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On Generalizations of the Close-to-Convex Functions Associated with $\mathfrak{q}$-Srivastava–Attiya Operator

Daniel Breaz [1], Abdullah A. Alahmari [2], Luminiţa-Ioana Cotîrlă [3,*] and Shujaat Ali Shah [4]

[1] Department of Mathematics, "1 Decembrie 1918" University of Alba Iulia, 510009 Alba Iulia, Romania
[2] Department of Mathematical Sciences, Faculty of Applied Science, Umm Al-Qura University, Makkah 21955, Saudi Arabia
[3] Department of Mathematics, Technical University of Cluj-Napoca, 400114 Cluj-Napoca, Romania
[4] Department of Mathematics and Statistics, Quaid-e-Awam University of Engineering, Science and Technology (QUEST), Nawabshah 67450, Pakistan
* Correspondence: luminita.cotirla@math.utcluj.ro

Abstract: The study of the $\mathfrak{q}$-analogue of the classical results of geometric function theory is currently of great interest to scholars. In this article, we define generalized classes of close-to-convex functions and quasi-convex functions with the help of the $\mathfrak{q}$-difference operator. Moreover, by using the $\mathfrak{q}$-analogues of a certain family of linear operators, the classes $K^{s}_{\mathfrak{q},b}(\mathfrak{h})$, $\widetilde{K}^{b}_{\mathfrak{q},s}(\mathfrak{h})$, $Q^{s}_{\mathfrak{q},b}(\mathfrak{h})$, and $\widetilde{Q}^{b}_{\mathfrak{q},s}(\mathfrak{h})$ are introduced. Several interesting inclusion relationships between these newly defined classes are discussed, and the invariance of these classes under the $\mathfrak{q}$-Bernadi integral operator was examined. Furthermore, some special cases and useful consequences of these investigations were taken into consideration.

Keywords: analytic functions; $\mathfrak{q}$-starlike functions; $\mathfrak{q}$-convex functions; $\mathfrak{q}$-close-to-convex functions; $\mathfrak{q}$-Srivastava–Attiya operator; $\mathfrak{q}$-multiplier transformation

MSC: 30C45; 30C50

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1. Introduction

Calculus that does not include the idea of limits is referred to as "$\mathfrak{q}$-calculus". Mathematicians have been paying much attention to $\mathfrak{q}$-calculus recently because of its applications in the study of, for example, $\mathfrak{q}$-deformed super-algebras, quantum groups, optimal control problems, fractal and multi-fractal measures, and chaotic dynamical systems. Since the concept of $\mathfrak{q}$-calculus was introduced, this idea has been applied in many research studies by various prominent scholars [1–5] in this branch of study. The application of $\mathfrak{q}$-calculus involving $\mathfrak{q}$-derivatives and $\mathfrak{q}$-integrals was initiated by F. Jackson of [6,7]. For further details, see [8–12].

We denote by $\mathbb{A}$ the class of analytic functions $\mathfrak{f}(\zeta)$ in the open unit disk $\mathbb{U} = \{\zeta : |\zeta| < 1\}$, which can be expressed as the series:

$$\mathfrak{f}(\zeta) = \zeta + \sum_{\kappa=2}^{\infty} a_{\kappa}\zeta^{\kappa}. \tag{1}$$

The subordination of two analytic functions $\mathfrak{f}_1$ and $\mathfrak{f}_2$ is denoted by $\mathfrak{f}_1 \prec \mathfrak{f}_2$ and defined as $\mathfrak{f}_1(\zeta) = \mathfrak{f}_2(w(\zeta))$, where $w(\zeta)$ is a Schwartz function in $\mathbb{U}$ (see [13]). Let S, S^*, C, K, and Q denote the subclasses of $\mathbb{A}$ of univalent, starlike, convex, close-to-convex, and quasi-convex functions, respectively.

In [6], Jackson introduced the $\mathfrak{q}$-difference operator $\eth_{\mathfrak{q}} : \mathbb{A} \to \mathbb{A}$, which is defined by

$$\eth_{\mathfrak{q}}\mathfrak{f}(\zeta) = \frac{\mathfrak{f}(\zeta) - \mathfrak{f}(\mathfrak{q}\zeta)}{(1-\mathfrak{q})\zeta}; \quad \mathfrak{q} \neq 1,\ \zeta \neq 0. \tag{2}$$

In particular, as $\mathfrak{q} \to 1^-$, we have $\eth_{\mathfrak{q}}\mathfrak{f}(\zeta) \to \mathfrak{f}'(\zeta)$, the usual derivative of the function $\mathfrak{f}(\zeta)$. One can see [14–19] for the important properties of this operator $\eth_{\mathfrak{q}}$. It was found that, for $\kappa \in \mathbb{N} = \{1, 2, 3, \ldots\}$ and $\zeta \in \mathbb{U}$,

$$\eth_{\mathfrak{q}}\left\{\sum_{\kappa=1}^{\infty} a_\kappa \zeta^\kappa\right\} = \sum_{\kappa=1}^{\infty} [\kappa]_{\mathfrak{q}} a_\kappa \zeta^{\kappa-1}, \tag{3}$$

where

$$[\kappa]_{\mathfrak{q}} = \frac{1-\mathfrak{q}^\kappa}{1-\mathfrak{q}} = 1 + \sum_{j=1}^{\kappa-1} \mathfrak{q}^j. \tag{4}$$

The following are some fundamental rules of the $\mathfrak{q}$-difference operator.

$$\eth_{\mathfrak{q}}(x\mathfrak{u}(\zeta) \pm y v(\zeta)) = x\eth_{\mathfrak{q}}\mathfrak{u}(\zeta) \pm y\eth_{\mathfrak{q}} v(\zeta) \tag{5}$$

$$\eth_{\mathfrak{q}}(\mathfrak{u}(\zeta)v(\zeta)) = \mathfrak{u}(\mathfrak{q}\zeta)\eth_{\mathfrak{q}}(v(\zeta)) + v(\zeta)\eth_{\mathfrak{q}}(\mathfrak{u}(\zeta)) \tag{6}$$

$$\eth_{\mathfrak{q}}\left(\frac{\mathfrak{u}(\zeta)}{v(\zeta)}\right) = \frac{\eth_{\mathfrak{q}}(\mathfrak{u}(\zeta))v(\zeta) - \mathfrak{u}(\zeta)\eth_{\mathfrak{q}}(v(\zeta))}{v(\mathfrak{q}\zeta)v(\zeta)}, \quad v(\mathfrak{q}\zeta)v(\zeta) \neq 0. \tag{7}$$

$$\eth_{\mathfrak{q}}(\log \mathfrak{u}(\zeta)) = \frac{\ln \mathfrak{q}\eth_{\mathfrak{q}}\mathfrak{u}(\zeta)}{(\mathfrak{q}-1)\mathfrak{u}(\zeta)}, \tag{8}$$

where $\mathfrak{u}, v \in \mathbb{A}$ and x and y are complex constants.

The $\mathfrak{q}$-starlike functions was introduced and studied by Ismail et al. [20]. This was the amazing breakthrough by which two different fields, $\mathfrak{q}$-calculus and the theory of analytic functions, were linked. After this interesting investigation, certain subclasses of analytic functions were defined and examined in terms of $\mathfrak{q}$-function theory by several scholars; we refer to [21–30]. The study of $\mathfrak{q}$-operators plays a vital role in the development of this field of study. Kanas and Raducanu [31] introduced the $\mathfrak{q}$-extension of the Ruscheweyh derivative operator, and the $\mathfrak{q}$-analogue of the Bernardi and Noor integral operators was defined by Noor et al. [32] and Arif et al. [33], respectively.

In [34], Shah and Noor discussed the $\mathfrak{q}$-extensions of the Srivastava–Attiya operator and the multiplier transformation. For $b \in \mathbb{C} \setminus \mathbb{Z}_0^-$, $s \in \mathbb{C}$ when $|\zeta| < 1$ and $Re(s) > 1$ when $|\zeta| = 1$, the operator $J^s_{\mathfrak{q},b} : \mathbb{A} \to \mathbb{A}$ is defined by

$$J^s_{\mathfrak{q},b}\mathfrak{f}(\zeta) = \psi_{\mathfrak{q}}(s, b; \zeta) * \mathfrak{f}(\zeta) = \zeta + \sum_{\kappa=2}^{\infty}\left(\frac{[1+b]_{\mathfrak{q}}}{[\kappa+b]_{\mathfrak{q}}}\right)^s a_\kappa \zeta^\kappa, \tag{9}$$

where

$$\psi_{\mathfrak{q}}(s, b; \zeta) = \zeta + \sum_{\kappa=2}^{\infty}\left(\frac{[1+b]_{\mathfrak{q}}}{[\kappa+b]_{\mathfrak{q}}}\right)^s \zeta^\kappa, \tag{10}$$

and "$*$" denotes convolution.

Special cases:

(i) If $\mathfrak{q} \to 1^-$, then the function $\psi_{\mathfrak{q}}(s, b; \zeta)$ reduces to the Hurwitz–Lerch zeta function and the operator $J^s_{\mathfrak{q},b}$ coincides with the Srivastava–Attiya operator; we refer to [35].

(ii) $J^1_{\mathfrak{q},0}\mathfrak{f}(\zeta) = \int_0^\zeta \frac{\mathfrak{f}(t)}{t}\eth_{\mathfrak{q}} t$ ($\mathfrak{q}$-Alexander operator).

(iii) $J^1_{\mathfrak{q},b}\mathfrak{f}(\zeta) = \frac{[1+b]_{\mathfrak{q}}}{\zeta^b}\int_0^\zeta t^{b-1}\mathfrak{f}(t)\eth_{\mathfrak{q}} t$ ($\mathfrak{q}$-Bernardi operator [32]).

(iv) $J^1_{\mathfrak{q},1}\mathfrak{f}(\zeta) = \frac{[2]_{\mathfrak{q}}}{\zeta}\int_0^\zeta \mathfrak{f}(t)\eth_{\mathfrak{q}} t$ ($\mathfrak{q}$-Libera operator [32]).

The operator $I^{b}_{\mathfrak{q},s} : \mathbb{A} \to \mathbb{A}$ is defined as

$$I^{b}_{\mathfrak{q},s}\mathfrak{f}(\zeta) = \zeta + \sum_{\kappa=2}^{\infty} \left(\frac{[\kappa + b]_{\mathfrak{q}}}{[1 + b]_{\mathfrak{q}}} \right)^{s} a_{\kappa}\zeta^{\kappa}, \tag{11}$$

where $\mathfrak{f} \in \mathbb{A}$, s is a real, and $b > -1$.

In particular, for $b = 0$ and non-negative real s, we obtain the Salagean $\mathfrak{q}$-difference operator [36].

We used (9) and (11) to obtain the following identities.

$$\zeta\mathfrak{d}_{\mathfrak{q}}\left(J^{s+1}_{\mathfrak{q},b}\mathfrak{f}(\zeta)\right) = \left(1 + \frac{[b]_{\mathfrak{q}}}{\mathfrak{q}^{b}}\right) J^{s}_{\mathfrak{q},b}\mathfrak{f}(\zeta) - \frac{[b]_{\mathfrak{q}}}{\mathfrak{q}^{b}} J^{s+1}_{\mathfrak{q},b}\mathfrak{f}(\zeta). \tag{12}$$

$$\zeta\mathfrak{d}_{\mathfrak{q}}\left(I^{b}_{\mathfrak{q},s}\mathfrak{f}(\zeta)\right) = \left(1 + \frac{[b]_{\mathfrak{q}}}{\mathfrak{q}^{b}}\right) I^{b}_{\mathfrak{q},s+1}\mathfrak{f}(\zeta) - \frac{[b]_{\mathfrak{q}}}{\mathfrak{q}^{b}} I^{b}_{\mathfrak{q},s}\mathfrak{f}(\zeta). \tag{13}$$

In 2017, Agarwal and Sahoo [21] generalized the classes $ST_{\mathfrak{q}}$ and $CV_{\mathfrak{q}}$ of functions of $\mathfrak{q}$-starlike and $\mathfrak{q}$-convex functions, respectively, as follows:

$$ST_{\mathfrak{q}}(\gamma) = \left\{ \mathfrak{f} \in \mathbb{A} : \left| \frac{\frac{\mathfrak{d}_{\mathfrak{q}}(\mathfrak{f}(\zeta))}{\mathfrak{f}(\zeta)} - \gamma}{1 - \gamma} - \frac{1}{1 - \mathfrak{q}} \right| < \frac{1}{1 - \mathfrak{q}} \right\} \tag{14}$$

and

$$CV_{\mathfrak{q}}(\gamma) = \{ \mathfrak{f} \in \mathbb{A} : \zeta\mathfrak{d}_{\mathfrak{q}}(\mathfrak{f}(\zeta)) \in ST_{\mathfrak{q}}(\gamma) \}, \tag{15}$$

where $\gamma \in [0, 1)$, $\mathfrak{q} \in (0, 1)$, and $\zeta \in \mathbb{U}$.

In [37], the class $K_{\mathfrak{q}}(\gamma)$ of $\mathfrak{q}-$close-to-convex functions of order γ was defined as the following.

$$K_{\mathfrak{q}}(\gamma) = \left\{ \mathfrak{f} \in \mathbb{A} : \left| \frac{\frac{\mathfrak{d}_{\mathfrak{q}}(\mathfrak{f}(\zeta))}{g(\zeta)} - \gamma}{1 - \gamma} - \frac{1}{1 - \mathfrak{q}} \right| < \frac{1}{1 - \mathfrak{q}} \right\}, \tag{16}$$

where $g \in ST_{\mathfrak{q}}(\gamma)$, $\gamma \in [0, 1)$, $\mathfrak{q} \in (0, 1)$, and $\zeta \in \mathbb{U}$.

It is noted that [37] $K_{\mathfrak{q}}(0) = K_{\mathfrak{q}}$, which is given as

$$K_{\mathfrak{q}} = \left\{ \mathfrak{f} \in \mathbb{A} : \left| \frac{\mathfrak{d}_{\mathfrak{q}}(\mathfrak{f}(\zeta))}{g(\zeta)} - \frac{1}{1 - \mathfrak{q}} \right| < \frac{1}{1 - \mathfrak{q}} \right\}, \tag{17}$$

equivalently, $\mathfrak{f} \in K_{\mathfrak{q}}$ if and only if

$$\frac{\mathfrak{d}_{\mathfrak{q}}(\mathfrak{f}(\zeta))}{g(\zeta)} \prec \frac{1 + \zeta}{1 - \mathfrak{q}\zeta}, \tag{18}$$

where $g \in ST_{\mathfrak{q}}(0) = ST_{\mathfrak{q}}$, $\mathfrak{q} \in (0, 1)$, and $\zeta \in \mathbb{U}$.

Shah and Noor in [34] defined the classes $ST^{s}_{\mathfrak{q},b}(\mathfrak{h})$ and $\widetilde{ST}^{b}_{\mathfrak{q},s}(\mathfrak{h})$, by using the operators given by (9) and (11), as the following.

Let Φ be the class of univalent convex functions $\mathfrak{h}$ with $\mathfrak{h}(0) = 1$ and $Re(\mathfrak{h}(\zeta)) > 0$ in $\mathbb{U}$. Then,

$$ST^{s}_{\mathfrak{q},b}(\mathfrak{h}) = \left\{ \mathfrak{f} \in \mathbb{A} : J^{s}_{\mathfrak{q},b}\mathfrak{f}(\zeta) \in ST_{\mathfrak{q}}(\mathfrak{h}) \right\}$$

and

$$\widetilde{ST}^{b}_{\mathfrak{q},s}(\mathfrak{h}) = \left\{ \mathfrak{f} \in \mathbb{A} : I^{b}_{\mathfrak{q},s}\mathfrak{f}(\zeta) \in ST_{\mathfrak{q}}(\mathfrak{h}) \right\},$$

where

$$ST_{\mathfrak{q}}(\mathfrak{h}) = \left\{ \mathfrak{f} \in \mathbb{A} : \frac{\zeta\mathfrak{d}_{\mathfrak{q}}\mathfrak{f}(\zeta)}{\mathfrak{f}(\zeta)} \prec \mathfrak{h}(\zeta) \right\}.$$

Motivated by the work in [34], we define the following;

Definition 1. *Let* $\mathfrak{f} \in \mathbb{A}$, $\mathfrak{h} \in \Phi$, *and* $\mathfrak{q} \in (0,1)$. *Then,* $\mathfrak{f} \in K_{\mathfrak{q}}(\mathfrak{h})$ *if and only if*

$$\frac{\zeta \partial_{\mathfrak{q}} \mathfrak{f}(\zeta)}{g(\zeta)} \prec \mathfrak{h}(\zeta),$$

for some $g \in ST_{\mathfrak{q}}(\mathfrak{h})$.

Definition 2. *Let* $\mathfrak{f} \in \mathbb{A}$, $\mathfrak{h} \in \Phi$, *and* $\mathfrak{q} \in (0,1)$. *Then,* $\mathfrak{f} \in K^{s}_{\mathfrak{q},b}(\mathfrak{h})$ *if and only if*

$$\frac{\zeta \partial_{\mathfrak{q}} J^{s}_{\mathfrak{q},b} \mathfrak{f}(\zeta)}{J^{s}_{\mathfrak{q},b} g(\zeta)} \prec \mathfrak{h}(\zeta),$$

for some $g \in ST^{s}_{\mathfrak{q},b}(\mathfrak{h})$ *with* $b \in \mathbb{C} \setminus \mathbb{Z}^{-}_{0}$, $s \in \mathbb{C}$ *when* $|\zeta| < 1$ *and* $Re(s) > 1$ *when* $|\zeta| = 1$.

Definition 3. *Let* $\mathfrak{f} \in \mathbb{A}$, $\mathfrak{h} \in \Phi$, $s \in \mathbb{R}$, $b > -1$, *and* $\mathfrak{q} \in (0,1)$. *Then,* $\mathfrak{f} \in \widetilde{K}^{b}_{\mathfrak{q},s}(\mathfrak{h})$ *if and only if*

$$\frac{\zeta \partial_{\mathfrak{q}} I^{b}_{\mathfrak{q},s} \mathfrak{f}(\zeta)}{I^{b}_{\mathfrak{q},s} g(\zeta)} \prec \mathfrak{h}(\zeta),$$

for some $g \in \widetilde{ST}^{b}_{\mathfrak{q},s}(\mathfrak{h})$.

Here, analogous to the above classes, we define

$$Q_{\mathfrak{q}}(\mathfrak{h}) = \{\mathfrak{f} \in \mathbb{A} : \zeta \partial_{\mathfrak{q}} \mathfrak{f}(\zeta) \in K_{\mathfrak{q}}(\mathfrak{h})\},$$

$$\mathfrak{f} \in Q^{s}_{\mathfrak{q},b}(\mathfrak{h}) \text{ if and only if } \zeta \partial_{\mathfrak{q}} \mathfrak{f}(\zeta) \in K^{s}_{\mathfrak{q},b}(\mathfrak{h})$$

and

$$\mathfrak{f} \in \widetilde{Q}^{b}_{\mathfrak{q},s}(\mathfrak{h}) \text{if and only if } \zeta \partial_{\mathfrak{q}} \mathfrak{f}(\zeta) \in \widetilde{K}^{b}_{\mathfrak{q},s}(\mathfrak{h}).$$

For different values of the parameters $\mathfrak{q}$, s, b, and $\mathfrak{h}$, the above newly defined classes reduce to the certain subclasses of analytic functions. For example:

(i) $K^{s}_{\mathfrak{q},b}\left(\frac{1+\{1-\gamma(1+\mathfrak{q})\}\zeta}{1-\mathfrak{q}\zeta}\right) = K^{s}_{\mathfrak{q},b}(\gamma)$ and $Q^{s}_{\mathfrak{q},b}\left(\frac{1+\{1-\gamma(1+\mathfrak{q})\}\zeta}{1-\mathfrak{q}\zeta}\right) = Q^{s}_{\mathfrak{q},b}(\gamma)$.

(ii) $K^{s}_{\mathfrak{q},b}\left(\frac{1+\zeta}{1-\mathfrak{q}\zeta}\right) = K^{s}_{\mathfrak{q},b}$ and $Q^{s}_{\mathfrak{q},b}\left(\frac{1+\zeta}{1-\mathfrak{q}\zeta}\right) = Q^{s}_{\mathfrak{q},b}$.

(iii) $K^{0}_{\mathfrak{q},b}(\mathfrak{h}) = K_{\mathfrak{q}}(\mathfrak{h}) = \widetilde{K}^{b}_{\mathfrak{q},0}(\mathfrak{h})$ and $Q^{0}_{\mathfrak{q},b}(\mathfrak{h}) = Q_{\mathfrak{q}}(\mathfrak{h}) = \widetilde{Q}^{b}_{\mathfrak{q},0}(\mathfrak{h})$.

(iv) $K_{\mathfrak{q}}\left(\frac{1+\{1-\gamma(1+\mathfrak{q})\}\zeta}{1-\mathfrak{q}\zeta}\right) = K_{\mathfrak{q}}(\gamma)$ (see [37]) and $Q_{\mathfrak{q}}\left(\frac{1+\{1-\gamma(1+\mathfrak{q})\}\zeta}{1-\mathfrak{q}\zeta}\right) = Q_{\mathfrak{q}}(\gamma)$.

(v) $K_{\mathfrak{q}}\left(\frac{1+\zeta}{1-\mathfrak{q}\zeta}\right) = K_{\mathfrak{q}}$ (see [37]) and $Q_{\mathfrak{q}}\left(\frac{1+\zeta}{1-\mathfrak{q}\zeta}\right) = Q_{\mathfrak{q}}$.

(vi) $\underset{\mathfrak{q}\to 1^{-}}{lim} K_{\mathfrak{q}} = K$ and $\underset{\mathfrak{q}\to 1^{-}}{lim} Q_{\mathfrak{q}} = Q$, the classes of close-to-convex and quasi-convex functions, respectively.

Shah and Noor [34] studied various interesting properties such as the inclusion results and an integral-related property for the generalized classes of $\mathfrak{q}$-starlike and $\mathfrak{q}$-convex functions. Inspired by these investigations, we organized this paper to discuss such types of results for the generalized classes of $\mathfrak{q}$-close-to-convex functions and $\mathfrak{q}$-quasi-convex functions. In the next section, firstly, we prove the q-analogue of the fundamental lemma and then, by using this lemma, our main results are examined.

2. Main Results

To discuss our main problems, we state some preliminary results as the following.

Lemma 1 ([38]). *Let $h(\zeta)$ be convex in $\mathbb{U}$ with $h(0) = 1$, and let $P : \mathbb{U} \to \mathbb{C}$ with $Re(P(\zeta)) > 0$ in $\mathbb{U}$. If $p(\zeta) = 1 + p_1\zeta + p_2\zeta^2 + \ldots$ is analytic in $\mathbb{U}$, then*

$$p(\zeta) + P(\zeta) \cdot \zeta p'(\zeta) \prec h(\zeta) \tag{19}$$

implies that $p(\zeta) \prec h(\zeta)$.

Lemma 2 ([39]). *Let $p(\zeta) = 1 + p_\kappa\zeta^\kappa + \ldots$, $\kappa \geq 1$, be analytic in the unit disc $\mathbb{U}$, and let $g(\zeta) = 1 + c_1\zeta + \ldots$ be analytic in $\overline{\mathbb{U}}$. If $p(\zeta)$ is not subordinate $g(\zeta)$, then there exists a real number $m \geq 1$, $\zeta_0 \in \mathbb{U}$, and $\varepsilon_0 \in \partial\mathbb{U}$ such that:*

(i) $p(|\zeta| < |\zeta_0|) \subset g(\mathbb{U})$.
(ii) $p(\zeta_0) = g(\varepsilon_0)$.
(iii) $arg(\zeta_0 \mathfrak{d}_\mathfrak{q} p(\zeta_0)) = arg(\varepsilon_0 \mathfrak{d}_\mathfrak{q} p(\varepsilon_0))$
(iv) $|\zeta_0 \mathfrak{d}_\mathfrak{q} p(\zeta_0)| = m|\varepsilon_0 \mathfrak{d}_\mathfrak{q} p(\varepsilon_0)|$.

Lemma 3. *Let $h(\zeta)$ be convex in $\mathbb{U}$ with $h(0) = 1$, and let $P : \mathbb{U} \to \mathbb{C}$ with $Re(P(\zeta)) > 0$ in $\mathbb{U}$. If $p(\zeta) = 1 + p_1\zeta + p_2\zeta^2 + \ldots$ is analytic in $\mathbb{U}$, then*

$$p(\zeta) + P(\zeta) \cdot \zeta \mathfrak{d}_\mathfrak{q} p(\zeta) \prec h(\zeta) \tag{20}$$

implies that $p(\zeta) \prec h(\zeta)$.

Proof. Suppose on the contrary that $p(\zeta) \not\prec h(\zeta)$. Then, by Lemma 2, there exists a real number $m \geq 1$, $\zeta_0 \in \mathbb{U}$, and $\varepsilon_0 \in \partial\mathbb{U}$ such that

$$p(\zeta_0) + P(\zeta_0) \cdot \zeta_0 \mathfrak{d}_\mathfrak{q} p(\zeta_0) = h(\varepsilon_0) + mP(\varepsilon_0) \cdot \varepsilon_0 \mathfrak{d}_\mathfrak{q} h(\varepsilon_0). \tag{21}$$

Since $Re(P(\zeta)) > 0$, we have

$$|\arg mP(\varepsilon_0)| < \frac{\pi}{2}$$

and $\varepsilon_0 \mathfrak{d}_\mathfrak{q} h(\varepsilon_0)$ is in the direction of the outer normal to the convex domain $h(\mathbb{U})$; therefore, the right-hand side of (21) is a complex number outside $h(\mathbb{U})$. That is,

$$p(\zeta_0) + P(\zeta_0) \cdot \zeta_0 \mathfrak{d}_\mathfrak{q} p(\zeta_0) \notin h(\mathbb{U}),$$

which is a contradiction to the hypothesis. We replaced $p_\varrho(\zeta) = p(\varrho\zeta)$ and $h_\varrho(\zeta) = h(\varrho\zeta)$, for $\varrho \in (0,1)$ to remove the restriction on the functions involved. Since the hypothesis is true, we obtain the required result by letting $\varrho \to 1^-$. □

Lemma 4 ([34]). *Let $\mathfrak{h} \in \Phi$. Then, for positive real s and $b \in \mathbb{N}$, we have*

$$ST^s_{\mathfrak{q},b}(\mathfrak{h}) \subset ST^{s+1}_{\mathfrak{q},b}(\mathfrak{h}).$$

Lemma 5 ([34]). *Let $\mathfrak{h} \in \Phi$. Then, for positive real s and $b \in \mathbb{N}$, we have*

$$\widetilde{ST}^b_{\mathfrak{q},s+1}(\mathfrak{h}) \subset \widetilde{ST}^b_{\mathfrak{q},s}(\mathfrak{h}).$$

2.1. Inclusion of Classes

Theorem 1. *Let $\mathfrak{h} \in \Phi$. Then, for positive real s and $b \in \mathbb{N}$,*

$$K^{s}_{\mathfrak{q},b}(\mathfrak{h}) \subset K^{s+1}_{\mathfrak{q},b}(\mathfrak{h}).$$

Proof. Let $\mathfrak{f} \in K^{s}_{\mathfrak{q},b}(\mathfrak{h})$. Then, by definition, there exists $g \in ST^{s}_{\mathfrak{q},b}(\mathfrak{h})$ such that

$$\frac{\zeta \mathfrak{d}_{\mathfrak{q}}\left(J^{s}_{\mathfrak{q},b}\mathfrak{f}(\zeta)\right)}{J^{s}_{\mathfrak{q},b}g(\zeta)} \prec \mathfrak{h}(\zeta). \tag{22}$$

Consider

$$\frac{\zeta \mathfrak{d}_{\mathfrak{q}}\left(J^{s+1}_{\mathfrak{q},b}\mathfrak{f}(\zeta)\right)}{J^{s+1}_{\mathfrak{q},b}g(\zeta)} = p(\zeta). \tag{23}$$

We note that $p(\zeta)$ is analytic in $\mathbb{U}$ with $p(0) = 1$.

From Identity (12), we can easily write

$$\frac{\zeta \mathfrak{d}_{\mathfrak{q}}\left(J^{s}_{\mathfrak{q},b}\mathfrak{f}(\zeta)\right)}{J^{s}_{\mathfrak{q},b}g(\zeta)} = \frac{\frac{\zeta \mathfrak{d}_{\mathfrak{q}}\left(\zeta \mathfrak{d}_{\mathfrak{q}}\left(J^{s+1}_{\mathfrak{q},b}\mathfrak{f}(\zeta)\right)\right)}{J^{s+1}_{\mathfrak{q},b}g(\zeta)} + \gamma_{\mathfrak{q}}\frac{\zeta \mathfrak{d}_{\mathfrak{q}}\left(J^{s+1}_{\mathfrak{q},b}\mathfrak{f}(\zeta)\right)}{J^{s+1}_{\mathfrak{q},b}g(\zeta)}}{\frac{\zeta \mathfrak{d}_{\mathfrak{q}}\left(J^{s+1}_{\mathfrak{q},b}g(\zeta)\right)}{J^{s+1}_{\mathfrak{q},b}g(\zeta)} + \gamma_{\mathfrak{q}}}, \tag{24}$$

where $\gamma_{\mathfrak{q}} = \frac{[b]_{\mathfrak{q}}}{q^{b}}$.

After the $\mathfrak{q}$-logarithmic differentiation of (23), we have

$$\frac{\zeta \mathfrak{d}_{\mathfrak{q}}\left(\zeta \mathfrak{d}_{\mathfrak{q}}\left(J^{s+1}_{\mathfrak{q},b}\mathfrak{f}(\zeta)\right)\right)}{J^{s+1}_{\mathfrak{q},b}g(\zeta)} = p(\zeta)R(\zeta) + \zeta \mathfrak{d}_{\mathfrak{q}}p(\zeta), \tag{25}$$

where $R(\zeta) = \frac{\zeta \mathfrak{d}_{\mathfrak{q}}\left(J^{s+1}_{\mathfrak{q},b}g(\zeta)\right)}{J^{s+1}_{\mathfrak{q},b}g(\zeta)}$.

From (24) and (25), we obtain

$$\frac{\zeta \mathfrak{d}_{\mathfrak{q}}\left(J^{s}_{\mathfrak{q},b}\mathfrak{f}(\zeta)\right)}{J^{s}_{\mathfrak{q},b}g(\zeta)} = p(\zeta) + \frac{\zeta \mathfrak{d}_{\mathfrak{q}}p(\zeta)}{R(\zeta) + \gamma_{\mathfrak{q}}}. \tag{26}$$

Consequently, from (22),

$$p(\zeta) + \frac{\zeta \mathfrak{d}_{\mathfrak{q}}p(\zeta)}{R(\zeta) + \gamma_{\mathfrak{q}}} \prec \mathfrak{h}(\zeta). \tag{27}$$

Since $g \in ST^{s}_{\mathfrak{q},b}(\mathfrak{h})$, by Lemma 4, we conclude $g \in ST^{s+1}_{\mathfrak{q},b}(\mathfrak{h})$. This implies $R(\zeta) \prec \mathfrak{h}(\zeta)$. Therefore, $Re(R(\zeta)) > 0$ in $\mathbb{U}$, and hence, $Re(\frac{1}{R(\zeta)+\gamma_{\mathfrak{q}}}) > 0$ in $\mathbb{U}$. Now, by applying Lemma 3, we obtain our required result. □

Theorem 2. *Let $\mathfrak{h} \in \Phi$. Then, for positive real s and $b \in \mathbb{N}$,*

$$Q^{s}_{\mathfrak{q},b}(\mathfrak{h}) \subset Q^{s+1}_{\mathfrak{q},b}(\mathfrak{h}).$$

Proof. Let $\mathfrak{f} \in Q^{s}_{\mathfrak{q},b}(\mathfrak{h})$. Then, due to the definition of the class $Q^{s}_{\mathfrak{q},b}(\mathfrak{h})$, we have $\zeta\mathfrak{d}_{\mathfrak{q}}\mathfrak{f}(\zeta) \in K^{s}_{\mathfrak{q},b}(\mathfrak{h})$. From Theorem 1, we know that $K^{s}_{\mathfrak{q},b}(\mathfrak{h}) \subset K^{s+1}_{\mathfrak{q},b}(\mathfrak{h})$, and this implies $\zeta(\mathfrak{d}_{\mathfrak{q}}\mathfrak{f}) \in K^{s+1}_{\mathfrak{q},b}(\mathfrak{h})$. Hence, again, due to the definition of the class $Q^{s+1}_{\mathfrak{q},b}(\mathfrak{h})$, we conclude $\mathfrak{f} \in Q^{s+1}_{\mathfrak{q},b}(\mathfrak{h})$. □

Corollary 1. *Let s be a positive real and $b \in \mathbb{N}$. Then, for $\mathfrak{h}(\zeta) = \frac{1+\{1-\gamma(1+\mathfrak{q})\}\zeta}{1-\mathfrak{q}\zeta}$ $(0 \le \gamma < 1)$,*

$$K^{s}_{\mathfrak{q},b}(\gamma) \subset K^{s+1}_{\mathfrak{q},b}(\gamma) \text{ and } Q^{s}_{\mathfrak{q},b}(\gamma) \subset Q^{s+1}_{\mathfrak{q},b}(\gamma)$$

Moreover, for $\mathfrak{h}(\zeta) = \frac{1+\zeta}{1-\mathfrak{q}\zeta}$,

$$K^{s}_{\mathfrak{q},b} \subset K^{s+1}_{\mathfrak{q},b} \text{ and } Q^{s}_{\mathfrak{q},b} \subset Q^{s+1}_{\mathfrak{q},b}.$$

One can prove the following result by using similar arguments as used before, together with the Lemma 5 and the identity (13).

Theorem 3. *Let $\mathfrak{h} \in \Phi$. Then, for positive real s and $b \in \mathbb{N}$,*

$$\widetilde{K}^{b}_{\mathfrak{q},s+1}(\mathfrak{h}) \subset \widetilde{K}^{b}_{\mathfrak{q},s}(\mathfrak{h})$$

and

$$\widetilde{Q}^{b}_{\mathfrak{q},s+1}(\mathfrak{h}) \subset \widetilde{Q}^{b}_{\mathfrak{q},s}(\mathfrak{h}).$$

2.2. Invariance of the Classes Under $\mathfrak{q}$-Bernardi Integral Operator

Theorem 4. *Let $\mathfrak{f} \in K^{s}_{\mathfrak{q},b}(\mathfrak{h})$. Then, $\mathfrak{f}_{\mathfrak{q},b}(\zeta) \in K^{s}_{\mathfrak{q},b}(\mathfrak{h})$, where*

$$\mathfrak{f}_{\mathfrak{q},b}(\zeta) = \frac{[1+b]_{\mathfrak{q}}}{\zeta^{b}} \int_{0}^{\zeta} t^{b-1}\mathfrak{f}(t)\mathfrak{d}_{\mathfrak{q}}t.$$

Proof. Let $\mathfrak{f} \in K^{s}_{\mathfrak{q},b}(\mathfrak{h})$. Then, we want to show that $\mathfrak{f}_{\mathfrak{q},b}(\zeta) \in K^{s}_{\mathfrak{q},b}(\mathfrak{h})$, where

$$\mathfrak{f}_{\mathfrak{q},b}(\zeta) = \frac{[1+b]_{\mathfrak{q}}}{\zeta^{b}} \int_{0}^{\zeta} t^{b-1}\mathfrak{f}(t)\mathfrak{d}_{\mathfrak{q}}t, \tag{28}$$

It was found in [34] that, for $g \in ST^{s}_{\mathfrak{q},b}(\mathfrak{h})$,

$$g_{\mathfrak{q},b}(\zeta) = \frac{[1+b]_{\mathfrak{q}}}{\zeta^{b}} \int_{0}^{\zeta} t^{b-1}g(t)\mathfrak{d}_{\mathfrak{q}}t \in ST^{s}_{\mathfrak{q},b}(\mathfrak{h}). \tag{29}$$

Consider

$$\frac{\zeta\mathfrak{d}_{\mathfrak{q}}\mathfrak{f}_{\mathfrak{q},b}(\zeta)}{g_{\mathfrak{q},b}(\zeta)} = p(\zeta), \tag{30}$$

where $p(\zeta)$ is analytic in $\mathbb{U}$ with $p(0) = 1$.

The following can be obtained from (28):

$$\zeta\mathfrak{d}_{\mathfrak{q}}\mathfrak{f}_{\mathfrak{q},b}(\zeta) = (1+\gamma_{\mathfrak{q}})\mathfrak{f}(\zeta) - \gamma_{\mathfrak{q}}\mathfrak{f}_{\mathfrak{q},b}(\zeta), \quad \left(\gamma_{\mathfrak{q}} = \frac{[b]_{\mathfrak{q}}}{\mathfrak{q}^{b}}\right). \tag{31}$$

The $\mathfrak{q}$-differentiation yields

$$(1+\gamma_{\mathfrak{q}})\mathfrak{d}_{\mathfrak{q}}\mathfrak{f}(\zeta) = \mathfrak{d}_{\mathfrak{q}}\left(\zeta\mathfrak{d}_{\mathfrak{q}}\mathfrak{f}_{\mathfrak{q},b}(\zeta)\right) + \gamma_{\mathfrak{q}}\mathfrak{d}_{\mathfrak{q}}\left(\mathfrak{f}_{\mathfrak{q},b}(\zeta)\right). \tag{32}$$

Similarly, from (29), we have

$$(1+\gamma_{\mathfrak{q}})g(\zeta) = \zeta\partial_{\mathfrak{q}}g_{\mathfrak{q},b}(\zeta) + \gamma_{\mathfrak{q}}g_{\mathfrak{q},b}(\zeta). \tag{33}$$

From (32) and (33), we obtain

$$\frac{\partial_{\mathfrak{q}}\mathfrak{f}(\zeta)}{g(\zeta)} = \frac{\partial_{\mathfrak{q}}\left(\zeta\partial_{\mathfrak{q}}\mathfrak{f}_{\mathfrak{q},b}(\zeta)\right) + \gamma_{\mathfrak{q}}\partial_{\mathfrak{q}}\left(\mathfrak{f}_{\mathfrak{q},b}(\zeta)\right)}{\zeta\partial_{\mathfrak{q}}g_{\mathfrak{q},b}(\zeta) + \gamma_{\mathfrak{q}}g_{\mathfrak{q},b}(\zeta)},$$

and equivalently,

$$\frac{\zeta\partial_{\mathfrak{q}}\mathfrak{f}(\zeta)}{g(\zeta)} = \frac{\frac{\zeta\partial_{\mathfrak{q}}\left(\zeta\partial_{\mathfrak{q}}\mathfrak{f}_{\mathfrak{q},b}(\zeta)\right)}{g_{\mathfrak{q},b}(\zeta)} + \gamma_{\mathfrak{q}}\frac{\zeta\partial_{\mathfrak{q}}\left(\mathfrak{f}_{\mathfrak{q},b}(\zeta)\right)}{g_{\mathfrak{q},b}(\zeta)}}{\frac{\zeta\partial_{\mathfrak{q}}g_{\mathfrak{q},b}(\zeta)}{g_{\mathfrak{q},b}(\zeta)} + \gamma_{\mathfrak{q}}}. \tag{34}$$

After the $\mathfrak{q}$-logarithmic differentiation of (30) and simple calculation,

$$\frac{\zeta\partial_{\mathfrak{q}}\left(\zeta\partial_{\mathfrak{q}}\mathfrak{f}_{\mathfrak{q},b}(\zeta)\right)}{g_{\mathfrak{q},b}(\zeta)} = p(\zeta).p_1(\zeta) + \zeta\partial_{\mathfrak{q}}p(\zeta), \tag{35}$$

where $p_1(\zeta) = \frac{\zeta\partial_{\mathfrak{q}}g_{\mathfrak{q},b}(\zeta)}{g_{\mathfrak{q},b}(\zeta)}$.

Substituting (35) in (34), we obtain

$$\frac{\zeta\partial_{\mathfrak{q}}\mathfrak{f}(\zeta)}{g(\zeta)} = p(\zeta) + \frac{\zeta\partial_{\mathfrak{q}}p(\zeta)}{p_1(\zeta)+\gamma_{\mathfrak{q}}}. \tag{36}$$

Since $\mathfrak{f} \in K^s_{\mathfrak{q},b}(\mathfrak{h})$, we can rewrite (36) as

$$p(\zeta) + \frac{\zeta\partial_{\mathfrak{q}}p(\zeta)}{p_1(\zeta)+\gamma_{\mathfrak{q}}} \prec \mathfrak{h}(\zeta).$$

From (29), we conclude that $Re(p_1(\zeta)) > 0$ in $\mathbb{U}$ implies $Re\left(\frac{1}{p_1(\zeta)+\gamma_{\mathfrak{q}}}\right) > 0$ in $\mathbb{U}$. Now, we use Lemma 3 to obtain $p(\zeta) \prec \mathfrak{h}(\zeta)$, and consequently, $\frac{\zeta\partial_{\mathfrak{q}}\mathfrak{f}_{\mathfrak{q},b}(\zeta)}{g_{\mathfrak{q},b}(\zeta)} \prec \mathfrak{h}(\zeta)$. Hence, $\mathfrak{f}_{\mathfrak{q},b}(\zeta) \in K^s_{\mathfrak{q},b}(\mathfrak{h})$. □

Upon using similar techniques to those in Theorem 2, the following result can be proven.

Theorem 5. *Let $\mathfrak{f} \in Q^s_{\mathfrak{q},b}(\mathfrak{h})$. Then, $\mathfrak{f}_{\mathfrak{q},b}(\zeta) \in Q^s_{\mathfrak{q},b}(\mathfrak{h})$, where $\mathfrak{f}_{\mathfrak{q},b}(\zeta)$ is defined by (28).*

Remark 1. *In particular, the classes $K^s_{\mathfrak{q},b}(\gamma)$, $Q^s_{\mathfrak{q},b}(\gamma)$, $K^s_{\mathfrak{q},b}$, and $Q^s_{\mathfrak{q},b}$ are invariant under the $\mathfrak{q}$-Bernardi integral operator. Using similar arguments, we can easily show that the classes $\widetilde{K}^b_{\mathfrak{q},s}(\mathfrak{h})$ and $\widetilde{Q}^b_{\mathfrak{q},s}(\mathfrak{h})$ will also be preserved under the $\mathfrak{q}$-Bernardi integral operator.*

3. Conclusions

In this field of study, several subclasses of close-to-convex functions associated with a certain family of linear operators were discussed. The inclusion results remain a very common investigation in such a situation. No one has studied this problem yet, in terms of $\mathfrak{q}$-calculus. Therefore, In this paper, we used the concept of a $\mathfrak{q}$-difference operator to define certain subclasses of univalent functions. Furthermore, various subclasses were introduced and studied by applying the $\mathfrak{q}$-Srivastava–Attiya operator and $\mathfrak{q}$-multiplier transformation operator. We investigated the inclusion results and the integral-preserving property for the newly defined classes. In the future, this work will motivate other authors to contribute in this direction for many generalized subclasses of $\mathfrak{q}$-close-to-convex univalent and multivalent functions.

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