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Asymptotics of Regular and Irregular Solutions in Chains of Coupled van der Pol Equations

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Abstract: Chains of coupled van der Pol equations are considered. The main assumption that motivates the use of special asymptotic methods is that the number of elements in the chain is sufficiently large. This allows moving from a discrete system of equations to the use of a continuity argument and obtaining an integro-differential boundary value problem as the initial model. In the study of the behaviour of all its solutions in a neighbourhood of the equilibrium state, infinite-dimensional critical cases arise in the problem of the stability of solutions. The main results include the construction of special families of quasi-normal forms, namely non-linear boundary value problems of either Schrödinger or Ginzburg–Landau type. Their solutions make it possible to determine the main terms of the asymptotic expansion of both regular and irregular solutions to the original system. The main goal is the study of chains with diffusion- and advective-type couplings, as well as fully connected chains.

Keywords: bifurcations; stability; normal forms; singular perturbations; dynamics

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1. Formulation of the Problem

The classical van der Pol equation

$$\ddot{u} + a\dot{u} + u = f(u, \dot{u}), \quad f(u, \dot{u}) = \dot{u}u^2$$

arises in many applied problems. Here, we consider a system of N coupled van der Pol equations

$$\ddot{u}_j + a\dot{u}_j + u = f(u_j, \dot{u}_j) + d\sum_{j=1}^{N} a_{ij}u_i, \; j = 1, \ldots N. \tag{1}$$

Systems of this type have been considered in the works of many authors (see, for example, [1–15]). We will assume that the coupling coefficients of each j element are the same for all j, i.e., $a_{ij} = a_{i-j}$, and that the chain is circular, i.e, the $j \pm N$ element coincides with the j element.

Let $x_j = 2\pi j N^{-1}$ $(j = 1, \ldots N)$ be uniformly distributed points on some circle with angular coordinate x_j. Function $u_j(t)$ is the value of $u(t, x)$ at $x = x_j$.

In this paper, we will consider several of the most common types of coupling, namely the following.

1. Diffusion chains. For such chains, we assume that

$$a_1 = a_{-1} = 1, \; a_0 = 2, \; a_j = 0 \text{ at } j = 2, \ldots N-2,$$

i.e., the second term on the right hand side of (1) has the form

$$d[u(t, x_{j+1}) - 2u(t, x_j) + u(t, x_{j-1})]. \tag{2}$$

Sometimes, in the case of $d < 0$, (2) is called anti-diffusion.

2. One-way coupled chains. They are determined by the relations

$$a_1 = 1,\ a_0 = -1,\ a_j = 0 \text{ at } j = 2, \dots N-1,$$

that means that in (1) we have

$$d[u(t, x_{j+1}) - u(t, x_{j-1})]. \tag{3}$$

Sometimes the coupling (3) is called advective. Another variant of the advective coupling is often used, when

$$a_1 = 1,\ a_{-1} = 1.$$

In this case, we have

$$d[u(t, x_{j+1}) - u(t, x_{j-1})].$$

3. Chains with two-way or semi-diffusion coupling. Here,

$$a_1 = a_{-1} = 1,\ a_0 = a_2 = \dots = a_{N-2} = 0$$

and the analogue of (2) and (3) has the form

$$d[u(t, x_{j+1}) + u(t, x_{j-1})]. \tag{4}$$

4. Fully coupled chains. In this case, we assume that

$$d \sum_{i=1}^{N} a_{ij} u_i = d \frac{1}{N} \sum_{i=1}^{N} a_i u(t, x_i).$$

The next assumption is central. We will assume that the number of N elements in (1) is sufficiently large, which means that the parameter $\varepsilon = 2\pi N^{-1}$ satisfies the relation

$$0 < \varepsilon \ll 1.$$

This condition allows us to naturally pass from the fixed variables $u(t, x_j)$ $(j = 1, \dots, N)$ to the continuous variable $x \in [0, 2\pi]$ of function $u(t, x)$. Then, system (1) can be written in a more general form as

$$\frac{\partial^2 u}{\partial t^2} + a\frac{\partial u}{\partial t} + u = f\Big(u, \frac{\partial u}{\partial t}\Big) + d \int\limits_{-\infty}^{+\infty} F(s, \varepsilon) u(t, x+s)\, ds, \tag{5}$$

with periodic boundary conditions

$$u(t, x + 2\pi) \equiv u(t, x). \tag{6}$$

Note that the last term on the right-hand side of (5) can also be written as an integral

$$d \int\limits_{0}^{2\pi} \widetilde{F}(s, \varepsilon) u(t, x+s)\, ds.$$

We will describe function $F(s, \varepsilon)$ in more detail. For the values $k = 0$ and $k = \pm 1$, we set

$$F_k(s) = \frac{1}{2\varepsilon\sigma\sqrt{\pi}} \exp\big(-(\varepsilon\sigma)^{-2}(s + \varepsilon k)^2\big),\ \sigma > 0 \quad \Bigg(\int\limits_{-\infty}^{+\infty} F_k(s)\, ds = 1\Bigg).$$

Note that for a fixed function $u(t,x)$, for $\sigma \to 0$, we have

$$\lim_{\sigma\to 0}\int_{-\infty}^{+\infty} F_k(s)u(t,x+s)ds = u(t,x+\varepsilon k).$$

Function

$$F(s,\varepsilon) = F_1(s) - 2F_0(s) + F_{-1}(s) \tag{7}$$

generalises the diffusion coupling, since it satisfies

$$\lim_{\sigma\to 0}\int_{-\infty}^{+\infty} F(s,\varepsilon)u(t,x+s)ds = u(t,x+\varepsilon) - 2u(t,x) + u(t,x-\varepsilon).$$

Note that the coupling of diffusion type (3) is the simplest difference approximation of the diffusion operator $\partial^2 u/\partial x^2$.

For semi-diffusion coupling, it is natural to assume that

$$F(s,\varepsilon) = F_1(s) + F_{-1}(s).$$

The expression (3) is the simplest difference approximation of the advection (transfer) operator $\partial u/\partial x$. At the same time, we will also consider a coupling "similar" to (4):

$$u(t,x+\varepsilon) - u(t,x-\varepsilon), \tag{8}$$

since it also approximates the advection operator to the same extent. In relation to the function $F(s,\varepsilon)$, for (3), the following expression arises:

$$F(s,\varepsilon) = F_1(s) - F_0(s),$$

whereas for (8)

$$F(s,\varepsilon) = F_1(s) - F_{-1}(s).$$

Note that the following relations

$$\int_{-\infty}^{+\infty} F_k(s)\exp(ims)ds = \cos(\varepsilon m)\exp(-\sigma^2(\varepsilon m)^2), \quad \left(\int_{-\infty}^{+\infty} F_k(s)ds = 1\right)$$

indicate the convenience of representing $F(s,\varepsilon)$ in terms of functions of the form $F_k(s)$. Functions $F(s,\varepsilon)$ of this type have been used in [16]. Similar expressions for $F(s)$, where $F(s) = Const \cdot \exp[-\sigma^2|s|]$, are also given in [17]; however, they are less convenient.

For fully coupled chains, for $N \gg 1$, the representation

$$d\frac{1}{N}\sum_{j=1}^{N} a_i u(t,x_i) = d\frac{1}{2\pi}\int_0^{2\pi}\varphi(s)u(t,x+s)ds \tag{9}$$

arises naturally. Note that the case when

$$\varphi(s) \equiv 1$$

seems to be the most important.

In this paper, we will study the behaviour of all the solutions to (5) and (6) with sufficiently small initial conditions (in the norm) under the constraints (7)–(9). More precisely, we will study the asymptotics as $\varepsilon \to 0$, for all $t \in (t_0,\infty)$, $x \in [0,2\pi]$ of functions that are sufficiently small and satisfy the boundary value problems (5) and (6) with a high degree of accuracy in ε (uniformly in t, x).

We fix the space of initial conditions

$$u(0,x) \in C_{[0,2\pi]}, \quad \frac{\partial u}{\partial t}\Big|_{t=0} \in C_{[0,2\pi]}$$

to be the phase space of the boundary value problem (5) and (6). In addition to the model in (5) and (6), we consider another model by replacing $u(t, x+s)$ with $\partial u(t, x+s)/\partial t$ in the integral of the right-hand side of (5), namely

$$\frac{\partial^2 u}{\partial t^2} + a\frac{\partial u}{\partial t} + u = f\Big(u, \frac{\partial u}{\partial t}\Big) + d\int_{-\infty}^{+\infty} F(s,\varepsilon)\frac{\partial}{\partial t}u(t, x+s)ds.$$

We shall comment on the similarities and differences between the solutions of such models.

When studying the local behaviour of solutions, the main focus is on the study of the properties of boundary value problems linearised at zero with 2π-periodic boundary conditions (6)

$$\frac{\partial^2 u}{\partial t^2} + a\frac{\partial u}{\partial t} + u = d\int_{-\infty}^{+\infty} F(s,\varepsilon)u(t, x+s)ds \tag{10}$$

and

$$\frac{\partial^2 u}{\partial t^2} + a\frac{\partial u}{\partial t} + u = d\int_{-\infty}^{+\infty} F(s,\varepsilon)\frac{\partial u(t, x+s)}{\partial t}ds.$$

In turn, the behaviour of solutions to these boundary value problems is determined by the location of the roots of the characteristic equation

$$\lambda^2 + a\lambda + 1 = d\int_{-\infty}^{+\infty} F(s,\varepsilon)\exp(iks)ds, \tag{11}$$

where $k = 0, \ \pm 1, \ \pm 2, \ \ldots$.

We now formulate some statements of a general plan which are analogues of Lyapunov's theorems on stability in the first approximation.

Proposition 1. *Let all roots of the characteristic Equation* (11) *have negative real parts and be separated from zero at $\varepsilon \to 0$. Then, the zero equilibrium of* (5) *and* (6) *is asymptotically stable, and all solutions from some sufficiently small ε-independent neighbourhood of the zero equilibrium state tend to zero as $t \to \infty$.*

Proposition 2. *Let the characteristic Equation* (11) *have a positive real part separated from zero at $\varepsilon \to 0$. Then, the zero equilibrium in* (5) *and* (6) *is unstable, and no attractors of this boundary value problem can exist in a sufficiently small ε-independent neighbourhood of the zero equilibrium state.*

Thus, only such critical cases need to be considered when the characteristic Equation (11) does not have roots with positive real parts separated from zero as $\varepsilon \to 0$, but there are roots whose real parts tend to zero as $\varepsilon \to 0$. Below, we study the local behaviour of solutions to the boundary value problem (5) and (6) in critical cases.

An important feature of these critical cases is the fact that an infinite number of roots of the corresponding characteristic equations tend to the imaginary axis as $\varepsilon \to 0$. Thus, we can say that critical cases of infinite dimension are realised. Critical cases of this type have been considered in [18–20]. The methodology of these works will be essentially used here.

The main result is the construction of special non-linear forms, the so-called quasi-normal forms (QNF), namely boundary value problems of parabolic type. They do not contain small parameters and their non-local dynamics determines the local behaviour of solutions to the original boundary value problems.

Solutions of quasi-normal form determine the principal terms of the asymptotic expansions of the original boundary value problems. As a rule, they are classical Ginzburg–Landau equations or integro-differential equations. Numerical methods are well developed for such equations. For the original boundary value problems, the use of numerical methods is difficult, since they are singularly perturbed and the solutions contain components that rapidly oscillate in the space or time variable. Therefore, despite the apparent complexity, quasi-normal forms are simpler objects, ready for further study using standard methods. In addition, the above quasi-normal forms allow one to immediately draw a conclusion, for example, about multistability (for quasi-normal forms in which continuum parameters are present) and about high sensitivity to parameter changes. The conclusion that many of the presented quasi-normal forms are characterised by complex irregular dynamics follows from the well-known results of the analysis of this type of system (see, for example, [21]).

The parameter σ in the formulas for $F(s,\varepsilon)$ has a clear meaning. It defines a set of chain elements that significantly affect each specific element. This effect is weaker the farther the elements are from each other. For $\sigma = 0$, additional critical cases arise when infinitely many roots of the characteristic equation, which correspond to harmonics with arbitrarily large numbers, have asymptotically small real parts. To describe the dynamic properties of the problem in this situation, the condition $\sigma \ll 1$ is considered. Note that the quasi-normal form in such cases acquire an additional spatial variable, which means that there is a tendency to complicate the oscillations.

Note that the condition that the real parts of the roots of the characteristic equation are nonpositive implies that

$$a \geq 0. \tag{12}$$

Below, when considering critical cases for fixed coefficients a_0 and d_0, these coefficients will vary, i.e., we will assume that

$$a = a_0 + \varepsilon^2 a_1, \quad d = d_0 + \varepsilon^2 d_2. \tag{13}$$

This is a quite important remark, since for $a > 0$, all solutions of the van der Pol equation tend to zero as $t \to \infty$, and, for $a_0 = 0$ and $a_1 < 0$, this equation has a stable limit cycle.

The solutions studied in this paper are conditionally divided into two types: regular and irregular. Regular solutions are solutions "well dependent" on the parameter ε, i.e., solutions for which the following asymptotic representation holds:

$$u(t, x+\varepsilon) = u(t,x) + \varepsilon \frac{\partial}{\partial x} u(t,x) + \frac{1}{2}\varepsilon^2 \frac{\partial^2 u}{\partial y^2} u(t,x) + \ldots.$$

Irregular solutions have a more complex structure, which consists of a superposition of functions smoothly (regularly) depending on the parameter ε and functions smoothly depending on the parameter ε^{-1}.

Note that for some types of couplings, only regular or only irregular solutions can exist or both can exist simultaneously.

Regarding the methodology, for finite-dimensional critical cases in the problem of the stability of solutions, as a rule, it is possible to substantiate statements about the existence of local invariant integral manifolds whose dimension is determined by the dimension of critical cases. The initial system on such manifolds is represented as special non-linear systems of ordinary differential equations, which are called normal forms. Their non-local solutions determine the local behaviour of all solutions to the original systems as $t \to \infty$.

In this paper, we study situations where the critical cases have infinite dimensions. It is difficult, and sometimes even impossible, to substantiate the existence of an invariant manifold if it does not coincide with the entire phase space. Nevertheless, it is possible to use the formalism of the method of normal forms, which is based on the use of the structure of "critical" solutions of linearised equations. Such constructions were successfully used, for example, in the works [22,23]. The fact that there is no quadratic non-linearity in the original van der Pol equation considerably simplifies the corresponding calculations, but is not fundamental. All constructions extend to non-linear second order equations with quadratic and cubic non-linearities of general form.

The main assumption that opens the way to the application of special asymptotic methods is that the number of elements N in the chain is sufficiently large. Thus, the small parameter $\varepsilon \sim N^{-1}$ arises in a natural way.

The main result is the construction of the so-called quasi-normal forms and continuum families of quasi-normal forms, which are special non-linear distributed boundary value problems that do not contain a small parameter. The non-local structure of the solutions of these quasi-normal forms determines the principal terms of the asymptotic expansions of the original problem.

The paper is organised as follows. The main content deals with the study of chains with diffusion-type couplings. Sections 2, 3 and Appendix A.1 are devoted to this. In Appendix A.2, the results are extended to equations in which, instead of the van der Pol non-linearity, there is a conservative non-linearity describing dislocations in a solid.

In Section 4, the dynamic properties of chains with advective coupling are studied, which is an equally interesting problem. In Section 5, we will discuss the asymptotics of solutions in fully connected chains. We note that, on one hand, the results of the above sections differ significantly from each other, and on the other hand, they serve as an important addition to the existing studies.

2. Asymptotic Behaviour of Solutions in Chains with Diffusion-Type Connections

The linearised zero boundary value problem for chains (5) and (6) with diffusion-type couplings has the form (10), (6), where $F(s,\varepsilon) = F_1(s) - 2F_0(s) + F_{-1}(s)$. Substituting $u = \exp(ikx + \lambda t)$ into these equations, we arrive at the characteristic equation

$$\lambda^2 + a\lambda + 1 = -4d\sin^2\left(\frac{z}{2}\right)\exp(-\delta^2 z^2), \tag{14}$$

where $z = \varepsilon k$, $k = 0, \pm 1, \pm 2, \ldots$. In order for this equation to have no roots with positive real parts, it is necessary and sufficient that the following condition is satisfied

$$1 + 4d\max_z\left(\sin^2\frac{z}{2}\cdot\exp(-\delta^2 z^2)\right) \geq 0. \tag{15}$$

In what follows, this condition is assumed to be satisfied.

2.1. Asymptotic Behaviour of Regular Solutions

Given that

$$a_0 > 0$$

for some fixed integer k, all roots of (14) have negative real parts separated from the imaginary axis as $\varepsilon \to 0$. Therefore, the critical case in the stability problem is realised only for

$$a_0 = 0.$$

Then, the roots $\lambda_k(\varepsilon)$ and $\bar{\lambda}_k(\varepsilon)$ are complex and

$$\lambda_k(\varepsilon) = i + \varepsilon\lambda_{1k} + o(\varepsilon^2), \quad \lambda_{1k} = -\frac{1}{2}a_1 + \frac{idk^2}{2}.$$

Equation (10) has a set of solutions

$$u(t,x,\varepsilon) = \sum_{k=-\infty}^{\infty} \xi_k(\tau)\exp(it), \tag{16}$$

and $\xi_k(\tau) = \exp\left[(\lambda_{1k} + o(\varepsilon^2))\tau\right]$, where $\tau = \varepsilon^2 t$ is a slow temporal variable. The expression (16) can be written as $u(t,x,\varepsilon) = \xi(\tau,x)$, where the Fourier coefficients $\xi(\tau,x)$ are equal to $\xi_k(\tau)$. Solutions to the non-linear boundary value problem (5) and (6) are then sought in the form of a formal series

$$u(t,x,\varepsilon) = \varepsilon\big(\xi(\tau,x)\exp(it) + \bar{c}c\big) + \varepsilon^3 u_3(t,\tau,x) + \dots. \tag{17}$$

The expression $\bar{c}c$ in (17) and below denotes the complex conjugate of the previous term. Here, $\xi(\tau,x)$ is an unknown function and $u_3(t,\tau,x)$ is 2π periodic in t and x. We substitute (17) into (5) and collect the coefficients of the same powers of ε. Then, for $u_3 = u_{30} + \bar{u}_{30}$, we arrive at

$$\frac{\partial^2 u_{30}}{\partial t^2} + u_{30} = \left[-ia_1\xi - 2i\frac{\partial\xi}{\partial\tau} + d\frac{\partial^2\xi}{\partial x^2} - i\xi|\xi|^2\right]\exp(it) + i\xi^3\exp(3it).$$

This equation is solvable with respect to $\xi(\tau,x)$ in the specified class of functions if and only if the equalities

$$2\frac{\partial\xi}{\partial\tau} = -id\frac{\partial^2\xi}{\partial x^2} - a_1\xi - \xi|\xi|^2, \quad \xi(\tau, x+2\pi) \equiv \xi(\tau,x) \tag{18}$$

are satisfied.

The boundary value problem (18) plays the role of a normal form. Its solutions bounded at $\tau \to \infty, x \in [0,2\pi]$ determine, according to (17), the asymptotics of regular solutions to the original boundary value problem (5) and (6).

At $a_1 < 0$, the boundary value problem (18) has an infinite number of periodic solutions $\rho_0 \exp\left[ikx + idk^2\tau/2\right]$. Note also the formula for $y(\tau,x) = |\xi(\tau,x)|^2$ is

$$\int_0^{2\pi}\left[\frac{\partial y}{\partial\tau} + a_1 y + y^2\right]dx = 0.$$

2.2. *Asymptotic Behaviour of Irregular Solutions*

We fix $\delta > 0$ arbitrarily and define the function

$$\gamma(\delta) = \left[1 + 4d\sin^2\left(\frac{\delta}{2}\right)\exp(-\delta^2\sigma^2)\right]^{1/2}.$$

Note that

$$\gamma'(\delta) = 2d\gamma^{-1}(\delta)\exp(-\delta^2\sigma^2)\sin\left(\frac{\delta}{2}\right)\cdot\left[\cos\frac{\delta}{2} - 2\delta\sigma^2\sin\frac{\delta}{2}\right].$$

Additionally, we assume that $\gamma'(\delta) \neq 0$, i.e.,

$$\delta \neq \pi n \quad \text{and} \quad ctg\frac{\delta}{2} \neq 2\delta\sigma^2. \tag{19}$$

We denote the value that completes the expression $\delta\varepsilon^{-1}$ to an integer by $\theta = \theta(\delta,\varepsilon) \in [0,1)$. Consider the set of integers

$$K_\varepsilon(\delta) = \{\delta\varepsilon^{-1} + \theta + m, \quad m = 0, \pm1, \pm2, \dots\}.$$

Thus, $\theta(\varepsilon)$ is a piecewise linear function, which, for $\varepsilon \to 0$, runs over all values from zero to one infinitely many times. For the values $k \in K_\varepsilon(\delta)$, we consider the asymptotics of the roots $\lambda_m^+(\varepsilon)$ and $\bar{\lambda}_m^+(\varepsilon)$ of the characteristic Equation (14). We obtain

$$\lambda_m^+(\varepsilon) = i\gamma(\delta) + i\varepsilon(\theta + m)\gamma'(\delta) + \varepsilon^2 \lambda_{m2}^+(\varepsilon),$$

$$\lambda_m^-(\varepsilon) = -i\gamma(\delta) + i\varepsilon(\theta + m)\gamma'(\delta) + \varepsilon^2 \lambda_{m2}^-(\varepsilon),$$

where $m = 0, \pm 1, \pm 2, \ldots$ and $\lambda_{m2}^\pm(\varepsilon)$ are some functions bounded as $\varepsilon \to 0$, whose explicit form is not required.

The linearised boundary value problem (10), (6) has solutions

$$\exp\left(i\left(\frac{\delta}{\varepsilon} + \theta + m\right)x + \lambda_m^\pm(\varepsilon)t\right).$$

Hence, it follows that the solutions to the same boundary value problem are the set of solutions

$$u = \sum_{k=-\infty}^{\infty} \Big(\xi_m \exp\left(i(\delta\varepsilon^{-1} + \theta + m)x + \lambda_m^+(\varepsilon)t\right) + \bar{c}c +$$
$$+ \eta_m \exp\left(i(\delta\varepsilon^{-1} + \theta + m)x + \lambda_m^-(\varepsilon)t\right) + \bar{c}c\Big),$$

where the quantities ξ_m and η_m are arbitrary. This expression can be rewritten in the form

$$\begin{aligned} u_1 = U_1(t, \tau, x, \varepsilon) = \exp\left[i(\delta\varepsilon^{-1} + \theta)x + i(\gamma(\delta) + \varepsilon\theta\gamma'(\delta))t\right](\xi(\tau, x^+) + \bar{c}c) + \\ + \exp\left[i(\delta\varepsilon^{-1} + \theta)x - i(\gamma(\delta) + \varepsilon\theta\gamma'(\delta))t\right](\eta(\tau, x^-) + \bar{c}c), \end{aligned} \tag{20}$$

where $\tau = \varepsilon^2 t$,

$$x^\pm = x \pm \varepsilon\gamma'(\delta)t,$$

and the Fourier coefficients of the functions $\xi(\tau, x)$ and $\eta(\tau, x)$ satisfy the following

$$\xi_m(\tau) = \xi_m \exp\left(\lambda_{m2}^+(\varepsilon)\tau\right), \quad \eta_m(\tau) = \eta_m \exp\left(\lambda_{m2}^-(\varepsilon)\tau\right).$$

Note that due to condition (19), the values $x^\pm$ do not coincide with x. Based on the obtained representation of solutions of the linear equation, we shall look for solutions to the non-linear boundary value problem (5) and (6) in the form

$$u(t, x, \varepsilon) = \varepsilon u_1 + \varepsilon^2 u_2 + \varepsilon^3 u_3. \tag{21}$$

Functions u_j are $2\pi\big(\gamma(\delta) + \varepsilon\theta\gamma'(\delta)\big)^{-1}$ periodic in t and 2π periodic in x. The expression for u_1 is the same as (20), with the only difference that now $\xi(\tau, x)$ and $\eta(\tau, x)$ are unknown complex amplitudes.

We substitute (21) into (5) and successively collect coefficients of the same powers of ε. In the first step, we arrive at an equation for u_1, whose solution is presented in (20). In the second step, we obtain the same equation for u_2 :

$$\frac{\partial^2 u_2}{\partial t^2} + u_2 = d \int_{-\infty}^{\infty} F(s, \varepsilon) u_2(t, \tau, x + s) ds.$$

We fix u_2 to be the function

$$u_2 = E^+(t, x, \varepsilon) f^+(t_1, \tau, x) + \bar{c}c + E^-(t, x, \varepsilon) f^-(t_1, \tau, x) + \bar{c}c, \tag{22}$$

where

$$E^\pm(t, \tau, x) = \exp\left(i(\delta\varepsilon^{-1} + \theta)x \pm i(\gamma(\delta) + \varepsilon\theta\gamma'(\delta))t\right), \quad t_1 = \varepsilon t,$$

and the expression $f(t_1, \tau, x)$ is not defined at this step. It will be chosen in the next step.

The next step is central. We obtain an equation for u_3 of the form

$$\frac{\partial^2 u_3}{\partial t^2} + u_3 - d \int\limits_{-\infty}^{\infty} F(s, \varepsilon) u_3(t, \tau, x+s) ds = E^+(t, \tau, x) B_0^+(t_1, x, \tau) +$$
$$+ E^-(t, \tau, x) B_0^-(t_1, x, \tau) + E^+(t, \tau, x) B^+(x^+, \tau) + E^-(t, \tau, x) B^-(x^-, \tau) + \bar{c}c +$$
$$+ E^3 B(t_1, x, x^+, x^-, \tau) + \bar{c}c.$$

Here, we use the notation

$$B_0^+(t_1, x, \tau) = -2i\gamma(\delta)\left[\left(\frac{\partial f^+}{\partial t_1} + \frac{\gamma'(\delta)}{\gamma(\delta)}\frac{\partial f^+}{\partial x}\right) + 2\xi\left(|\eta|^2 - M(|\eta|^2)\right)\right],$$
$$B_0^-(t_1, x, \tau) = 2i\gamma(\delta)\left[\left(\frac{\partial f^-}{\partial t_1} - \frac{\gamma'(\delta)}{\gamma(\delta)}\frac{\partial f^-}{\partial x}\right) - 2\eta\left(|\xi|^2 - M(|\xi|^2)\right)\right],$$

$$B^+(x^+, \tau) = -i\gamma(\delta)\frac{\partial \xi}{\partial \tau} - i\gamma''(\delta)\left(-\frac{\partial^2 \xi}{\partial x^2} + 2i\theta\frac{\partial \xi}{\partial x} + \theta^2\xi\right) - i\gamma(\delta) a_1 \xi -$$
$$- \xi(|\xi|^2 + 2M(|\eta|^2)),$$
$$B^-(x^-, \tau) = i\gamma(\delta)\frac{\partial \eta}{\partial \tau} + i\gamma''(\delta)\left(-\frac{\partial^2 \eta}{\partial x^2} + 2i\theta\frac{\partial \eta}{\partial x} + \theta^2\eta\right) + i\gamma(\delta) a_1 \eta -$$
$$- \eta(|\eta|^2 + 2M(|\xi|^2)).$$

In these formulas

$$M(\varphi(x)) = \frac{1}{2\pi}\int\limits_0^{2\pi} \varphi(x) dx.$$

The expression for $B(t_1, x, x^+, x^-, \tau)$ is not given, since it will not be used to find ξ and η, only when deriving a formula to determine $u(t, x, \varepsilon)$ in (21). Note that $B_0^{\pm}$ and $B^{\pm}$ have different arguments. Therefore, to solve Equation (22) in the specified class of functions, it is necessary to require that the equalities

$$B_0^{\pm} \equiv B^{\pm} \equiv 0$$

are satisfied. We consider the equalities $B_0^{\pm} = 0$ as equations for the unknown functions $f^{\pm}(t_1, x)$. Its solutions can be written explicitly. One can verify that

$$f^+(t_1, x, \tau) = -(2\gamma'(\delta))^{-1}\xi(\tau, x^+) \int\limits_0^{x^-} \left(|\eta(\tau, x)|^2 - M(|\eta(\tau, x)|^2)\right) dx,$$
$$f^-(t_1, x, \tau) = (2\gamma'(\delta))^{-1}\eta(\tau, x^-) \int\limits_0^{x^+} \left(|\xi(\tau, x)|^2 - M(|\xi(\tau, x)|^2)\right) dx. \quad (23)$$

Then, from the equalities $B^{\pm} = 0$, we obtain that ξ and η satisfy the following

$$2i\gamma(\delta_0)\frac{\partial \xi}{\partial \tau} = \gamma''(\delta_0)\left[\frac{\partial^2 \xi}{\partial x^2} + 2i\theta\frac{\partial \xi}{\partial x} - \theta^2\xi\right] - i\gamma(\delta_0) a_1 \xi - i\gamma(\delta_0)\xi(|\xi|^2 + 2M(|\eta|^2)), \quad (24)$$

$$-2i\gamma(\delta_0)\frac{\partial \eta}{\partial \tau} = \gamma''(\delta_0)\left[\frac{\partial^2 \eta}{\partial x^2} - 2i\theta\frac{\partial \eta}{\partial x} - \theta^2\eta\right] + i\gamma(\delta_0) a_1 \eta + i\gamma(\delta_0)\eta(|\eta|^2 + 2M(|\xi|^2)), \quad (25)$$

$$\xi(\tau, x+2\pi) \equiv \xi(\tau, x), \quad \eta(\tau, x+2\pi) \equiv \eta(\tau, x). \tag{26}$$

The main result is that the boundary value problem (24)–(26) plays the role of a normal form for (5) and (6), when considering solutions with modes from the set $K_\varepsilon(\delta) \cap K_\varepsilon(-\delta)$. In order to formulate the corresponding result more precisely, we introduce some notation. We arbitrarily fix $\theta_0 \in [0, 1)$. By $\varepsilon_n = \varepsilon_n(\delta, \theta_0)$, we denote a sequence $\varepsilon_n \to 0\,(n \to \infty)$, where the value of $\theta_0(\varepsilon_n, \delta)$ does not change.

Theorem 1. *We fix arbitrarily $\theta_0 \in [0, 1)$. Let $\xi(\tau, x), \eta(\tau, x)$ be a solution of the boundary value problem (24)–(26). Then, for sufficiently small values of $\varepsilon = \varepsilon_n(\delta, \theta_0)$, the function (21) for $\xi = \xi(\tau, x^+), \eta = \eta(\tau, x^-)$ satisfies the boundary value problem (5) and (6) up to $o(\varepsilon_n^3)$.*

We note again that, in this section, we have studied the asymptotic behaviour of solutions whose modes belong to the set $K(\delta)$. It is natural to call the value $\delta\varepsilon^{-1} + \theta$ the base mode for such solutions.

2.3. Quasinormal Form in the Case $\gamma'(\delta) = 0$

Let the condition

$$\gamma'(\delta_0) = 0 \tag{27}$$

be satisfied for some value $\delta = \delta_0$. For example, δ_0 could be $\delta_0 = 2\pi n_0$, where $n_0 > 0$ and is an integer. The expression $2\pi n_0 \varepsilon^{-1}$ is an integer because $\varepsilon = 2\pi N^{-1}$, which means $\theta(2\pi n_0, \varepsilon) = 0$.

Under condition (27), the constructions are substantially simplified. This follows from the fact that $x^+ = x^- = x$. In the asymptotic representation (21), we have $u_2 = 0$, and hence $f^\pm = 0$. Here, we restrict ourselves to presenting the final boundary value problem for the amplitudes $\xi(\tau, x)$ and $\eta(\tau, x)$:

$$2i\gamma(\delta_0)\frac{\partial \xi}{\partial \tau} = \gamma''(\delta_0)\left[\frac{\partial^2 \xi}{\partial x^2} + 2i\theta\frac{\partial \xi}{\partial x} - \theta^2 \xi\right] - i\gamma(\delta_0)a_1\xi - i\gamma(\delta_0)\xi(|\xi|^2 + 2|\eta|^2), \tag{28}$$

$$-2i\gamma(\delta_0)\frac{\partial \eta}{\partial \tau} = \gamma''(\delta_0)\left[\frac{\partial^2 \eta}{\partial x^2} - 2i\theta\frac{\partial \eta}{\partial x} - \theta^2 \eta\right] + i\gamma(\delta_0)a_1\eta + i\gamma(\delta_0)\eta(2|\xi|^2 + |\eta|^2), \tag{29}$$

$$\xi(\tau, x+2\pi) \equiv \xi(\tau, x), \quad \eta(\tau, x+2\pi) \equiv \eta(\tau, x). \tag{30}$$

The final result is the assertion of Theorem 1. It should be noted that in the case of $\delta_0 = 2\pi n_0$ in (28)–(30), and hence in Theorem 1, the value θ is zero.

2.4. Quasinormal Form in the Case $\gamma(\delta) = 0$

Let $\gamma(\delta_0) = 0$ be satisfied for some $\delta_0 > 0$. In this case, we conclude that

$$1 + 4d\sin^2\frac{\delta_0}{2} \cdot \exp(-\delta^2\sigma^2) = 0. \tag{31}$$

Hence, taking into account condition (15), we conclude that $\gamma'(\delta_0) = 0, \gamma''(\delta_0) < 0$ and δ_0 is the first positive root of the equation

$$4ctg\frac{\delta_0}{2} = \sigma^2\delta_0.$$

Finding δ_0, from (31) we find the value of the coefficient d. For coefficient a in (5), it is convenient to take the equality $a = \varepsilon a_1$.

In the case of (31), the situation is even more simplified. In the analogue of the asymptotic representation (21), we have

$$u(t,x,\varepsilon) = \varepsilon^{1/2}\Big(\xi(t_1,x)\exp(i(\delta_0\varepsilon^{-1}+\theta)x) + \bar{cc}\Big) + \varepsilon^{3/2}u_3(t_1,x), \quad t_1 = \varepsilon t. \tag{32}$$

Substituting (32) into (5), after some calculations we obtain the quasi-normal form

$$\begin{gathered}\frac{\partial^2\xi}{\partial t_1^2} + a_1\frac{\partial\xi}{\partial t_1} = -\gamma''(\delta_0)\left(\frac{\partial^2\xi}{\partial x^2} - i\theta\frac{\partial\xi}{\partial x} - \theta^2\xi\right) - 2|\xi|^2\frac{\partial\xi}{\partial t_1} - \xi^2\frac{\partial\bar{\xi}}{\partial t_1}, \\ \xi(\tau,x+2\pi) \equiv \xi(\tau,x)\end{gathered} \tag{33}$$

as the main result. Note that the linear and non-linear components in (33) differ significantly from the above quasi-normal forms.

In Appendix A.1, we consider the question of constructing a CNF for finding the amplitudes of multi-frequency solutions.

3. Equations with a Small Parameter σ

In this section, we assume that the parameter σ appearing in the definition of $F(s,\varepsilon)$ is sufficiently small:

$$\sigma = \varepsilon\sigma_1. \tag{34}$$

Therefore, in the case under consideration, the function $F_0(s,\varepsilon)$ is essentially "closer" to the δ-function.

In addition, here we study the influence of variations in the value of N on the asymptotic behaviour of the solutions. We fix an arbitrary integer value c, and let the number of elements in (1) be $N+c$.

3.1. Asymptotic Behaviour of Solutions with One Base Mode

We set

$$\mu = \frac{2\pi}{N+c}.$$

On the right-hand side of Equation (5), the parameter ε is replaced by μ; that is, instead of $F(s,\varepsilon)$, we have $F(s,\mu)$:

$$\frac{\partial^2 u}{\partial t^2} + \varepsilon a_1\frac{\partial u}{\partial t} + u + f(u,\dot{u}) = d\int_{-\infty}^{+\infty} F(s,\mu)u(t,x+s)ds. \tag{35}$$

Considering that $\varepsilon = 2\pi N^{-1}$, we obtain

$$\mu = \varepsilon\left[1 - \frac{c}{2\pi}\varepsilon + \frac{c^2}{(2\pi)^2}\varepsilon^2 + \ldots\right].$$

The boundary value problem (35), (6) linearised at zero has the characteristic equation

$$\lambda^2 + \varepsilon a_1\lambda + 1 = -4d\sin^2\left(\frac{z}{2}\right)\exp(-\varepsilon^2\sigma_1^2 z^2), \quad z = \mu k, \quad k = 0, \pm1, \pm2, \ldots. \tag{36}$$

As in the previous section, we consider the question of constructing the asymptotes of solutions to (35), (6) based on modes from $K(\delta) = \{\delta\varepsilon^{-1} + \theta + m,\ m = 0, \pm1, \pm2, \ldots, \delta > 0\}$. We consider the most important case when δ is not an integer multiple of 2 $pi : \delta \neq 2\pi_j$.

The roots $\lambda_m(\varepsilon)$ $(m = 0, \pm1, \pm2, \ldots)$ of the characteristic Equation (36) corresponding to modes from $K(\delta)$ have amplitude

$$\lambda_m(\varepsilon) = i\gamma(\delta) + \varepsilon\lambda_{m1} + \varepsilon^2\lambda_{m2} + \ldots, \quad \gamma(\delta) = \left(1 + 4\sin^2\frac{\delta}{2}\right)^{1/2},$$

where

$$\lambda_{m1} = id(2\gamma(\delta))^{-1}\sin(\delta)(m+\theta-\delta c(2\pi)^{-1}),$$

$$\lambda_{m2} = -\frac{1}{2}a_1 - iA_0 - iA_1(m+\theta-\delta c(2\pi)^{-1}) - iA_2(m+\theta-\delta c(2\pi)^{-1})^2.$$

The coefficients A_0, A_1 and A_2 satisfy

$$A_0 = 2d\sigma^2\delta^2\Big(\sin\frac{\delta}{2}\Big)^2\cdot\gamma^{-1}(\delta),$$
$$A_1 = dc\sin\delta\cdot(4\pi\gamma(\delta))^{-1},$$
$$A_2 = d[(\sin\delta)^2 - 2\gamma^2(\delta)\cos\delta](8\gamma(\delta))^{-3}.$$

The solution of the linearised equation for (35) corresponding to the root $\lambda_m(\varepsilon)$ can be written as

$$u_m(t,x,\varepsilon) = \xi_m(\tau,x^+)E^+ + \eta_m(\tau,x^-)E^-,$$

where $\tau=\varepsilon^2 t$, $x^\pm = x\pm 2\varepsilon d$,

$$E^\pm = \exp\big[i(\delta\varepsilon^{-1}+\theta-c\delta(2\pi)^{-1})x \pm i(\gamma(\delta)+\varepsilon(2\gamma(\delta))^{-1}\cdot(\theta-\delta c(2\pi)^{-1})\sin\delta)t\big],$$
$$\xi_m(\tau) = \xi_m\exp\big((\lambda_{m2}+O(\varepsilon))\tau\big),\quad \eta_m(\tau)=\eta_m\exp\big((\overline{\lambda}_{m2}+O(\varepsilon))\tau\big),$$

where ξ_m and η_m are arbitrary complex constants. As in the previous sections, we seek the asymptotes of the solutions to the non-linear boundary value problem (5) and (6) in the form

$$\begin{aligned} u = \;& \varepsilon^{1/2}\Big(\xi(\tau,x)E^+ + \bar{cc} + \eta(\tau,x)E^- + \bar{cc}\Big) + \varepsilon\Big(f^+E^+ + \bar{cc} + f^-E^- + \bar{cc}\Big) + \\ & +\varepsilon^{3/2}u_3(t,\tau,x) + \ldots . \end{aligned} \tag{37}$$

The functions $f^\pm = f^\pm(t_1,\tau,x)$ are defined by the same formulas as in (23), where $t_1=\varepsilon t$ and

$$\gamma(\delta) = \Big(1+4\sin^2\frac{\delta}{2}\Big)^{1/2}.$$

We substitute (37) into (5). We collect the coefficients at $\varepsilon^{3/2}$, and in order to determine the known amplitudes ξ and η, we obtain the boundary value problem

$$\begin{aligned}\frac{\partial\xi}{\partial\tau} =& -\frac{1}{2}a_1\xi - iA_0\xi - iA_1\Big(\theta - c\delta(2\pi)^{-1} - \frac{i\partial}{\partial x}\Big)\xi - iA_2\Big(\theta - c\delta(2\pi)^{-1} - \frac{i\partial}{\partial x}\Big)^2\xi - \\ &-\frac{1}{2}\xi\big(|\xi|^2 + 2M(|\eta|^2)\big),\end{aligned}$$

$$\begin{aligned}\frac{\partial\eta}{\partial\tau} =& -\frac{1}{2}a_1\eta + iA_0\eta + iA_1\Big(\theta - c\delta(2\pi)^{-1} - \frac{i\partial}{\partial x}\Big)\eta + iA_2\Big(\theta - c\delta(2\pi)^{-1} - \frac{i\partial}{\partial x}\Big)^2\eta - \\ &-\frac{1}{2}\eta\big(|\eta|^2 + 2M(|\xi|^2)\big),\end{aligned}$$

$$\xi(\tau,x+2\pi)\equiv\xi(\tau,x),\quad \eta(\tau,x+2\pi)\equiv\eta(\tau,x).$$

The main result is that this boundary value problem plays the role of a normal form. For every fixed $\theta_0\in[0,1)$, its non-local solutions bounded as $\tau\to\infty$, $x\in[0,2\pi]$, allow us to construct asymptotes for sufficiently small functions $\varepsilon_n=\varepsilon_n(\theta_0)$ that satisfy the original boundary value problem (5) and (6) up to $o(\varepsilon^{3/2})$.

Remark 1. *The roles of the coefficients c and σ_1 are determined by the above formulas. Under the conditions of the next Section 3.2, these coefficients play a much more important role.*

3.2. The Case of an Infinite Set of Basic Modes

Significantly new and interesting points arise when considering the asymptotics of solutions containing various infinite sets of basic modes. Here, we emphasise an important class of solutions whose base modes belong to the set

$$K_\infty(\delta) = \{\delta\varepsilon^{-1} + \theta + 2\pi n\varepsilon^{-1} + m,\ m, n = 0, \pm 1, \pm 2, \ldots\}.$$

The roots $\lambda_{mn}(\varepsilon)$ of the characteristic equation corresponding to these modes have the asymptotics

$$\lambda_{mn}(\varepsilon) = i\gamma(\delta) + \varepsilon\lambda_{mn1} + \varepsilon^2\lambda_{mn2},$$

where

$$\lambda_{mn1} = ib\left(m - cn + \theta - \frac{\delta c}{2\pi}\right), \quad b = (\gamma(\delta))^{-1}\sin\delta,$$
$$\lambda_{mn2} = -\frac{1}{2}a_1 + B_1\left((m-cn)^2 + 2(m-cn)\left(\theta - \frac{\delta c}{2\pi}\right) + \left(\theta - \frac{\delta c}{2\pi}\right)^2\right) + B_2(\delta + 2\pi n)^2,$$
$$B_1 = -i(2\gamma(\delta))^{-1}\left((\gamma(\delta))^{-2}\sin^2\delta - \cos\delta\right), \quad B_2 = -i2(\gamma(\delta))^{-1}\sigma_1^2\sin^2\frac{\delta}{2}.$$

The solutions of the linear boundary value problem $u_{mn}(t,x,\varepsilon)$ corresponding to $\lambda_{mn}(\varepsilon)$ can be written as

$$u_{mn}(t,x,\varepsilon) = \xi_{mn}(\tau)\exp\left[i(\delta\varepsilon^{-1} + \theta)x + imx^+ + 2\pi iny^+ + i\left(\gamma(\delta) + \varepsilon b\left(\theta - \frac{\delta c}{2\pi}\right)\right)t\right],$$
$$x^\pm = x \pm \varepsilon bt, \quad y^\pm = y \mp \varepsilon cbt, \quad y = 2\pi\varepsilon^{-1}x.$$

Therefore, in order to construct the asymptotes of the solutions to the boundary value problem, we use the expressions

$$u(t,x,\varepsilon) = U(t,x,\varepsilon) + O(\varepsilon^2) \tag{38}$$

and

$$\begin{aligned} U(t,x,\varepsilon) = \ & \varepsilon^{1/2}\Big(\xi(\tau,x^+,y^+)E_c^+ + \bar{c}c + \eta(\tau,x^-,y^-)E_c^- + \bar{c}c\Big) + \\ & +\varepsilon\Big(f_c^+(t_1,x,y)E_c^+ + \bar{c}c + f_c^-(t_1,x,y)E_c^- + \bar{c}c\Big) + \\ & +\varepsilon^{3/2}\Big(H_1^+(\tau,x,y)E_c^+ + \bar{c}c + H_1^-(\tau,x,y)E_c^- + \bar{c}c\Big) + \\ & +H_3^+(\tau,x,y)(E_c^+)^3 + \bar{c}c + H_3^-(\tau,x,y)(E_c^-)^3 + \bar{c}c \end{aligned} \tag{39}$$

where $t_1 = \varepsilon t$ and $E_c^\pm$ is given by

$$E_c^\pm = \exp\left[i\left(\delta\varepsilon^{-1} + \theta - \frac{\delta c}{2\pi}\right)x \pm i\left(\gamma(\delta) + \varepsilon b\left(\theta - \frac{\delta c}{2\pi}\right)\right)t\right].$$

Substituting (38) and (39) into (5), after simple calculations, we obtain equations for $H_1^\pm$ and $H_3^\pm$. The functions $H_3^\pm$ are determined from these equations, and the solvability conditions for $H_1^\pm$ allow us to determine $f_c^\pm$ and obtain equations for $\xi(\tau,x,y)$ and $\eta(\tau,x,y)$. The difficulty lies in the fact that the equations for determining $H_1^\pm$ include functions whose spatial arguments x^+, y^+ and x^-, y^- are different; for some of these are x^+ and y^+, while for others x^- and y^-. In non-linearity, these arguments are present in different factors. The purpose of the constructions being carried out is to obtain systems of boundary value problems for determining the unknown functions ξ and η with the same arguments.

In order to have arbitrariness in the choice of functions $f^{\pm}$, we consider the question of the solvability in the class of 2π-periodic functions in x and y of the equation

$$\frac{\partial g(x,y)}{\partial x} - c\frac{\partial g(x,y)}{\partial y} = p(x,y), \tag{40}$$

where $p(x,y)$ is 2π periodic in x and y. To do this, we introduce some notation. By D, J and J_0, we denote [24] operators defined on continuously differentiable functions $v(x,y)$ of two variables x and y, acting according to the rules

$$\begin{aligned} Dv(x,y) &= \frac{\partial v}{\partial x} - c\frac{\partial v}{\partial y}, \\ Jv(x,y) &= \int\limits_0^x v(s, cx+y-cs)ds, \\ J_0 v(x,y) &= \frac{1}{2\pi}\int\limits_0^{2\pi} v(s, cx+y-cs)ds. \end{aligned}$$

Under the condition

$$J_0 p(x,y) = 0$$

there exists a periodic solution of Equation (40) in both variables, which is given by

$$g(x,y) = Jp(x,y).$$

Note that an arbitrary 2π-periodic function $p(x,y)$ in x and y can be represented as

$$p = J_0 p - (1 - J_0)p$$

and the following is satisfied

$$J_0(p - J_0 p) = 0.$$

Below, we shall need the following relations, which follow from these definitions:

$$Dv(cx+y) = 0, \quad D(J - J_0)v(x,y) = DJv(x,y) = v(x,y).$$

The cubic non-linearity $R = (i\gamma\xi E_c^+ + \bar{c}c + i\gamma\eta E_c^- + \bar{c}c)\cdot(\xi E_c^+ + \bar{c}c + \eta E_c^- + \bar{c}c)^2$ for $E_c^{\pm}$ implies the following

$$R = -i\gamma(\delta)\xi(|\xi|^2 + 2|\eta|^2) - i\gamma(\delta)\eta(|\eta|^2 + 2|\xi|^2).$$

It is convenient to represent the above expression as a sum of four terms

$$\begin{aligned} R_1 &= -i\gamma(\delta)\xi(|\xi|^2 + 2J_0|\eta|^2), \quad R_2 = -i\gamma(\delta)\xi(2|\eta|^2 - 2J_0|\eta|^2), \\ R_3 &= -i\gamma(\delta)\eta(|\eta|^2 + 2J_0|\xi|^2), \quad R_4 = -i\gamma(\delta)\eta(2|\xi|^2 - 2J_0|\xi|^2). \end{aligned}$$

The functions R_1 and R_3 depend only on x^+, y^+ and x^-, y^-, respectively, while both of the functions R_2 and R_4 depend on all arguments. We manage the arbitrariness in the choice of $f^{\pm}$ in such a way as to "remove" the terms R_2 and R_4 in the equation for $H_1^{\pm}$. From here, we arrive at:

$$2i\gamma(\delta)\frac{\partial f^+}{\partial t_1} = 2i(\sin\delta)Df^+ - i\gamma(\delta)\xi\big(2|\eta|^2 - 2J_0|\eta|^2\big), \tag{41}$$

$$-2i\gamma(\delta)\frac{\partial f^-}{\partial t_1} = 2i(\sin\delta)Df^- - i\gamma(\delta)\eta\big(2|\xi|^2 - 2J_0|\xi|^2\big). \tag{42}$$

In (41), we set $f^+(t_1,x,y) = \xi(\tau,x^+,y^+) \cdot g(t_1,x,y)$. Then, we obtain

$$2i\gamma(\delta)\frac{\partial g^+}{\partial t_1} - 2i(\sin\delta) \cdot Dg^+ = -2i\gamma(\delta)\big(|\eta(\tau,x^-,y^-)|^2 - J_0|\eta(\tau,x^-,y^-)|^2\big).$$

Then, for $x = x^- - bt_1$, $y = y^- + cbt_1$, we arrive at the equation for $g^+ = g^+(x,y)$:

$$bDg^+ = |\eta(\tau,x,y)|^2 - J_0(|\eta(\tau,x,y)|^2).$$

Hence, we conclude that

$$g^+(x,y) = b^{-1}\big(J(|\eta(\tau,x,y)|^2 - J_0|\eta(\tau,x,y)|^2)\big).$$

Analogously, we find that $f^-(t_1,x,y) = \eta(\tau,x^-,y^-) \cdot g(t_1,x,y)$ and

$$g^-(x,y) = -b^{-1}\big(J(|\xi(\tau,x,y)|^2 - J_0|\xi(\tau,x,y)|^2)\big).$$

Let us formulate the main result. Consider the boundary value problem

$$\begin{aligned}\frac{\partial\xi}{\partial\tau} = \ & \left(\delta - i\frac{\partial}{\partial y}\right)^2\xi + iB_1\left[-D^2 - 2i\left(\theta - \frac{c\delta}{2\pi}\right)D + \left(\theta - \frac{c\delta}{2\pi}\right)^2\right]\xi + \\ & + iB_2\left(\theta - \frac{c\delta}{2\pi} - iD\right)\xi - \frac{1}{2}a_1\xi - \frac{1}{2}\xi\big(|\xi|^2 + 2J_0(|\eta|^2)\big),\end{aligned} \tag{43}$$

$$\begin{aligned}\frac{\partial\eta}{\partial\tau} = \ & \left(\delta - i\frac{\partial}{\partial y}\right)^2\eta - iB_1\left[-D^2 - 2i\left(\theta - \frac{c\delta}{2\pi}\right)D + \left(\theta - \frac{c\delta}{2\pi}\right)^2\right]\eta - \\ & - iB_2\left(\theta - \frac{c\delta}{2\pi} - iD\right)\eta - \frac{1}{2}a_1\eta - \frac{1}{2}\eta\big(|\eta|^2 + 2J_0(|\xi|^2)\big),\end{aligned} \tag{44}$$

$$\xi(\tau,x+2\pi,y) \equiv \xi(\tau,x,y) \equiv \xi(\tau,x,y+2\pi), \tag{45}$$

$$\eta(\tau,x+2\pi,y) \equiv \eta(\tau,x,y) \equiv \eta(\tau,x,y+2\pi). \tag{46}$$

Recall that

$$Df = \frac{\partial f}{\partial x} - c\frac{\partial f}{\partial y}.$$

By $\varepsilon_k(\theta_0)$ $(k = k_0, k_0+1,\ldots)$, we denote a sequence such that $\varepsilon_k(\theta_0) \to 0$ for $k \to \infty$ and $\theta(\varepsilon_k(\theta_0)) = \theta_0$.

Theorem 2. *Let* (34) *be satisfied. We fix arbitrarily $\delta \neq 2\pi k\,(k = 0,1,\ldots), \theta_0 \in [0,1)$ and an integer c. In addition, let $\mu = N + c$ and let $\big(\xi(\tau,x,y), \eta(\tau,x,y)\big)$ for their derivatives with respect to τ and let their second order derivatives with respect to x and y be bounded functions as $\tau \to \infty$, $x \in [0,2\pi]$, $y \in [0,2\pi]$. Moreover, let $\big(\xi(\tau,x,y), \eta(\tau,x,y)\big)$ be a solution to the boundary value problem* (43)–(46) *for $\theta = \theta_0$. Then, the function $U(t,x,\varepsilon_k)$, for $\tau = \varepsilon_k(\theta_0)t$, $x^\pm = x \pm \varepsilon_k bt$, $y^\pm = y \mp \varepsilon_k cbt$, and $y = 2\pi\varepsilon_k^{-1}x$ satisfies the boundary value problem* (5) *and* (6) *up to $o(\varepsilon_k^3)$.*

3.3. Examples

We consider two cases. In the first of them, we assume that $K_\infty = \{2\pi n\varepsilon^{-1} + m;\ m,n = 0,\pm1,\pm2,\ldots\}$ and $\sigma = \varepsilon\sigma_1$, $\delta = \varepsilon^2 t$, $y = 2\pi\varepsilon^{-1}x, a = \varepsilon^2 a_1$, $d = d_0 + \varepsilon^2 d_1$ and $4d_0 > -1$. Then, the final quasi-normal form is

$$\begin{aligned}&\frac{\partial\xi}{\partial\tau} = -\frac{a_1}{2}\xi - i\frac{d_0}{8}D^2\xi - i\frac{d_0\sigma_1^2}{2}\frac{\partial^2\xi}{\partial y^2} - \frac{1}{2}\xi|\xi|^2,\\ &\xi(\tau,x+2\pi,y) \equiv \xi(\tau,x,y) \equiv \xi(\tau,x,y+2\pi).\end{aligned}$$

The function $\xi(\tau, x, y)$ determines the asymptotes of the solutions of (5) and (6) according to the formula

$$u(t, x, \varepsilon) = \varepsilon^{1/2}\big(\xi(\tau, x, y)\exp(it) + \bar{c}c\big) + O(\varepsilon^{3/2}).$$

It is interesting to compare this result with the situation considered in Appendix A.1.2 (see Formula (A8)).

In the second case, we assume that

$$d_0 = -\frac{1}{4}.$$

It is convenient to assume that the value of N is even. Otherwise, it suffices to replace the integer c by $c+1$. Then, the expression $\pi\varepsilon^{-1}$ is an integer, and hence $\theta = 0$.

Consider the basic set of modes $K_\infty = \{\pi(2n+1)\varepsilon^{-1} + m;\ m, n = 0, \pm1, \pm2, \dots\}$. Here, we assume that $\sigma = \varepsilon\sigma_1$, $a = \varepsilon a_1$, $t_1 = \varepsilon t$ and $d = d_0 + \varepsilon^2 d_1$. As a result, we arrive at the real quasi-normal form

$$\frac{\partial^2\xi}{\partial t_1^2} + a_1\frac{\partial\xi}{\partial t_1} + d_1\xi = 4^{-2}D^2\xi + \sigma_1^2\frac{\partial^2\xi}{\partial y^2} - \xi^2\frac{\partial\xi}{\partial t_1},$$
$$\xi(\tau, x+2\pi, y) \equiv \xi(\tau, x, y) \equiv -\xi(\tau, x, y+\pi)$$

where $u(t, x, \varepsilon) = \varepsilon^{1/2}\xi(\tau, x, y) + O(\varepsilon^{3/2})$. The obtained result differs significantly from the case considered in Section 2.4.

In Appendix A.2, the results obtained are applied to the problem of dislocations in a solid.

4. Advective Coupling

Here, we consider the equation

$$\frac{\partial^2 u}{\partial t^2} + a\frac{\partial u}{\partial t} + u = f\left(u, \frac{\partial u}{\partial t}\right) + d\int_{-\infty}^{\infty}(F_\varepsilon(s) - F_{-\varepsilon}(s))u(t, x+s)ds \tag{47}$$

with boundary conditions (6).

The equation linearised at zero has the form

$$\frac{\partial^2 u}{\partial t^2} + a\frac{\partial u}{\partial t} + u = d\int_{-\infty}^{\infty}(F_\varepsilon(s) - F_{-\varepsilon}(s))u(t, x+s)ds. \tag{48}$$

We investigate the roots of the characteristic equation for (48)

$$\lambda^2 + a\lambda + 1 = 2idg(z)\sin z, \tag{49}$$

where $g(z) = \exp\big(-\sigma^2 z^2\big)$, $z = \varepsilon k$, $k = 0, \pm1, \pm2, \dots$. Let the following nondegeneracy condition be satisfied

$$a > 0. \tag{50}$$

We consider separately two cases depending on the value of parameter δ. First, in Section 4.1, we will focus on "average" values of this parameter, i.e., the value $\delta > 0$ is supposed to be somehow fixed. In Section 4.2, we will consider the case of sufficiently small values of δ.

4.1. The Case of "Average" Values of σ

For $d = 0$, all roots of (49) have negative real parts. We denote the smallest (if it exists) value of the parameter d by $d_0^+ > 0$, for which there exists a value z_0^+ such that, for $d = d_0^+$

and $z = z_0^+$, Equation (49) has a purely imaginary root $\lambda = i\omega$ $(\omega > 0)$. We denote the largest (if it exists) value of the parameter d by $d_0^- < 0$, for which there exists z_0^- such that, for $d = d_0^-$ and $z = z_0^-$, Equation (49) has a purely imaginary root $\lambda = i\omega$ $(\omega > 0)$. Let z_0 be the first positive root of equation $tgz = (2\sigma z)^{-1}$. We set $g_0 = \max_z g(z) \sin z$. Then, $g_0 = g(z_0) \sin z_0$.

Lemma 1. *We fix arbitrarily $a_0 > 0$. We have:*

$$\omega = 1,\ d_0^+ = \frac{1}{2} a g_0^{-1},\ d_0^- = -\frac{1}{2} a g_0^{-1},\ z_0^+ = z_0,\ z_0^- = -z_0.$$

Lemma 2. *Assume that the following relations hold:*

$$d_0^- < d < d_0^+.$$

Then, all roots of (49) have negative real parts and are separated from zero as $\varepsilon \to 0$.

Lemma 3. *Let one of the followings relations,*

$$d_0 > d_0^+ \text{ or } d_0 < d_0^-,$$

hold. Then, Equation (49) has a positive real part separated from zero as $\varepsilon \to 0$.

The proofs of these Lemmas are simple, and thus we omit them.

From Lemmas 1–3, it follows that, for $d = d_0^\pm$, in the boundary value problem (48), (6), the critical cases are realised in the stability problem. In (47) we set

$$a = a_0 + \varepsilon^2 a_1, \quad d = d_0^+ + \varepsilon^2 d_1. \tag{51}$$

Let us find the asymptotes as $\varepsilon \to 0$ of all the roots $\lambda_m^\pm(\varepsilon)$ and $\overline{\lambda}_m^\pm(\varepsilon)$ $(m = 0, \pm 1, \pm 2, \ldots)$ of Equations (49) whose real parts tend to zero. We first introduce some more notation. Let $\Theta = \Theta(\varepsilon, z) \in [0, 1)$ be the value that completes the expression $|z|\varepsilon^{-1}$. By g^0, we denote the expression

$$g^0 = \frac{d^2}{dz^2}(g(z) \sin z)\Big|_{z=z_0}.$$

Note that $g^0 < 0$. Consider the set of integer values $k = z_0\varepsilon^{-1} + \Theta + m$ $(m = 0, \pm 1, \pm 2, \ldots)$. Then, substituting them in (49), we have the equality $z = z_0 + \varepsilon(\Theta + m)$.

Lemma 4. *The assympotic equalities*

$$\lambda_m^+(\varepsilon) = i + \varepsilon^2 (L_0(\Theta + m)^2 + L_1) + O(\varepsilon^4), \tag{52}$$

where

$$L_0 = (a_0 + 4)^{-1}(ia_0 - 2) 2 d_0 g^0,$$
$$L_1 = (a_0 + 4)^{-1}(ia_0 - 2)(2 d_0 g_0 - a_1),$$

hold.

Note that formula (52) does not change when the value d_0^+ in (51) is replaced by d_0^-.

Equation (48), for $m = 0, \pm 1, \pm 2, \ldots$, has the solution

$$u_m(t, x, \varepsilon) = \exp\left[i(z_0^\pm \varepsilon^{-1} + \Theta + m)x + (i + \varepsilon^2(L_0(\Theta + m)^2 + L_1) + O(\varepsilon^4))t\right].$$

At the next stage, we introduce the non-linear boundary value problem

$$\frac{\partial \xi}{\partial \tau} = L_0 \frac{\partial^2 \xi}{\partial x^2} + 2iL_0\Theta \frac{\partial \xi}{\partial x} + (L_1 - L_0\Theta^2)\xi + \sigma\xi|\xi|^2, \tag{53}$$

$$\xi(\tau, x + 2\pi) \equiv \xi(\tau, x), \tag{54}$$

where $\sigma = -(a_0^2 + 4)^{-1}(2 + ia_0)$. Moreover, we define function $U_0(\tau, x)$ by

$$U_0(\tau, x) = \xi^3 iA \exp\big(i\big(3\big(z_0\varepsilon^{-1} + \Theta\big)\big)x + 3it\big) - \overline{\xi}^{\,3} i\overline{A} \exp\big(-i\big(3\big(z_0\varepsilon^{-1} + \Theta\big)x\big) - 3it\big),$$

where $A = -(a_0^2 + 4)^{-1}(2 + ia_0)$. Below, by $\varepsilon_n = \varepsilon_n(\Theta_0)$ we denote a sequence $\varepsilon_n \to 0$, on which $A = -(a_0^2 + 4)^{-1}(2 + ia_0)$. We formulate the main result of this section.

Theorem 3. *Let conditions (51) be satisfied. We arbitrarily fix $\Theta_0 \in [0, 1)$. Let $\xi(\tau, x)$ be a bounded, for $\tau \to \infty$, $x \in [0, 2\pi]$, solution to the boundary problem (53) and (54) for $\Theta = \Theta_0$. Then, the function*

$$\begin{aligned} u_0(t, x, \varepsilon) =& \varepsilon(\xi(\tau, x) \exp(i(z_0\varepsilon^{-1} + \Theta)x + it) + \\ &+ \overline{\xi}(\tau, x) \exp(-i(z_0\varepsilon^{-1} + \Theta)x - it)) + \varepsilon^3 U_0(\tau, x), \end{aligned}$$

for $\varepsilon = \varepsilon_n(\Theta_0)$, satisfies the boundary value problem (47), (6) up to $o(\varepsilon_n^3(\Theta_0))$.

4.2. Advective Connection for Small Values of the Parameter σ

Here, we assume that the parameter σ appearing in $g(z)$ is sufficiently small. For some σ_1, the condition

$$\sigma = \varepsilon\sigma_1 \tag{55}$$

is satisfied. In this case, we have $\omega = 1$, $d_0^+ = a_0/2$, $d_0^- = -a_0/2$. The main difference is that for each z, the equality $g(z) = 1 - \varepsilon^2\sigma_1^2 z^2 + O(\varepsilon^4)$ holds. Thus, the values of $z_0^{\pm}$ are not uniquely determined: $z_0^+ = 2\pi n + \pi/2$, $z_0^- = 2\pi n - \pi/2$ $(n = 0, \pm 1, \pm 2, \ldots)$.

Let us make one simplifying assumption. Let the number of elements of the considered chain N be a multiple of four. Then, the values of $z_n^{\pm}\varepsilon^{-1}$ are integers for all n, and therefore $\Theta(\varepsilon, z_n^{\pm}) = 0$.

Asymptotic formulas similar to (52) for the roots of $\lambda_{m,n}^+(\varepsilon)$, $\overline{\lambda}_{m,n}^+(\varepsilon)$, $(m, n = 0, \pm 1, \pm 2, \ldots)$, whose real parts tend to zero as $\varepsilon \to 0$, have the form

$$\lambda_{mn}^+(\varepsilon) = i + \varepsilon^2(a_0^2 + 4)^{-1}(2 + ia_0)\left[d_1 - a_1 - d_0\left(\frac{1}{2}m^2 + \sigma^2\left(2\pi n + \frac{\pi}{2}\right)^2 2\right)\right] + O(\varepsilon^4).$$

These roots correspond to solutions of the linear boundary value problem

$$u_{m,n}(t, x) = \exp\left[i\left(\left(2\pi n + \frac{\pi}{2}\right)\varepsilon^{-1} + m\right)x + (i + O(\varepsilon^2))t\right].$$

Therefore, we seek formal solutions of (47) in the form

$$\begin{aligned} u(t, x, \varepsilon) =& \varepsilon\big(\xi(\tau, x, y) \exp(i\pi(2\varepsilon)^{-1}x + it) + \overline{\xi}(\tau, x, y) \exp(-i\pi(2\varepsilon)^{-1}x - it)\big) + \\ &+ \varepsilon^3 U(t, \tau, x, y), \quad \tau = \varepsilon^2 t, \; y = n\varepsilon^{-1}x, \end{aligned} \tag{56}$$

where $U(t, \tau, x, y)$ is 2π periodic with respect to t and x and 1-periodic with respect to y.

Substitute (56) into (47). After standard operations, we obtain an equation for $U(t,\tau,x,y)$. From the condition of its solvability in the indicated class of functions, we arrive at a boundary value problem for determining the amplitude $\xi(\tau,x,y)$:

$$\frac{\partial\xi}{\partial\tau} =(a_0^2+4)^{-1}(2+ia_0)\Big[-d_0\Big(\frac{1}{2}\frac{\partial^2\xi}{\partial x^2}+\sigma^2\frac{\partial^2\xi}{\partial y^2}+i\sigma^2\frac{\pi}{2}\frac{\partial\xi}{\partial y}\Big)+ + \Big(d_1-a_1-d_0\sigma^2\frac{\pi^2}{4}\Big)\xi-\xi|\xi|^2\Big], \tag{57}$$

$$\xi(\tau,x+2\pi,y)\equiv\xi(\tau,x,y)\equiv\xi(\tau,x,y+1). \tag{58}$$

Thus, we justify the following result.

Theorem 4. *Let* (50), (51) *and* (55) *be satisfied and* $\xi_0(\tau,x,y)$ *be bounded as* $\tau\to\infty$, $x\in[0,2\pi]$, $y\in[0,1]$, *by the solution of the boundary value problem* (57) *and* (58). *Then, function* (56) *for* $\xi=\xi_0(\tau,x,y)$ *satisfies the boundary value problem* (47), (6) *up to* $o(\varepsilon^3)$.

Note that Equations (53) and (57) are equations of Ginzburg–Landau type. It is known (see, for example, [21]) that their solutions can have complex structures, including irregular. In this respect, Equation (57) is much more complicated than (53) because it contains two spatial variables.

5. Fully Coupled Chains of van der Pol Equations

It suffices to consider the most important example of such chains of the form

$$\ddot{u}_j+\varepsilon^2a_1\dot{u}_j+u_j-\dot{u}u^2=d\frac{1}{N}\sum_{k+1}^{n}u_k,\quad j=1,\dots,N.$$

For sufficiently large values of N, we pass to the boundary value problem

$$\frac{\partial^2u}{\partial t^2}+\varepsilon^2a_1\frac{\partial u}{\partial t}+u-u^2\frac{\partial u}{\partial t}=d\int_0^1u(t,s)ds, \tag{59}$$

$$u(t,x+1)\equiv u(t,x). \tag{60}$$

The behaviour of the solutions of this boundary value problem differs significantly from the cases considered above. Here, we briefly dwell on the consideration of two cases depending on the value of the coefficient d.

5.1. The Case of Small Values of the Coefficient d

We assume that for some fixed value d, the condition

$$d=\varepsilon d_1. \tag{61}$$

For $\varepsilon=0$, the linearised boundary value problem

$$\frac{\partial^2u}{\partial t^2}+u=0,\quad u(t,x+1)\equiv u(t,x)$$

has the same characteristic equation,

$$\lambda^2+1=0, \tag{62}$$

for all modes $\exp(2\pi ikx)$, $k=0,\pm1,\pm2,\dots$. Therefore, under condition (61), we consider the critical case of an infinite set of pairs of purely imaginary roots with resonances $1:1:\dots$.

We seek solutions of the boundary value problem (59) and (60) based on solutions of (62) in the form

$$u(t,x,\varepsilon) = U(t,x,\varepsilon) + O(\varepsilon^4), \tag{63}$$

$$\begin{aligned} U(t,x,\varepsilon) = \quad & \varepsilon\big(\xi(\tau,x)\exp(it) + \bar{c}c\big) + \varepsilon^3\big(u_{31}(\tau,x)\exp(it) + \bar{c}c + \\ & + u_{33}(\tau,x)\exp(3it) + \bar{c}c\big), \quad \tau = \varepsilon^2 t. \end{aligned} \tag{64}$$

Substituting (63) and (64) into (59) and performing standard operations, we find an expression for u_{33}, and from the condition that the equation is solvable with respect to u_{31}, we obtain a boundary value problem for determining the unknown amplitude $\xi(\tau,x)$. As a result, we arrive at the relations

$$\frac{\partial\xi}{\partial\tau} = -\frac{1}{2}a_1\xi - \frac{id}{2}\int\limits_0^1 \xi(\tau,s)ds - \frac{1}{2}\xi|\xi|^2, \tag{65}$$

$$\xi(\tau,x+1) \equiv \xi(\tau,x). \tag{66}$$

$$u_{31}(\tau,x) = 0, \quad u_{33}(\tau,x) = -\frac{i}{8}\xi^3(\tau,x).$$

The next statement says that this boundary value problem plays the role of a quasi-normal form for (59) and (60).

Theorem 5. *Let* (61) *be satisfied and the boundary value problem* (65) *and* (66) *have a bounded solution* $\xi(\tau,x)$*, for* $\tau \to \infty$, $x \in [0,1]$*. Then, function* $U(t,\tau,x)$ *satisfies the original boundary value problem* (59) *and* (60) *up to* $O(\varepsilon^4)$*.*

5.2. Quasi-Normal Form for "Average" Values of the Parameter d

A boundary value problem linearised at zero for (59) and (60),

$$\frac{\partial^2 u}{\partial t^2} + \varepsilon^2 a_1\frac{\partial u}{\partial t} + u = d\int\limits_0^1 u(t,s)ds, \quad u(t,x+1) \equiv u(t,x), \tag{67}$$

has the characteristic equation

$$\lambda_k^2 + \varepsilon^2 a_1\lambda_k + 1 = \begin{cases} d, & \text{if } k = 0, \\ 0 & \text{if } k \neq 0. \end{cases}$$

Therefore, there is a pair of roots $\lambda_0(\varepsilon)$ and $\bar{\lambda}_0(\varepsilon)$, where $\lambda_0(\varepsilon) = i\omega_0 + O(\varepsilon)$ and $\omega_0 = (1-d)^{1/2}$, and infinitely many identical pairs of roots $\lambda_k(\varepsilon)$ and $\bar{\lambda}_k(\varepsilon)$ and $\lambda_k(\varepsilon) = i + O(\varepsilon)$ $(k = \pm1, \pm2, \ldots)$. These roots correspond to solutions of the linear boundary value problem (67)

$$u_k(t,x,\varepsilon) = \exp(2\pi ikx + \lambda_k(\varepsilon)t).$$

Following the above method, we look for solutions to the non-linear boundary value problem (59) and (60) in the form $u(t,x,\varepsilon) = U(t,x,\varepsilon) + O(\varepsilon^4)$, where

$$U(t,x,\varepsilon) = \varepsilon\big(\xi(\tau,x)\exp(it) + \bar{c}c + \eta(\tau)\exp(i\omega_0 t) + \bar{c}c\big) + \varepsilon^3 u_3(t,\tau,x). \tag{68}$$

It is important to keep in mind that the Fourier coefficient of function $\xi(\tau,x)$ for a zero-mode harmonic is zero, so

$$M(\xi(\tau,x)) \equiv \int\limits_0^1 \xi(\tau,x)dx = 0.$$

We substitute (68) into (59). After standard operations, we arrive at a system of equations for determining the unknown amplitudes $\xi(\tau, x)$ and $\eta(\tau)$:

$$\frac{\partial \xi}{\partial \tau} = -\frac{1}{2}a_1\xi - \frac{1}{2}\left[\xi|\xi|^2 - M(\xi|\xi|^2) + 2\xi|\eta|^2\right], \tag{69}$$

$$\xi(\tau, x+1) \equiv \xi(\tau, x). \tag{70}$$

$$\frac{\partial \eta}{\partial \tau} = -\frac{1}{2}a_1\eta - \frac{1}{2}\eta\left[|\eta|^2 + 2M(|\xi|^2)\right]. \tag{71}$$

Let us formulate the final result.

Theorem 6. *Let $d < 1$ and the boundary value problem (69)–(71) have a bounded solution $\xi(\tau, x), \eta(\tau)$ as $\tau \to \infty$, $x \in [0, 1]$. Then, function $U(t, \tau, x, \varepsilon)$ satisfies the boundary value problem (59) and (60) up to $O(\varepsilon^4)$.*

In Appendix A.3, the results obtained are applied to the problem of vibrations of pedestrian bridges.

6. Conclusions

Non-linear integro-differential boundary value problems that arise in the study of various chains of coupled van der Pol equations were investigated. Critical cases are singled out in the problem of the stability of the equilibrium state. An important conclusion is that these critical cases have infinite dimensions. A special technique was developed for studying the local behaviour of solutions in critical cases based on the construction of quasi-normal forms for finding the amplitude of solutions to the original boundary value problem. The above quasi-normal forms can be conditionally divided into three groups. The first group includes continuum families of equations depending on some parameters of Schrödinger type, in which the linear part is the same as in the Schrödinger equation. The number of parameters included in such quasi-normal forms is determined by the number of basic—asymptotically large—modes of the studied classes of solutions. Accordingly, the critical cases in the problem of stability are almost naturally called continual here. For an infinitely large number of such modes, an equation of the Schrödinger type arises with two spatial variables. Such quasi-normal forms are characteristic of chains with diffusion-type couplings. They are discussed in Sections 2 and 3 and Appendix A.1.

The second type of quasi-normal forms describes solutions that include one basic mode. It is determined from the condition of the presence of a "point" critical case in the problem under study. Such quasi-normal forms are Ginzburg–Landau-type boundary value problems. Here, we can talk about complex dynamics which are characteristic of the Ginzburg–Landau equation with one or two spatial variables. Hence, it follows that in the original problem, the structure of the solutions can be complex. Quasi-normal forms of the second group are presented in Section 4.

Quasi-normal forms of the third group are typical for problems describing fully connected chains of equations. These quasi-normal forms are special non-linear integro-differential equations. Section 5 shows that they can have interesting families of solutions which are discontinuous in the spatial variable.

Note that in each of the above problems, we study the asymptotes of both regular (i.e., solutions that smoothly depend on a small parameter) and irregular solutions that have regular components, as well as solutions that smoothly depend on some large parameter. In the latter case, this leads to the appearance of solutions rapidly oscillating in the spatial variable.

In this work, with the help of solutions to quasi-normal forms, functions are constructed that satisfy the original boundary value problem with high accuracy. Even in regular cases, we are not talking about the asymptotes of exact solutions. The same conclusion also applies to the works of other authors (see, for example, [25]). From the point of view of problems of mathematical physics, the obtained conclusions are sufficient. In some cases, more accurate results can be obtained. For example, in the case when a quasi-normal form has a rough solution periodic in τ and some nondegeneracy-type conditions are satisfied, then it is possible to substantiate the existence of an exact solution (torus) for the original boundary value problem of the same stability and with the same asymptotes as the cycle in the quasi-normal form.

The proposed methods can also be used to study chains with other types of couplings with more general non-linearities, as well as with Neumann- or Dirichlet-type boundary conditions. In this connection, we note the problem of dislocations in a solid given as an application.

Note that under the condition $\gamma'(\delta) \neq 0$, where $\gamma(\delta)$ is the oscillation frequency of solutions with base mode $\delta\varepsilon^{-1}$, the solutions contain different spatial variables $x^{\pm}$ and $y^{\pm}$. In order to obtain a quasi-normal form with the same spatial variables and spatial derivatives, certain efforts had to be made. Auxiliary functions $f^{\pm}$ were introduced and special partial differential equations were solved to determine these functions.

It is worth noting that, under the condition $\gamma(\delta) = 0$, i.e., in the non-oscillatory case, the corresponding quasi-normal forms differ significantly from the quasi-normal forms in the case of $\gamma(\delta) \neq 0$ (see Formula (33) and the formulas in Section 3.3). Note that oscillations in chains can also occur when the van der Pol equation itself has only a stable stationary solution.

It may be of interest to study chains of coupled van der Pol equations in which the integral term is replaced by

$$d \int\limits_{-\infty}^{\infty} F(s,\varepsilon) \frac{\partial u(t, x+s)}{\partial t} ds.$$

The proposed methods also extend to the study of solutions with an arbitrary number of basic modes, including those in the presence of resonance relations. In particular, the role of resonance relations is illustrated by considering problems with an infinite number of basic modes.

We dwell separately on the role of the parameters σ, θ and c. The parameter σ characterises the depth of the connection between the elements of the chain. For a large σ, this relationship is significant only between neighbouring elements, while for a small σ, the influence of relatively distant elements increases. If for a relatively large σ, the quasi normal form contains derivatives with respect to only one spatial variable, then for a small σ, differential operators contain derivatives with respect to two spatial variables. Obviously, the complexity of the solutions in the latter case increases.

Many of the above quasi-normal forms contain the parameter θ, which varies from 0 to 1 depending on the number of elements, N, in the chain. For different θ, the properties of the solutions of quasi-normal forms can change significantly [26]. Therefore, as N increases, an infinite process of direct and inverse bifurcations in the quasi-normal form is possible, and hence it is possible in the original system.

The integer parameter c also shows the changes in the properties of solutions when the number of elements in the chain changes from N to $N + c$. It is worth noting that this parameter is the coefficient of the second spatial derivative in problems with an infinite number of basic modes of solutions.

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Appendix A

Appendix A.1. Construction of a CNF for Finding the Amplitudes of Multifrequency Solutions

In this section, we briefly describe the new points that arise when the asymptotes of solutions to the boundary value problem (5) and (6) are based on several periodic solutions to the linear boundary value problem (10) and (6).

Appendix A.1.1. Dual Frequency Quasiperiodic Solutions

We fix arbitrary positive distinct values of δ_1 and δ_2, and let $\gamma'(\delta_j) \neq 0$ $(j = 1, 2)$. Thus, $\delta_{1,2}$ do not match integer multiples of 2π. We consider the question of the asymptotes of the solutions (5), (6) based on the modes $K(\delta_1)$ and $K(\delta_2)$, $\big(K(\delta) = \{\delta\varepsilon^{-1} + \theta + m,\ m = 0, \pm1, \ldots\}\big)$.

We set $x_j^{\pm} = x \pm \varepsilon\gamma'(\delta_j)t$ and

$$E_j^{\pm} = E_j^{\pm}(t, x, \varepsilon) = \exp\left[i\left(\frac{\delta_j}{\varepsilon} + \theta_j\right)x + i\big(\gamma(\delta_j) + \varepsilon\theta_j\gamma'(\delta_j)\big)t\right].$$

An analogue of the asymptotic expression (21) is the equality

$$\begin{aligned} u = \ & \textstyle\sum_{j=1,2}\Big(\varepsilon^{1/2}\Big(\xi_j(\tau, x_j^+)E_j^+ + \bar{cc} + \eta_j(\tau, x_j^-)E_j^- + \bar{cc}\Big) + \\ & + \varepsilon\Big(f_j^+(t_1, \tau, x)E_j^+ + \bar{cc} + f_j^-(t_1, \tau, x)E_j^- + \bar{cc}\Big) + \ldots\Big), \end{aligned} \tag{A1}$$

where $t_1 = \varepsilon t, \tau = \varepsilon^2 t$,

$$f_1^+(t_1, x, \tau) = -(2\gamma'(\delta_1))^{-1}\xi_1(\tau, x_1^+)\Big(\int_0^{x_1^-}(|\eta_1(\tau, x)|^2 - M(|\eta_1(\tau, x)|^2))dx\Big) +$$

$$+ \int_0^{x_2^-}(|\eta_2(\tau, x)|^2 - M(|\eta_2(\tau, x)|^2))dx + \int_0^{x_2^+}(|\xi_2(\tau, x)|^2 - M(|\xi_2(\tau, x)|^2))dx,$$

$$f_2^+(t_1, x, \tau) = -(2\gamma'(\delta_2))^{-1}\xi_2(\tau, x_2^+)\Big(\int_0^{x_1^-}(|\eta_1(\tau, x)|^2 - M(|\eta_1(\tau, x)|^2))dx\Big) +$$

$$+ \int_0^{x_2^-}(|\eta_2(\tau, x)|^2 - M(|\eta_2(\tau, x)|^2))dx + \int_0^{x_1^+}(|\xi_1(\tau, x)|^2 - M(|\xi_1(\tau, x)|^2))dx,$$

$$f_1^-(t_1, x, \tau) = (2\gamma'(\delta_1))^{-1}\eta_1(\tau, x_1^-)\Big(\int_0^{x_1^+}(|\xi_1(\tau, x)|^2 - M(|\xi_1(\tau, x)|^2))dx\Big) +$$

$$+ \int_0^{x_2^+}(|\eta_2(\tau, x)|^2 - M(|\eta_2(\tau, x)|^2))dx + \int_0^{x_2^-}(|\eta_2(\tau, x)|^2 - M(|\eta_2(\tau, x)|^2))dx,$$

$$f_2^-(t_1, x, \tau) = (2\gamma'(\delta_2))^{-1}\eta_2(\tau, x_2^-)\Big(\int_0^{x_1^+}(|\xi_1(\tau, x)|^2 - M(|\xi_1(\tau, x)|^2))dx\Big) +$$

$$+ \int_0^{x_2^+}(|\xi_2(\tau, x)|^2 - M(|\xi_2(\tau, x)|^2))dx + \int_0^{x_1^-}(|\eta_1(\tau, x)|^2 - M(|\eta_1(\tau, x)|^2))dx.$$

The final boundary value problem for finding $\xi_{1,2}$ and $\eta_{1,2}$ has the form

$$2\frac{\partial \xi_j}{\partial \tau} = \begin{array}{l} -id\gamma(\delta_j)\left[\frac{\partial^2\xi_j}{\partial x^2} + 2i\theta_j\frac{\partial \xi_j}{\partial x} - \theta_j^2\xi_j\right] - \gamma(\delta_j)a_1\xi_j - \xi_j\big(|\xi_j|^2 + \\ +2M(|\eta_1|^2 + |\eta_2|^2 + |\xi_{j+1}|^2)\big), \end{array} \tag{A2}$$

$$2\frac{\partial \eta_j}{\partial \tau} = \begin{array}{l} id\gamma(\delta_j)\left[\frac{\partial^2\eta_j}{\partial x^2} - 2i\theta_j\frac{\partial \eta_j}{\partial x} - \theta_j^2\eta_j\right] - \gamma(\delta_j)a_1\eta_j - \eta_j\big(|\eta_j|^2 + \\ +2M(|\xi_1|^2 + |\xi_2|^2 + |\eta_{j+1}|^2)\big), \end{array} \tag{A3}$$

$$\xi_j(\tau, x + 2\pi) \equiv \xi_j(\tau, x), \quad \eta_j(\tau, x + 2\pi) \equiv \eta_j(\tau, x), \quad j = 1, 2 \tag{A4}$$

where $\xi_{j+1}\,(\eta_{j+1})$ means that for $j = 1$, we have $\xi_{j+1}\,(\eta_{j+1}) = \xi_2\,(\eta_2)$, and for $j = 2$, we have $\xi_{j+1}\,(\eta_{j+1}) = \xi_1\,(\eta_1)$.

Additional difficulties arise when formulating a statement similar to Theorem 1. These difficulties are related to the choice of a sequence $\varepsilon_m \to 0$, such that the values of both quantities $\theta_j(\varepsilon)$ can be replaced by fixed values θ_{j0} $(j = 1, 2)$. Let us illustrate this choice in more detail.

We fix arbitrarily θ_{10} and θ_{20} from the interval $[0, 1]$. By $\varepsilon_m = \varepsilon_m(\theta_{10})$, we denote a sequences such that $\varepsilon_m \to 0\,(m \to \infty)$ and $\theta_1(\varepsilon_m) = \theta_{10}$. Let $\Omega(\theta_{10})$ be the set of all limit points of the sequence $\theta_2(\varepsilon_m)$ and $\theta_{20} \in \Omega(\theta_{10})$. Note that there are situations when $\Omega(\theta_{10})$ consists of a single point, and it is possible that $\Omega(\theta_{10}) = [0, 1]$. Then, for some subsequence ε_{m_k} of sequence ε_m, the conditions $\theta_1(\varepsilon_{m_k}) = \theta_{10}$, $\theta_2(\varepsilon_{m_k}) \to \theta_{20}$ are satisfied for $k \to \infty$. Thus, for $\varepsilon_{m_k} \to 0$, the solutions of the boundary value problem (5) and (6) are determined using the above formulas.

Theorem A1. *Let the boundary value problem (A2)–(A4) have bounded solutions $\xi_j(\tau, x), \eta_j(\tau, x)$, as $\tau \to \infty$, $x \in [0, 2\pi]$, for some fixed values of $\theta_j = \theta_{j0}$. Then, the function (A1) satisfies the boundary value problem (5) and (6) up to $o(\varepsilon_{m_k})$ for $m_k \to \infty$.*

Remark A1. *For $\gamma'(\delta_j) = 0\,(j = 1, 2)$, the formulas are substantially simplified. We obtain a boundary value problem only for $\xi_1(\tau, x)$ and $\xi_2(\tau, x)$. Here, $x_1^{\pm}$ and $x_2^{\pm}$ coincide with x.*

Remark A2. *The question of constructing the asymptotics of solutions based on modes from an arbitrary number of sets $K(\delta_1), \ldots, K(\delta_s)$ is considered similarly.*

Appendix A.1.2. The Case of an Infinite Set of Basic Modes

It is of interest to consider an infinite set of basic modes, $K = \bigcup\limits_{j=1}^{\infty} K(\delta_j)$. We first consider the case when

$$\delta_j = 2\pi_j \quad (j = 0, 1, \ldots). \tag{A5}$$

Note that for such δ_j, the value of θ_j is zero. The main part of the solutions to (5) and (6) under the condition (A5) is based on the expression

$$u = \sum_{m,n=0}^{\infty} \xi_{mn}(\tau) \exp\big(i(2\pi n\varepsilon^{-1} + m)x + it\big) = \xi(\tau, x, y)\exp(it),\ \tau = \varepsilon^2 t,\ y = 2\pi\varepsilon^{-1}x$$

and the periodicity conditions

$$\xi(\tau, x + 2\pi, y) \equiv \xi(\tau, x, y) \equiv \xi(\tau, x, y + 2\pi) \tag{A6}$$

are satisfied. In the situation under consideration, there is an infinite resonance $1 : 1 : 1 : \ldots$.

We seek solutions of the non-linear Equation (5) in the form

$$u = \begin{array}{l} \varepsilon^{1/2}\big(\xi(\tau,x,y)\exp(it) + \bar{c}c\big) + \varepsilon^{3/2}\big(u_{31}(\tau,x,y)\exp(it) + \bar{c}c + \\ +u_{32}(\tau,x,y)\exp(3it) + \bar{c}c\big) + O(\varepsilon^2). \end{array} \tag{A7}$$

We substitute the expression (A7) into (5). After standard calculations, at the third step (collecting the coefficients at $\varepsilon^{3/2}$), we obtain equations for u_{31} and u_{32}. The second of these functions is found and the condition for the solvability of the equation with respect to u_{31} leads to the appearance of an equation for determining $\xi(\tau,x,y)$. In order to write the corresponding equation in a convenient form, we take into account the equalities

$$\begin{aligned} & d\int\limits_{-\infty}^{\infty} F(s,\varepsilon)\xi(\tau,x+s,y+2\pi\varepsilon^{-1}s)ds = \\ & = d\exp(it)\int\limits_{-\infty}^{\infty} F(s,\varepsilon)\sum_{m,n=-\infty}^{\infty}\xi_{mn}(\tau)\cdot\exp\big(im(x+s)+i2\pi\varepsilon^{-1}n(x+s)\big)ds = \\ & = 4d\exp(it)\sum_{m,n=-\infty}^{\infty}\xi_{mn}(\tau)\sin^2\left(\frac{\varepsilon m}{2}\right)\exp\big(-\sigma^2(2\pi n)^2+O(\varepsilon)\big) = \\ & = d\exp(it)\frac{\partial^2}{\partial x^2}\big(R(\rho)\xi\big), \end{aligned}$$

where $R(\rho) = \exp(-\sigma^2\rho^2)$, $\rho = \partial/\partial y$. Then, the boundary value problem for finding the amplitude $\xi(\tau,x,y)$ can be formally written using the infinite differentiation operator with respect to y in the form

$$\frac{\partial\xi}{\partial\tau} = id\frac{\partial^2}{\partial x^2}R(\rho)\xi + \frac{1}{2}a_1\xi - \frac{1}{2}\xi|\xi|^2. \tag{A8}$$

Finally, we have the following.

Theorem A2. *Let the boundary value problem* (A8), (A6) *have a bounded solution* $\xi(\tau,x,y)$ *for* $\tau\to\infty$, $x\in[0,2\pi]$, $y\in[0,2\pi]$. *Then, the function* (A7) *satisfies the original boundary value problem* (5), (6) *up to* $o(\varepsilon^{3/2})$.

Appendix A.2. Application to the Problem of Dislocations in a Solid

Assume we have the simplest crystal structure, consisting of layers of atoms located at some distance from each other. J. Frenkel and T. Kontorova proposed a mathematical model describing the behaviour of a point defect in the crystal structure of a solid [27]. These defects are called dislocations. At present, dislocations are understood as a more complex imperfection of the crystal structure than any of the point defects [28].

As a reference equation for describing vibrations of an isolated atom, a conservative second-order equation is used

$$m\ddot{u} + \alpha\sin u = 0.$$

In [27], the equation of motion of the j-th atom in the lattice,

$$m\ddot{u}_j + \alpha\sin u_j = \beta[u_{j+1} - 2u_j + u_{j-1}], \quad j = 1,\ldots,N, \tag{A9}$$

was proposed in order to describe dislocations in a solid. Here, m,α and β are positive coefficients, whereas $u_j = u_j(t,x_j)$ is the deviation of the j-th atom from the equilibrium position.The function $u_j(t,x_j)$ satisfies the periodic boundary conditions $u_{N+1} = u_1, u_0 =$

u_N. The values of x_j are the angular coordinates of the corresponding point on some circle. It is assumed that the number of atoms, N, is sufficiently large, i.e., $N \gg 1$. Therefore,

$$\varepsilon = \frac{2\pi}{N} \ll 1. \tag{A10}$$

Passing to a continuous mass distribution, from (A9), after obvious renormalisations and replacement of $\sin u$ by a more general function $f(u)$, we obtain the equation

$$\dot{u} + f(u) = d(u(t, x+\varepsilon) - 2u + u(t, x-\varepsilon)), \tag{A11}$$

where $d > 0$,

$$u(t, x+2\pi) \equiv u(t, x), \tag{A12}$$

and for $f(u)$ we have

$$f(u) = au + bu^3 + \varphi(y) \quad \text{and} \quad \varphi(y) = o(y^4) \quad \text{as} \quad y \to 0.$$

It is natural to consider Equation (A11) with a more general coupling

$$\frac{\partial^2 u}{\partial t^2} + f(u) = d\int\limits_{-\infty}^{\infty} \big(F_\varepsilon(s,\varepsilon) - 2F_0(s,\varepsilon) + F_{-\varepsilon}(s,\varepsilon)\big)u(t, x+s)ds. \tag{A13}$$

In [29], one can find interesting results for the case when the coefficient α in (A9) is sufficiently small. We also refer to the papers [25,30], in which the Fermi–Pasta–Ulam problem similar to Equation (A9) was considered.

We now consider the problem, under condition (A10), of the behaviour of all solutions to the boundary value problem (A12) and (A13) with initial conditions from some sufficiently small (and ε-independent) neighbourhood of the zero equilibrium state. The behaviour of solutions of the linearised boundary value problem

$$\frac{\partial^2 u}{\partial t^2} + au = d\int\limits_{-\infty}^{\infty} \big(F_\varepsilon(s,\varepsilon) - 2F_0(s,\varepsilon) + F_{-\varepsilon}(s,\varepsilon)\big)u(t, x+s)ds, \tag{A14}$$

for a 2π periodic in x function $u(t,x)$ plays an important role. Its characteristic equation has the form

$$\lambda^2 + a = -4d\sin^2\frac{z}{2}, \quad z = \varepsilon k, \quad k = 0, \pm 1, \pm 2, \dots. \tag{A15}$$

In what follows, we assume that

$$a = 1 + \varepsilon^2 a_1.$$

All roots in (A15) are purely imaginary, so (A14) has a critical case of infinite dimension. We use the same technique as in Section 3 and Appendix A.1. Repeating the corresponding constructions for (A12) and (A13), we obtain the final quasi-normal forms for finding the asymptotics of solutions. The only difference between these forms—boundary value problems—and (24)–(26) and (A2)–(A4) is that instead of cubic non-linearity

$$-\frac{1}{2}\xi(|\xi|^2 + 2|\eta|^2), \quad \Big(-\frac{1}{2}\eta(|\eta|^2 + 2|\xi|^2)\Big)$$

with a real coefficient, the same non-linearity appears with a purely imaginary coefficient

$$-\frac{3}{2\gamma(\delta)}ib\xi(|\xi|^2 + 2|\eta|^2), \quad \Big(-\frac{3}{2\gamma(\delta)}ib\eta(|\eta|^2 + 2|\xi|^2)\Big),$$

and $-ia_1\xi/(2\gamma(\delta))$ appears instead of $-a_1\xi/2$. In this case, it is possible to completely split the corresponding boundary value problems from two coupled equations with respect to ξ and η into two independent equations. In order to so, we make the Lyapunov substitutions

$$\xi(\tau,x,y)=\xi_1(\tau,x,y)\exp\left(\frac{i}{2\gamma(\delta)}a_1\tau+\frac{3bi}{2\gamma(\delta)}\int\limits_0^\tau J_0(|\eta(s,x,y)|^2)ds\right),$$
$$\eta(\tau,x,y)=\eta_1(\tau,x,y)\exp\left(\frac{-i}{2\gamma(\delta)}a_1\tau-\frac{3bi}{2\gamma(\delta)}\int\limits_0^\tau J_0(|\xi(s,x,y)|^2)ds\right).$$

Note that these formulas imply that

$$\xi_1(\tau,x,y)=\xi(\tau,x,y)\exp\left(\frac{-ia_1}{2\gamma(\delta)}\tau-\frac{3bi}{2\gamma(\delta)}\int\limits_0^\tau J_0(|\eta_1(s,x,y)|^2)ds\right),$$
$$\eta_1(\tau,x,y)=\eta(\tau,x,y)\exp\left(\frac{ia_1}{2\gamma(\delta)}\tau+\frac{3bi}{2\gamma(\delta)}\int\limits_0^\tau J_0(|\xi_1(s,x,y)|^2)ds\right).$$

Thus, the boundary value problems for determining the amplitudes for rapidly oscillating modes for irregular solutions constitute a system of two independent equations of the Schrödinger type.

Appendix A.3. Applications to the Problem of Vibrations of Pedestrian Bridges

Appendix A.3.1. Formulation of the Problem

In [31], in connection with the study of the stability of pedestrian suspension bridges, a model was proposed that takes into account the influence of pedestrians on structural vibrations

$$\begin{gathered}\ddot{u}_j+\lambda(\dot{u}_j^2+u_j^2-\varepsilon)\dot{u}_j+\omega^2u_j=-\ddot{y},\\ \ddot{y}+2h\dot{y}+\Omega^2y=-\frac{r}{N}\sum_{i=1}^N\ddot{u}_i,\end{gathered}\tag{A16}$$

where $j=1,\ldots,N$. Here, the value u_j determines the deviation of the "pedestrian" from the bridge and y specifies the deviation of the bridge. All parameters of this "walker–bridge" model are positive. They are described in [31,32]. A number of interesting results on the dynamic properties of this type of model based on studies of synchronisation phenomena are given in [33–38]. The known results for this problem refer only to systems with a small number of elements. In this paper, quasi-normal forms are obtained for the most interesting cases with a large number of elements (pedestrians).

In this paper, we present several analytical results on the collective behaviour of a chain of coupled oscillators (A16).

The values of $u_j(t)$ can be associated with the values of functions of two variables $u(t,x_j)$. Here, $x_j\in[0,1]$ are points uniformly distributed on some circle with angular coordinate $x_j=2\pi N^{-1}j$. With this definition of x_j, periodic boundary conditions with respect to the variable x arise in a natural way. Note that one could also consider points x_j uniformly distributed on the segment $[0,1]$. Then, it is more natural to use Neumann-type boundary conditions. Since this case of a segment does not differ significantly from the case of a circle, we restrict ourselves to considering the case of periodic boundary conditions.

There are two main assumptions that open the way to the application of analytical methods. First, we assume that the number of oscillators (pedestrians) is large enough, i.e., $N \gg 1$. This gives grounds to move from a discrete system with respect to $u(t, x_j)$, $y(t)$ to a continuous spatially distributed boundary value problem for $u(t, x)$, $y(t)$

$$\begin{gathered} \frac{\partial^2 u}{\partial t^2} + \lambda\left(u^2 + \left(\frac{\partial u}{\partial t}\right)^2 - \varepsilon\right)\frac{\partial u}{\partial t} + \omega^2 u = -\frac{d^2 y}{dt^2}, \\ \frac{d^2 y}{dt^2} + 2h\frac{dy}{dt} + \Omega^2 y = -r\int_0^1 \frac{\partial^2 u(t,s)}{\partial t^2} ds, \end{gathered} \tag{A17}$$

$$u(t, x+1) \equiv u(t, x). \tag{A18}$$

The second limitation is that the ε parameter is small enough:

$$0 < \varepsilon \ll 1. \tag{A19}$$

Note that, under this condition, the van der Pol equation

$$\ddot{u} + \lambda[\dot{u}^2 + u^2 - \varepsilon]\dot{u} + \omega^2 u = 0$$

has a stable cycle $u_0(t, \varepsilon) = \varepsilon^{1/2}\rho_0 \cos(\omega t(1 + O(\varepsilon))) + O(\varepsilon^{3/2})$ with period $2\pi(\omega + O(\varepsilon))^{-1}$, where $\rho_0 = (3\omega^2 + 1)^{-1/2}$.

Under condition (A19), consider the behaviour of all solutions to the boundary value problem (A17) and (A18) with initial conditions from some sufficiently small ε-independent neighbourhood of the zero equilibrium state.

We introduce some notation. Let

$$M(v(x)) = \int_0^1 v(x)dx.$$

In (A17) and (A18), set

$$u(t, x) = u_0(t) + u_1(t, x), \quad M(u_1) = 0.$$

As a result, we arrive at the system

$$\begin{cases} \dfrac{\partial^2 u_0}{\partial t^2} + \lambda M\left(\left(u^2 + \left(\dfrac{\partial u}{\partial t}\right)^2\right)\dfrac{\partial u}{\partial t}\right) - \lambda\varepsilon\dfrac{du_0}{dt} + \omega^2 u_0 = -\dfrac{d^2 y}{dt^2}, \\ \dfrac{d^2 y}{dt^2} + 2h\dfrac{dy}{dt} + \Omega^2 y = -r\dfrac{d^2 u_0}{dt^2}, \end{cases} \tag{A20}$$

$$\frac{\partial^2 u_1}{\partial t^2} + \lambda\left[\left(u^2 + \left(\frac{\partial u}{\partial t}\right)^2\right)\frac{\partial u}{\partial t} - M\left(\left(u^2 + \left(\frac{\partial u}{\partial t}\right)^2\right)\frac{\partial u}{\partial t}\right)\right] - \lambda\varepsilon\frac{\partial u_1}{\partial t} + \omega^2 u_1 = 0. \tag{A21}$$

Taking into account boundary condition (A18), we have

$$u_1(t, x+1) \equiv u_1(t, x). \tag{A22}$$

When studying the local dynamics of solutions, an important role is played by the behaviour of solutions of linearised systems for $\varepsilon = 0$ that are linear in u_0, u_1 and y:

$$\begin{cases} \dfrac{d^2 u_0}{dt^2} + \omega^2 u_0 = -\dfrac{d^2 y}{dt^2}, \\ \dfrac{d^2 y}{dt^2} + 2h\dfrac{dy}{dt} + \Omega^2 y = -r\dfrac{d^2 u_0}{dt^2}, \end{cases} \tag{A23}$$

$$\frac{\partial^2 u_1}{\partial t^2} + \omega^2 u_1 = 0. \tag{A24}$$

Let us consider two cases separately, when the parameter r is small and when it is not.

Appendix A.3.2. First Case

Let the parameter r be small, i.e., for some fixed value r_1 we have

$$r = \varepsilon r_1. \tag{A25}$$

The boundary value problem (A22)–(A24) implements the critical case of an infinite set of pairs of purely imaginary roots $\pm i\omega$. They correspond to periodic solutions

$$u_k(t,x) = \exp(i\omega t + 2\pi i k x), \quad y_k(t,x) = 0 \quad (k = 0, \pm 1, \pm 2, \ldots). \tag{A26}$$

We use the technique for constructing quasi-normal forms developed in [20,38]. We seek the asymptotes of the solutions to the boundary value problem (A20)–(A22) based on solution (A26). To do this, we use the formal asymptotic representation

$$\begin{aligned} u(t,x) &= \varepsilon^{1/2}\Big(\xi(\tau,x)\exp(i\omega t) + \bar{\xi}(\tau,x)\exp(-i\omega t)\Big) + \varepsilon^{3/2} u_3(t,\tau,x) + \ldots, \\ y(t) &= \varepsilon^{3/2} y_3(t,\tau) + \ldots. \end{aligned} \tag{A27}$$

where $\tau = \varepsilon t$ is a slow temporal variable, the dependence on x is 1-periodic, $\xi(\tau,x)$ are unknown amplitudes and functions u_3 and y_3 are $2\pi/\omega$ periodic in t.

We substitute (A27) into (A20) and (A21) and equate the coefficients of the same powers of ε. For $\varepsilon^{1/2}$, we obtain an identity, and by collecting the coefficients of $\varepsilon^{3/2}$, we arrive at a system of equations for u_3, y_3. The condition for the solvability of this system in the indicated class of functions is the following equation

$$\frac{\partial \xi}{\partial \tau} = \frac{1}{2}\lambda \xi + \gamma \int_0^1 \xi(\tau,s)ds + b\xi|\xi|^2 \tag{A28}$$

together with the boundary conditions

$$\xi(\tau, x+1) \equiv \xi(\tau,x). \tag{A29}$$

For the coefficients γ and b, we have

$$\gamma = r_1\omega^2\Big[2(\Omega^2 - \omega^2 + 2i\omega h)\Big]^{-1}, \quad b = -\frac{1}{2}\lambda(3\omega^2+1).$$

The following theorem plays a central role; it states that the boundary value problem (A28) and (A29) is a quasi-normal form.

Theorem A3. *Let the condition* (A25) *be satisfied and the boundary value problem* (A28) *and* (A29) *have a bounded solution* $\xi(\tau,x)$ *as* $\tau \to \infty$, $x \in [0,1]$. *Then, the* 2π *periodic in* x *functions*

$$\begin{aligned} u(t,x,\varepsilon) = \ & \varepsilon^{1/2}\Big(\xi(\tau,x)\exp(i\omega t) + \bar{\xi}(\tau,x)\exp(-i\omega t)\Big) + \\ & + \varepsilon^{3/2}\tfrac{\lambda i}{8}(1-\omega^2)\Big(\xi^3(\tau,x)\exp(3i\omega t) - \bar{\xi}^3(\tau,x)\exp(-3i\omega t)\Big), \end{aligned} \tag{A30}$$

$$\begin{aligned} y(t,x,\varepsilon) = \ & \varepsilon^{3/2} r\omega^2 M\Big(\big[\Omega^2 - \omega^2 + 2ih\omega\big]^{-1}\xi(\tau,x)\exp(i\omega t) + \\ & + \big[\Omega^2 - \omega^2 - 2ih\omega\big]^{-1}\bar{\xi}(\tau,x)\exp(-i\omega t)\Big) \end{aligned} \tag{A31}$$

satisfy the original system (A17) *up to* $o(\varepsilon^{3/2})$.

Consider the question of constructing exact solutions to the boundary value problem (A28) and (A29). Set $\gamma = (\gamma_1 + i\gamma_2)\lambda/2,\ b_0 = 3\omega^2 + 1\ (b_0 > 0)$.

Under the condition $\lambda/2 + \gamma_1 > 0$, we have infinitely many periodic solutions

$$\xi_0(\tau, x) = \left((1+\gamma_1)b_0^{-1}\right)^{1/2} \exp\left(i\gamma_2 \frac{1}{2}\lambda\tau\right), \quad \xi_k(\tau, x) = b_0^{-1/2} \exp(i2\pi k x),$$

$(k = \pm 1, \pm 2, \ldots)$.

It is more interesting to construct solutions that are periodic in τ and piecewise constant in the spatial variable. For example, we fix an arbitrary (finite) number of intervals from the segment $[0,1]$ with a total length of $1/2$ and set $\xi(\tau, x) = \left((1+\gamma_1)b_0^{-1}\right)^{1/2} \exp\left(i\gamma_2\lambda\tau/2\right)$, whereas for other x values from $[0,1]$, we set $\xi(\tau, x) = -\left((1+\gamma_1)b_0^{-1}\right)^{1/2} \exp\left(i\gamma_2\lambda\tau/2\right)$.

One can construct families of $4\pi(\lambda\gamma_2)^{-1}$-periodic in τ and 1-periodic piecewise-continuous in x solutions $\xi(\tau, x, \alpha, k_1, k_2) = \rho(x, \alpha, k_1, k_2) \exp(i\gamma_2\lambda\tau/2)$, where

$$\rho(x, \alpha, k_1, k_2) = \begin{cases} \left((1+\gamma_1)b_0^{-1}\right)^{1/2} \exp\left(i2\pi\alpha^{-1}k_1 x\right),\ x \in (0, \alpha),\ k_1 = \pm 1, \pm 2, \ldots, \\ \left((1+\gamma_1)b_0^{-1}\right)^{1/2} \exp\left(i2\pi(-\alpha)^{-1}k_2 x\right),\ x \in (\alpha, 1),\ k_2 = \pm 1, \pm 2, \ldots. \end{cases}$$

More interesting are the cycles consisting of two steps with different amplitudes in the interval $[0,1]$. To construct them, we fix arbitrarily the parameters $\alpha \in (0,1)$ and $\varphi_{1,2} \in [0, 2\pi]$. Let

$$\xi_0(\tau, x) = \rho(x)\exp(i\delta\tau), \quad \rho(x) = \begin{cases} \rho_1 \exp(i\varphi_1),\ x \in [0, \alpha], \\ \rho_2 \exp(i\varphi_2),\ x \in [\alpha, 1]. \end{cases}$$

Substitute this expression into (A28). Then, we obtain a system of four algebraic equations in five real variables $\rho_1, \rho_2, \delta, \alpha$ and $\varphi = \varphi_2 - \varphi_1 \in [0, 2\pi]$

$$B\begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} = \delta \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix}, \tag{A32}$$

where

$$B = \begin{pmatrix} \alpha\gamma_2 & -(1-\alpha)(\gamma_1\cos\varphi - \gamma_2\sin\varphi) \\ -\alpha\gamma_1\sin\varphi + \alpha\gamma_2\cos\varphi & (1-\alpha)\gamma_2 \end{pmatrix},$$

$$b_0\rho_1^3 = (1+\alpha\gamma_1)\rho_1 + (1-\alpha)(\gamma_1\cos\varphi - \gamma_2\sin\varphi)\rho_2, \tag{A33}$$

$$b_0\rho_2^3 = (1+(1-\alpha)\gamma_1)\rho_2 + \alpha(\gamma_1\cos\varphi + \gamma_2\sin\varphi)\rho_1. \tag{A34}$$

The condition for the eigenvalues δ_+ and δ_- and the corresponding eigenvectors in (A32) to be real is

$$4\alpha(1-\alpha)\sin^2\varphi \le \gamma_2^2(\gamma_1^2 + \gamma_2^2)^{-1}. \tag{A35}$$

Then,

$$\delta_\pm = \frac{1}{2}\gamma_2 \pm \left[\gamma_2^2 - 4\alpha(1-\alpha)\sin^2\varphi \cdot (\gamma_1^2 + \gamma_2^2)\right]^{1/2}$$

and

$$\rho_2^\pm = c_\pm \rho_1^\pm, \text{ where } c_\pm = (\delta_\pm - \alpha\gamma_2)[(1-\alpha)(\gamma_2\cos\varphi + \gamma_1\sin\varphi)]^{-1}. \tag{A36}$$

Given (A35) and (A36), the expressions (A33) and (A34) take the form

$$b_0(\rho_1^{\pm})^2 = R_1^{\pm}, \text{ where } R_1^{\pm} = 1 + \alpha\gamma_1 + (1-\alpha)c_{\pm}(\gamma_1 \cos\varphi - \gamma_2 \sin\varphi), \tag{A37}$$

$$b_0(\rho_1^{\pm})^2 = R_2^{\pm}, \text{ where } R_2^{\pm} = \big[(1 + (1-\alpha)\gamma_1)c_{\pm} + \alpha(\gamma_1 \cos\varphi + \gamma_2 \sin\varphi)\big]c_{\pm}^{-3}. \tag{A38}$$

We fix the parameter $\varphi \in [0, 2\pi]$ arbitrarily. Denote by $\Phi_{\pm}(\varphi)$ the set of all values $\alpha \in [0,1]$ for which Equation (A35) holds and $R_j^{\pm} \geq 0$ $(j = 1,2)$. Equating the right parts in (A37) and (A38), we arrive at

$$R_1^{\pm} = R_2^{\pm},$$

which we consider as an equation with respect to $\alpha_{\pm} = \alpha_{\pm}(\varphi)$. In the case when the root $\alpha_{\pm}(\varphi)$ of this equation exists and belongs to the set $\Phi_{\pm}(\varphi)$, we determine all elements of the stepwise periodic solution $\rho(x)\exp(i\delta\tau)$ of the boundary value problem (A28) and (A29).

Numerical experiments allowed to establish that, for certain values of the coefficients in (A28), there are one-parametric families of such stepwise periodic solutions.

Appendix A.3.3. Second Case

Here, we consider the situation when the parameter $r \neq 0$ and is somehow fixed. We assume that all roots of the characteristic equation

$$(\lambda^2 + \omega^2)(\lambda^2 + 2h\lambda + \Omega^2) - r\lambda^4 = 0$$

for the linear system (A23) have negative real parts. Then, the boundary value problem (A24), (A22) has infinitely many periodic solutions (A26), where the index k takes the values $\pm 1, \pm 2, \ldots$. Due to the fact that $k \neq 0$ in (A27), there is an additional condition

$$M(\xi(\tau, x)) = 0.$$

Substituting (A27) into (A17) and (A18) and collecting the coefficients of the same powers of ε, we obtain a system of equations with respect to $2\pi/\omega$-periodic in t functions u_3 and y_3. From the solvability condition for this system, we arrive at the equation

$$2\frac{d\xi}{d\tau} = \lambda\xi - \lambda(1 + 3\omega^2)(\xi|\xi|^2 - M(\xi|\xi|^2)) \tag{A39}$$

with conditions

$$\xi(\tau, x+1) \equiv \xi(\tau, x), \quad M(\xi(\tau, x)) = 0. \tag{A40}$$

Theorem A4. *Let the condition $r \neq 0$ be satisfied and the boundary value problem* (A39) *and* (A40) *have a bounded solution $\xi(\tau, x)$ as $\tau \to \infty$, $x \in [0,1]$. Then, the 2π periodic in x functions* (A30) *and $y(t, x, \varepsilon) = 0$ satisfy the original system* (A17) *up to $o(\varepsilon^{3/2})$.*

Thus, the resulting boundary value problem is a quasi-normal form in the situation under consideration.

For example, functions $(1 + 3\omega^2)^{-1/2}\exp(i2\pi kx)$, $k = \pm 1, \pm 2, \ldots$, are periodic solutions to (A39) and (A40).

The equilibrium states for (A39) and (A40) are the family of step functions

$$\xi(x) = \begin{cases} (1-\alpha)\Big(\alpha(1+3\omega^2)^{1/2}\Big)^{-1}, & x \in [0, \alpha], \\ (1+3\omega^2)^{-1/2}, & x \in (\alpha, 1], \end{cases} \tag{A41}$$

depending on the parameter $\alpha \in (0,1)$.

Remark A3. *The stepwise solutions constructed above allow an asymptotic study of their stability. We do not dwell here on this. We only note that some results on the stability of solutions of the form* (A41) *are given in [38].*

Remark A4. *In a more general case, when in the original system* (A17) *the left side of the first equation contains, for example, the term* γu^3*, we arrive at a quasi-normal form that differs from* (A39) *only in the presence of one more purely imaginary term,* $3i\lambda\gamma\xi|\xi|^2$*. This leads to the fact that, instead of a family of equilibrium states in* (A39) *and* (A40)*, continuum families of solutions periodic in* τ *appear with different periods.*

Remark A5. *When considering the construction of three, four, etc., stepwise solutions with different amplitudes on the segment* $[0,1]$*, multiparametric families of such solutions arise.*

An important conclusion is that the dynamic properties of the boundary value problems (A28) and (A29), (A39) and (A40) are quite rich.

We note that similarly we consider the quasi-linear case when the first equation in (A17) is replaced by

$$\frac{\partial^2 u}{\partial t^2} + \omega^2 u + \varepsilon f\left(u, \frac{\partial u}{\partial t}\right) = -\frac{d^2 y}{dt^2}.$$

In this case, the quasi-normal form analogous to (A28) has the form

$$\begin{aligned} &2i\omega\frac{\partial \xi}{\partial \tau} = g(\xi) - M(g(\xi)), \\ &g(\xi) = \frac{\omega}{2\pi}\int_0^{2\pi/\omega} f\Big(\xi\exp(i\omega t) + \bar{\xi}\exp(-i\omega t),\ i\omega\xi\exp(i\omega t) - \\ &\qquad\qquad - i\omega\bar{\xi}\exp(-i\omega t)\Big)\exp(-i\omega t)dt \end{aligned} \tag{A42}$$

and for $u(t,\tau,x),\ y(t,\tau)$, we have the asymptotic representations

$$\begin{aligned} u(t,\tau,x) &= \xi(\tau,x)\exp(i\omega t) + \bar{\xi}(\tau,x)\exp(-i\omega t) + \varepsilon u_1(t,\tau,x) + \ldots, \\ y(t,\tau) &= \varepsilon y_1(t,\tau) + \ldots. \end{aligned}$$

It is possible to choose the function f in such a way, for example, in the form of a polynomial in u and $\partial u/\partial t$ of degree 5, such that the oscillations have cluster character; the boundary value problem (A42), (A40) had stepwise solutions such that different "steps" oscillated with different periods in t.

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