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Article

Variable Besov–Morrey Spaces Associated with Operators

Khedoudj Saibi

Department of Mathematics, Zhejiang Normal University, Jinhua 321004, China; khedoudjsaibi@zjnu.edu.cn or saibi.khedoudj@yahoo.com

Abstract: Let (X, d, μ) be a space of homogenous type and L be a non-negative self-adjoint operator on $L^2(X)$ with heat kernels satisfying Gaussian upper bounds. In this paper, we introduce the variable Besov–Morrey space associated with the operator L and prove that this space can be characterized via the Peetre maximal functions. Then, we establish its atomic decomposition.

Keywords: variable exponents; Besov–Morrey spaces; metric measure; heat kernel; maximal characterization; atomic characterizations

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1. Introduction

The theory of function spaces with variable exponents, traced back to Orlicz [1], has gained a lot of attention since the development emerged from the pioneering work of Kováčik and Rákosníck [2]. In particular, the Morrey space with a variable exponent over open sets of $\mathbb{R}^n$ was introduced by [3], and the boundedness of the maximal operator on this space was proved in the same paper under the log-Hölder condition of the variable exponents. Independently, Kokilashvili and Meskhi [4] studied the boundedness of the fractional maximal operator and fractional integral operator on the variable exponent Morrey spaces defined over spaces of homogeneous type in the sense of Coifman and Weiss [5].

Besov spaces were also extended to the variable case. Indeed, by means of the variable sequence space $l_{q(\cdot)}(L_{p(\cdot)})$, Almeida and Hästö [6] introduced the variable Besov space $B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ and proved that the definition of this space is independent of the choice of the basis functions and some basic properties, and gave Sobolev-type embeddings. Recently, Besov spaces were further generalized to the Morrey-type with variable exponents. In particular, Almeida and Caetano [7] introduced the Besov–Morrey space with variable exponents via the Morrey sequence space $l_{q(\cdot)}(L_{p(\cdot),u(\cdot)})$ and proved some elementary properties for this space.

Over the last decades, the theory of function spaces associated with different operators attracted great interest and has become a fruitful research topic. Mainly, Auscher, Duong, and McIntosh [8] introduced the Hardy spaces $H^1_L(\mathbb{R}^n)$ associated with the operator L, where L is a linear operator on $L^2(\mathbb{R}^n)$ which generates an analytic semigroup $\{e^{-tL}\}_{t>0}$, whose kernels have pointwise Gaussian upper bounds. Kerkyacharian and Petroshev initially introduced Besov and Triebel–Lizorkin spaces associated with operators on a homogeneous-type space in [9], where they proved embedding theorems, heat kernel characterization, and frame decomposition. Such spaces are associated with a non-negative self-adjoint operator whose heat kernels satisfy Gaussian upper bounds, the Hölder continuity, and Markov property. Hu [10] gave their characterization by means of the Peetre-type maximal functions and proved their atomic decompositions. Very recently, Zhuo and Yang [11] generalized the results obtained in [9] to the variable case. More precisely, they introduced the variable Besov space associated with heat kernels and proved several characterizations of this space, such as the Peetre maximal functions characterizations, heat kernel characterizations, and the frame decomposition.

It is well known that Besov spaces include many important function spaces, such as Lebesgue, Hardy, and Sobolev spaces. Thus, it is worthwhile to generalize and extend these spaces to more general settings. Therefore, in this article, we aim to extend the Besov spaces associated with operators to a more general framework. Indeed, we introduce and study the Morrefied version of the space investigated in [11]. More precisely, we firstly aim to study the variable Besov–Morrey space on spaces of homogeneous-type $B^{\alpha(\cdot),L}_{p(\cdot),u(\cdot),q(\cdot)}(X)$, with a measure satisfying the doubling condition, associated with the non-negative self-adjoint operator L on $L^2(X)$, whose heat kernels satisfy the small-time Gaussian upper bound and the Hölder continuity. We introduce the definition of our space by means of the Littlewood–Paley-type decomposition and establish its characterizations by means of the Peetre maximal functions to conclude that different choices of the basis functions in our definition produce equivalent quasi-norms and yield to the same space. Our second aim is to prove the atomic decomposition of the variable Besov–Morrey spaces associated with the operator L.

We finish this introduction by describing the layout of this paper. In Section 2, we give some notions, definitions, and properties. In Section 3, we introduce the variable Besov–Morrey space associated with the operator L by means of a vector-valued inequality (Theorem 1), establishing its Peetre-type maximal functions characterizations and atomic decompositions.

As usual, throughout the paper, we denote by $\mathbb{N}$ and $\mathbb{Z}$ the set of non-negative integers and the set of integers, respectively, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Additionally, by C, c, $C_1, \cdots$, we denote positive constants independent of the parameters, which can differ from occurrence to occurrence. The symbols $A \lesssim B$ and $A \asymp B$ are used for the inequality $A \leq CB$ and the compound inequality $cA \leq B \leq CA$, respectively.

2. Preliminaries

In this section, we give some notions, notations, and definitions, and we describe the assumptions required for the operator L.

The space (X, d) is assumed to be a locally compact metric measure space and μ a positive regular Borel measure (see, for instance, p. 965 in [12], for the definition and more details about the regular Borel measure), satisfying the doubling condition, i.e., there exists a positive constant c_0 such that

$$\mu(B(x, 2r)) \leq c_0 \mu(B(x, r)) \text{ for all } x \in X \text{ and } r > 0, \tag{1}$$

where $B(x, r)$ is the ball with center x and radius r, i.e., $B(x, r) := \{y \in X : d(x, y) < r\}$. We call the triplet (X, d, μ) a space of homogeneous type in the sense of Coifman and Weiss [5].

It is easy to show that (1) implies

$$\mu(B(x, \kappa r)) \leq c_0 \kappa^n \mu(B(x, r)) \text{ for all } x \in X, r > 0 \text{ and } \kappa > 1, \tag{2}$$

where $n = \log_2 c_0 > 0$.

For any $x, y \in X$ and $r \in (0, \infty)$, we have $B(x, r) \subset B(y, r + d(x, y))$; then, $\mu(B(x, r)) \leq \mu(B(y, r + d(x, y)))$. Thus, by (2), we have

$$\mu(B(x, r)) \leq c_0 \left(1 + \frac{d(x, y)}{r}\right)^n \mu(B(y, r)), \tag{3}$$

where c_0 is as in (1).

Let L be a non-negative self-adjoint operator, with a dense domain in $L^2(X)$ denoted by $\mathrm{Dom}(L)$. The heat semigroup $\{e^{-tL}\}_{t\geq 0}$ arising from L is the family of the integral operators associated with the heat kernels $\{p_t\}_{t\geq 0}$ defined for any function $f \in L^2(X)$ by:

$$e^{-tL}f(x) := \int_X p_t(x,y)f(y)d\mu(y) \quad \text{for any} \quad x \in X.$$

We assume that there exist positive constants C_1 and C_2 such that the kernels $\{p_t\}_{t\geq 0}$ satisfy the following:

- Small-time Gaussian upper bound:

$$|p_t(x,y)| \leq \Big[\mu(B(x,\sqrt{t}))\mu(B(y,\sqrt{t}))\Big]^{-\frac{1}{2}} C_1 e^{-\frac{C_2 d(x,y)}{t}}, \tag{4}$$

- Hölder continuity: There exists a $\tau \in (0,\infty)$ such that for any $x, y_1, y_2 \in X$ and $t \in (0,1]$ such that $d(x,y) < t$,

$$\begin{aligned} &|p_t(x,y_1) - p_t(x,y_2)| \\ &\quad \leq C_1 \left(\frac{d(x,y)}{\sqrt{t}}\right)^{\tau} \Big(\mu(B(x,\sqrt{t}))\mu(B(y,\sqrt{t}))\Big)^{-\frac{1}{2}} C_1 e^{-\frac{C_2 d(x,y)}{t}}. \end{aligned} \tag{5}$$

- Markov property: For any $x \in X$ and $t \in (0,1]$,

$$\int_X p_t(x,y)d\mu(y) = 1. \tag{6}$$

Now, we recall some notions and definitions related to the variable function spaces. A variable exponent is a measurable function $p(\cdot) : X \to (0,\infty]$. We set

$$p_- := \operatorname{ess\,inf}_{x\in X} p(x) \quad \text{and} \quad p_+ := \operatorname{ess\,sup}_{x\in X} p(x)$$

and

$$\mathcal{P}(X) := \Big\{p(\cdot) \text{ variable exponent} : 0 < p_- \leq p_+ < \infty\Big\}.$$

Let $p(\cdot) \in \mathcal{P}(X)$. The variable Lebesgue space $L^{p(\cdot)}(X)$ consists of all measurable functions $f : X \to \mathbb{R}$, such that $\varrho_{p(\cdot)}(f) < \infty$, where

$$\varrho_{p(\cdot)}(f) := \int_X |f(x)|^{p(x)} d\mu(x),$$

equipped with the Luxemburg quasi-norm

$$\|f\|_{L^{p(\cdot)}(X)} := \inf\left\{\lambda > 0 : \int_X \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} d\mu(x) \leq 1\right\}.$$

One can easily show that for any $s \in (0,\infty)$, we have

$$\||f|^s\|_{L^{p(\cdot)}(X)} = \|f\|^s_{L^{sp(\cdot)}(X)}. \tag{7}$$

Let $p(\cdot), u(\cdot) \in \mathcal{P}(X)$ such that $p(x) \leq u(x)$, the Morrey space with variable exponents denoted by $M_{p(\cdot),u(\cdot)}(X)$ is defined as the set of all measurable functions f, such that $\|f\|_{M_{p(\cdot),u(\cdot)}(X)} < \infty$, where

$$\|f\|_{M_{p(\cdot),u(\cdot)}(X)} := \sup_{x\in X, r>0} [V_r(x)]^{\frac{1}{u(x)} - \frac{1}{p(x)}} \|f\chi_{B(x,r)}\|_{L_{p(\cdot)}(X)};$$

here, and hereafter, $V_r(x)$ denotes the measure of the ball $B(x,r)$. It is easy to see that the above norm can be written as

$$\|f\|_{M_{p(\cdot),u(\cdot)}(X)} := \sup_{x\in X, r>0} \inf\left\{\lambda > 0 : \varrho_{p(\cdot)}\left([V_r(x)]^{\frac{1}{u(x)}-\frac{1}{p(x)}} \frac{f}{\lambda}\chi_{B(x,r)}\right) \le 1\right\}.$$

By (7), we can show that for $s \in (0,\infty)$ and $f \in M_{p(\cdot),u(\cdot)}(X)$, we have

$$\|f\|^s_{M_{p(\cdot),u(\cdot)}(X)} = \||f|^s\|_{M_{\frac{p(\cdot)}{s},\frac{u(\cdot)}{s}}(X)}.$$

A variable exponent $p(\cdot)$ is said to satisfy the locally log-Hölder continuity condition, and we write $p(\cdot) \in H^{\log}_{\mathrm{loc}}(X)$ if there exists a positive constant $c_{\log}(p)$, such that for any $x, y \in X$,

$$|p(x) - p(y)| \le \frac{c_{\log}(p)}{\log(e + 1/d(x,y))}; \tag{8}$$

additionally, we say that $p(\cdot)$ satisfies the globally log-Hölder condition, and we denote by $p(\cdot) \in H_{\log}(X)$ if it further satisfies the log-Hölder decay condition with respect to a base point x_0, i.e., there exists $p_\infty \in \mathbb{R}$ and $c_\infty(p) \ge 0$ such that

$$|p(x) - p_\infty| \le \frac{c_\infty(p)}{\log(e + d(x,y))}.$$

We recall that for any $f \in L^1_{\mathrm{loc}}(X)$, the Hardy–Littlewood maximal operator M is defined for all $x \in X$ by setting

$$M(f)(x) := \sup_{B\ni x} \frac{1}{|B|}\int_B |f(y)|\,dy, \tag{9}$$

where the supremum is taken over all balls B of X containing x. The next lemma presents the boundedness of the Hardy–Littlewood maximal operator on the Morrey space with variable exponents on homogeneous spaces; for the proof, we refer the reader to Theorem 1 in [13].

Lemma 1. *Let (X,d,μ) be a space of homogeneous type and $p(\cdot) : X \to [1,\infty)$ and $u(\cdot) : X \to [1,\infty)$ such that $u_+ < \infty$, $1/p(\cdot), 1/u(\cdot) \in H_{\log}(X)$ and $p(\cdot) \le u(\cdot)$. Then, the Hardy–Littlewood maximal function is bounded on $M_{p(\cdot),u(\cdot)}(X)$.*

Definition 1. *Let $p(\cdot), u(\cdot), q(\cdot) \in \mathcal{P}(X)$ such that $p(x) \le u(x)$. Then, the mixed Morrey sequence spaces $l_{q(\cdot)}(M_{p(\cdot),u(\cdot)}(X))$ are the set of all sequences $\{f_i\}_{i\in\mathbb{N}_0}$ of measurable functions on X, such that $\varrho_{l_{q(\cdot)}(M_{p(\cdot),u(\cdot)}(X))}(\{f_i\}_{i\in\mathbb{N}_0}) < \infty$, where*

$$\begin{aligned}&\varrho_{l_{q(\cdot)}(M_{p(\cdot),u(\cdot)}(X))}(\{f_i\}_{i\in\mathbb{N}_0})\\ &\qquad := \sum_{i\ge 0} \sup_{x\in X, r>0} \inf\left\{\beta_i > 0 : \varrho\left(\frac{[V_r(x)]^{\frac{1}{u(x)}-\frac{1}{p(x)}} f_i \chi_{B(x,r)}}{\beta_i^{\frac{1}{q(\cdot)}}}\right) \le 1\right\}.\end{aligned}$$

For any sequence $\{f_i\}_{i\in\mathbb{N}_0}$ in $l_{q(\cdot)}(M_{p(\cdot),u(\cdot)}(X))$, its norm in this space is given by

$$\|\{f_i\}_{i\in\mathbb{N}}\|_{l_{q(\cdot)}(M_{p(\cdot),u(\cdot)}(X))} := \inf\left\{\lambda > 0 : \varrho_{l_{q(\cdot)}(M_{p(\cdot),u(\cdot)}(X))}\left(\frac{\{f_i\}_{i\in\mathbb{N}_0}}{\lambda}\right) \le 1\right\}. \tag{10}$$

We say that a function $\varrho : X \to [0,\infty]$ is the following:

1. $\varrho(0) = 0$.
2. $\varrho(\lambda x) = \varrho(x)$ for all $x \in X, \lambda \in \mathbb{K}$ with $|\lambda| = 1$.

3. ϱ is convex.
4. ϱ is left-continuous.
5. $\varrho(\lambda x) = 0$ for all $\lambda > 0$ implies $x = 0$.

If, in addition,

$$\varrho(x) = 0 \text{ implies } x = 0,$$

then ϱ is called a modular. For more details about semimodulars and modulars, we refer the reader to Chapter 2 in [14].

Remark 1.

1. *By a similar argument used for Theorem 3.7 in [7], one can show that* $\varrho_{l_{q(\cdot)}(M_{p(\cdot),u(\cdot)}(X))}$ *defines a semimodular on* X *and a modular if* $q_+ < \infty$*; here, and hereafter,* $q_+ := \text{ess}\sup_{x \in X} q(x)$.
2. *It can be shown that* $\|\cdot\|_{l_{q(\cdot)}(M_{p(\cdot),u(\cdot)}(X))}$ *defines a quasi-norm, and it is a norm in the following particular cases:*
   (a) *Case (1):* $p(x) \geq 1$ *and* $q(x) = \gamma$ *for any* $x \in X$*, where* $\gamma \in (0, \infty]$;
   (b) *Case (2):* $1 \leq q(x) \leq p(x) \leq u(x) \leq \infty$;
   (c) *Case (3):* $\frac{1}{p(x)} + \frac{1}{q(x)} \leq 1$.
3. *It is easy to show that for any* $s \in (0, \infty)$,

$$\left\| \{|f_i|^s\}_{i \in \mathbb{N}_0} \right\|_{l_{\frac{q(\cdot)}{s}}(M_{\frac{p(\cdot)}{s},\frac{u(\cdot)}{s}}(X))} = \left\| \{f_i\}_{i \in \mathbb{N}_0} \right\|^s_{l_{q(\cdot)}(M_{p(\cdot),u(\cdot)}(X))}.$$

The next lemma gives the relationship between the semimodular and the quasi-norm.

Lemma 2. *Let* $p(\cdot), u(\cdot), q(\cdot) \in \mathcal{P}(X)$ *such that* $p(x) \leq u(x)$*. If* $q_+ < \infty$ *or* $\varrho_{l_{q(\cdot)}(M_{p(\cdot),u(\cdot)}(X))}(\lambda\{f_i\}_i) > 0$*, then*

$$\begin{aligned} &\|\{f_i\}_{i \in \mathbb{N}_0}\|_{l_{q(\cdot)}(M_{p(\cdot),u(\cdot)}(X))} \\ &\quad \leq \max\left\{ \varrho_{l_{q(\cdot)}(M_{p(\cdot),u(\cdot)}(X))}(\{f_i\}_{i \in \mathbb{N}_0})^{\frac{1}{q_-}}, \varrho_{l_{q(\cdot)}(M_{p(\cdot),u(\cdot)}(X))}(\{f_i\}_{i \in \mathbb{N}_0})^{\frac{1}{q_+}} \right\} \end{aligned}$$

Proof. It can be easily shown that the right-hand side satisfies the inequality appearing in (10) and, taking into account that we are dealing with a modular, when $q_+ < \infty$. □

Next, we prove a convolution inequality which is considered as a replacement of the maximal inequalities and is used to prove the results in the next section.

Let $\delta, \sigma \in (0, \infty)$, define

$$\eta_{\delta,\sigma}(x, y) := \Big(V_\delta(x) V_\delta(y)\Big)^{-\frac{1}{2}} \left(1 + \frac{d(x, y)}{\delta}\right)^{-\sigma}.$$

For any $f \in L^1_{\text{loc}}(X)$, we define the operator $\eta_{\delta,\sigma}$ for any $x \in X$ by

$$\eta_{\delta,\sigma}(f)(x) := \int_X \eta_{\delta,\sigma}(x, y) f(y) d\mu(y).$$

The next result is given in [11].

Lemma 3. *Let* $\alpha(\cdot) \in H^{\log}_{\text{loc}}(X) \cap L^\infty(X)$*,* $\delta \in (0, 1), \sigma \in (n, \infty)$ *and* $l \in [c_{\log}(\alpha), \infty)$*. Then, there exists a positive constant* C *such that for any* $x, y \in X$,

$$\varepsilon^{-j\alpha(x)} |\eta_{\delta,\sigma+l}(f)(x)| \leq C \eta_{\delta,\sigma}\Big(\varepsilon^{-j\alpha(\cdot)}|f|\Big)(x).$$

The next theorem is a generalization of the convolution inequality ([7], Theorem 4.6) to homogeneous-type spaces.

Theorem 1. *Let (X, d, μ) be a space of homogeneous type and $p(\cdot) : X \to [1, \infty)$, $u(\cdot) : X \to [1, \infty)$ and $q(\cdot) : X \to [0, \infty)$ such that $p(\cdot) \le u(\cdot)$, $1/p(\cdot), 1/u(\cdot) \in H_{\log}(X)$, $1/q(\cdot) \in H^{\log}_{\mathrm{loc}}(X)$, $\delta \in (0, 1]$ and $\sigma \in (n + c_{\log}(1/q), \infty)$, where n is as in (2). Then, there exists a positive constant C, such that for any $\{f_i\}_{i\in\mathbb{N}_0} \subset L^1_{\mathrm{loc}}(X)$,*

$$\left\| \{\eta_{\delta^i,\sigma}(f_i)\}_{i\in\mathbb{N}_0} \right\|_{l_{q(\cdot)}(M_{p(\cdot),u(\cdot)}(X))} \le C \left\| \{f_i\}_{i\in\mathbb{N}_0} \right\|_{l_{q(\cdot)}(M_{p(\cdot),u(\cdot)}(X))}.$$

Proof. We assume that $\|\{f_i\}_{i\in\mathbb{N}_0}\|_{l_{q(\cdot)}(M_{p(\cdot),u(\cdot)}(X))} = 1$, and we prove that

$$\left\| \{\eta_{\delta^i,\sigma}(f_i)\}_{i\in\mathbb{N}_0} \right\|_{l_{q(\cdot)}(M_{p(\cdot),u(\cdot)}(X))} \lesssim 1.$$

The latter is equivalent to

$$\inf\left\{ \lambda > 0 : \varrho_{l_{q(\cdot)}(M_{p(\cdot),u(\cdot)}(X))}\left(\frac{1}{\lambda} \{\eta_{\delta^i,\sigma}(f_i)\}_{i\in\mathbb{N}_0} \right) \le 1 \right\} \lesssim 1.$$

Then, it is enough to prove that there exist some constants $c > 0$, such that

$$\varrho_{l_{q(\cdot)}(M_{p(\cdot),u(\cdot)}(X))}\left(\{c\eta_{\delta^i,\sigma}(f_i)\}_{i\in\mathbb{N}_0} \right) \le 1,$$

which is equivalent to

$$\sum_{i\in\mathbb{N}_0} \sup_{x\in X, r>0} \inf\left\{ \lambda_i > 0 : \varrho_{p(\cdot)}\left(\frac{[V_r(x)]^{\frac{1}{u(x)} - \frac{1}{p(x)}} c|\eta_{\delta^i,\sigma}(f_i)|}{\lambda_i^{\frac{1}{q(\cdot)}}} \chi_{B(x,r)} \right) \le 1 \right\} \le 1.$$

We claim that

$$\begin{aligned} &\sup_{x\in X, r>0} \inf\left\{ \lambda_i > 0 : \varrho_{p(\cdot)}\left(\frac{[V_r(x)]^{\frac{1}{u(x)} - \frac{1}{p(x)}} c|\eta_{\delta^i,\sigma}(f_i)|}{\lambda_i^{\frac{1}{q(\cdot)}}} \chi_{B(x,r)} \right) \le 1 \right\} \\ &\le \sup_{x\in X, r>0} \inf\left\{ \beta_i > 0 : \varrho_{p(\cdot)}\left(\frac{[V_r(x)]^{\frac{1}{u(x)} - \frac{1}{p(x)}} |f_i|}{\beta_i^{\frac{1}{q(\cdot)}}} \chi_{B(x,r)} \right) \le 1 \right\} + 2^{-i}. \end{aligned}$$

Then, by taking the sum over i, we obtain

$$\sum_{i\in\mathbb{N}_0} \sup_{x\in X, r>0} \inf\left\{ \lambda_i > 0 : \varrho_{p(\cdot)}\left(\frac{[V_r(x)]^{\frac{1}{u(x)} - \frac{1}{p(x)}} c|\eta_{\delta^i,\sigma}(f)|}{\lambda_i^{\frac{1}{q(\cdot)}}} \chi_{B(x,r)} \right) \le 1 \right\} \le 3$$

which means

$$\varrho_{l_{q(\cdot)(M_{p(\cdot),u(\cdot)}(X))}}(c\{\eta_{\delta^i,\sigma}(f_i)\}_{i\in\mathbb{N}_0}) \le 3.$$

Then, by the connection between the semimodular and the quasi-norm, we obtain

$$\left\| c\{\eta_{\delta^i,\sigma}(f_i)\}_{i\in\mathbb{N}_0} \right\|_{l_{q(\cdot)}(M_{p(\cdot),u(\cdot)}(X))} \le 3^{\frac{1}{q_-}}$$

and the result follows by homogeneity.

We return to the claim. One can easily see that it is a consequence of

$$\varrho_{p(\cdot)}\left(\frac{c[V_r(x)]^{\frac{1}{u(x)}-\frac{1}{p(x)}}|\eta_{\delta^i,\sigma}(f_i)|\chi_{B(x,r)}}{\Lambda_i^{\frac{1}{q(\cdot)}}}\right)\leq 1,$$

where

$$\Lambda_i := \sup_{x\in X, r>0}\inf\left\{\beta_i>0:\varrho_{p(\cdot)}\left(\frac{[V_r(x)]^{\frac{1}{u(x)}-\frac{1}{p(x)}}|f_i|}{\beta_i^{\frac{1}{q(\cdot)}}}\chi_{B(x,r)}\right)\leq 1\right\}+2^{-i}.$$

It is easy to see that

$$\sup_{x\in X, r>0}\varrho_{p(\cdot)}\left(\frac{[V_r(x)]^{\frac{1}{u(x)}-\frac{1}{p(x)}}c|f_i|}{\Lambda_i^{\frac{1}{q(\cdot)}}}\chi_{B(x,r)}\right)\leq 1,$$

which implies that

$$\left\|\Lambda^{-\frac{1}{q(\cdot)}}f_i\right\|_{M_{p(\cdot),u(\cdot)}(X)}\leq 1.$$

Note that for any $\delta\in(0,\infty)$,

$$\begin{aligned}|\eta_{\delta^i,\sigma}(f_i)(x)| &= \left|\int_X \eta_{\delta^i,\sigma}(x,y)f(y)d\mu(y)\right| \\ &\leq \sum_{j=0}^{\infty}\int_{U_j(B(x,\delta^i))}|\eta_{\delta^i,\sigma}(x,y)f(y)|d\mu(y) \\ &= \int_{U_0(B(x,\delta^i))}|\eta_{\delta^i,\sigma}(x,y)f(y)|d\mu(y) \\ &\quad + \sum_{j=1}^{\infty}\int_{U_j(B(x,\delta^i))}|\eta_{\delta^i,\sigma}(x,y)f(y)|d\mu(y),\end{aligned}$$

where $U_0(B(x,\delta^i))=B(x,\delta^i)$ and for any $j\in\mathbb{N}$,

$$U_j(B(x,\delta^i))=B(x,2^j\delta^i)\backslash B(x,2^{j-1}\delta^i).$$

By (3), we have

$$\begin{aligned}&\int_{U_0(B(x,\delta^i))}|\eta_{\delta^i,\sigma}(x,y)f(y)|d\mu(y) \\ &\qquad = \int_{B(x,\delta^i)}\Big(V_{\delta^i}(x)V_{\delta^i}(y)\Big)^{-\frac{1}{2}}\Big(1+\frac{d(x,y)}{\delta^i}\Big)^{-\sigma}|f(y)|d\mu(y) \\ &\qquad \lesssim \int_{B(x,\delta^i)}\Big(V_{\delta^i}(x)\Big)^{-1}\Big(1+\frac{d(x,y)}{\delta^i}\Big)^{\frac{n}{2}}|f(y)|d\mu(y) \\ &\qquad \lesssim M(f)(x).\end{aligned}\tag{11}$$

For any $j \in \mathbb{N}$, we have

$$\begin{aligned}\int_{U_j(B(x,\delta^i))} &|\eta_{\delta^i,\sigma}(x,y)f(y)|d\mu(y) \\ &= \int_{U_j(B(x,\delta^i))} \Big(V_{\delta^i}(x)V_{\delta^i}(y)\Big)^{-\frac{1}{2}}\Big(1+\frac{d(x,y)}{\delta^i}\Big)^{-\sigma}|f(y)|d\mu(y) \\ &\lesssim \int_{U_j(B(x,\delta^i))} 2^{-j\sigma}\Big(V_{\delta^i}(x)V_{\delta^i}(y)\Big)^{-\frac{1}{2}}|f(y)|d\mu(y).\end{aligned}$$

Note that

$$\begin{aligned}V_{2^j\delta^i}(x) &= (V_{2^j\delta^i}(x))^{\frac{1}{2}}(V_{2^j\delta^i}(x))^{\frac{1}{2}} \\ &\lesssim 2^{\frac{jn}{2}}(V_{\delta^i}(x))^{\frac{1}{2}}\Big(1+\frac{d(x,y)}{2^j\delta^i}\Big)^{\frac{n}{2}}(V_{2^j\delta^i}(y))^{\frac{1}{2}} \\ &\lesssim 2^{jn}(V_{\delta^i}(x))^{\frac{1}{2}}(V_{\delta^i}(y))^{\frac{1}{2}}.\end{aligned}$$

Then,

$$\begin{aligned}\int_{U_j(B(x,\delta^i))} &|\eta_{\delta^i,\sigma}(x,y)f(y)|d\mu(y) \\ &\lesssim \int_{B(x,2^j\delta^i)} 2^{-j\sigma}\Big(V_{\delta^i}(x)V_{\delta^i}(y)\Big)^{-\frac{1}{2}}\frac{V_{2^j\delta^i}(x)}{V_{2^j\delta^i}(x)}|f(y)|d\mu(y) \\ &\lesssim 2^{-j(\sigma-n)}\frac{1}{V_{2^j\delta^i}(x)}\int_{B(x,2^j\delta^i)}|f(y)|d\mu(y),\end{aligned} \tag{12}$$

Then, adding the summation on j of (12) to (11), we obtain

$$|\eta_{\delta^i,\sigma}(f_i)(x)| \lesssim M(f)(x) \tag{13}$$

for any $x \in X$ and $\sigma \in (n,\infty)$. Thus, by Lemma 3 and (13), we have

$$\begin{aligned}\left\|\frac{\eta_{\delta^i,\sigma}(f_i)}{\Lambda^{\frac{1}{q(\cdot)}}}\right\|_{M_{p(\cdot),u(\cdot)}(X)} &\le \left\|\eta_{\delta^i,\sigma-c_{\log}(1/q)}(\Lambda^{-\frac{1}{q(\cdot)}}f_i)\right\|_{M_{p(\cdot),u(\cdot)}(X)} \\ &\le \left\|M(\Lambda^{-\frac{1}{q(\cdot)}}f_i)\right\|_{M_{p(\cdot),u(\cdot)}(X)} \\ &\le C\left\|\Lambda^{-\frac{1}{q(\cdot)}}f_i\right\|_{M_{p(\cdot),u(\cdot)}(X)},\end{aligned}$$

where we used Lemma 3 in the first inequality and Lemma 1 in the third inequality. The proof is complete. □

3. Besov–Morrey Space Associated with Operators

In this section, we give the definition of the Besov–Morrey space with a variable exponent associated with the operator L and prove that this space can be characterized by Peetre maximal functions and admit atomic decomposition. To this end, let us recall some notions and definitions.

If $\mu(X) = \infty$, the test function space denoted by $D(L)$ is defined as the set of all functions $\phi \in \cap_{m\in\mathbb{Z}_+}\mathrm{Dom}(L^m)$, such that for any $m,\gamma \in \mathbb{N}_0$,

$$\mathcal{P}_{m,\gamma}(\phi) := \sup_{x\in X}[1+d(x,x_0)]^{\gamma}|L^m\phi(x)| < \infty.$$

If $\mu(X) < \infty$, the test function space denoted by $D(L)$ is defined as the set of all functions

$$\mathcal{P}_m(\phi) := \|L^m \phi\|_{L^2(X)}.$$

The distribution space $D'(L)$ is the set of all continuous linear functionals on $D(L)$ equipped with the weak$-\star$ topology.

Similarly to [9], let $\varepsilon \in (0,1)$, $\varphi_0, \varphi \in C^\infty(\mathbb{R}_+)$, such that

$$\text{supp}\,\varphi_0 \subset [0, \varepsilon^{-1}], \varphi_0^{(2\nu+1)}(0) = 0,\ \forall \nu \geq 0,\ |\varphi_0(\lambda)| \geq c \text{ for any } \lambda \in [0, \varepsilon^{-\frac{3}{4}}], \tag{14}$$

and

$$\text{supp}\,\varphi \subset [\varepsilon, \varepsilon^{-1}], |\varphi(\lambda)| \geq c \text{ for any } \lambda \in [\varepsilon^{\frac{3}{4}}, \varepsilon^{-\frac{3}{4}}]. \tag{15}$$

Then, for any $\lambda \geq 0$

$$\sum_{i=0}^{\infty} |\varphi(2^{-i}\lambda)| \geq c > 0.$$

In the sequel, we set $\varphi_i(\lambda) = \varphi(\varepsilon^i \lambda)$.

By ([9], Corollary 3.5), for any $i \in \mathbb{N}_0$, $\varphi_i(\sqrt{L})$ is an integral operator with kernel $\varphi_i(\sqrt{L})(x,\cdot) \in D(L)$ for any given $x \in X$; thus, we may consider

$$\varphi_i(\sqrt{L})(f)(x) := \int_X f(y)\varphi_i(\sqrt{L})(x,y)d\mu(y).$$

Definition 2. *Let $p(\cdot), u(\cdot), q(\cdot) \in \mathcal{P}(X)$ such that $p(x) \leq u(x)$, $\alpha(\cdot) \in C^{\log}_{\text{loc}}(X) \cap L^\infty(X)$, and let L be a non-negative self-adjoint operator whose domain is dense in $L^2(X)$, satisfying (4)–(6). The variable Besov–Morrey space $B^{\alpha(\cdot),L}_{p(\cdot),u(\cdot),q(\cdot)}$ consists of the set of all $f \in D'(L)$, such that*

$$\|f\|_{B^{\alpha(\cdot),L}_{p(\cdot),u(\cdot),q(\cdot)}(X)} := \left\| \left\{ \varepsilon^{-i\alpha(\cdot)} \varphi_i(\sqrt{L}) \right\}_{i \in \mathbb{N}_0} \right\|_{l_{q(\cdot)}(M_{p(\cdot),u(\cdot)}(X))} < \infty,$$

where φ_0 and φ satisfy (14) and (15).

Remark 2. *If $p(\cdot) = u(\cdot)$, then the variable Besov–Morrey space $B^{\alpha(\cdot),L}_{p(\cdot),u(\cdot),q(\cdot)}$ dates back to the variable Besov space studied in [11].*

3.1. Peetre Maximal Function Characterizations

We present the Peetre maximal function characterizations of the Besov–Morrey space $B^{\alpha(\cdot),L}_{p(\cdot),u(\cdot),q(\cdot)}(X)$, from which we conclude that the definition of our space is independent of the choice of φ_0 and φ appearing in Definition 2.

Let ϕ_0, ϕ be two functions in $S(\mathbb{R}_+)$ and $a > 0$. For any $f \in D'(L)$, $\alpha \in \mathcal{P}(X)$ and $i \in \mathbb{N}_0$. The Peetre maximal functions $\left(\phi_i(\sqrt{L})\right)^*_a(f)$ and $\left(\phi_i(\sqrt{L})\right)^*_{a,\alpha(\cdot)}(f)$ are defined, respectively, by setting for any $x \in X$,

$$\left(\phi_i(\sqrt{L})\right)^*_a(f)(x) := \sup_{y \in X}[1 + \delta^{-i}d(x,y)]^{-a}|\phi_i(\sqrt{L})f(y)|$$

and

$$\left(\phi_i(\sqrt{L})\right)^*_{a,\alpha(\cdot)}(f)(x) := \sup_{y \in X}[1 + \delta^{-i}d(x,y)]^{-a}\delta^{i\alpha(\cdot)}|\phi_i(\sqrt{L})f(y)|.$$

The first main result for this section is given in the following theorem.

Theorem 2. *Let $p(\cdot), u(\cdot)$ and $q(\cdot)$ be as in Theorem 1, $\alpha(\cdot) \in H^{\log}_{\mathrm{loc}}(X) \cap L^\infty(X)$, and ψ_0, ψ be in $C^\infty(\mathbb{R}_+)$, satisfying (14) and (15). Let*

$$a \in \left(\frac{3n}{\min\{1, p_-, q_-\}} + c_{\log}(\alpha) + c_{\log}(1/q), \infty \right). \tag{16}$$

Then, the following holds:

1. *$f \in B^{\alpha(\cdot),L}_{p(\cdot),u(\cdot),q(\cdot)}(X)$ if and only if $f \in D'(L)$ and $\|f\|^*_{B^{\alpha(\cdot),L}_{p(\cdot),u(\cdot),q(\cdot)}(X)} < \infty$, where*

$$\|f\|^*_{B^{\alpha(\cdot),L}_{p(\cdot),u(\cdot),q(\cdot)}(X)} := \left\| \left\{ \varepsilon^{-i\alpha(\cdot)} \left(\psi_i(\sqrt{L}) \right)^*_a (f) \right\}_{i \in \mathbb{N}_0} \right\|_{l_{q(\cdot)}(M_{p(\cdot),u(\cdot)}(X))}.$$

2. *$f \in B^{\alpha(\cdot),L}_{p(\cdot),u(\cdot),q(\cdot)}(X)$ if and only if $f \in D'(L)$ and $\|f\|^{**}_{B^{\alpha(\cdot),L}_{p(\cdot),u(\cdot),q(\cdot)}(X)} < \infty$, where*

$$\|f\|^{**}_{B^{\alpha(\cdot),L}_{p(\cdot),u(\cdot),q(\cdot)}(X)} := \left\| \left\{ \varepsilon^{-i\alpha(\cdot)} \left(\psi_i(\sqrt{L}) \right)^*_{a,-\alpha(\cdot)} (f) \right\}_{i \in \mathbb{N}_0} \right\|_{l_{q(\cdot)}(M_{p(\cdot),u(\cdot)}(X))}.$$

Before giving the proof of Theorem 2, we present the following lemma, which comes from Corollary 3.5 in [9], and Lemma 2.1 in [15].

Lemma 4.

(1) *Let $f \in C^\infty(\mathbb{R}_+)$, such that for any $\nu, r \geq 0$ and $\lambda \geq 0$, $|f^{(\nu)}(\lambda)| \leq C_{\nu,r}(1+\lambda)^{-r}$ and $f^{(2\nu+1)}(0) = 0$ for $\nu \geq 0$. Then, for any $\delta > 0$, $f(\delta\sqrt{L})$ is an integral operator with kernel $f(\delta\sqrt{L})(x,y)$, such that for any $x, y \in X$,*

$$\left| f(\delta\sqrt{L})(x,y) \right| \leq C_\sigma \eta_{\delta,\sigma}(x,y).$$

(2) *For any even function $\varphi \in S(\mathbb{R}_+)$, the kernel $\varphi(\sqrt{L})(x,y)$ of the operator $\varphi(\sqrt{L})$ belongs to $D(L)$ as a function of $x \in X$ or $y \in X$ for any given $y \in X$ or $x \in X$.*

(3) *Let $\sigma \in (n, \infty)$. Then, for any $s, t \in (0, \infty)$ and $x, y \in X$,*

$$\begin{aligned} &\int_X \eta_{\delta,\sigma}(x,z)\eta_{\beta,\sigma}(z,y)d\mu(z) \\ &\qquad \leq \frac{c_0^2 2^{n+\sigma+1}}{1-2^{n-\sigma}} \max\left\{ (\delta^{-1}\beta)^n, (\beta^{-1}\delta)^n \right\} \eta_{\max\{\delta,\beta\},\sigma}(x,y). \end{aligned}$$

(4) *Let $\sigma \in (n, \infty)$. Then, for any $\delta \in (0, \infty)$ and $x \in X$,*

$$\int_X \eta_{\delta,\sigma}(x,y)d\mu(y) \leq \frac{2^n c_0}{1-2^{n-\sigma}}.$$

The following lemma is just ([11], Lemma 3.8).

Lemma 5. *Let ϕ_0, ϕ be two functions in $S(\mathbb{R}_+)$, satisfying (14) and (15) with $\varepsilon \in (0,1)$ and L as in Definition 2. Then, for any given $r \in (0,1]$, $a \in (n/2r, \infty)$ and $N \in (n, \infty)$, there exists a positive constant C depending on a, r, and N, such that, for any $i \in \mathbb{N}_0$, $f \in D'(L)$ and $x \in X$,*

$$\left(\phi_i(\sqrt{L}) \right)^*_a (f)(x) \leq C \left[\sum_{k=0}^{\infty} \delta^{k(Nr-n)} \eta_{\delta^i, ar-n/2} \left(\left| \phi_{i+k}(\sqrt{L})(f) \right|^r \right)(x) \right]^{\frac{1}{r}}.$$

Proof of Theorem 2. We start by (1). We divide the proof into two steps.

First step: We need to prove that

$$\left\|\left\{\varepsilon^{-j\alpha(\cdot)}\varphi_j(\sqrt{L})(f)\right\}_{j\in\mathbb{N}_0}\right\|_{l_{q(\cdot)}(M_{p(\cdot),u(\cdot)}(X))} \lesssim \left\|\left\{\varepsilon^{-i\alpha(\cdot)}\left(\psi_i(\sqrt{L})\right)^*_a(f)\right\}_{i\in\mathbb{N}_0}\right\|_{l_{q(\cdot)}(M_{p(\cdot),u(\cdot)}(X))}. \tag{17}$$

By ([16], Lemma 6.10), there exist functions $\{\widetilde{\psi}_0,\widetilde{\psi}\}\in C^\infty(\mathbb{R}_+)$, satisfying (14) and (15) such that for any $\lambda\in\mathbb{R}_+$,

$$\sum_{i=0}^{\infty}\widetilde{\psi}_i(\lambda)\psi_i(\lambda)=1.$$

Then,

$$\varphi_j(\lambda)=\sum_{i=0}^{\infty}\varphi_j(\lambda)\widetilde{\psi}_i(\lambda)\psi_i(\lambda)=\sum_{i=j-2}^{i=j+2}\varphi_j(\lambda)\widetilde{\psi}_i(\lambda)\psi_i(\lambda),$$

where $\widetilde{\psi}_{-1}=\psi_{-1}(\lambda)=0$. By ([9], Proposition 5.5(b)), for any $f\in D'(L)$ and $x\in X$, we have

$$\begin{aligned}\varphi_j(\sqrt{L})(f)(x)&=\sum_{i=j-2}^{i=j+2}\varphi_j(\sqrt{L})\widetilde{\psi}_i(\sqrt{L}))\psi_i(\sqrt{L})(f)(x)\\&=\sum_{i=j-2}^{i=j+2}\int_X\theta_{i,j}(x,y)\psi_i(\sqrt{L})(f)(y)d\mu(y),\end{aligned}$$

where $\theta_{i,j}(\cdot,\cdot)$ is the kernel of the operator $\varphi_j(\sqrt{L}))\widetilde{\psi}_i(\sqrt{L})$. From Lemma 4, we have

$$|\varphi_j(\sqrt{L})(x,y)|\lesssim\eta_{\varepsilon^i,\sigma}(x,y)\text{ for any }x,y\in X$$

and

$$|\widetilde{\psi}_i(\sqrt{L})(y,u)|\lesssim\eta_{\varepsilon^i,\sigma}(y,u)\text{ for any }y,u\in X,$$

where $\sigma>a+n$. Then, by Lemma 4 (3), we obtain

$$\begin{aligned}|\theta_{i,j}(x,y)|&=\left|\int_X\varphi_j(\sqrt{L})(x,y)\widetilde{\psi}_i(\sqrt{L})(y,u)d\mu(y)\right|\\&\lesssim\int_X\eta_{\varepsilon^j,\sigma}(x,y)\eta_{\varepsilon^i,\sigma}(y,u)d\mu(y)\lesssim\eta_{\varepsilon^i,\sigma}(x,u).\end{aligned}$$

Therefore,

$$\begin{aligned}|\varphi_j(\sqrt{L})(f)(x)|&\lesssim\sum_{i=j-2}^{i=j+2}\int_X\eta_{\varepsilon^i,\sigma}(x,y)|\psi_i(\sqrt{L})(f)(y)|d\mu(y)\\&=\sum_{i=j-2}^{i=j+2}\int_X\eta_{\varepsilon^i,\sigma-a}(x,y)\left[1+\varepsilon^i d(x,y)\right]^{-a}|\psi_i(\sqrt{L})(f)(y)|d\mu(y).\end{aligned}$$

Thus, by the definition of $\left(\psi_i(\sqrt{L})\right)_a^*(f)(x)$ and Lemma 4 (4), we obtain

$$
\begin{aligned}
|\varphi_j(\sqrt{L})(f)(x)| &\lesssim \sum_{i=j-2}^{i=j+2} \left(\psi_i(\sqrt{L})\right)_a^*(f)(x) \int_X \eta_{\varepsilon^i,\sigma-a}(x,y)d\mu(y) \\
&\lesssim \sum_{i=j-2}^{i=j+2} \left(\psi_i(\sqrt{L})\right)_a^*(f)(x) \\
&\asymp \sum_{k=-2}^{k=2} \left(\psi_{j+k}(\sqrt{L})\right)_a^*(f)(x).
\end{aligned} \tag{18}
$$

Then, by Remark 1 (2), we have

$$
\begin{aligned}
&\left\| \left\{ \varepsilon^{-j\alpha(\cdot)} \varphi_j(\sqrt{L})(f) \right\}_{j\in\mathbb{N}_0} \right\|_{l_{q(\cdot)}(M_{p(\cdot),u(\cdot)}(X))} \\
&\quad \lesssim \left\| \left\{ \sum_{k=-2}^{k=2} \varepsilon^{-j\alpha(\cdot)} \left(\psi_{j+k}(\sqrt{L})\right)_a^*(f) \right\}_{j\in\mathbb{N}_0} \right\|_{l_{q(\cdot)}(M_{p(\cdot),u(\cdot)}(X))} \\
&\quad \lesssim \sum_{k=-2}^{k=2} \left\| \left\{ \varepsilon^{-j\alpha(\cdot)} \left(\psi_{j+k}(\sqrt{L})\right)_a^*(f) \right\}_{j\in\mathbb{N}_0} \right\|_{l_{q(\cdot)}(M_{p(\cdot),u(\cdot)}(X))} \\
&\quad \lesssim \left\| \left\{ \varepsilon^{-i\alpha(\cdot)} \left(\psi_i(\sqrt{L})\right)_a^*(f) \right\}_{j\in\mathbb{N}_0} \right\|_{l_{q(\cdot)}(M_{p(\cdot),u(\cdot)}(X))}.
\end{aligned} \tag{19}
$$

Second step: we prove the converse part, i.e., we show that for any $f \in D'(L)$

$$
\begin{aligned}
&\left\| \left\{ \varepsilon^{-i\alpha(\cdot)} \left(\psi_i(\sqrt{L})\right)_a^*(f) \right\}_{i\in\mathbb{N}_0} \right\|_{l_{q_{(\cdot)}}(M_{p(\cdot),u(\cdot)}(X))} \\
&\quad \lesssim \left\| \left\{ \varepsilon^{-j\alpha(\cdot)} \varphi_j(\sqrt{L})(f) \right\}_{j\in\mathbb{N}_0} \right\|_{l_{q_{(\cdot)}}(M_{p(\cdot),u(\cdot)}(X))}.
\end{aligned}
$$

It suffices to show that for any $f \in D'(L)$,

$$
\begin{aligned}
&\left\| \left\{ \varepsilon^{-i\alpha(\cdot)} \left(\psi_i(\sqrt{L})\right)_a^*(f) \right\}_{i\in\mathbb{N}_0} \right\|_{l_{q_{(\cdot)}}(M_{p(\cdot),u(\cdot)}(X))} \\
&\quad \lesssim \left\| \left\{ \varepsilon^{-j\alpha(\cdot)} \psi_j(\sqrt{L})(f) \right\}_{j\in\mathbb{N}_0} \right\|_{l_{q_{(\cdot)}}(M_{p(\cdot),u(\cdot)}(X))}
\end{aligned} \tag{20}
$$

Then we use (17) to obtain

$$
\begin{aligned}
&\left\| \left\{ \varepsilon^{-i\alpha(\cdot)} \left(\psi_i(\sqrt{L})\right)_a^*(f) \right\}_{i\in\mathbb{N}_0} \right\|_{l_{q_{(\cdot)}}(M_{p(\cdot),u(\cdot)}(X))} \\
&\quad \lesssim \left\| \left\{ \varepsilon^{-j\alpha(\cdot)} \psi_j(\sqrt{L})(f) \right\}_{j\in\mathbb{N}_0} \right\|_{l_{q_{(\cdot)}}(M_{p(\cdot),u(\cdot)}(X))} \\
&\quad \lesssim \left\| \left\{ \varepsilon^{-i\alpha(\cdot)} \left(\varphi_i(\sqrt{L})\right)_a^*(f) \right\}_{i\in\mathbb{N}_0} \right\|_{l_{q_{(\cdot)}}(M_{p(\cdot),u(\cdot)}(X))} \\
&\quad \lesssim \left\| \left\{ \varepsilon^{-j\alpha(\cdot)} \varphi_j(\sqrt{L})(f) \right\}_{j\in\mathbb{N}_0} \right\|_{l_{q_{(\cdot)}}(M_{p(\cdot),u(\cdot)}(X))},
\end{aligned} \tag{21}
$$

where in the last inequality we used (20) again.

We return to prove (20). By Lemma 5, for any $r \in (0,1]$ and $N \in (\frac{n}{r} + |\alpha_+|, \infty)$, we have

$$\left(\psi_i(\sqrt{L})\right)_a^*(f)(x) \lesssim \left[\sum_{k=0}^{\infty} \varepsilon^{k(Nr-n)} \eta_{\varepsilon^i, ar-n/2}\left(|\psi_{k+i}(\sqrt{L})(f)|^r\right)(x)\right]^{\frac{1}{r}}.$$

Since a is as in (16), there exists $r \in (0, \min\{1, \frac{p_-}{2}, \frac{q_-}{2}\})$ such that

$$a > \frac{3n}{2r} + c_{\log}(\alpha) + c_{\log}(1/q).$$

Then,

$$\begin{aligned}
&\left\|\left\{\varepsilon^{-i\alpha(\cdot)}\left(\psi_i(\sqrt{L})\right)_a^*(f)\right\}_{i\in\mathbb{N}_0}\right\|_{l_{q(\cdot)}(M_{p(\cdot),u(\cdot)}(X))} \\
&\quad\lesssim \left\|\left\{\varepsilon^{-i\alpha(\cdot)}\left[\sum_{k=0}^{\infty}\varepsilon^{k(Nr-n)}\right.\right.\right. \\
&\qquad\left.\left.\left.\times\, \eta_{\varepsilon^i, ar-n/2}\left(|\psi_{k+i}(\sqrt{L})(f)|^r\right)\right]^{\frac{1}{r}}\right\}_{i\in\mathbb{N}_0}\right\|_{l_{q(\cdot)}(M_{p(\cdot),u(\cdot)}(X))},
\end{aligned}$$

by Remark 1 (3), we have

$$\begin{aligned}
&\left\|\left\{\varepsilon^{-i\alpha(\cdot)}\left(\psi_i(\sqrt{L})\right)_a^*(f)\right\}_{i\in\mathbb{N}_0}\right\|_{l_{q(\cdot)}(M_{p(\cdot),u(\cdot)}(X))} \\
&\quad\lesssim \left\|\left\{\varepsilon^{-ir\alpha(\cdot)}\sum_{k=0}^{\infty}\varepsilon^{k(Nr-n)}\right.\right. \\
&\qquad\left.\left.\times\, \eta_{\varepsilon^i, ar-n/2}\left(|\psi_{k+i}(\sqrt{L})(f)|^r\right)\right\}_{i\in\mathbb{N}_0}\right\|^{\frac{1}{r}}_{l_{q(\cdot)/r}(M_{p(\cdot)/r,u(\cdot)/r}(X))}
\end{aligned}$$

Thus, by Remark 1 (2) and Lemma 3, we obtain

$$\begin{aligned}
&\left\|\left\{\varepsilon^{-i\alpha(\cdot)}\left(\psi_i(\sqrt{L})\right)_a^*(f)\right\}_{i\in\mathbb{N}_0}\right\|_{l_{q(\cdot)}(M_{p(\cdot),u(\cdot)}(X))} \\
&\quad\lesssim \left\{\sum_{k=0}^{\infty}\varepsilon^{k(Nr-n+r\alpha_-)}\left\|\left\{\eta_{\varepsilon^i, ar-n/2-rc_{\log}(\alpha)}\right.\right.\right. \\
&\qquad\left.\left.\left.\times\left(\varepsilon^{-(k+i)r\alpha(\cdot)}|\psi_{k+i}(\sqrt{L})(f)|^r\right)\right\}_{i\in\mathbb{N}_0}\right\|_{l_{q(\cdot)/r}(M_{p(\cdot)/r,u(\cdot)/r}(X))}\right\}^{\frac{1}{r}} \\
&\quad\lesssim \left\{\sum_{k=0}^{\infty}\varepsilon^{k(Nr-n+r\alpha_-)}\right.
\end{aligned}$$

We apply Theorem 1, and we obtain

$$\begin{aligned}
&\left\|\left\{\varepsilon^{-i\alpha(\cdot)}\left(\psi_i(\sqrt{L})\right)_a^*(f)\right\}_{i\in\mathbb{N}_0}\right\|_{l_{q(\cdot)}(M_{p(\cdot),u(\cdot)}(X))} \\
&\qquad\left.\times\left\|\left\{\varepsilon^{-(k+i)r\alpha(\cdot)}|\psi_{k+i}(\sqrt{L})(f)|^r\right\}_{i\in\mathbb{N}_0}\right\|_{l_{q(\cdot)/r}(M_{p(\cdot)/r,u(\cdot)/r}(X))}\right\}^{\frac{1}{r}}.
\end{aligned}$$

Since $N \in (n/r + |\alpha_-|, \infty)$, we have

$$\left\| \left\{ \varepsilon^{-i\alpha(\cdot)} \left(\psi_i(\sqrt{L}) \right)_a^* (f) \right\}_{i \in \mathbb{N}_0} \right\|_{l_{q(\cdot)}(M_{p(\cdot),u(\cdot)}(X))} \lesssim \left\| \left\{ \varepsilon^{-i\alpha(\cdot)} |\psi_i(\sqrt{L})(f)| \right\}_{i \in \mathbb{N}_0} \right\|_{l_{q(\cdot)}(M_{p(\cdot),u(\cdot)}(X))}.$$

Thus, by (19) and (21), the proof of (1) is complete.

Next, we prove (2).

First Step: We firstly show that for any $f \in B^{\alpha(\cdot),L}_{p(\cdot),u(\cdot),q(\cdot)}(X)$,

$$\left\| \left\{ \varepsilon^{-i\alpha(\cdot)} \left(\psi_i(\sqrt{L}) \right)_{a,-\alpha(\cdot)}^* (f) \right\}_{i \in \mathbb{N}_0} \right\|_{l_{q(\cdot)}(M_{p(\cdot),u(\cdot)}(X))} \lesssim \left\| \left\{ \varepsilon^{-j\alpha(\cdot)} \varphi_j(\sqrt{L})(f) \right\}_{j \in \mathbb{N}_0} \right\|_{l_{q(\cdot)}(M_{p(\cdot),u(\cdot)}(X))}.$$

To show this, let $r \in (0, \min\{1, \frac{p_-}{2}, \frac{q_-}{2}\})$, $a \in (3n/2r + c_{\log}(\alpha) + c_{\log}(1/q), \infty)$. Then, from Lemmas 3 and 5, we know that for any $i \in \mathbb{N}_0$ and $x \in X$,

$$\left([1 + \varepsilon^{-i} d(x,y)]^{-a} \varepsilon^{-i\alpha(\cdot)} |\varphi_i(\sqrt{L}) f(y)| \right)^r \lesssim \sum_{k=0}^{\infty} \varepsilon^{k(Nr-n)} \eta_{\varepsilon^i, ar-n/2-rc_{\log}(\alpha)} \left(\varepsilon^{-ir\alpha(\cdot)} |\psi_{i+k}(\sqrt{L}(f)|^r \right)(x),$$

where $N \in (n/r, \infty)$. By Remark 1 (2), we have

$$\begin{aligned} &\left\| \left\{ \varepsilon^{-i\alpha(\cdot)} \left(\psi_i(\sqrt{L}) \right)_{a,-\alpha(\cdot)}^* (f) \right\}_{i \in \mathbb{N}_0} \right\|_{l_{q(\cdot)}(M_{p(\cdot),u(\cdot)}(X))} \\ &\lesssim \left\| \left\{ \sum_{k=0}^{\infty} \varepsilon^{k(Nr-n)} \times \eta_{\varepsilon^i, ar-n/2-rc_{\log}(\alpha)} \left(\varepsilon^{-ir\alpha(\cdot)} |\psi_{i+k}(\sqrt{L})(f)|^r \right) \right\}_{i \in \mathbb{N}_0} \right\|^{\frac{1}{r}}_{l_{q(\cdot)/r}(M_{p(\cdot)/r,u(\cdot)/r}(X))} \end{aligned}$$

Then, applying Remark 1 (2) and Theorem 1, we have

$$\begin{aligned} &\left\| \left\{ \varepsilon^{-i\alpha(\cdot)} \left(\psi_i(\sqrt{L}) \right)_{a,-\alpha(\cdot)}^* (f) \right\}_{i \in \mathbb{N}_0} \right\|_{l_{q(\cdot)}(M_{p(\cdot),u(\cdot)}(X))} \\ &\lesssim \left[\sum_{k=0}^{\infty} \varepsilon^{k(Nr-n)} \left\| \left\{ \varepsilon^{-ir\alpha(\cdot)} |\psi_{i+k}(\sqrt{L})(f)|^r \right\}_{i \in \mathbb{N}_0} \right\|_{l_{q(\cdot)/r}(M_{p(\cdot)/r,u(\cdot)/r}(X))} \right]^{\frac{1}{r}} \\ &\lesssim \left\| \left\{ \varepsilon^{-j\alpha(\cdot)} \psi_j(\sqrt{L})(f) \right\}_{j \in \mathbb{N}_0} \right\|_{l_{q(\cdot)}(M_{p(\cdot),u(\cdot)}(X))}. \end{aligned}$$

By this and a similar argument to the one used in (21), we have

$$\|f\|^{**}_{B^{\alpha(\cdot),L}_{p(\cdot),u(\cdot),q(\cdot)}(X)} \lesssim \|f\|_{B^{\alpha(\cdot),L}_{p(\cdot),u(\cdot),q(\cdot)}(X)}. \tag{22}$$

Second Step: We prove that for any $f \in D'(L)$ such that $\|f\|^{**}_{B^{\alpha(\cdot),L}_{p(\cdot),u(\cdot),q(\cdot)}(X)} < \infty$, we have

$$\|f\|_{B^{\alpha(\cdot),L}_{p(\cdot),u(\cdot),q(\cdot)}(X)} \lesssim \|f\|^{**}_{B^{\alpha(\cdot),L}_{p(\cdot),u(\cdot),q(\cdot)}(X)}.$$

By the first inequality in (18) and Lemma 3, we know that for any $j \in \mathbb{Z}_+$, $\sigma \in (n + a + c_{\log}(\alpha), \infty)$ and for any $x \in X$,

$$\begin{aligned}\varepsilon^{-j\alpha(\cdot)}|\varphi_j(\sqrt{L})(f)(x)| &\lesssim \varepsilon^{-j\alpha(\cdot)} \sum_{i=j-2}^{i=j+2} \int_X \eta_{\varepsilon^i,\sigma}(x,y)|\psi_i(\sqrt{L})(f)(y)|d\mu(y) \\ &\lesssim \sum_{i=j-2}^{i=j+2} \int_X \eta_{\varepsilon^i,\sigma-c_{\log}(\alpha)}(x,y)\varepsilon^{-j\alpha(y)}|\psi_i(\sqrt{L})(f)(y)|d\mu(y),\end{aligned}$$

then, by the definition of $\left(\psi_i(\sqrt{L})\right)^*_{a,-\alpha(\cdot)}(f)(x)$ and Lemma 4 (4), we obtain

$$\begin{aligned}\varepsilon^{-j\alpha(\cdot)}|\varphi_j(\sqrt{L})(f)(x)| &\lesssim \sum_{i=j-2}^{i=j+2} \left(\psi_i(\sqrt{L})\right)^*_{a,-\alpha(\cdot)}(f)(x) \\ &\quad\times \int_X \eta_{\varepsilon^i,\sigma-a-c_{\log}(\alpha)}(x,y)d\mu(y) \\ &\lesssim \sum_{i=j-2}^{i=j+2} \left(\psi_i(\sqrt{L})\right)^*_{a,-\alpha(\cdot)}(f)(x).\end{aligned}$$

Then, by a similar argument used for (19), we can obtain

$$\|f\|^{**}_{B^{\alpha(\cdot),L}_{p(\cdot),u(\cdot),q(\cdot)}(X)} \lesssim \|f\|_{B^{\alpha(\cdot),L}_{p(\cdot),u(\cdot),q(\cdot)}(X)}. \tag{23}$$

Thus, by (22) and (23), the proof of (2) is complete, which ends the proof of the theorem. □

3.2. Atomic Characterization

In this subsection, we assume that the measure μ satisfies the uniformly bounded condition, that is,

$$\sup_{x\in X} \mu(B(x,1)) < \infty.$$

We establish the atomic decomposition of $B^{\alpha(\cdot),L}_{p(\cdot),u(\cdot),q(\cdot)}(X)$. Let us begin by recalling the following lemma concerning the properties of the Christ's dyadic cubes [17] on the space of homogeneous type.

Lemma 6. *There exists a collection $\{Q^i_\alpha : i \in \mathbb{Z}, \alpha \in I_i\}$ of open subsets of X, where I_i is some index set (possibly finite) and a constant $\delta \in (0,1)$ and $A_1, A_2 > 0$, such that the following holds:*

1. *$\mu(X \setminus \cup_{\alpha \in I_i} Q^i_\alpha) = 0$ for each fixed i and $Q^i_\alpha \cap Q^i_\beta = \emptyset$ if $\alpha \neq \beta$;*
2. *For α, β, i, j with $j \geq i$, either $Q^j_\alpha \subset Q^i_\beta$ or $Q^j_\alpha \cap Q^i_\beta = \emptyset$;*
3. *For each (α, i) and $j < i$, there exists a unique β such that $Q^i_\alpha \subset Q^j_\beta$;*
4. *$\mathrm{diam}(Q^i_\alpha) \leq A_1\delta^i$, where $\mathrm{diam}(Q^i_\alpha) := \sup\{d(x,y) : x, y \in Q^i_\alpha\}$;*
5. *Each Q^i_α contains some ball $B(z^i_\alpha, A_2\delta^i)$, where $z^i_\alpha \in X$.*

Without loss of generality, we assume that $\delta = 1/2$. We denote by $\mathcal{D}$ the family of all dyadic cubes on X. For $i \in \mathbb{Z}$, we set $\mathcal{D}_i = \{Q^i_\alpha : \alpha \in I_i\}$, so that $\mathcal{D} = \cup_{i\in\mathbb{Z}}\mathcal{D}_i$.

Definition 3. *Let $p(\cdot), u(\cdot), q(\cdot), \alpha(\cdot)$ be positive functions on X such that $p(\cdot) \le u(\cdot)$ for almost every $x \in X$. The sequence space $b^{\alpha(\cdot),q(\cdot)}_{p(\cdot),u(\cdot)}$ is defined to be the set of all sequences $\lambda = \{\lambda_Q\}_{Q \in \cup_{j \ge 0} \mathcal{D}_j}$, such that $\|\lambda\|_{b^{\alpha(\cdot),q(\cdot)}_{p(\cdot),u(\cdot)}} < \infty$, where*

$$\|\lambda\|_{b^{\alpha(\cdot),q(\cdot)}_{p(\cdot),u(\cdot)}} := \left\| \left\{ \delta^{-j\alpha(\cdot)} \sum_{Q \in \mathcal{D}_j} |\lambda_Q| [\mu(Q)]^{-1/2} \chi_Q \right\}_{j \in \mathbb{N}_0} \right\|_{l_{q(\cdot)}(M_{p(\cdot),u(\cdot)}(X))}.$$

Definition 4. *Let $K, S \in \mathbb{N}_0$, and let Q be a dyadic cube in $\mathcal{D}_j$ with $j \in \mathbb{N}_0$. If $j \ge 1$, we say that a function $a_Q \in L^2(X)$ is a $(K,S)-$atom for Q if a_Q satisfies the following conditions for $m = K$ and $m = -S$:*

1. $a_Q \in \mathrm{Dom}\,(L^m)$;
2. $\mathrm{supp}\,(L^m a_Q) \subset B(z_Q, (A_1+1)2^{-i})$;
3. $\sup_{x \in X} |L^m a_Q(x)| \le \delta^{2jm} [\mu(Q)]^{-1/2}$.

If $j = 0$, a function $a_Q \in L^2(X)$ is a $(K,S)-$atom for Q if a_Q satisfies the above conditions only for $m = K$.

Lemma 7. *Let $K, S \in \mathbb{N}_0$, Q be a dyadic cube in D_j, $j \in \mathbb{N}_0$, a_Q be a $(K,S)-$atom for Q, and $N > 0$ be arbitrarily large. Suppose that $\phi_0, \phi \in S(\mathbb{R}_+)$ such that $(\cdot)^{-\max\{S,K\}}\phi(\cdot) \in S(\mathbb{R}_+)$. Then,*

$$|\varphi(\delta^i \sqrt{L}) a_Q(x)| \le \begin{cases} c' \delta^{2(i-j)K} [\mu(Q)]^{1/2} \eta_{\delta^i,N}(x, z_Q), & \text{if } i > j, \\ c' \delta^{2(j-i)S} [\mu(Q)]^{1/2} \eta_{\delta^i,N}(x, z_Q), & \text{otherwise.} \end{cases}$$

Theorem 3. *Let $p(\cdot), u(\cdot), q(\cdot) \in \mathcal{P}(X)$ such that $p(x) \le u(x)$, $\alpha(\cdot) \in C^{\log}_{\mathrm{loc}}(X) \cap L^\infty(X)$, and let L be a non-negative self-adjoint operator whose domain is dense in $L^2(X)$, satisfying (4)–(6). Let $K, S \in \mathbb{N}_0$ such that*

$$2K > \alpha_+ \quad \text{and} \quad 2S > \frac{4n}{\min\{1, p_-, q_-\}} + c_{\log}(\alpha) + c_{\log}(1/q) + \alpha_-. \tag{24}$$

Then, there exists a constant $C > 0$ such that for any sequence of $(K,S)-$atoms $\{a_Q\}_{Q \in \cup_{j \in \mathbb{N}_0} \mathcal{D}_j}$,

$$\left\| \sum_{j \in \mathbb{N}_0} \sum_{Q \in \mathcal{D}_j} \lambda_Q a_Q \right\|_{B^{\alpha(\cdot),L}_{p(\cdot),u(\cdot),q(\cdot)}(X)} \le C \|\lambda\|_{b^{\alpha(\cdot),q(\cdot)}_{p(\cdot),u(\cdot)}}.$$

Conversely, there exists a positive constant C, such that for any function $f \in B^{\alpha(\cdot),L}_{p(\cdot),u(\cdot),q(\cdot)}(X)$, there exist a sequence of $(K,S)-$atoms $\{a_Q\}_{Q \in \cup_{i \ge 0} \mathcal{D}_i}$ and a sequence of complex scalars $\lambda = \{\lambda_Q\}_{Q \in \cup_{i \in \mathbb{N}_0} \mathcal{D}_i}$ such that

$$f = \sum_{j=0}^{\infty} \sum_{Q \in \mathcal{D}_j} \lambda_Q a_Q,$$

where the sum converges in $D'(L)$. Moreover,

$$\|\lambda\|_{b^{\alpha(\cdot),q(\cdot)}_{p(\cdot),u(\cdot)}} \le C \|f\|_{B^{\alpha(\cdot),L}_{p(\cdot),u(\cdot),q(\cdot)}(X)}. \tag{25}$$

Before giving the proof of the above theorem, we give the following lemma, which plays a crucial role in the proof of the second part. For the proof, we refer to [10], Lemma 4.7.

Lemma 8. *Let $M \in \mathbb{N}$ (resp. $M = 0$). There exists a function $\psi \in S(\mathbb{R}_+)$, such that the following holds:*

1. *$\lambda^{-M}\psi(\lambda) \in S(\mathbb{R}_+)$.*
2. *There exists $\varepsilon \in (0,1)$ such that $|\psi(\lambda)| > 0$ on $(\varepsilon, \varepsilon^{-1})$ (resp.$|\psi(\lambda)| > 0$ on $(0, \varepsilon^{-1})$).*
3. *For all integers $\tau \geq -M$ and for all $j \in \mathbb{N}_0$,*

$$\operatorname{supp} K_{(\delta^{2j}L)^\tau \psi(\delta^j\sqrt{L})} \subset \{(x,y) \in X \times X : d(x,y) < \delta^j\},$$

where $K_{(\delta^{2j}L)^\tau \psi(\delta^j\sqrt{L})}$ is the kernel of the operator $(\delta^{2j}L)^\tau \psi(\delta^j\sqrt{L})$.

4. *For every integer $\tau \geq -M$, there exists a constant c depending on τ, such that for all $j \in \mathbb{N}_0$,*

$$\left|K_{(\delta^{2j}L)^\tau \psi(\delta^j\sqrt{L})}(x,y)\right| \leq c\left[V_{\delta^j}(x)\right]^{-1}.$$

We also present the following technical lemma, whose proof is similar to the one of Lemma 6.1 in [7], by establishing the appropriate changes. Indeed, it is a generalization to the homogeneous spaces case.

Lemma 9. *Let $p(\cdot), p(\cdot), p(\cdot) \in \mathcal{P}(X)$ such that $p(\cdot) \leq u(\cdot)$. Let $\varepsilon > 0$ and $\delta \in (0,1)$. For any sequence $\{f_i\}_{i\in\mathbb{N}_0}$ of non-negative measurable functions on X, we denote*

$$F_j(x) := \sum_{i=0}^{\infty} \delta^{|j-i|\varepsilon} f_i(x),\ x \in X,\ j \in \mathbb{N}_0.$$

Then,

$$\left\|\{F_j\}_{j\in\mathbb{N}_0}\right\|_{l_{q(\cdot)}(M_{p(\cdot),u(\cdot)}(X))} \lesssim \left\|\{f_i\}_{i\in\mathbb{N}_0}\right\|_{l_{q(\cdot)}(M_{p(\cdot),u(\cdot)}(X))}.$$

Proof of Theorem 3. Let $K, S \in \mathbb{N}_0$, satisfying (24) and $\varphi_0, \varphi \in S(\mathbb{R}_+)$, satisfying (14), (15), and $(\cdot)^{-M}\phi(\cdot) \in S(\mathbb{R}_+)$ with $M \geq \max\{K, S\}$. We decompose the summation as follows:

$$\begin{aligned}
&\delta^{-i\alpha(x)}\left|\varphi_i(\sqrt{L})\left(\sum_{j=0}^{\infty}\sum_{Q\in\mathcal{D}_j}\lambda_Q a_Q\right)(x)\right| \\
&\quad \leq \delta^{-i\alpha(x)}\sum_{j=0}^{i}\sum_{Q\in\mathcal{D}_j}|\lambda_Q||\varphi_i(\sqrt{L})a_Q| + \sum_{j=i}^{\infty}\sum_{Q\in\mathcal{D}_j}|\lambda_Q||\varphi_i(\sqrt{L})a_Q|,
\end{aligned}$$

We apply Lemma 7 to obtain

$$\begin{aligned}
&\delta^{-i\alpha(x)}\left|\varphi_i(\sqrt{L})\left(\sum_{j=0}^{\infty}\sum_{Q\in\mathcal{D}_j}\lambda_Q a_Q\right)(x)\right| \\
&\quad \lesssim \delta^{-i\alpha(x)}\sum_{j=0}^{i}\sum_{Q\in\mathcal{D}_j}|\lambda_Q|\delta^{2(i-j)K}\eta_{\delta^j,N}(x,z_Q) \\
&\qquad + \delta^{-i\alpha(x)}\sum_{j=i}^{\infty}\sum_{Q\in\mathcal{D}_j}|\lambda_Q|\delta^{2(j-i)S}\eta_{\delta^i,N}(x,z_Q).
\end{aligned}$$

Then

$$\begin{aligned}
&\delta^{-i\alpha(x)}\left|\varphi_i(\sqrt{L})\left(\sum_{j=0}^{\infty}\sum_{Q\in\mathcal{D}_j}\lambda_Q a_Q\right)(x)\right| \\
&\quad \lesssim \sum_{j=0}^{i}\delta^{(j-i)\alpha(x)+2(i-j)K}\sum_{Q\in\mathcal{D}_j}\delta^{-j\alpha(x)}|\lambda_Q|\eta_{\delta^j,N}(x,z_Q) \\
&\qquad + \sum_{j=i}^{\infty}\delta^{(j-i)\alpha(x)+2(j-i)S}\sum_{Q\in\mathcal{D}_j}\delta^{-j\alpha(x)}|\lambda_Q|\eta_{\delta^i,N}(x,z_Q),
\end{aligned}$$

where $N \in \left(\frac{3n}{r} + c_{\log}(1/q) + c_{\log}(\alpha), 2S - \frac{n}{r} + \alpha_- + rc_{\log}(\alpha)\right)$ such that $2S > \frac{4n}{r} + c_{\log}(1/q)$ $+c_{\log}(\alpha) + \alpha_+$, where $r \in (0, \min\{1, p_-, q_-\})$ Now, we set

$$S_0 = \{Q \in \mathcal{D}_j : d(z_Q, x) < A_1\delta^{i\wedge j}\},$$

$$S_m = \{Q \in \mathcal{D}_j : A_1\delta^{1-m}\delta^{i\wedge j} \leq d(z_Q, x) < A_1\delta^{-m}\delta^{i\wedge j}\},\ m \in \mathbb{N},$$

$$B_m = \{z \in X : d(x, z) < A_1\delta^{-(1+m)}\delta^{i\wedge j}\},\ m \in \mathbb{N},$$

where A_1 is as in Lemma 6. We have

$$\sum_{Q\in\mathcal{D}_j} \delta^{-j\alpha(x)}|\lambda_Q|[\mu(Q)]^{\frac{-1}{2}}(1 + \delta^{-(i\wedge j)}d(z_Q, x))^{-N}$$
$$\lesssim \left(\sum_{m=0}^{\infty}\sum_{Q\in S_m} \delta^{-jr\alpha(x)}|\lambda_Q|^r[\mu(Q)]^{\frac{-r}{2}}(1 + A_1\delta^{1-m})^{-Nr}\right)^{\frac{1}{r}}.$$

We multiply and divide by $\mu(Q)$ to obtain

$$\sum_{Q\in\mathcal{D}_j} \delta^{-j\alpha(x)}|\lambda_Q|[\mu(Q)]^{\frac{-1}{2}}(1 + \delta^{-(i\wedge j)}d(z_Q, x))^{-N}$$
$$\lesssim \left(\sum_{m=0}^{\infty} \delta^{mrN}\int_X \sum_{Q\in S_m} \delta^{-jr\alpha(x)}|\lambda_Q|^r[\mu(Q)]^{\frac{-r}{2}}[\mu(Q)]^{-1}\chi_Q(z)d\mu(z)\right)^{\frac{1}{r}}.$$

Now, we multiply and divide by $\eta_{\delta^{i\wedge j}, \frac{Nr}{2}}(x, z)$ to

$$\sum_{Q\in\mathcal{D}_j} \delta^{-j\alpha(x)}|\lambda_Q|[\mu(Q)]^{\frac{-1}{2}}(1 + \delta^{-(i\wedge j)}d(z_Q, x))^{-N}$$
$$\lesssim \left(\sum_{m=0}^{\infty} \delta^{mrN}\int_{B_m} \sum_{Q\in S_m} \delta^{-jr\alpha(x)}|\lambda_Q|^r[\mu(Q)]^{\frac{-r}{2}}\chi_Q(z)\eta_{\delta^{i\wedge j}, \frac{Nr}{2}}(x, z)\right.$$
$$\left.\times (1 + \delta^{-(i\wedge j)}d(z, x))^{\frac{Nr}{2}}\frac{[\mu(B(x, \delta^{i\wedge j}))\mu(B(z, \delta^{i\wedge j}))]^{1/2}}{\mu(Q)}d\mu(z)\right)^{\frac{1}{r}}.$$

Note that

$$\frac{[\mu(B(x, \delta^{i\wedge j}))\mu(B(z, \delta^{-i\wedge j}))]^{1/2}}{\mu(Q)} \lesssim \frac{\delta^{-mn}\mu(B(z_Q, A_1\delta^{-m-2}\delta^{i\wedge j}))}{\mu(B(z_Q, A_2\delta^j))}$$
$$\lesssim \delta^{-(3m+j-i\wedge j)n}.$$

Then,

$$\sum_{Q\in\mathcal{D}_j} \delta^{-j\alpha(x)}|\lambda_Q|[\mu(Q)]^{\frac{-1}{2}}(1 + \delta^{-(i\wedge j)}d(z_Q, x))^{-N}$$
$$\lesssim \left(\sum_{m=0}^{\infty} \delta^{m[\frac{Nr}{2} - 3n]}\delta^{(-j+i\wedge j)n}\right.$$
$$\left.\times \int_{B_m} \sum_{Q\in S_m} \delta^{-jr\alpha(x)}|\lambda_Q|^r[\mu(Q)]^{\frac{-r}{2}}\chi_Q(z)\eta_{i\wedge j, \frac{Nr}{2}}(x, z)d\mu(z)\right)^{\frac{1}{r}}$$

Then, by Lemma 3, we have

$$\begin{aligned}
&\sum_{Q\in\mathcal{D}_j} \delta^{-j\alpha(x)}|\lambda_Q|[\mu(Q)]^{\frac{-1}{2}}(1+\delta^{-(i\wedge j)}d(z_Q,x))^{-N} \\
&\lesssim \Bigg(\sum_{m=0}^{\infty} \delta^{m[\frac{Nr}{2}-3n]}\delta^{(-j+i\wedge j)n} \\
&\qquad \eta_{i\wedge j,\frac{Nr}{2}-rc_{\log}(\alpha)}\Big(\sum_{Q\in S_m} \delta^{-jr\alpha(\cdot)}|\lambda_Q|^r[\mu(Q)]^{\frac{-r}{2}}\chi_Q(\cdot)\Big)\Bigg)^{\frac{1}{r}}
\end{aligned}$$

Thus,

$$\begin{aligned}
&\delta^{-i\alpha(x)}\left|\varphi(2^{-i}\sqrt{L})\Big(\sum_{j=0}^{\infty}\sum_{Q\in\mathcal{D}_j}\lambda_Q a_Q\Big)(x)\right| \\
&\lesssim \sum_{j=0}^{i}\delta^{(j-i)\alpha(x)+2(i-j)K}\left[\eta_{\delta^j,\frac{Nr}{2}-rc_{\log}(\alpha)}\left(\sum_{Q\in\mathcal{D}_j}(\delta^{-jr\alpha(\cdot)}|\lambda_Q|^r[\mu(Q)]^{-\frac{r}{2}}\chi_Q)\right)\right]^{\frac{1}{r}} \\
&\quad+\sum_{j=i}^{\infty}\delta^{(j-i)\alpha(x)+(j-i)(2S-\frac{N}{2})+\frac{(i-j)n}{r}} \\
&\qquad\times\left[\eta_{\delta^i,\frac{Nr}{2}-rc_{\log}(\alpha)}\left(\sum_{Q\in\mathcal{D}_j}(\delta^{-jr\alpha(\cdot)}|\lambda_Q|^r[\mu(Q)]^{-\frac{r}{2}}\chi_Q)\right)\right]^{\frac{1}{r}}.
\end{aligned}$$

Note that

$$\begin{aligned}
\eta_{\delta^i,\frac{Nr}{2}-rc_{\log}(\alpha)}(x,z) &= (V_{\delta^i}(x)V_{\delta^i}(y))^{-1/2}\Big[1+\delta^{-i}d(x,y)\Big]^{-\frac{Nr}{2}+rc_{\log}(\alpha)} \\
&\lesssim \delta^{(j-i)(rc_{\log}(\alpha)-\frac{Nr}{2})}\big(V_{\delta^j}(x)V_{\delta^j}(y)\big)^{-1/2}\Big[1+\delta^{-j}d(x,y)\Big]^{-\frac{Nr}{2}+rc_{\log}(\alpha)}.
\end{aligned}$$

So,

$$\begin{aligned}
&\delta^{-i\alpha(x)}\left|\varphi(2^{-i}\sqrt{L})\Big(\sum_{j=0}^{\infty}\sum_{Q\in\mathcal{D}_j}\lambda_Q a_Q\Big)(x)\right| \\
&\le \sum_{j=0}^{\infty}\Theta(j-i)\left[\eta_{\delta^{i\wedge j},\frac{Nr}{2}-rc_{\log}(\alpha)}\left(\sum_{Q\in\mathcal{D}_j}(\delta^{-jr\alpha(\cdot)}|\lambda_Q|^r[\mu(Q)]^{-\frac{r}{2}}\chi_Q)\right)\right]^{\frac{1}{r}}.
\end{aligned}$$

where the map $\Theta:\mathbb{Z}\to\mathbb{R}$ is defined by

$$\Theta(i)=\begin{cases}\delta^{i(2S+\alpha_{-}-\frac{n}{r}-\frac{N}{2}-\frac{Nr}{2}+rc_{\log}(\alpha))}, & \text{if } i>0\\ \delta^{-i(2K-\alpha_{+})}, & \text{otherwise.}\end{cases}$$

Then, by Lemma 9, we have

$$
\begin{aligned}
&\left\| \sum_{j\in\mathbb{N}_0} \sum_{Q\in\mathcal{D}_j} \lambda_Q a_Q \right\|_{B^{\alpha(\cdot),L}_{p(\cdot),u(\cdot),q(\cdot)}(X)} \\
&\quad \lesssim \left\| \left\{ \sum_{j\in\mathbb{N}_0} \Theta(j-i) \left[\eta_{\delta^{i\wedge j},\frac{Nr}{2}-rc_{\log}(\alpha)} \right.\right.\right. \\
&\qquad \left.\left.\left. \times \left(\sum_{Q\in\mathcal{D}_j} (\delta^{-jr\alpha(\cdot)} |\lambda_Q|^r [\mu(Q)]^{-\frac{r}{2}} \chi_Q) \right) \right]^{\frac{1}{r}} \right\}_{i\in\mathbb{N}_0} \right\|_{l_{q(\cdot)}(M_{p(\cdot),u(\cdot)}(X))}
\end{aligned}
$$

We apply Lemma 1 to obtain

$$
\begin{aligned}
&\left\| \sum_{j\in\mathbb{N}_0} \sum_{Q\in\mathcal{D}_j} \lambda_Q a_Q \right\|_{B^{\alpha(\cdot),L}_{p(\cdot),u(\cdot),q(\cdot)}(X)} \\
&\quad \lesssim \left\| \left\{ \eta_{\delta^{j},\frac{Nr}{2}-rc_{\log}(\alpha)} \right.\right. \\
&\qquad \left.\left. \times \left(\sum_{Q\in\mathcal{D}_j} (\delta^{-jr\alpha(\cdot)} |\lambda_Q|^r [\mu(Q)]^{-\frac{r}{2}} \chi_Q) \right) \right\}_{j\in\mathbb{N}_0} \right\|^{\frac{1}{r}}_{l_{q(\cdot)/r}(M_{p(\cdot)/r,u(\cdot)/r}(X))} \\
&\quad \lesssim \left\| \left\{ \sum_{Q\in\mathcal{D}_j} (\delta^{-j\alpha(\cdot)} |\lambda_Q| [\mu(Q)]^{-\frac{1}{2}} \chi_Q) \right\}_{j\in\mathbb{N}_0} \right\|_{l_{q(\cdot)}(M_{p(\cdot),u(\cdot)}(X))}.
\end{aligned}
$$

We now pass to showing the converse part. Let $K, S \in \mathbb{N}_0$; we choose $\psi_0, \psi \in \mathcal{S}([0,\infty))$, satisfying Lemma 8 with $M \geq S$. In particular, the couple $\{\psi_0, \psi\}$ satisfies (14) and (15). Then, there exist two functions $\varphi_0, \varphi \in \mathcal{S}([0,\infty))$, such that

$$
\sum_{i\in\mathbb{N}_0} \psi_i(\lambda)\varphi_i(\lambda) = 1 \text{ for any } \lambda \in [0,\infty). \tag{26}
$$

It follows that

$$
f = \sum_{i\in\mathbb{N}_0} \psi_i(\sqrt{L})\varphi_i(\sqrt{L}) f \text{ for any } f \in D'. \tag{27}
$$

For $Q \in \mathcal{D}_0$, we set

$$
\widetilde{\lambda}_Q := [\mu(Q)]^{1/2} (\sup_{y\in Q} |\varphi_0(\sqrt{L}) f(y)|) \sup_{x\in X} |K_{(\delta^{2i}L)^k\psi_0}(x,y)| d\mu(y),
$$

$$
\widetilde{a}_Q := \begin{cases} \frac{1}{\widetilde{\lambda}_Q} \int_Q K_{\psi_0(\sqrt{L})}(x,y)\varphi_0(\sqrt{L}) f(y) d\mu(y), & \text{if } \lambda_Q \neq 0 \\ 0, & \text{otherwise.} \end{cases}
$$

If $Q \in \mathcal{D}_i$ with $i \geq 1$, we set

$$
\begin{aligned}
\widetilde{\lambda}_Q :=& [\mu(Q)]^{1/2} (\sup_{y\in Q} |\varphi_i(\sqrt{L}) f(y)|) \\
& \times \max_{m\in\{K,-S\}} \sup_{x\in X} \int_Q |K_{(\delta^{2i}L)^m\psi_i(\sqrt{L})}(x,y)| d\mu(y),
\end{aligned}
$$

$$
\widetilde{a}_Q := \begin{cases} \frac{1}{\widetilde{\lambda}_Q} \int_Q K_{\psi_i(\sqrt{L})}(\cdot,y)\varphi_j(\sqrt{L}) f(y) d\mu(y), & \text{if } \lambda_Q \neq 0 \\ 0, & \text{otherwise.} \end{cases}
$$

Then, by (27), we have

$$f = \sum_{i\in\mathbb{N}_0} \int_X K_{\psi_i(\sqrt{L})}(\cdot,y)\varphi_i(\sqrt{L})f(y)d\mu(y)$$
$$= \sum_{i\in\mathbb{N}_0} \sum_{Q\in\mathcal{D}_i} \int_Q K_{\psi_i(\sqrt{L})}(\cdot,y)\varphi_i(\sqrt{L})f(y)d\mu(y)$$
$$= \sum_{i\in\mathbb{N}_0} \sum_{Q\in\mathcal{D}_i} \widetilde{\lambda}_Q \widetilde{a}_Q,$$

where the sum converges in $D'(L)$. It is easy to see that

$$\widetilde{a}_Q = \frac{1}{\widetilde{\lambda}_Q}\psi_i(\sqrt{L})[(\varphi_i(\sqrt{L})f)\chi_Q],$$

Then, $\widetilde{a}_Q \in D(L^K)\cap D(L^{-S})$ (resp. $\widetilde{a}_Q \in D(L^K)$) whenever $Q \in \mathcal{D}_i$ (resp. $Q \in \mathcal{D}_0$). Moreover,

$$L^m\widetilde{a}_Q = \frac{1}{\widetilde{\lambda}_Q} L^m\psi_i(\sqrt{L})[(\varphi_i(\sqrt{L})f)\chi_Q]$$
$$= \frac{\delta^{-2im}}{\widetilde{\lambda}_Q}\int_Q K_{(\delta^{2i}L)^m\psi_i(\sqrt{L})}(\cdot,y)\varphi_i(\sqrt{L})f(y)d\mu(y)$$

holds for $m \in \{K,-S\}$ (resp., $M = K$). Then, by Lemma 8, we deduce that for any $Q \in \cup_{i\in\mathbb{N}_0}\mathcal{D}_i$, $\widetilde{a}_Q$ is a $(K,S)-$atom up to a multiple constant independent of Q.

Now, for any $Q \in \cup_{i\in\mathbb{N}_0}\mathcal{D}_i$, we define

$$\lambda_Q = c\widetilde{\lambda}_Q, \ \ a_Q = \widetilde{a}_Q,$$

where c is a sufficiently large constant independent of Q. Then, a_Q is a $(K,S)-$atom and

$$f = \sum_{i=0}^{\infty}\sum_{Q\in\mathcal{D}_i} \lambda_Q a_Q,$$

where the sum converges in $D'(L)$. It remains to show (25). Indeed, by Lemma 8, we have, for any $Q \in \cup_{i\in\mathbb{N}_0}\mathcal{D}_i$,

$$|\lambda_i| \lesssim [\mu(Q)]^{1/2}(\sup_{y\in Q}|\varphi_i(\sqrt{L})f(y)|)\sup_{d(x,z_Q)\le(A_1+1)2^{-i}}\int_Q [V_{2^{-i}}(x)]^{-1}d\mu(y)$$
$$\lesssim [\mu(Q)]^{1/2}(\sup_{y\in Q}|\varphi_i(\sqrt{L})f(y)|).$$

Taking a satisfying (16), then

$$\sum_{Q\in\mathcal{D}_i}\delta^{-k\alpha(x)}|\lambda_Q|[\mu(Q)]^{-1/2}\chi_Q(x) \lesssim \sum_{Q\in\mathcal{D}_i}\sup_{y\in Q}\delta^{-i\alpha(x)}|\varphi_i(\sqrt{L})f(y)|$$
$$\lesssim \sup_{y\in B(x,2A_1\delta^{-i})}\delta^{-i\alpha(x)}|\varphi_i(\sqrt{L})f(y)|$$
$$\lesssim \sup_{y\in X}\frac{\delta^{-i\alpha(x)}|\varphi_i(\sqrt{L})f(y)|}{(1+\delta^{-i}d(x,y))^a}$$
$$= \delta^{-i\alpha(x)}\left(\varphi_i(\sqrt{L})\right)_a^* f(x).$$

Then, by Theorem 2, we obtain

$$\|\lambda\|_{b^{\alpha(\cdot),q(\cdot)}_{p(\cdot),u(\cdot)}} \leq C\|f\|_{B^{\alpha(\cdot),L}_{p(\cdot),u(\cdot),q(\cdot)}(X)}.$$

This finishes the proof of the theorem. □

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