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On Symmetric Weighing Matrices

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Abstract: Symmetric weighing matrices are an important class of combinatorial designs and their constructions remains unknown even for some small orders. The existence of symmetric weighing matrices becomes an even more interesting challenge as these matrices possess both beautiful mathematical properties and many applications. In this paper, we present two original construction methods for symmetric weighing matrices. The suggested methods lead to two infinite families of symmetric weighing matrices. The first consists of symmetric weighing matrices $W(2^p \times 15, 2^q \times 25)$ for all $q < p$ and $p = 1, 2, \ldots$ while the second is an infinite family of symmetric weighing matrices $W(2^{p+1} \times 15, 2 + 2^q \times 25)$ for all $q \leq p$ and $p = 1, 2, \ldots$. These matrices are constructed by combining together a circulant and a negacyclic matrix with identical first row. The first of the infinite families includes a symmetric weighing matrix of order 30 and weight 25. The results presented here are new and have never been reported.

Keywords: sequences; autocorrelation; construction; circulant matrix; negacyclic matrix

MSC: 62K05; 62K15

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1. Introduction and Preliminaries

A weighing matrix $W = W(n,k)$ is a square matrix of order n with entries 0, ± 1 having k non-zero entries per row and column and whose inner product of distinct rows (or columns) is zero. Hence, W satisfies $WW^T = W^TW = kI_n$. The number k is called the *weight* of W. If a weighing matrix $W = W(n,k)$ is equal to its transpose W^T then W is called *symmetric weighing matrix* of order n and weight k. Note that W^T can be looked upon as the design matrix of a chemical balance weighing design. It was shown by [1,2] that if the variance of the errors of the weights obtained by individual weighings is σ^2 (it is assumed the balance is not biased and the errors are mutually independent and normal), then using a weighing matrix $W(n,k)$ to design an experiment to weigh n objects will give a variance of σ^2/k. Under standard linear model assumptions, following [3], it is not hard to check that such a design is universally optimal in the class of all n-observation chemical balance weighing designs, with n objects, such that at most k objects are used in each weighing. More applications of weighing matrices and further results can be found in [4]. A great review of symmetric weighing matrices of order n and weight $k = n - 1$, known as symmetric conference matrices, is given by [5]. Weighing matrices constructed from two to circulant matrices were extensively studied by the excellent review paper of [6]. Recently, negacyclic matrices were extensively applied to construct weighing matrices in [7]. Note that the matrices presented in that paper are not necessarily symmetric. Hence, the study of weighing matrices is important, among others, from the perspective of optimal statistical design theory. In the present paper, we propose a method to obtain families of symmetric weighing matrices.

For the construction of the new infinite classes of weighing matrices we need the following definitions and notation.

Let $A = (a_0, a_1, \ldots, a_{n-1})$ be a sequence of length n. The *non-periodic autocorrelation function* $N_A(s)$ (abbreviated as NPAF) of the above sequence is defined as

$$N_A(s) = \sum_{i=0}^{n-s-1} a_i a_{i+s}, \qquad s = 0, 1, \ldots, n-1, \tag{1}$$

and the *periodic autocorrelation function* $P_A(s)$ (abbreviated as PAF), is defined, reducing $i+s$ modulo n, as

$$P_A(s) = \sum_{i=0}^{n-1} a_i a_{i+s}, \qquad s = 0, 1, \ldots, n-1. \tag{2}$$

Circulant matrices of order n are polynomials in the shift matrix

$$S = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & & 1 \\ 1 & 0 & 0 & & 0 \end{pmatrix}.$$

For example, a circulant matrix with first row $A = (a_0, a_1, a_2, \ldots, a_{n-1})$ is the matrix $a_0 S^0 + a_1 S^1 + a_2 S^2 + \ldots + a_{n-1} S^{n-1} = \sum_{i=0}^{n-1} a_i S^i$.

Negacyclic matrices of order n are polynomials in the negashift matrix

$$N = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & & 1 \\ -1 & 0 & 0 & & 0 \end{pmatrix}.$$

For example, a negacyclic matrix with first row $A = (a_0, a_1, a_2, \ldots, a_{n-1})$ is the matrix $a_0 N^0 + a_1 N^1 + a_2 N^2 + \ldots + a_{n-1} N^{n-1} = \sum_{i=0}^{n-1} a_i N^i$. Note that the negashift matrix N is equal to the shift matrix S multiplied by a diagonal matrix $D = diag(-1, 1, 1, \ldots, 1)$ of order n with -1 at the $(1,1)$-position and 1 elsewhere in the diagonal (i.e., $N = SD$). More details on shift matrix and the negashift matrix can be found in [8].

Notation. We use the following notations throughout this paper.

1. A *circulant matrix* of order n with first row $A = (a_0, a_1, \ldots, a_{n-1})$ will be denoted as $Circ(A)$, while a *negacyclic matrix* of order n with first row $A = (a_0, a_1, \ldots, a_{n-1})$ will be denoted as $NCycl(A)$.
2. We use $P = (p_{ij})$ to denote the $n \times n$ *monomial matrix* whose elements satisfy $p_{ij} = \begin{cases} (-1)^{i+1}, & when\ \ i = j \\ 0, & otherwise. \end{cases} \quad i, j = 1, 2, \ldots, n$
3. If $A = (a_1, a_2, \ldots, a_n)$ and $B = (b_1, b_2, \ldots, b_m)$ are two sequences of lengths n and m, respectively, then the *concatenate sequence* C is defined as

$$C = (A, B) = (c_1, c_2, \ldots, c_{n+m}) = (a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_m).$$

If $A = (0)$ is a sequence of length 1 then we write

$$C = (A, B) = (0, B) = (c_1, c_2, \ldots, c_{m+1}) = (0, b_1, b_2, \ldots, b_m).$$

4. We use $-$ to denote -1 and $+$ to denote $+1$.
5. By D^T we denote the transpose of the matrix D.

6. We use $0_{n\times m}$ to denote the $n \times m$ matrix with all its elements equal to 0. If no confusion is caused, a square matrix of order n with all its entries equal to zero will be denoted by 0_n.

Additional details on the above definitions and notation can be found in [8].

In this paper, we construct two infinite families of symmetric weighing matrices. In Section 2, a symmetric weighing matrix of order 30 and weight 25 is constructed by combining together a circulant and a negacyclic matrix with identical first row. In Section 3, we generalise the results of Section 2 and we present two infinite families of symmetric weighing matrices. The first family is $W(2^p \times 15, 2^q \times 25)$ for all $q < p$ and $p = 1, 2, \ldots$ while the second is $W(2^{p+1} \times 15, 2 + 2^q \times 25)$ for all $q < p$ and $p = 1, 2, \ldots$. Open problems for symmetric weighing matrices includes the construction of symmetric weighing matrices $W(n, w)$ for a number of different values as shown in Table 2.88 of [9]. With our infinite classes we eliminate the value $W(30, 25)$ by proving and presenting existence.

2. A Symmetric Weighing Matrix of Order 30 and Weight 25

In this section, we present a method that combines a circulant and a negacyclic matrix to construct a symmetric weighing matrix. This method was applied to search for symmetric weighing matrices of order 30 and weight 25. The validity of the method is proved in Theorem 1 while the new symmetric weighing matrices $W(30, 25)$ are given in Corollary 1.

Theorem 1. *Let $A = (a_1, a_2, \ldots, a_{2n-1})$ be a sequence of length $2n - 1$ and entries from the set $\{0, 1, -1\}$. The sequence A satisfies:*

(i) $a_i = (-1)^{i+n} a_{2n-i}$, *for all* $i = 1, 2 \ldots, 2n - 1$,
(ii) $N_A(2s) = N_A(2n - 2s)$, *for all* $s = 1, 2, \ldots, n - 1$,

and we set $B = (0, A)$. Then the matrix

$$D = (Circ(B) + NCycl(B))P/2 + [(Circ(B) + NCycl(B))P/2]^T$$

is a symmetric weighing matrix of order $2n$ and weight $w = \sum_{i=1}^{2n-1} |a_i|$.

Proof. To prove that D is a symmetric weighing matrix we have to show that D is a symmetric matrix and satisfy $DD^T = D^T D = wI_{2n}$. From the definition of matrix D it is obvious that $D^T = D$ and thus D is a symmetric matrix. Moreover, due to the construction it is clear that there are exactly w non-zero elements in each row and column of D. If we show that the columns and rows are pairwise orthogonal the result will follow.

Suppose that the rows (and columns) of D are enumerated by $1, 2, \ldots, 2n$. We shall work with rows since similar results can be derived for columns as well. We skip many pages of routine calculations for the inner product of two rows and we end up to the following results. The inner product $< R_i, R_j >$, of two rows R_i and R_j with even indices difference (i.e., $|i - j| = 2s$) is equal to $N_A(2s) - N_A(2n - 2s)$ and this is equal to zero from property (ii). The inner product $< R_i, R_j >$, of two rows R_i and R_j with odd indices difference (i.e., $|i - j| = 2t - 1$), for some integer t, is a multiple of $(a_\ell - (-1)^{\ell+n} a_{2n-\ell})$ for some $\ell \in \{1, \ldots, 2n - 1\}$ and this is equal to zero from property (i). The result follows.

The smallest example that we are able to generate from the above infinite family is given in Corollary 1. We chose this small example to be able to visualise the results and illustrate the methodology and by presenting the full matrices that are needed for the construction. The other symmetric weighing matrices that are generated from the infinite families given here, would be too big to be explicitly presented in this paper. □

Corollary 1. *There exist a symmetric weighing matrix of order* 30 *and weight* 25.

Proof. Use any of the sequences given in the Appendix A and apply Theorem 1. □

We shall illustrate the construction of a symmetric weighing matrix of order 30 and weight 25 by the next analytical example.

Example 1. *We select the first sequence given in Appendix A, that is* $A = (++++0++-$ $-0-+--++---0-++-0-+-+)$. *We set* $B = (0, A) = (0++++0++--0-$ $+--++---0-++-0-+-+)$ *as it is described in Theorem 1. Note that the symbols* $+$ *and* $-$ *are as denoted in the notation (point 4.). The next step is to construct the circulant* $Circ(B)$ *and the negacyclic matrix* $NCycl(B)$ *of order* 30 *with first row* B. *We also need the monomial matrix* P *of order* 30 *which is as defined in the notation. Finally, the desirable symmetric weighing matrix of order* 30 *and weight* 25 *is* $D = (Circ(B) + NCycl(B))P/2 + [(Circ(B) + NCycl(B))P/2]^T$. *It is easy to verify that* D *is symmetric, satisfies* $DD^T = D^TD = 25I_{30}$ *and thus* D *is the desirable symmetric weighing matrix of order* 30 *and weight* 25. *Matrices* $Circ(B)$, $NCycl(B)$ *and* D *are given explicitly in the Appendix B, at the end of the paper.*

In the next section we give some infinite families of symmetric weighing matrices.

3. Infinite Families of Symmetric Weighing Matrices

The following Theorem gives a doubling method that retains the symmetry of the matrices.

Theorem 2. *Let* $W_0 = W(n, w)$ *be a symmetric weighing matrix of order* n *and weight* w. *Then there exists a symmetric weighing matrix* (i) $W_t = W(n2^t, w2^t)$ *and* (ii) $Q_t = W(n2^t, w)$ *for all* $t = 1, 2, \ldots$.

Proof. Set $Q_0 = W_0$ and

$$(i)\ W_t = \begin{pmatrix} W_{t-1} & W_{t-1} \\ W_{t-1} & -W_{t-1} \end{pmatrix} \text{ and } (ii)\ Q_t = \begin{pmatrix} Q_{t-1} & 0_{n2^{t-1}} \\ 0_{n2^{t-1}} & Q_{t-1} \end{pmatrix}, t = 1, 2, \ldots.$$

With simple matrix calculations using W_t and Q_t as defined above we have that

$$W_t = W_t^T,\ \ Q_t = Q_t^T,\ \ W_tW_t^T = W_t^2 = w2^tI_{n2^t},\ \ and\ \ Q_tQ_t^T = Q_t^2 = wI_{n2^t}.$$

The results follow. □

Using the results of Theorem 2 we obtain the following result.

Corollary 2. *Let* $W_0 = W(n, w)$ *be a symmetric weighing matrix of order* n *and weight* w. *Then there exists an infinite family of symmetric weighing matrices* $W(n2^p, w2^q)$ *of order* $n2^p$ *and weight* $w2^q$ *for all* $q \leq p$ *and* $p = 0, 1, \ldots$.

Proof. For $p = q$ the result follows by applying Theorem 2(i) with $t = p$. For $q < p$ we apply Theorem 2(i) with $t = 1, 2, \ldots, q$ to obtain a symmetric weighing matrix $W_q = W(n2^q, w2^q)$. Then we set $Q_q = W_q$ and we apply Theorem 2(ii) with $t = q+1, \ldots, p$ to obtain the desirable symmetric weighing matrix $Q_p = W(n2^p, w2^q)$.

By using the new symmetric weighing matrices of order 30 and weight 25 we present our first infinite family of symmetric weighing matrices in the next example. □

Example 2. *Let* $W_0 = W(30, 25)$ *be any of the new symmetric weighing matrices of order* 30 *and weight* 25 *which are constructed in Corollary 1. We use* W_0 *in Corollary 2 and we obtain an infinite family of symmetric weighing matrices* $W(2^p \times 15, 2^q \times 25)$, $q < p$, $p = 1, 2, \ldots$.

The next Theorem doubles the order and extends the weight of a symmetric weighing matrix and also conserves the symmetry of the generated matrix.

Theorem 3. *Let $Z_0 = W(n, w)$ be a symmetric weighing matrix with zero diagonal. Then there exists an infinite family of symmetric weighing matrices $Z_p = W(n2^p, w2^q + 2)$ for any $q \leq p$ and $p = 1, 2, \ldots$.*

Proof. Using the given Z_0 we can construct a symmetric weighing matrix $Z_{p-1} = W(n2^{p-1}, w2^{q-1})$ by applying Corollary 2. The matrix Z_0 has zero diagonal and, due to the construction, the symmetric weighing matrix Z_{p-1} also has zero diagonal. Set

$$Z_p = \begin{pmatrix} Z_{p-1} + I_{n2^{p-1}} & Z_{p-1} - I_{n2^{p-1}} \\ Z_{p-1} - I_{n2^{p-1}} & -Z_{p-1} - I_{n2^{p-1}} \end{pmatrix}.$$

The matrix Z_p is a square matrix of order $n2^p$ with $w2^q + 2$ non-zero elements in each row and column. We have that $Z_p^T = Z_p$ and so Z_p is a symmetric matrix. The only thing that is left to be shown is that $Z_p Z_p^T = Z_p^2 = (w2^q + 2) I_{n2^p}$. We have

$$Z_p^2 = \begin{pmatrix} 2Z_{p-1} + 2I_{n2^{p-1}} & 0_{n2^{p-1}} \\ 0_{n2^{p-1}} & 2Z_{p-1} + 2I_{n2^{p-1}} \end{pmatrix} = (w2^q + 2) I_{n2^p}.$$

Thus, Z_p is the desirable symmetric weighing matrix of order $n2^p$ and weight $w2^q + 2$, for all $q \leq p$ and $p = 1, 2, \ldots$. □

In the next example we give a second infinite family of symmetric weighing matrices by using Theorem 3 and the new symmetric weighing matrices given in Corollary 1.

Example 3. Let $Z_0 = W(30, 25)$ be any of the new symmetric weighing matrices given in Corollary 1. We use Z_0 in Theorem 3 and we obtain an infinite family of symmetric weighing matrices $W(2^{p+1} \times 15, 2 + 2^q \times 25)$ for all $q \leq p$ and $p = 1, 2, \ldots$. The proof (and the construction) follows as the proof of Theorem 3.

4. Conclusions

Constructing weighing designs is a considerably hard and interesting problem. In addition, the existence of symmetric weighing matrices is even more interesting challenge both for their beautiful mathematical properties and their applications. Symmetric weighing matrices have important properties and applications in many fields. For example, they are used in statistical experimental designs to minimise the variance in weight experiments. They can also be used in the engineering of spectrometers, image scanners and optical multiplexing systems (see for example [10,11]. This is because of the design of these instruments, which involves an optical mask and two detectors that measure the intensity of light. A symmetric weighing matrix is directly applicable here as the result of the subtraction of the measurements from the two detectors in the system results in three cases that correspond to weighing matrix levels of 1, 0 and −1, respectively. For more details on such applications we refer to [11]. The state of the art for weighing matrices and symmetric weighing matrices can be found in the excellent reference by Craigen and Kharaghani (2007). In our view, combining circulant and negacirculant matrices provides additional tools for generating structures such as weighing matrices and orthogonal designs. The presented methodology may be possible to be expanded to the case of more complicated designs such as the variable based orthogonal designs (see for example [4]) and symmetric or skew-symmetric structures (see for example Craigen, R. and Kharaghani, H. (2007)).

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Appendix A. Sequences Needed for the Proof of Corollary 1

Any of the sequences given here can be used to generate the desirable symmetric weighting matrix of order 30 and weight 25. These sequences were generated by a computer search. The results presented here are not exhaustive and not necessarily inequivalent. Any of these pairs of sequences can be used to illustrate the construction methodology as specified at Example 1.

Seq. No.	Sequence A
1	$++++0++--0-+--++---0-++-0-+-+$
2	$+++--+++-------+-+-+--+--++-+$
3	$+++---++-+-+---+------++-++-+$
4	$++-+++++---++--+-++-+-+---+$
5	$++-++-++++-+-++-----+-+++---+$
6	$++-+0++-+0-+---+---0+++-0---+$
7	$+-++-++------+---+-+-++---+++$
8	$+-++--+--+-+-+-------+++--+++$
9	$+-+-0-++-0---++--+-0--++0++++$
10	$+---+++-+-----++-+-++++-++-++$
11	$+---+-+-++-+--++----+++++-++$
12	$+---0-+++0---+---+-0+-++0+-++$

Appendix B. Matrices for Example 1

In these appendices we explicitly present the generate matrices needed to illustrate the provided methodology. Even though this is just an example of the infinitely many matrices generated by the suggested methods, the dimensions of the symmetric weighing matrix $SW(30, 25)$ still allows for the matrices to be fully displaced in the paper. This provides an additional illustration of how circulant and negacyclic matrices are generated and how they are then combined into the $SW(30, 25)$.

$$Circ(B) = \begin{pmatrix}
0++++0++--0-+--++---0-++-0-+-+ \\
+0++++0++--0-+--++---0-++-0-+- \\
-+0++++0++--0-+--++---0-++-0-+ \\
+-+0++++0++--0-+--++---0-++-0- \\
-+-+0++++0++--0-+--++---0-++-0 \\
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+++0++--0-+--++---0-++-0-+-+0+ \\
++++0++--0-+--++---0-++-0-+-+0
\end{pmatrix}$$

$$NCycl(B) = \begin{pmatrix}
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\end{pmatrix}$$

$$D = (Circ(B) + NCycl(B))P/2 + [(Circ(B) + NCycl(B))P/2]^T =$$

$$\begin{pmatrix}
0&-&+&-&+&0&+&-&-&+&0&+&+&+&-&-&+&+&-&+&0&+&+&-&-&0&-&-&-&- \\
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\end{pmatrix}$$

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