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Quantum Theory of Scattering of Nonclassical Fields by Free Electrons

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Abstract: At present, there is no non-perturbative theory of scattering of nonclassical electromagnetic waves by free electrons that describes the scattering process completely with the help of quantum physics. In this paper, such a theory is presented, which takes into account the statistics and the number of scattered photons. This theory is completely analytical for an arbitrary number of electrons in the system and, in a particular case, is equivalent to the previous theory of scattering as the number of incident photons tends to infinity. It is shown that this theory can differ greatly from the previously known theory of Thomson scattering in the non-perturbative case and at relatively small numbers of incident photons. In addition, this theory is applicable to the scattering of ultrashort pulses by free electrons.

Keywords: quantum scattering theory; nonclassical fields; non-perturbative theory; Thomson scattering; photons; electrons; analytical solution; Schrödinger equation; photon statistics

MSC: 81Uxx; 81Vxx; 35P25

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1. Introduction

The scattering of electromagnetic waves by free charged particles is currently well studied using classical physics and is being studied using quantum physics [1–6]. If we consider the energy of incident photons as $\hbar\omega \ll mc^2$, then such scattering is called Thomson scattering [1] (low-energy limit of Compton scattering), where ω is the electromagnetic field frequency, $\hbar$ is Planck's constant, m is the electron mass and c is the speed of light in a vacuum. Thomson scattering is one of the most important fundamental processes in electrodynamics [7]. This scattering is found in astrophysics [8], can serve as an important diagnostic for measuring high-temperature plasma [2,3] and is the basis for diffraction analyses of matter [6,9–11], as well as the basis for creating sources of high-energy X-ray radiation controlled by accelerators (Thomson or Compton sources) [12,13]. At present, in connection with the development of quantum optics, the problem of scattering of nonclassical electromagnetic fields by free electrons is of interest and relevant [14].

It is well known that the most complete theory describing the interaction of radiation with matter is based on quantum physics, i.e., where the radiation field, the substance with which it interacts and the scattered radiation are described completely by quantum physics. In the case of radiation scattering by free electrons, there is no complete quantum solution to this problem. Usually, such a solution is limited to the quantum consideration of electrons and scattered radiation [4,6,9], but the incident field is usually given classically. This is quite understandable, since it is mathematically difficult to consider the problem taking into account the quantum components of the incident radiation, and this problem has not been solved so far. For a long time, it was believed that there could be no significant corrections due to the quantum nature of electromagnetic fields. At a qualitative level, it seems that with a small number of photons, i.e., in the case of weak interactions, the dependence of the scattered radiation on the intensity should be linear, and therefore coinciding with the Thomson formula.

In this work, based on the exact solution of the Schrödinger equation, an analytical solution was found for a system of free electrons in a quantized field of incident and scattered radiation. Based on this solution, the probabilities of detecting m scattered photons and their statistics are obtained for given statistics and a given number of incident photons. The solution obtained when the number of incident photons tends to infinity coincides with the previously well-known scattering theory based on the concept of an external classical field. Furthermore, the result obtained can be extended to the case of interaction of electrons with ultrashort pulses (including nonclassical ones), and in this case, as the number of incident photons tends to infinity, our theory also passes into the previously known approach. At high intensities or a relatively small number of incident photons, our theory may differ significantly from the case of an external classical field using the Thomson formula, i.e., the quantum effects will be significant.

2. Solving Quantum Equations

Consider a multi-electron system interacting with a quantized electromagnetic field. Let us represent the electromagnetic field in terms of the transverse vector potential $\mathbf{A}$ in the Coulomb gauge $div\mathbf{A} = 0$ [15,16]. In this case, the Hamiltonian of such a system will consist of an incident electromagnetic field and a scattered one, as well as a multielectron system interacting with these fields. Let us first consider a two-mode electromagnetic field, where the mode with index $i = 1$ (with frequency ω_1 and polarization $\mathbf{u}_1$) corresponds to the incident electromagnetic field, and the mode with index $i = 2$ (with frequency ω_2 polarization $\mathbf{u}_2$) corresponds to the scattered field. It should be added that the consideration of the incident field as a two-mode field is quite obvious from the point of view of quantum electrodynamics. It is well known that in the simplest case, scattering from the point of view of quantum electrodynamics is described as the annihilation of an incident photon of a given mode and the birth of another photon of a given mode, after which summation is performed over all final states of the produced photon [17]. In quantum electrodynamics, this process is considered from the point of view of perturbation theory and is usually limited to the first, second and, very rarely, the third order of perturbation theory. Here, we consider the exact solution of this problem. In this case, the Hamiltonian of the entire system can be represented as (the atomic system of units to be used will be $\hbar = 1$, $|e| = 1$ and $m = 1$, where e is the electron charge):

$$\left\{ \hat{H}_1 + \hat{H}_2 + \frac{1}{2}\sum_a \left(\hat{\mathbf{p}}_a + \frac{1}{c}\hat{\mathbf{A}}_a \right)^2 \right\} \Psi = i\frac{\partial \Psi}{\partial t}, \tag{1}$$

where $\hat{H}_i = \omega_i \hat{a}_i^\dagger \hat{a}_i$ is the Hamiltonian for the 1st and 2nd modes; $\hat{a}_i$ and $\hat{a}_i^\dagger$ are the photon annihilation and creation operators for the i mode, respectively; $\hat{\mathbf{p}}_a$ is the electron momentum operator with the number a; $\hat{\mathbf{A}}_a = \hat{\mathbf{A}}_{1,a} + \hat{\mathbf{A}}_{2,a}$, where $\hat{\mathbf{A}}_{i,a} = \sqrt{\frac{2\pi c^2}{\omega_i V}} (\mathbf{u}_i \hat{a}_i e^{i\mathbf{k}_i \mathbf{r}_a} + \mathbf{u}_i^\dagger \hat{a}_i^\dagger e^{-i\mathbf{k}_i \mathbf{r}_a})$ this is the i mode vector potential acting on the a electron [15,16]; sum $\sum_a$ in Equation (1) is taken over all electrons of the system under consideration; and V is the volume of quantization of the electromagnetic field.

The solution to Equation (1) can be found analytically (in order not to mix mathematics with physics, the solution of this equation is given separately in the Supplementary Materials). As a result, we can find the probability of detecting the system in the final states n and m in the first and second modes, respectively, when the system transitions from the initial Fock state s_1, s_2 in the form $P_n = \langle |c_{n,s_1+s_2-n}|^2 \rangle$, where

$$
\begin{aligned}
c_{n,m} &= \sum_{k=0}^{s_1+s_2} A^{s_1,s_2}_{k,s_1+s_2-k} A^{*n,m}_{k,s_1+s_2-k} e^{-2ik\,\arccos\left(\sqrt{1-R}\sin\phi\right)}, \\
A^{n,m}_{k,p} &= \frac{\mu^{k+n}\sqrt{k!p!}}{(1+\mu^2)^{\frac{s_1+s_2}{2}}\sqrt{n!m!}} P_k^{(-(1+s_1+s_2),p-n)}\left(-\frac{2+\mu^2}{\mu^2}\right), \\
\mu &= \sqrt{1+\frac{1-R}{R}\cos^2\phi} - \cos\phi\sqrt{\frac{1-R}{R}},
\end{aligned} \tag{2}
$$

and $P^{\alpha,\beta}_{\gamma}(x)$ are Jacobi polynomials. The condition $n+m = s_1+s_2$ is satisfied, i.e., the number of photons is stored in the system [18]. The symbol $\langle ... \rangle$ means averaging over all electronic states, the coefficient R refers to the reflection coefficient and ϕ is the phases, which will be equal to

$$
\begin{aligned}
R &= \frac{\sin^2\left(\Omega t/2\sqrt{1+\epsilon^2}\right)}{(1+\epsilon^2)}, \ \cos\phi = -\epsilon\sqrt{\frac{R}{1-R}}, \\
\Omega &= \frac{4\pi\left|\mathbf{u}_1\mathbf{u}_2\sum_a e^{i\Delta\mathbf{k}\mathbf{r}_a}\right|}{\omega_1 V}, \ \epsilon = \frac{\omega_2-\omega_1}{\Omega},
\end{aligned} \tag{3}
$$

where $\Delta\mathbf{k} = \mathbf{k}_2 - \mathbf{k}_1$ is the recoil momentum during scattering and $\left|\sum_a e^{i\Delta\mathbf{k}\mathbf{r}_a}\right|$ is the modulus, which can also be represented in another form $\sqrt{\sum_{a,b}\cos(\Delta\mathbf{k}\Delta\mathbf{r})}$, where $\Delta\mathbf{r} = \mathbf{r}_a - \mathbf{r}_b$. It can be seen that the dependence on the coordinates, as well as the dependence on the momentum, is determined only by the difference. In Equation (2), no matter what value of $\phi \in (0, \pi/2)$ we choose, the value P_n will not depend on ϕ.

It is well known that scattering in quantum electrodynamics is the process of the creation of a photon from a vacuum state upon the annihilation of an incident photon [15,16]. In our case, the vacuum state will be the second mode, i.e., we assume that $s_2 = 0$, and the external incident field will be determined by the number of photons s_1 (hereinafter, for simplicity, we will rename $s_1 = s$). In this case, the probability P_m in its simplest form can be represented

$$
P_m = \frac{s!}{m!(s-m)!}\langle R^m(1-R)^{s-m}\rangle. \tag{4}
$$

Equation (4) is quite simple and obeys the well-known expansion in Newton's binomial (remember that the condition $s = m+n$ is satisfied). The most important thing in this expression is the determined value of the coefficient R. Despite the fact that the system under study is multi-parametric, the entire dependence was reduced to a single reflection coefficient R, which contains all the quantities of this problem.

3. Results

We have obtained the probability of the incident radiation mode interacting with the scattered radiation mode. In reality, in the vacuum state, there are an infinite number of modes, each mode of which interacts with the mode of the incident radiation via Equation (4). In this case, the number of produced photons will be determined not only by the number of produced photons with a certain mode, but also by the total number of produced photons from all modes. Let us denote the number of photons produced from the $\mathbf{k}$ mode as $m_{\mathbf{k}}$, and represent the dependence of the R coefficient on $\mathbf{k}$ as $R = R(\mathbf{k})$. Before presenting the average scattered photon energy $\bar{\varepsilon}$, it should be added that we must take into account the Bose statistics for such scattered photons. In order to do this, we first consider the case when photons of one $\mathbf{k}_0$ mode with a given polarization are incident, then $\bar{\varepsilon} = \sum_{\mathbf{k}'}\sum_{M_{\mathbf{k}}}\prod_{\mathbf{k}} P_{m_{\mathbf{k}}}\omega_{\mathbf{k}'} m_{\mathbf{k}'}$, where the sum $\mathbf{k}'$ is taken over all modes, including $\mathbf{k}' = \mathbf{k}$, and the sum over $M_{\mathbf{k}}$ is taken over all quantum numbers $m_{\mathbf{k}}$ and all modes $\mathbf{k}$, i.e., $M_{\mathbf{k}} = m_1, m_2, m_3 ....$ This way of finding $\bar{\varepsilon}$ will take into account all possible combinations of photons determined by Bose statistics. As a result, the average energy of scattered photons will be $\bar{\varepsilon}$, and we obtain

$$\bar{\varepsilon} = \sum_{\mathbf{k}} \sum_{m_{\mathbf{k}}=0}^{s} m_{\mathbf{k}} \omega_{\mathbf{k}} P_{m_{\mathbf{k}}} = s \int \omega_{\mathbf{k}} \langle R(\mathbf{k}) \rangle \frac{d^3\mathbf{k} V}{(2\pi)^3}, \tag{5}$$

where $\sum_{\mathbf{k}}$ is the summation over all modes of scattered radiation, including the summation over polarizations. It should be added that the first term in Equation (5) is well known in quantum electrodynamics scattering calculations [17]. Equation (5) is applicable for the scattering of s photons of one mode $\mathbf{k}_0$ in the Fock state. Following the general rules of probability, in order to take into account the different statistics of incident photons with different modes, which in the general case should obey the Bose statistics, it is necessary that

$$\bar{\varepsilon} = \sum_{\mathbf{k}_0} \sum_{s_{\mathbf{k}_0}} \mathcal{P}_{s_{\mathbf{k}_0}} s_{\mathbf{k}_0} \int \omega_{\mathbf{k}} \langle R(\mathbf{k}) \rangle \frac{d^3\mathbf{k} V}{(2\pi)^3}, \tag{6}$$

where $\mathcal{P}_{s_{\mathbf{k}_0}}$ is the probability of detecting $s_{\mathbf{k}_0}$ photons in the $\mathbf{k}_0$ mode. It should be added that the coefficient R depends on the frequency $\omega_0 = |\mathbf{k}_0|c$ (previously, we denoted it as ω_1). If we consider monochromatic incident photons with frequency ω_0, then $\sum_{\mathbf{k}_0} \to 1$.

Let us calculate $\bar{\varepsilon}$, and for this we take into account that $\epsilon \gg 1$, except for the case when the frequencies of the incident and scattered photons coincide. This is a direct consequence of the fact that the quantity $\Omega/\omega \ll 1$. Using this condition, one can obtain that

$$\frac{d\bar{\varepsilon}}{d\Omega_{\mathbf{n}}} = \frac{1}{\pi^2 c^3} \sum_{\mathbf{k}_0} \bar{s}_{\mathbf{k}_0} \omega_{\mathbf{k}_0}^2 \left\langle \left| \sum_a e^{i\Delta\mathbf{k}_0 \mathbf{r}_a} (\mathbf{u}_0 \times \mathbf{n}) \right| F(x) \right\rangle,$$
$$x = \frac{4\pi t}{\omega_{\mathbf{k}_0} V} \left| (\mathbf{u}_0 \times \mathbf{n}) \sum_a e^{i\Delta\mathbf{k}_0 \mathbf{r}_a} \right| \tag{7}$$

where $\Omega_{\mathbf{n}}$ is the solid angle into which scattering occurs, $\bar{s}_{\mathbf{k}_0} = \sum_{s_{\mathbf{k}_0}} \mathcal{P}_{s_{\mathbf{k}_0}} s_{\mathbf{k}_0}$, $\Delta\mathbf{k}_0 = \omega_0/c(\mathbf{n} - \mathbf{n}_0)$ is the electron recoil momentum during elastic photon scattering and the unit vectors $\mathbf{n} = \mathbf{k}/|\mathbf{k}|$, $\mathbf{n}_0 = \mathbf{k}_0/|\mathbf{k}_0|$. The function $F(x)$ is an important dependence in our theory, not just in the case of coherent incident radiation, and is shown in Figure 1. It does not have a precise analytical form and is calculated as

$$F(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2 - \pi H_{-1}(x\sqrt{1+\epsilon^2})}{1+\epsilon^2} d\epsilon,$$
$$F(x) = \begin{cases} \frac{\pi}{4}x, x \ll 1 \\ 1 - \frac{\cos(x)}{x}, x \gg 1, \end{cases} \tag{8}$$

where $H_\nu(x)$ is a Struve function. Furthermore, the function $F(x)$ has a very accurate analytical approximation (error $\ll 1\%$) in the form

$$F(x) = \frac{\pi}{4} \frac{x}{1 + 0.1263x^{3.95}} + \frac{0.1044x^{3.657}}{1 + 0.1044x^{3.657}} \left(1 - \frac{\cos(x)}{x}\right). \tag{9}$$

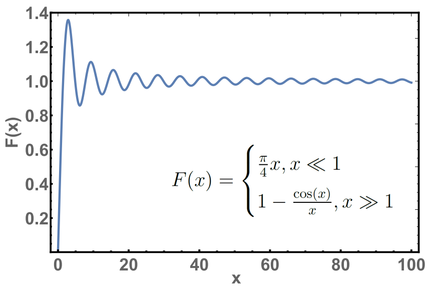

Figure 1. The dependence of the function $F(x)$ on the dimensionless parameter x is presented.

Next, we consider one of the special cases, which is related to Thomson scattering by classical incident radiation. It is well known that the quantum electromagnetic field volume V can be represented in terms of the electric field amplitude [15,16] as $\frac{E_0^2}{8\pi}V = \sum_{\mathbf{k}_0} \bar{s}_{\mathbf{k}_0}\omega_{\mathbf{k}_0}$. Let the number of incident photons approach infinity, i.e., $V \to \infty$. Indeed, an infinite number of photons can exist only in an infinite volume (of course, if we assume that the electric field E_0 is not infinite). Then, we obtain the dependence $F(x) = \frac{\pi}{4}x$, and as a result we also obtain

$$\frac{d\bar{\varepsilon}}{d\Omega_{\mathbf{n}}} = \frac{E_0^2 t}{8\pi c^3}\sum_{\mathbf{k}_0}\left|(\mathbf{u}_0\times\mathbf{n})\sum_a e^{i\Delta\mathbf{k}_0\mathbf{r}_a}\right|^2 P_{\varepsilon_{\mathbf{k}_0}}, \tag{10}$$

where $P_{\varepsilon_{\mathbf{k}_0}} = \frac{\bar{s}_{\mathbf{k}_0}\omega_{\mathbf{k}_0}}{\sum_{\mathbf{k}_0}\bar{s}_{\mathbf{k}_0}\omega_{\mathbf{k}_0}}$ is the probability of detecting photons with energy $\bar{s}_{\mathbf{k}_0}\omega_{\mathbf{k}_0}$. For example, in the case of linearly polarized monochromatic incident radiation in one direction, one can obtain

$$\frac{d^2\bar{\varepsilon}}{dt d\Omega_{\mathbf{n}}} = \frac{E_0^2}{8\pi c^3}\langle\left|(\mathbf{u}_0\times\mathbf{n})\sum_a e^{i\Delta\mathbf{k}_0\mathbf{r}_a}\right|^2\rangle. \tag{11}$$

Of course, Equation (11) is easy to change in the case of non-linearly polarized light, for example, for unpolarized light, $|\mathbf{u}_0\times\mathbf{n}|^2 \to \overline{|\mathbf{u}_0\times\mathbf{n}|^2} = \frac{1}{2}(1+(\mathbf{n}_0\mathbf{n})^2)$. Equation (11) coincides with the expression for the scattering of a classical electromagnetic field by free electrons [7,17]. In the case of one electron in the system, the average dependence in Equation (11) becomes $\langle ... \rangle = 1$ and we get the well-known Thomson formula.

Equation (10) can be extended to the case of using classical ultrashort laser pulses (USPs). In the USP scattering theory, the pulse interaction time is equal to the pulse duration τ, i.e., $t = \tau$ [9,19–22]. In our case, the time t is the time of interaction of the incident radiation with electrons. Thus, if we assume in our theory that $t = \tau$, and $P_{\varepsilon_{\mathbf{k}_0}}$ are no longer equal to one, then we will get an extension of our theory to the case of interaction with USPs. Indeed, usually the frequency distribution in USPs is concentrated near the carrier frequency with dispersions of the order of $\sim 1/\tau$, i.e., $(\omega_{\mathbf{k}_0} - \omega_c) \lesssim 1/\tau$. Usually, when talking about USPs, this represents the dependence of the electric field on time, i.e., the form of the USPs. Equation (10) can also be easily represented in this form if we consider the expression $E_0^2\tau P_{\varepsilon_{\mathbf{k}_0}}$. This expression can be expressed in another form: $E_0^2\tau P_{\varepsilon_{\mathbf{k}_0}} = \left|E_0\sqrt{\tau P_{\varepsilon_{\mathbf{k}_0}}}\right|^2$, where $E_0\sqrt{\tau P_{\varepsilon_{\mathbf{k}_0}}} = \frac{1}{\sqrt{2\pi\tau}}\int_{-\infty}^{\infty}E(t)e^{i\omega_{\mathbf{k}_0}t}dt$. Then, $E(t) = \frac{\tau}{\sqrt{2\pi}}\int_{-\infty}^{\infty}E(\omega_{\mathbf{k}_0})e^{-i\omega_{\mathbf{k}_0}t}d\omega_{\mathbf{k}_0}$ (designation introduced $E(\omega_{\mathbf{k}_0}) = E_0\sqrt{P_{\varepsilon_{\mathbf{k}_0}}}$). For example, if we choose a Gaussian frequency distribution $P_{\varepsilon_{\mathbf{k}_0}} = \frac{1}{\pi}e^{-\frac{1}{\pi}(\omega_{\mathbf{k}_0}-\omega_c)^2\tau^2}$ ($\int P_{\varepsilon_{\mathbf{k}_0}}d(\omega_{\mathbf{k}_0}\tau) = 1$), then $E(t) = E_0 e^{-i\omega_c t}e^{-\frac{\pi}{2}(\frac{t}{\tau})^2}$. It can be observed that as $\tau \to \infty$, we obtain the well-known expression for the electric field of a plane wave $E(t) = E_0 e^{-i\omega_c t}$. As a result, the scattered energy of the USP will be

$$\frac{d\bar{\varepsilon}}{d\Omega_{\mathbf{n}}} = \frac{\tau}{8\pi c^3}\int\sum_{\mathbf{k}_0}|E(\omega_{\mathbf{k}_0})|^2\langle\left|(\mathbf{u}_0\times\mathbf{n})\sum_a e^{i\Delta\mathbf{k}_0\mathbf{r}_a}\right|^2\rangle. \tag{12}$$

Equation (12) completely coincides with the quantum theory of scattering of classical ultrashort pulses [9].

As an example, consider the case of scattering of monochromatic photons by one electron, but with any statistics of the incident photons. In this case, we normalize the scattered energy to the energy ($\bar{\varepsilon}_s$) at $F(x\to\infty)\to 1$ (let us call $\bar{\varepsilon}_s$ the saturation energy). As a result, we obtain

$$\bar{\varepsilon} = \bar{\varepsilon}_s\delta_s,\ \bar{\varepsilon}_s = \frac{\hbar e^2}{m}\frac{\bar{s}\omega_0^2}{c^3}(CGS\ s.u.),$$
$$\delta_s = \int\frac{|\mathbf{u}_0\times\mathbf{n}|F(x|\mathbf{u}_0\times\mathbf{n}|)}{\pi^2}d\Omega_{\mathbf{n}}, \tag{13}$$

where $x = \frac{E_0^2 t}{2\omega_0^2 \bar{s}}$ (ω_0 is the frequency of the incident radiation) and $\bar{s} = \sum_s \mathcal{P}_s s$ is the average number of photons with statistics $\mathcal{P}_s$, see Figure 2. It should be said that the parameter x can be expressed in terms of the known pondermotive energy $x = \frac{2U_p t}{\bar{s}}$ (where $U_p = \frac{2\pi I}{c\omega_0^2}$ is the ponderomotive energy and I is the intensity of the incident radiation).

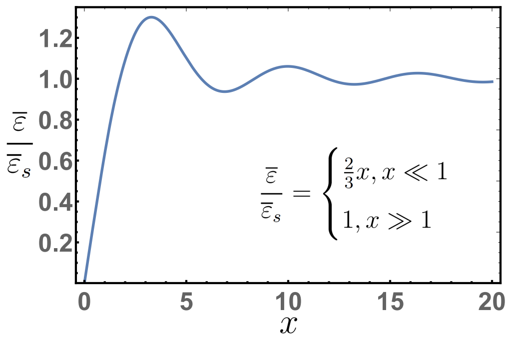

Figure 2. The normalized energy $\frac{\bar{\varepsilon}}{\bar{\varepsilon}_s}$ depending on the dimensionless parameter x.

The physical meaning of the energy $\bar{\varepsilon}_s$ is clear based on the fact that such an energy is obtained at $t \to \infty$; therefore, it is called the saturation energy. In addition, the energy of $\bar{\varepsilon}_s$ can be represented as $\bar{\varepsilon}_s = \varepsilon_0 \alpha \frac{\hbar\omega_0}{mc^2}$, where $\alpha = \frac{e^2}{\hbar c}$ is the fine-structure constant and $\varepsilon_0 = \hbar\omega_0\bar{s}$ is the energy of the incident radiation. It can be seen from this expression that the scattered energy is always many times lower than the incident radiation energy ($\alpha \approx 1/137$, $\frac{\hbar\omega_0}{mc^2} \ll 1$), i.e., $\bar{\varepsilon}_s \ll \varepsilon_0$. It is also seen that there is a maximum value of the scattering energy. The value of this energy is easy to calculate and will be $\bar{\varepsilon}_{max} = 1.301\bar{\varepsilon}_s$ for $x = 3.287$. There is also a connection between the Thomson scattering energies $\bar{\varepsilon}_T$ and $\bar{\varepsilon}_s$ considered here. It is easy to show that the relation is $\bar{\varepsilon}_T = 2/3\bar{\varepsilon}_s x$.

4. Conclusions

In this work, a general theory of radiation scattering by free electrons was developed. This theory is completely quantized and takes into account the statistics and the number of incident and scattered photons. In this theory, when the number of incident photons tends to infinity, the theory is equivalent to the previously known theory of scattering, including the theory of scattering of ultrashort pulses. The theory presented here has been developed in a general form and takes into account an arbitrary number of electrons in the system and also has a simple analytical form. It should be added that the theory presented here is not the first of this kind of research (e.g., [14] and the references therein), but the first where the incident and scattered electromagnetic field (including nonclassical) for an arbitrary number of electrons is taken into account nonperturbatively.

A significant advantage of this theory is that it takes into account an arbitrary number of electrons in the system (in the general case, charged particles). It is well known that scattering can be coherent or incoherent, which is considered in our theory. In other words, depending on the spatial distribution of the electron density, either coherent or incoherent scattering can dominate or both types of scattering can be comparable. This is usually an important factor in the scattering of radiation in a plasma. If we take into account the quantum nature of the incident radiation, i.e., when the number of photons is not infinite, then, obviously, the role of coherence and incoherence may differ from previously known cases. Thus, the scattering of nonclassical radiation (with a finite number of incident photons or their nonclassical statistics) by a multielectron system is one of the applications of this theory. It should be added that the result obtained is based on the solution of Equation (1) and the final result has a simple analytical form, which simplifies the use of the developed theory. It can also be seen that, quite "beautifully" from a mathematical point of view, when the number of incident photons tends to infinity, our result turns into the one previously known in scattering theory (see Equations (10) and (11)), which confirms the correctness of the solution to Equation (1) and the developed theory.

It should be added that this theory is suitable for any electromagnetic fields where no relativistic effects arise when it interacts with electrons, i.e., for fields where $\hbar\omega \ll mc^2$ and $\frac{|e|E_0}{m\omega c} \ll 1$. Thus, this theory can be applied to both classical and nonclassical electromagnetic fields. It should be added that this theory will also work for quantum entangled photons, squeezed light or single photons. To do this, it suffices to determine the statistical properties of these fields $\mathcal{P}_{s_{\mathbf{k}_0}}$ and, as a result, the average number of incident photons $\bar{s}_{\mathbf{k}_0} = \sum_{s_{\mathbf{k}_0}} \mathcal{P}_{s_{\mathbf{k}_0}} s_{\mathbf{k}_0}$, see Equation (7). How the scattering will depend on such "nonclassical" properties will be determined by the specific case and is one of the applications of this theory.

In general, this theory will expand the limits of using the scattering of electromagnetic waves in matter, since this theory moves into the area of quantum optics. Indeed, quantum optics studies the quantum properties of the interaction of radiation with matter, and this theory points to just such properties.

Supplementary Materials: The following supporting information can be downloaded at: https://www.mdpi.com/article/10.3390/math11092094/s1, File S1: Quantum theory of scattering of nonclassical fields by free electrons. References [7,15–18,23–27] are cited in the Supplementary Materials.

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