -\beta_1\beta_2 \mathbf{u}_1\mathbf{u}_2 \sum_a \sin(\mathbf{k}_1 \mathbf{r}_a)\sin(\mathbf{k}_2 \mathbf{r}_a). \quad \text{(S2)}$$

Let us estimate which expressions in the equation Eq.(S1) can be ignored due to the smallness of the parameter β_i^2/ω_i. To do this, it is necessary to consider the Eq.(S1) in the case when the effect of the electromagnetic field on the charges will be maximum, i.e. for $\mathbf{k}_i \mathbf{r}_a \to 0$. It should be added that this condition coincides with the dipole approximation. In this case, the Eq.(S1) has an exact solution. We add that this solution can be found using [6], where a similar problem was solved.

In the case when the interaction is maximum, i.e. when $\mathbf{k}_i\mathbf{r}_a \to 0$ then Eq.(S1) of the manuscript will be

$$\hat{H} = \frac{1}{2}\sum_{i=1}^{2}\omega_i^{'}\left(Q_i^2 - \frac{\partial^2}{\partial Q_i^2}\right) + \frac{1}{2}\sum_a \hat{\mathbf{p}}_a^2 + \hat{\alpha}Q_1 + \hat{\beta}Q_2 + BQ_1Q_2,$$
$$\hat{\alpha} = \beta_1\mathbf{u}_1\sum_a \hat{\mathbf{p}}_a,\ \hat{\beta} = \beta_2\mathbf{u}_2\sum_a \hat{\mathbf{p}}_a. \tag{S3}$$

A similar Hamiltonian was considered in [6]. In that work, two free particles were considered; in our work, we do not limit the number of free particles. Despite this, one can see that the results obtained in [6] can be easily extended to the case of any number of particles. Following a similar change of variables, which are presented in [6], one can obtain the Hamiltonian Eq.(S3) in the form

$$\hat{H} = \frac{1}{2}\sum_{i=1}^{2}\omega_i^{''}\left(y_i^2 - \frac{\partial^2}{\partial y_i^2}\right) + \frac{1}{2}\sum_a \hat{\mathbf{p}}_a^2 + \left(\hat{\mathbf{A}}y_1 + \hat{\mathbf{B}}y_2\right)\sum_a \hat{\mathbf{p}}_a,$$
$$\hat{\mathbf{A}} = \sqrt{\frac{\omega_1^{'}}{\omega_1^{''}}}\left(\beta_1\mathbf{u}_1\cos\theta - \sqrt{\frac{\omega_2^{'}}{\omega_1^{'}}}\beta_2\mathbf{u}_2\sin\theta\right),\ \hat{\mathbf{B}} = \sqrt{\frac{\omega_2^{'}}{\omega_2^{''}}}\left(\sqrt{\frac{\omega_1^{'}}{\omega_2^{'}}}\beta_1\mathbf{u}_1\sin\theta + \beta_2\mathbf{u}_2\cos\theta\right),$$
$$\tan\theta = \frac{\epsilon}{|\epsilon|}\sqrt{\epsilon^2+1} - \epsilon,\ \epsilon = \frac{\omega_2^{'} - \omega_1^{'}}{2B},$$
$$\omega_1^{''} = \omega_1^{'}\sqrt{1 - \sqrt{\frac{\omega_2^{'}}{\omega_1^{'}}}\frac{B}{\omega_1^{'}}\tan\theta},\ \omega_2^{''} = \omega_2^{'}\sqrt{1 + \sqrt{\frac{\omega_1^{'}}{\omega_2^{'}}}\frac{B}{\omega_2^{'}}\tan\theta},$$
$$y_1 = \sqrt{\frac{\omega_1^{''}}{\omega_1^{'}}}\left(Q_1\cos\theta - \sqrt{\frac{\omega_1^{'}}{\omega_2^{'}}}Q_2\sin\theta\right),\ y_2 = \sqrt{\frac{\omega_2^{''}}{\omega_2^{'}}}\left(\sqrt{\frac{\omega_2^{'}}{\omega_1^{'}}}Q_1\sin\theta + Q_2\cos\theta\right). \tag{S4}$$

Next, a method for solving the Schrödinger equation with Hamilton Eq.(S4) is presented in [6]. Let's write the result of this solution

$$p_{i,l} = \left|\sum_{k,n,m} C^{i,l}_{k,i+l-k} e^{-\frac{i}{\hbar}E_{n,m}t} A^{n_0,m_0}_{n,m} B^{n,m}_{k,i+l-k}\right|^2, E_{n,m} = \omega_1^{''}\left(n + \frac{1}{2}\right) + \omega_2^{''}\left(m + \frac{1}{2}\right), \tag{S5}$$

where $p_{i,l}$ is the probability of detecting i and l photons in the first and second modes, respectively, if the initial number of photons in the Fock state in these modes is n_0, m_0, respectively.

$$
B_{k,p}^{n,m} = (-1)^{(k-n)\theta(k-n)}(-1)^{(p-m)\theta(p-m)}\sqrt{\frac{k!}{n!}}^{sgn(n-k)}\sqrt{\frac{p!}{m!}}^{sgn(m-p)} e^{-\frac{1}{4}\left({\alpha'}^2+{\beta'}^2\right)}
$$
$$
\times\left(\frac{\alpha'}{\sqrt{2}}\right)^{|n-k|}\left(\frac{\beta'}{\sqrt{2}}\right)^{|m-p|} L_{(n+k-|n-k|)/2}^{|n-k|}\left(\left(\frac{\alpha'}{\sqrt{2}}\right)^2\right) L_{(m+p-|m-p|)/2}^{|m-p|}\left(\left(\frac{\beta'}{\sqrt{2}}\right)^2\right),
$$
$$
A_{n,m}^{n_0,m_0} = (-1)^{(n_0-n)\theta(n_0-n)}(-1)^{(m_0-m)\theta(m_0-m)}\sqrt{\frac{n!}{n_0!}}^{sgn(n_0-n)}\sqrt{\frac{m!}{m_0!}}^{sgn(m_0-m)} e^{-\frac{1}{4}\left({\alpha'}^2+{\beta'}^2\right)}
$$
$$
\times\left(\frac{\alpha'}{\sqrt{2}}\right)^{|n_0-n|}\left(\frac{\beta'}{\sqrt{2}}\right)^{|m_0-m|} L_{(n_0+n-|n_0-n|)/2}^{|n_0-n|}\left(\left(\frac{\alpha'}{\sqrt{2}}\right)^2\right) L_{(m_0+m-|m_0-m|)/2}^{|m_0-m|}\left(\left(\frac{\beta'}{\sqrt{2}}\right)^2\right),
$$
$$
C_{k,p}^{i,l} = \frac{\mu^{i+k}\sqrt{p!k!}}{(1+\mu^2)^{\frac{k+p}{2}}\sqrt{i!l!}} P_k^{(-(1+p+k),p-i)}\left(-\frac{2+\mu^2}{\mu^2}\right), \quad \text{(S6)}
$$

where $L_a^b(z)$ are Laguerre polynomials, $\theta(x)$ is the Haveside tetta function, $sgn(x)$ is the signum function, and the coefficients $\alpha' = \frac{\mathbf{A}\mathbf{p}_0}{\omega_1'}$, $\beta' = \frac{\mathbf{B}\mathbf{p}_0}{\omega_2'}$ ($\mathbf{p}_0$ is the total initial momentum of all electrons), $P_a^{(b,c)}(x)$ is the Jacobi polynomial, and $\mu = \tan\theta$. Let's estimate the dimensionless parameters α' and β'. You can see that the order of these parameters will be the same $\sim \sqrt{\frac{\beta^2}{\omega}}\sqrt{\frac{E_0}{\omega}}$ (E_0 is the kinetic energy of the electrons). The first parameter was evaluated at the beginning of the paper, where it was said that $\sqrt{\frac{\beta^2}{\omega}} \ll 1$. The second parameter $\sqrt{\frac{E_0}{\omega}}$ in the general case can also be considered small, or at least such that $\sim \sqrt{\frac{\beta^2}{\omega}}\sqrt{\frac{E_0}{\omega}} \ll 1$. As a result, the niche parameters are $\alpha' \ll 1$ and $\beta' \ll 1$. In this case, you can see that the result is independent of $\frac{1}{2}\sum_a \hat{\mathbf{p}}_a^2 + \hat{\alpha}Q_1 + \hat{\beta}Q_2$ in Eq.(S3) and these terms can be ignored in the more general case when $\mathbf{k}_i\mathbf{r}_a$ can be an arbitrary value.

This estimate shows that the statistical properties of photons in an electromagnetic field for a small parameter β_i^2/ω_i do not depend on the operator $\hat{\mathbf{p}}_a$. This is quite an obvious statement if we assume that free electrons do not absorb photons, which means that when they are scattered, the movement of electrons in an electromagnetic field cannot affect the statistics of scattered photons. It should be added that this analysis and our problem are only suitable for Thomson scattering $\hbar\omega \ll mc^2$. As a result, it is necessary to solve the following field equation $\hat{H}\Phi = i\frac{\partial\Phi}{\partial t}$, where

$$
\hat{H} = \frac{1}{2}\sum_{i=1}^{2}\omega_i'\left(Q_i^2 - \frac{\partial^2}{\partial Q_i^2}\right) + \sum_{i,j=1}^{2} A_{i,j}Q_i\frac{\partial}{\partial Q_j} + BQ_1Q_2 + C\frac{\partial}{\partial Q_1}\frac{\partial}{\partial Q_2}. \quad \text{(S7)}
$$

The solution of this kind of differential equations has not previously been found in the literature. The underlying problem is the diagonalization of this equation. If we assume that we have a dipole interaction, then in Eq.(S7) only the coefficient B is preserved, while the coefficients $A_{i,j}$ and C will be equal to zero. In this case, the method of diaganization of the Hamiltonian is well known, it is the change of variables, eg [7–9]. The method of diaganization of the Hamiltonian when there is only one of the coefficients $A_{i,j}$ was developed in [10] through a unitary transformation. In our task, we combined these two methods into one with some additions. To diagonalize the Hamiltonian Eq.(S7) we first make a change of variables in the form $Q_1/\sqrt{\omega_1'} = x\cos\theta + y\sin\theta$, $Q_2/\sqrt{\omega_2'} = (-x\sin\theta + y\cos\theta)(1+\delta)$, where θ and δ are some unknown coefficients. At the second stage of diaganization, we carry out a unitary transformation over the Hamiltonian now depending on the variables $\{x, y\}$, i.e. $\hat{H} = \hat{H}(x, y)$. This means that we represent $\Phi = \hat{S}^{-1}\Phi'$, where $\Phi' = \hat{S}\Phi$. This wave function Φ' will correspond to the Hamiltonian $\hat{H}' = \hat{S}\hat{H}(x, y)\hat{S}^{-1}$, and the conditions $\hat{H}\Phi = E\Phi$ and $\hat{H}'\Phi' = E\Phi'$, where E is the energy eigenvalue. We choose the unitary operator $\hat{S}$ in the form $\hat{S} = e^{i\gamma\frac{\partial}{\partial x}\frac{\partial}{\partial y}}e^{i\alpha xy}$ [10], where γ and α are some coefficients. Thus, we get the Hamiltonian $\hat{H}'$ which has an analytical form, where there are 4 unknown coefficients $\theta, \delta, \gamma, \alpha$. You can also see from Eq.(S7) that we also have 4 mixed dependencies, which means that 4 mixed dependencies will be on $\{x, y\}$. As a result, having compiled a system of fourth-order equations and equating the coefficients for mixed dependences to zero, we can reduce the Hamiltonian $\hat{H}'$ to a diagonal form.

Let us pose the problem of diaganalyzing the Hamiltonian Eq.(S7). We make the change of variables $Q_1/\sqrt{\omega_1'} = x\cos\theta + y\sin\theta$, $Q_2/\sqrt{\omega_2'} = (-x\sin\theta + y\cos\theta)(1+\delta)$, where θ and δ are some unknown coefficients. As a result, we get the Hamiltonian

$$\hat{H} = \frac{1}{2}\left(\omega_{1,x}'^2 x^2 - a'\frac{\partial^2}{\partial x^2}\right) + \frac{1}{2}\left(\omega_{2,y}'^2 y^2 - b'\frac{\partial^2}{\partial y^2}\right) + \sum_{i,j=1}^{2} A_{i,j}' x_i \frac{\partial}{\partial x_j} + B' xy + C'\frac{\partial}{\partial x}\frac{\partial}{\partial y},$$

$$\omega_{1,x}'^2 = \omega_1'^2\cos^2\theta + \omega_2'^2(1+\delta)^2\sin^2\theta - B\sqrt{\omega_1'\omega_2'}\sin 2\theta(1+\delta),$$

$$\omega_{2,y}'^2 = \omega_2'^2\cos^2\theta(1+\delta)^2 + \omega_1'^2\sin^2\theta + B\sqrt{\omega_1'\omega_2'}\sin 2\theta(1+\delta),$$

$$A_{1,1}' = A_{1,1}\cos^2\theta + A_{2,2}\sin^2\theta - \frac{1}{2}\sin 2\theta\left(\frac{A_{1,2}}{1+\delta}\sqrt{\frac{\omega_1'}{\omega_2'}} + \sqrt{\frac{\omega_2'}{\omega_1'}}A_{2,1}(1+\delta)\right),$$

$$A_{2,2}' = A_{1,1}\sin^2\theta + A_{2,2}\cos^2\theta + \frac{1}{2}\sin 2\theta\left(\frac{A_{1,2}}{1+\delta}\sqrt{\frac{\omega_1'}{\omega_2'}} + \sqrt{\frac{\omega_2'}{\omega_1'}}A_{2,1}(1+\delta)\right),$$

$$A_{1,2}' = \frac{1}{2}(A_{1,1} - A_{2,2})\sin 2\theta + \frac{A_{1,2}}{1+\delta}\sqrt{\frac{\omega_1'}{\omega_2'}}\cos^2\theta - \sqrt{\frac{\omega_2'}{\omega_1'}}A_{2,1}(1+\delta)\sin^2\theta,$$

$$A_{2,1}' = \frac{1}{2}(A_{1,1} - A_{2,2})\sin 2\theta - \frac{A_{1,2}}{1+\delta}\sqrt{\frac{\omega_1'}{\omega_2'}}\sin^2\theta + \sqrt{\frac{\omega_2'}{\omega_1'}}A_{2,1}(1+\delta)\cos^2\theta,$$

$$B' = \frac{1}{2}\sin 2\theta\left(\omega_1'^2 - \omega_2'^2(1+\delta)^2\right) + B\sqrt{\omega_1'\omega_2'}\left(\cos^2\theta(1+\delta) - \sin^2\theta(1+\delta)\right),$$

$$C' = -\frac{\sin 2\theta}{2} + \frac{\sin 2\theta}{2(1+\delta)^2} + \frac{C}{\sqrt{\omega_1'\omega_2'}}\frac{\cos 2\theta}{1+\delta},$$

$$a' = \cos^2\theta + \frac{\sin^2\theta}{(1+\delta)^2} + \frac{\sin 2\theta}{1+\delta}\frac{C}{\sqrt{\omega_1'\omega_2'}}, \quad b' = \sin^2\theta + \frac{\cos^2\theta}{(1+\delta)^2} - \frac{\sin 2\theta}{1+\delta}\frac{C}{\sqrt{\omega_1'\omega_2'}}, \tag{S8}$$

where $x_1 = x, x_2 = y$.

We need to find a solution to the Schrödinger equation $\hat{H}\Phi = i\frac{\partial\Phi}{\partial t}$, where the Hamiltonian $\hat{H}$ is determined from Eq.(S8). Next, we perform a unitary transformation over the desired wave function $\Phi = \hat{S}^{-1}\Phi'$, where $\Phi' = \hat{S}\Phi$. This wave function Φ' will correspond to the Hamiltonian $\hat{H}' = \hat{S}\hat{H}(x, y)\hat{S}^{-1}$, and the conditions $\hat{H}\Phi = E\Phi$ and $\hat{H}'\Phi' = E\Phi'$, where E is the energy eigenvalue. We choose the unitary operator $\hat{S}$ in the form $\hat{S} = e^{i\gamma \frac{\partial}{\partial x}\frac{\partial}{\partial y}} e^{i\alpha xy}$, where γ and α are some coefficients. Having carried out all the calculations, we can see that the Hamiltonian $\hat{H}'$ has a finite form (the action of the operators gives a zero value at the 3rd stage). To carry out such calculations, we use the well-known expansion

$$e^{\hat{X}}\hat{Y}e^{-\hat{X}} = \hat{Y} + \left[\hat{X}, \hat{Y}\right] + \frac{1}{2!}\left[\hat{X}, \left[\hat{X}, \hat{Y}\right]\right] + \frac{1}{3!}\left[\hat{X}, \left[\hat{X}, \left[\hat{X}, \hat{Y}\right]\right]\right] + \dots .$$

As a result, the Hamiltonian $\hat{H}'$ can be reduced to a diagonal form (under the condition $\beta_i^2/\omega_i \ll 1$ described in the paper) if the unknown coefficients are (to simplify the notation and further calculations, we rename $\theta = \theta_1$)

$$\alpha = \epsilon_1 - \frac{\epsilon_1}{|\epsilon_1|}\sqrt{1+\epsilon_1^2},\ \gamma = \frac{\epsilon_1}{2|\epsilon_1|}\frac{1}{\sqrt{1+\epsilon_1^2}},\ \tan 2\theta_1 = 2\frac{\frac{4\pi}{V}|\mathbf{u}_1\mathbf{u}_2|}{\omega_2^2-\omega_1^2}\sum_a \cos(\Delta\mathbf{k}\mathbf{r}_a)$$
$$\epsilon_1 = \frac{\omega_2^2-\omega_1^2}{2\cos 2\theta_1 \frac{4\pi}{V}|\mathbf{u}_1\mathbf{u}_2\sum_a \sin(\Delta\mathbf{k}\mathbf{r}_a)|},\ \delta = \frac{C}{\omega_1}\cot 2\theta_1 + i\frac{A_{1,1}+A_{2,2}}{2\omega_1\epsilon_1 \sin 2\theta_1}. \tag{S9}$$

Passing for convenience to the dimensionless variables $\{x, y\}$ (this is also taken into account in the coefficients α and β in Eq.(S9)), we obtain the Hamiltonian in the diagonal form

$$\hat{H}' = \frac{\Omega_1}{2}\left(x^2 - \frac{\partial^2}{\partial x^2}\right) + \frac{\Omega_2}{2}\left(y^2 - \frac{\partial^2}{\partial y^2}\right) + (A'_{1,1} - i\alpha C)x\frac{\partial}{\partial x} + (A'_{2,2} - i\alpha C)y\frac{\partial}{\partial y},$$
$$\Omega_1 = \sqrt{\Omega_0+\sigma},\ \Omega_2 = \sqrt{\Omega_0-\sigma},\ \sigma = i\frac{\epsilon_1}{|\epsilon_1|}\sqrt{1+\epsilon_1^2}(A'_{1,2} - A'_{2,1}),$$
$$\Omega_0 = \omega_0^2 a'b' + i\omega_0\epsilon_1(A'_{1,2} + A'_{2,1}),\ \omega_0 = i\sqrt{\frac{A'_{2,1}\omega'^2_{1,x} - A'_{1,2}\omega'^2_{2,y}}{a'A'_{1,2} - b'A'_{2,1}}}. \tag{S10}$$

Given that $a/\Omega_1 \ll 1$ and $b/\Omega_2 \ll 1$ (since $a \sim \beta^2$ and $b \sim \beta^2$, and $\Omega_1 \sim \omega$ and $\Omega_2 \sim \omega$) we get the solution

$$\Phi'_k(x) = C_k e^{-x^2/2}H_k(x),\ \Phi_m(y) = C_p e^{-x^2/2}H_p(y),$$
$$E_k = \Omega_1\left(k+\frac{1}{2}\right) + c_1,\ E_p = \Omega_2\left(p+\frac{1}{2}\right) + c_2, \tag{S11}$$

where $H_k(x)$ are Hermite polynomials, c_1 and c_2 non-essential constants, E_k and E_p energy. Find the total energy $E_{k,p} = E_k + E_p$. We take into account that $\sigma/\Omega_1 \ll 1$ and $\sigma/\Omega_2 \ll 1$. We take into account that $\sigma/\Omega_1 \ll 1$ and $\sigma/\Omega_2 \ll 1$. Expanding in a series in terms of this small parameter and discarding constant values (which do not affect the quantities under study), we get $E_{k,p} = \Omega_0(k+p) + \frac{\sigma}{2\omega_1}(k-p)$ (Here we have taken into account that $\Omega_0 = \omega_1$ to the nearest β^2/ω). Further, we will use this energy $E_{k,p}$, although, as will be shown below, the number of photons in the system remains $k + p = const$, and it means that the first term in the energy is a constant value and can be ignored. Consider the parameter $\frac{\sigma}{\omega_1}$. It is easy to show that it will be equal to

$$\frac{\sigma}{\omega_1} = \Omega\sqrt{1+\epsilon^2},\ \Omega = \frac{4\pi\left|\mathbf{u}_1\mathbf{u}_2\sum_a e^{i\Delta\mathbf{k}\mathbf{r}_a}\right|}{\omega_1 V},\ \epsilon = \frac{\omega_2-\omega_1}{\Omega}. \tag{S12}$$

As a result, the general solution of our problem, without choosing the initial conditions, will look like

$$\Phi^{'}(x,y,t)=\sum_{k,p}A_{k,p}\Phi^{'}_{k}(x)\Phi^{'}_{p}(y)e^{-iE_{k,p}t}, \tag{S13}$$

where $A_{k,p}$ are expansion coefficients. To find $\Phi(x,y,t)=e^{-i\alpha xy}e^{-i\gamma\frac{\partial}{\partial x}\frac{\partial}{\partial y}}\Phi^{'}(x,y,t)$. As a result, we get

$$\Phi(x,y,t)=\sum_{k,p}A_{k,p}e^{-iE_{k,p}t}\Phi_{k,p}(x,y),\ \ \Phi_{k,p}(x,y)=e^{-i\alpha xy}e^{-i\gamma\frac{\partial}{\partial x}\frac{\partial}{\partial y}}\Phi^{'}_{k}(x)\Phi^{'}_{p}(y),$$
$$\Phi(Q_1,Q_2,t)=\Phi(x,y,t),\ \ x=Q_1\cos\theta-Q_2\sin\theta,\ \ y=Q_1\sin\theta+Q_2\cos\theta. \tag{S14}$$

Also, the wave function $\Phi(Q_1,Q_2,t)$ can be expanded in terms of eigenfunctions of the noninteracting system $|\Phi(Q_1,Q_2,t)\rangle=\sum_{n,m}c_{n,m}|n\rangle|m\rangle e^{-i\varepsilon_{n,m}t}$, where $p_{n,m}=|c_{n,m}|^2$ is the probability of finding in the first and second mode n and m photons, respectively. Using Eq.(S14) and this expansion, it is easy to show that

$$c_{n,m}=\sum_{k,p}A^{s_1,s_2}_{k,p}A^{*n,m}_{k,p}e^{-iE_{k,p}t},\ \ A^{s_1,s_2}_{k,p}=\langle\Phi_{k,p}(Q_1,Q_2)|s_1,s_2\rangle, \tag{S15}$$

where $|s_1,s_2\rangle=|\Phi(Q_1,Q_2,t=0)\rangle$, and s_1,s_2 is the initial number of photons (before interaction) in the Fock state in the first and second modes, respectively.

At first glance, the expression $A^{s_1,s_2}_{k,p}$ in Eq.(S15) is not analytically calculated, because we do not even know the analytical form of the function $\Phi_{k,p}(x,y)$, see Eq. (S14). The expression $\Phi_{k,p}(x,y)$ can be represented not as an operator action on $\Phi^{'}_{k}(x)\Phi^{'}_{p}(y)$, but in integral form. To do this, we need to represent $\Phi^{'}_{k}(x)$ through the Fourier integral, i.e. $\Phi^{'}_{k}(x)=\frac{(-i)^n}{\sqrt{2\pi}}C_n\int_{-\infty}^{\infty}e^{-\frac{p^2}{2}}H_n(p)e^{ipx}dp$. As a result, we get

$$\Phi_{k,p}(x,y)=e^{-i\alpha xy}e^{-i\gamma\frac{\partial}{\partial x}\frac{\partial}{\partial y}}\Phi^{'}_{k}(x)\Phi^{'}_{p}(y)=$$
$$\frac{(-i)^nC_nC_m}{\sqrt{2\pi}\sqrt{1+\alpha\gamma}}\int_{-\infty}^{\infty}e^{-\frac{p^2}{2}}H_n(p)e^{ix\left(\frac{p}{\sqrt{1+\alpha\gamma}}-\alpha y\right)}e^{-\frac{1}{1+\alpha\gamma}\left(y+\frac{\gamma p}{\sqrt{1+\alpha\gamma}}\right)^2}H_m\Big(\frac{y+\frac{\gamma p}{\sqrt{1+\alpha\gamma}}}{\sqrt{1+\alpha\gamma}}\Big)dp. \tag{S16}$$

It can be seen that the function $\Phi_{k,p}(x,y)$ is representable only in integral form, and the Fourier transform $\Phi_{k,p}(p,y)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\Phi_{k,p}(x,y)e^{ipx}dx$ of it is an analytic function, we obtain

$$\Phi_{k,p}(p,y)=C_kC_pi^ne^{-\frac{\xi}{2}(p-\alpha y)^2}H_k\Big(\sqrt{\xi}(p-\alpha y)\Big)e^{-\frac{\xi}{2}(y+\alpha p)^2}H_p\Big(\sqrt{\xi}(y+\alpha p)\Big),\ \ \xi=\frac{1}{1+\alpha^2}. \tag{S17}$$

The coefficient $A^{s_1,s_2}_{k,p}$ in Eq.(S15) can be calculated in another way using Eq.(S17). For this, we note that in Eq.(S14)

$$\Phi(p,y,t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\Phi(x,y,t)e^{ipx}dx=\sum_{k,p}A_{k,p}e^{-iE_{k,p}t}\Phi_{k,p}(p,y). \tag{S18}$$

From Eq.(S18) one can see (similarly to Eq.(S15)) that

$$c_{n,m} = \sum_{k,p} A^{s_1,s_2}_{k,p} A^{*n,m}_{k,p} e^{-iE_{k,p}t}, \;\; A^{s_1,s_2}_{k,p} = \langle \Phi_{k,p}(p,y) | \Phi(p,y,t=0) \rangle. \tag{S19}$$

Find $\Phi(p,y,t=0)$ from initial conditions

$$\Phi(p,y,t=0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Phi(x,y,0) e^{ipx} dx = \sum_{k_1,p_1} (-i)^{k_1} B^{s_1,s_2}_{k_1,p_1}(\theta_1) |k_1,p_1\rangle, \tag{S20}$$

where $|k_1,p_1\rangle = |k_1\rangle p_1\rangle$ these are Fock states, and $|k_1\rangle$ depends on the variable p, $B^{s_1,s_2}_{k_1,p_1}(\theta_1) = \langle \Phi_{k,p}(x,y)|s_1,s_2\rangle$ (of course considering that $x = Q_1 \cos\theta_1 - Q_2 \sin\theta_1,\; y = Q_1 \sin\theta_1 + Q_2 \cos\theta_1$). Next, you can see that the function $\Phi_{k,p}(p,y) = C_k C_p i^k e^{-\frac{p'^2}{2}} H_k(p') e^{-\frac{y'^2}{2}} H_p(y')$ has exactly the same structure as $\Phi_{k,p}(x,y)$ if you notice that $p' = p\cos\theta_2 - y \sin\theta_2,\; y' = y\cos\theta_2 + p\sin\theta_2$ ($\tan\theta_2 = \alpha$). As a result, then we get by substituting Eq.(S24) into Eq.(S21) (for clarity, let's add $A^{s_1,s_2}_{k,p} = A^{s_1,s_2}_{k,p}(\Theta)$)

$$A^{s_1,s_2}_{k,p}(\Theta) = \sum_{k_1,p_1} (-i)^{k_1} B^{s_1,s_2}_{k_1,p_1}(\theta_1) B^{*k,p}_{k_1,p_1}(\theta_2). \tag{S21}$$

The properties of the coefficient $B^{n,m}_{k,p}$ have been well studied before, see eg. [7, 8, 10] and it is equal to

$$B^{n,m}_{k,p}(\theta) = \frac{\mu^{k+n}\sqrt{k!p!}}{(1+\mu^2)^{\frac{s_1+s_2}{2}}\sqrt{n!m!}} P_k^{(-(1+s_1+s_2),p-n)}\left(-\frac{2+\mu^2}{\mu^2}\right), \;\; \mu = \tan\theta, \tag{S22}$$

where $P^{\alpha,\beta}_{\gamma}(x)$ are Jacobi polynomials and the condition $n+m = k+p$ is satisfied. From the properties of this coefficient, one can immediately say that the number of photons will be conserved $s_1 + s_2 = n + m$. This is an important conclusion of this theory. Further, it can be shown that there is a certain relation between the two angles θ_1 and θ_2, namely

$$\epsilon = \frac{\epsilon_1\epsilon_2}{\sqrt{1+\epsilon_1^2+\epsilon_2^2}}, \;\; \tan 2\theta_1 = \frac{1}{\epsilon_1}, \;\; \tan 2\theta_2 = \frac{1}{\epsilon_2}. \tag{S23}$$

Using the properties of the Jacobi polynomials and Eq.(S23), one can show that the angle Θ in Eq.(S21) satisfies the condition $\tan 2\Theta = 1/\epsilon$. Moreover, it turns out that the coefficients $B^{n,m}_{k,p}(\theta) = A^{n,m}_{k,p}(\theta)$. As a result, we get that

$$A^{n,m}_{k,p}(\Theta) = \frac{\mu^{k+n}\sqrt{k!p!}}{(1+\mu^2)^{\frac{s_1+s_2}{2}}\sqrt{n!m!}} P_k^{(-(1+s_1+s_2),p-n)}\left(-\frac{2+\mu^2}{\mu^2}\right), \;\; \mu = \tan\Theta. \tag{S24}$$

It was shown in [7] that Eq.(S21) can be represented in a more convenient form by expressing it in terms of the reflection coefficient and the phase shift.

As a result, we can find the probability of detecting the system in the final states n and m in the first and second modes, respectively, when the system transitions from the initial Fock state s_1, s_2 in the form $P_n = \langle |c_{n,s_1+s_2-n}|^2 \rangle$, where

$$
\begin{aligned}
c_{n,m} &= \sum_{k=0}^{s_1+s_2} A^{s_1,s_2}_{k,s_1+s_2-k} A^{*n,m}_{k,s_1+s_2-k} e^{-2ik\ \arccos\left(\sqrt{1-R}\sin\phi\right)}, \\
A^{n,m}_{k,p} &= \frac{\mu^{k+n}\sqrt{k!p!}}{(1+\mu^2)^{\frac{s_1+s_2}{2}}\sqrt{n!m!}} P_k^{(-(1+s_1+s_2),p-n)}\left(-\frac{2+\mu^2}{\mu^2}\right), \\
\mu &= \sqrt{1+\frac{1-R}{R}\cos^2\phi} - \cos\phi\sqrt{\frac{1-R}{R}},
\end{aligned}
\tag{S25}
$$

where $P^{\alpha,\beta}_\gamma(x)$ are Jacobi polynomials and the condition $n+m = s_1+s_2$ is satisfied, i.e. the number of photons is stored in the system. The coefficient R and ϕ have the meaning of the reflection coefficient, and ϕ are the phases, which will be equal

$$
\begin{aligned}
R &= \frac{\sin^2\left(\Omega t/2\sqrt{1+\epsilon^2}\right)}{(1+\epsilon^2)},\ \cos\phi = -\epsilon\sqrt{\frac{R}{1-R}}, \\
\Omega &= \frac{4\pi\left|\mathbf{u}_1\mathbf{u}_2\sum_a e^{i\Delta\mathbf{k}\mathbf{r}_a}\right|}{\omega_1 V},\ \epsilon = \frac{\omega_2-\omega_1}{\Omega},
\end{aligned}
\tag{S26}
$$

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