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Article

Sharp Coefficient Bounds for a New Subclass of Starlike Functions of Complex Order γ Associated with Cardioid Domain

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Abstract: In this study, by using the concepts of subordination, we define a new family $\mathcal{R}(M, N, \lambda, \gamma)$ of starlike functions of complex order γ connected with the cardioid domain. The main contribution of this article consists of the derivations of sharp inequality, considering the functions belonging to the family $\mathcal{R}(M, N, \lambda, \gamma)$ of starlike functions in $\mathcal{U}$. Particularly, sharp bounds of the first two Taylor–Maclaurin coefficients, sharp estimates of the Fekete–Szegö-type functionals, and coefficient inequalities are investigated for this newly defined family $\mathcal{R}(M, N, \lambda, \gamma)$ of starlike functions. Furthermore, for the inverse function and the $\log\left(\frac{g(z)}{z}\right)$ function, we investigate the same types of problems. Several well-known corollaries are also highlighted to show the connections between prior research and the new findings.

Keywords: analytic functions; subordination; convex and starlike functions; Fibonacci numbers; shell-like curve; cardioid domain

MSC: 05A30; 30C45; 11B65; 47B38

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1. Introduction and Preliminaries

Suppose $\mathcal{A}$ represents the collection of all analytic functions $g(z)$ in the open unit disc

$$\mathcal{U} = \{z : |z| < 1\},$$

which are normalized by

$$g(0) = 0 \text{ and } g'(0) = 1.$$

Thus, the form given in (1) can be used to express any function $g \in \mathcal{A}$:

$$g(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

The class of functions from $\mathcal{A}$ that are univalent in an open unit disc is denoted by $\mathcal{S}$.

Coefficients of functions, Taylor series representations, and their associated functional inequalities are of major interest in the theory of analytic and univalent functions. The Fekete–Szegö inequality is one of the most significant and useful functional inequalities. There are a number of results that have been proven regarding the maximization of the non-linear functional $\left|a_3 - \mu a_2^2\right|$ or other classes and subclasses of univalent functions, and these type of problems are called Fekete–Szegö problems (see [1]). If $g \in \mathcal{S}$ and it is of the form (1), then

$$\left|a_3 - \mu a_2^2\right| \le \left\{ \begin{array}{ccc} 3-4\mu & \text{if} & \mu \le 0 \\ 1+2\exp\left(\frac{2\mu}{\mu-1}\right) & \text{if} & 0 \le \mu < 1 \\ 4\mu-3 & \text{if} & \mu \ge 1 \end{array} \right\}$$

and the result $\left|a_3 - \mu a_2^2\right|$ is sharp (see [1]). There is a long history of the Fekete–Szegö problem in literature and for complex number μ.

The function g is said to be subordinate to the function f, written symbolically as

$$g(z) \prec f(z), \ \ z \in \mathcal{U},$$

if there exists a function w such that

$$g(z) = f(u(z)), \, z \in \mathcal{U}$$

where $|u(z)| < 1$ and $u(0) = 0, z \in \mathcal{U}$. Furthermore, if the function f is univalent in $\mathcal{U}$, then it follows that $g(0) = f(0)$ and $g(\mathcal{U}) \subset f(\mathcal{U})$.

The area of function theory was established in 1851. This field first gained attention as a potential area for future research in 1916 when Bieberbach [2] investigated the coefficient conjecture. De Branges [3] proved this idea in 1985. Many of the top researchers of the day attempted to prove or disprove this Bieberbach hypothesis between 1916 and 1985. As a result, they found a large number of normalized univalent function subfamilies belonging to class $\mathcal{S}$ that are associated with various image domains. The most fundamental and important subclasses of the set $\mathcal{S}$ are represented by the families of starlike ($\mathcal{S}^*$) and convex ($\mathcal{C}$) functions, respectively.

The familiar class of starlike functions in $\mathcal{U}$, denoted by $\mathcal{S}^*$, consists of function $g \in \mathcal{S}^*$ and satisfies the following condition:

$$Re\left(\frac{zg^{'}(z)}{g(z)}\right) > 0, z \in \mathcal{U}.$$

The class of convex functions in $\mathcal{U}$, denoted by $\mathcal{C}$, consists of function $g \in \mathcal{C}$ and satisfies the following condition

$$1 + Re\left(\frac{zg^{''}(z)}{g^{'}(z)}\right) > 0, \, z \in \mathcal{U}.$$

The above two classes can be written in terms of subordination, as follows:

$$\mathcal{S}^* = \left\{ g \in \mathcal{A} : \frac{zg^{'}(z)}{g(z)} \prec \frac{1+z}{1-z} \right\}$$

and

$$\mathcal{C} = \left\{ g \in \mathcal{A} : 1 + \frac{zg^{''}(z)}{g^{'}(z)} \prec \frac{1+z}{1-z} \right\}.$$

Ma and Minda [4] gave the generalization of $\mathcal{S}^*$ and $\mathcal{C}$ as follows:

$$\mathcal{S}^*(\varphi) = \left\{ g \in \mathcal{A} : \frac{zg^{'}(z)}{g(z)} \prec \varphi(z) \right\}$$

and

$$\mathcal{C}(\varphi) = \left\{ g \in \mathcal{A} : 1 + \frac{zg^{''}(z)}{g^{'}(z)} \prec \varphi(z) \right\},$$

where $\varphi(z)$ is a real part function that is positive and is normalized by the rule

$$\varphi(0) = 1, \text{ and } \varphi^{'}(0) > 0$$

and φ maps $\mathcal{U}$ onto a starlike region with respect to 1 and symmetric with respect to the real axis. For these classes of analytic functions, Ma and Minda [4] discussed a number of particular results, including distortion, growth, and covering theorems. As a special case of class $\mathcal{A}$ of normalized analytic functions, various subfamilies of class $\mathcal{A}$ have been examined recently. Many subfamilies of class $\mathcal{A}$ of normalised analytic functions have been examined recently as a particular instance of the class $\mathcal{S}^*(\varphi)$; for example, Janowski starlike class $\mathcal{S}^*(M, N)$ was investigated in [5], class $\mathcal{S}^*_L$ was studied in [6] by Sokól and Stankiewicz, class $\mathcal{S}^*_{\sin}$ was investigated by Cho et al. [7], class $\mathcal{S}^*(e^z)$ was studied in [8], and $\mathcal{S}^*_{\tan}$ was studied in [9]. For a more recent study about sharp estimates, see the following articles [10–18].

Ravichandran et al. [19] gave an extension of the above two classes in the following way:

$$\mathcal{S}^*(\gamma, \varphi) = \left\{ g \in \mathcal{A} : 1 + \frac{1}{\gamma}\left(\frac{zg^{'}(z)}{g(z)} - 1\right) \prec \varphi(z), \ \ \gamma \in \mathbb{C}\backslash\{0\} \right\}$$

and

$$\mathcal{C}(\gamma, \varphi) = \left\{ g \in \mathcal{A} : 1 + \frac{1}{\gamma}\left(\frac{zg^{''}(z)}{g^{'}(z)}\right) \prec \varphi(z), \ \ \gamma \in \mathbb{C}\backslash\{0\} \right\}.$$

These types of functions are referred to be Ma–Minda starlike and convex functions of order γ, $(\gamma \in \mathbb{C}\backslash\{0\})$, respectively.

The image of $\mathcal{U}$ under every $f \in \mathcal{S}$ contains a disk of radius $\frac{1}{4}$, and thus every $g \in \mathcal{S}$ has an inverse defined as:

$$g^{-1}(g(z)) = z, \ \ z \in \mathcal{U}$$

and

$$g(g^{-1}(w)) = w, \ |w| < r_0(g), \ r_0(g) \geq \frac{1}{4}.$$

The series of g^{-1} is given as:

$$g^{-1}(w) = w + A_2w^2 + A_3w^3 + A_4w^4 ..., \tag{2}$$

where

$$\begin{aligned} A_2 &= -a_2, \\ A_3 &= (2a_2^2 - a_3) \end{aligned} \tag{3}$$

and

$$A_4 = -(5a_2^3 - 5a_2a_3 + a_4).$$

The logarithmic coefficients T_n of a function $g \in \mathcal{S}$ are defined by

$$\log \frac{g(z)}{z} = 2\sum_{n=2}^{\infty} T_n z^n. \tag{4}$$

On the basis of the geometrical interpretation of their image domains, numerous subclasses of analytic functions have been defined and investigated using the concepts of subordination. Some interesting geometrical classes have been defined when the domain is the right half of a plane [20], a circular disc [21],an oval or petal-type domain [22], a conic domain [23,24], a leaf-like domain [25], a generalized conic domain [26], and, most importantly, a shell-like curve [27–30].

The function
$$p(z)=\frac{1+\tau^2z^2}{1-\tau z-\tau^2z^2}, \tag{5}$$
is essential for the shell-like shape, where
$$\tau=\frac{1-\sqrt{5}}{2}.$$

The image of the unit circle under the function p gives the conchoid of Maclaurin, that is,
$$p\left(e^{i\varphi}\right)=\frac{\sqrt{5}}{2(3-2\cos\varphi)}+i\frac{\sin\varphi(4\cos\varphi-1)}{2(3-2\cos\varphi)(1+\cos\varphi)},\ \ 0\le\varphi<2\pi.$$

The function given in (5) has the following series representation:
$$1+\sum_{n=1}^{\infty}(u_{n-1}+u_{n+1})\tau^n z^n,$$
where
$$u_n=\frac{(1-\tau)^n-\tau^n}{\sqrt{5}},$$
where u_n produces a Fibonacci series of coefficient constants that are more closely related to the Fibonacci numbers.

Taking inspiration from the idea of circular disc and shell-like curves, Malik et al. [31] defined new domain for analytic functions named the cardioid domain. A new class of analytic functions is defined, associated with the cardioid domain (for more detail, see [31]).

Definition 1 ([31]). *Assume that $CP[M,N]$ represents the class of functions p defined by the subordination relation*
$$p(z)\prec\widetilde{p}(M,N,z),$$
where $\widetilde{p}(M,N,z)$ is defined by
$$\widetilde{p}(M,N,z)=\frac{2M\tau^2z^2+(M-1)\tau z+2}{2N\tau^2z^2+(N-1)\tau z+2} \tag{6}$$
with $-1<N<M\le 1$ and $\tau=\frac{1-\sqrt{5}}{2}$, $z\in\mathcal{U}$.

The explanation of the function $\widetilde{p}(M,N,z)$ in geometric terms might be helpful in understanding the class $CP[M,N]$. If we denote
$$R_{\widetilde{p}}\left(M,N;e^{i\theta}\right)=u$$
and
$$I_{\widetilde{p}}\left(M,N;e^{i\theta}\right)=v$$
then the image $\widetilde{p}\left(M,N,e^{i\theta}\right)$ of the unit circle is a cardioid-like curve defined by
$$\begin{aligned} u &= \frac{4+(M-1)(N-1)\tau^2+4MN\tau^4+2\lambda\cos\theta+4(M+N)\tau^2\cos2\theta}{4+(N-1)^2\tau^2+4N^2\tau^4+4(N-1)\left(\tau+N\tau^3\right)\cos\theta+8N\tau^2\cos2\theta}, \\ v &= \frac{(M-N)\left(\tau-\tau^3\right)\sin\theta+2\tau^2\sin2\theta}{4+(N-1)^2\tau^2+4N^2\tau^4+4(N-1)\left(\tau+N\tau^3\right)\cos\theta+8N\tau^2\cos2\theta}, \end{aligned} \tag{7}$$

where

$$\lambda = (M+N-2)\tau + (2MN-M-N)\tau^3, \quad -1 < N < M \leq 1, \tau = \frac{1-\sqrt{5}}{2}, \, 0 \leq \theta \leq 2\pi.$$

Moreover, we observe that

$$\widetilde{p}(M,N,0) = 1 \text{ and } \widetilde{p}(M,N,1) = \frac{MN + 9(M+N) + 1 + 4(N-M)\sqrt{5}}{N^2 + 18N + 1}.$$

According to (7), the cusp of the cardioid-like curve is provided by

$$\zeta(M,N) = \widetilde{p}\left(M,N;e^{\pm i \arccos\left(\frac{1}{4}\right)}\right) = \frac{2MN - 3(M+N) + 2 + (M-N)\sqrt{5}}{2(N^2 - 3N + 1)}.$$

The image of each inner circle is a nested cardioid-like curve if the open unit disc $\mathcal{U}$ is considered a collection of concentric circles with the origin at the center. As a result, the open unit disc $\mathcal{U}$ is mapped onto a cardioid region by the function $\widetilde{p}(M,N,z)$. This means that $\widetilde{p}(M,N;\mathcal{U})$ is a cardioid domain. See [31] for a graphical study of the geometry of the cardioid domain.

The recent paper [31] inspired us to adopt this strategy to define a new subclass of generalized subordinate functions of complex order γ associated with the cardioid domain.

Definition 2. *Let the function g of the form (1) be in the class $\mathcal{R}(M,N,\lambda,\gamma)$ if the following conditions are satisfied:*

$$1 + \frac{1}{\gamma}\left(\frac{zg'(z) + \lambda z^2 g''(z)}{(1-\lambda)g(z) + \lambda z g'(z)} - 1\right) \prec \widetilde{p}(M,N;z),$$

where $0 \leq \lambda \leq 1$, $\gamma \in \mathbb{C}\backslash\{0\}$, and $\widetilde{p}(M,N;z)$ is given by (6).

Alternatively, $g \in \mathcal{R}(M,N,\lambda,\gamma)$ when the function

$$1 + \frac{1}{\gamma}\left(\frac{zg'(z) + \lambda z^2 g''(z)}{(1-\lambda)g(z) + \lambda z g'(z)} - 1\right)$$

takes its values from the cardioid domain $\widetilde{p}(M,N;z)$.

Definition 3. *Let the function g of the form (1) be in the class $\mathcal{R}(M,N,\gamma)$ if the following conditions are satisfied:*

$$1 + \frac{1}{\gamma}\left(\frac{zg''(z)}{g'(z)}\right) \prec \widetilde{p}(M,N;z),$$

where $\gamma \in \mathbb{C}\backslash\{0\}$ and $\widetilde{p}(M,N;z)$ is given by (6), and $\mathcal{R}(M,N,\gamma)$ is the class of convex functions of order γ related to the cardioid domain.

Definition 4. *Let the function g of the form (1) be in the class $\mathcal{C}(M,N)$ if the following conditions are satisfied:*

$$1 + \frac{zg''(z)}{g'(z)} \prec \widetilde{p}(M,N;z),$$

where $(M,N;z)$ is given by (6) and $\mathcal{R}(M,N)$ is the class of convex functions related to the cardioid domain.

Remark 1. *For $\gamma = 1$ and $\lambda = 0$ in Definition (2), we obtained known class $\mathcal{R}(M,N)$ of starlike functions associated with the cardioid domain proven by Malik et al. in [32].*

Remark 2. *For $M = 1, N = -1, \gamma = 1,$ and $\lambda = 0$ in Definition (2), then $\mathcal{R}(M, N, \lambda, \gamma) = SL$ and this class is defined as starlike functions associated with Fibonacci numbers, introduced and studied by Sokół in [30].*

Remark 3. *For $M = 1, N = -1, \gamma = 1,$ and $\lambda = 1$ in Definition (2), then $(M, N, \lambda, \gamma) = C$, and this family is defined as a class of convex functions connected with Fibonacci numbers.*

2. Set of Lemmas

By utilizing the following lemmas, we will demonstrate our findings.

Lemma 1 ([31]). *Let the function $\widetilde{p}(M, N; z)$ be defined by (6). Then:*

(i) For the disc $|z| < \tau^2$, the function $\widetilde{p}(M, N; z)$ is univalent.
(ii) If $p(z) \prec \widetilde{p}\,(M, N; z)$, then $Rep(z) > \alpha$, where

$$\alpha = \frac{2(M + N - 2)\tau + 2(2MN - M - N)\tau^3 + 16(M + N)\tau^2\eta}{4(N - 1)(\tau + N\tau^3) + 32N\tau^2\eta},$$

where

$$\eta = \frac{4 + \tau^2 - N^2\tau^2 - 4N^2\tau^4 - (1 - N\tau^2)\chi(N)}{4\tau(1 + N^2\tau^2)},$$

$$\chi(N) = \sqrt{5(2N\tau^2 - (N - 1)\tau + 2)(2N\tau^2 + (N - 1)\tau + 2)}$$

and

$$-1 < N < M \leq 1, \text{ and } \tau = \frac{1 - \sqrt{5}}{2}.$$

(iii) If $\widetilde{p}(M, N; z) = 1 + \sum\limits_{n=1}^{\infty} \widetilde{p}_n z^n$, then

$$\widetilde{p}_n = \begin{cases} (M - N)\frac{\tau}{2} & \text{for } n = 1, \\ (M - N)(5 - N)\frac{\tau^2}{2^2} & \text{for } n = 2, \\ \frac{1-N}{2}\tau p_{n-1} - N\tau^2 p_{n-2} & \text{for } n = 3, 4, 5, \ldots \end{cases} \tag{8}$$

where

$$-1 < N < M \leq 1.$$

(iv) Let $p(z) \prec \widetilde{p}(M, N; z)$ and be of the form $p(z) = 1 + \sum\limits_{n=1}^{\infty} p_n z^n$. Then,

$$\left|p_2 - v p_1^2\right| \leq \frac{(M - N)|\tau|}{4} \max\{2, |\tau(v(M - N) + N - 5)|\}, \quad v \in \mathbb{C}. \tag{9}$$

Lemma 2 ([33]). *Let $p \in P$, such that $p(z) = 1 + \sum\limits_{n=1}^{\infty} c_n z^n$. Then,*

$$\left|c_2 - \frac{v}{2}c_1^2\right| \leq \max\{2, 2|v - 1|\} = \left\{ \begin{array}{cc} 2 & \mathit{if}\ 0 \leq v \leq 2, \\ 2|v - 1|, & \mathit{elsewhere} \end{array} \right\} \tag{10}$$

and

$$|c_n| \leq 2, \ \mathit{for} \ \ n \geq 1. \tag{11}$$

Lemma 3 ([34]). *Let the function g given by*

$$g(z) = \sum_{n=1}^{\infty} b_n z^n$$

be convex in $\mathcal{U}$. Also let the function f given by

$$f(z) = \sum_{n=1}^{\infty} a_n z^n$$

be analytic in $\mathcal{U}$. If

$$f(z) \prec g(z),$$

then

$$|a_n| < |b_1|, \quad n = 1, 2, 3 \ldots$$

Motivated by the recent studies of starlike functions associated with the cardioid domain [31,32], we define a class of generalized subordinate functions of complex order γ connected with cardioid domains. We investigate sharp coefficient estimates of Taylor series and Fekete-Szegő problems for certain generalized subordinate functions of complex order γ associated with the cardioid domain. Additionally, similar problems are discovered for the inverse function and for $\log \frac{g(z)}{z}$.

3. Main Results

In this section, the Taylor–Maclaurin initial coefficients for the functions belonging to $\mathcal{R}(M, N, \lambda, \gamma)$ are computed.

Theorem 1. *Let $g \in \mathcal{R}(M, N, \lambda, \gamma)$ be given by (1), $-1 \leq N < M \leq 1$. Then*

$$\begin{aligned} |\rho_2| &\leq \frac{|\gamma|(M-N)|\tau|}{2(1+\lambda)}, \\ |\rho_3| &\leq \frac{|\gamma|(M-N)|\tau|^2}{8(1+2\lambda)} \left\{ \frac{\gamma M}{(1+\lambda)^2} - \frac{(\gamma+1)N}{(1+\lambda)^2} + 5 \right\}. \end{aligned}$$

These findings are sharp.

Proof. Let $g \in \mathcal{R}(M, N, \gamma, \lambda)$ and be of the form (1). Then,

$$1 + \frac{1}{\gamma} \left(\frac{zg'(z) + \lambda z^2 g''(z)}{(1-\lambda)g(z) + \lambda z g'(z)} - 1 \right) \prec \widetilde{p}(M, N; z), \tag{12}$$

where

$$\widetilde{p}(M, N, z) = \frac{2M\tau^2 z^2 + (M-1)\tau z + 2}{2N\tau^2 z^2 + (N-1)\tau z + 2}.$$

If we write

$$D(g(z)) = (1-\lambda)g(z) + \lambda z g'(z)$$

then Equation (12) becomes

$$1 + \frac{1}{\gamma} \left(\frac{zD'(g(z))}{D(g(z))} - 1 \right) \prec \widetilde{p}(M, N; z)$$

and

$$a_n = (1 + \lambda(n-1))\rho_n \tag{13}$$

for

$$D(g(z)) = z + \sum_{n=2}^{\infty} a_n z^n.$$

By applying the concept of subordination, there exists a function u with

$$u(0) = 0 \text{ and } |u(z)| < 1$$

such that

$$1 + \frac{1}{\gamma}\left(\frac{zD'(g(z))}{D(g(z))} - 1\right) = \widetilde{p}(M, N; u(z)). \tag{14}$$

Let

$$\begin{aligned} u(z) &= \frac{p(z)-1}{p(z)+1} \\ &= \frac{c_1 z + c_2 z^2 + c_3 z^3 + \ldots}{2 + c_1 z + c_2 z^2 + \ldots} \\ &= \frac{1}{2}c_1 z + \frac{1}{2}\left(c_2 - \frac{1}{2}c_1^2\right)z^2 + \frac{1}{2}\left(c_3 - c_1 c_2 + \frac{1}{4}c_1^3\right)z^3 + \cdots . \end{aligned} \tag{15}$$

Since $\widetilde{p}(M, N; z) = 1 + \sum\limits_{n=1}^{\infty} \widetilde{p}_n z^n$, then

$$\begin{aligned} &\widetilde{p}(M, N; u(z)) \\ &= 1 + \widetilde{p}_1\left\{\frac{1}{2}c_1 z + \frac{1}{2}\left(c_2 - \frac{1}{2}c_1^2\right)z^2 \ldots\right\} + \widetilde{p}_2\left\{\frac{1}{2}c_1 z + \frac{1}{2}\left(c_2 - \frac{1}{2}c_1^2\right)z^2 \ldots\right\} + \ldots \\ &= 1 + \frac{\widetilde{p}_1 c_1}{2}z + \left(\frac{1}{2}\left(c_2 - \frac{1}{2}c_1^2\right)\widetilde{p}_1 + \frac{\widetilde{p}_2 c_1^2}{4}\right)z^2 + \ldots. \end{aligned} \tag{16}$$

Also consider the function

$$\widetilde{p}(M, N; z) = \frac{2M\tau^2 z^2 + (M-1)\tau z + 2}{2N\tau^2 z^2 + (N-1)\tau z + 2}.$$

Letting $\tau z = \beta$, then

$$\begin{aligned} \widetilde{p}(M, N, z) &= \frac{2M\beta_0^2 + (M-1)\beta + 2}{2N\beta_0^2 + (N-1)\beta + 2} \\ &= \frac{M\beta_0^2 + \frac{(M-1)}{2}\beta + 1}{N\beta_0^2 + \frac{(N-1)}{2}\beta + 1} \\ &= \left(M\beta_0^2 + \frac{(M-1)}{2}\beta + 1\right)\left[1 + \frac{1}{2}(1-N)\beta + \left(\frac{N^2 - 6N + 1}{4}\right)\beta_0^2 + \ldots\right] \\ &= 1 + \frac{1}{2}(M-N)\beta + \frac{1}{4}(M-N)(5-N)\beta_0^2 + \ldots. \end{aligned}$$

This implies that

$$\widetilde{p}(M, N; z) = 1 + \frac{1}{2}(M-N)\tau z + \frac{1}{4}(M-N)(5-N)\tau^2 z^2 + \ldots. \tag{17}$$

It is simple to observe from (16) that

$$\begin{aligned} &\widetilde{p}(M, N; u(z)) \\ &= 1 + \frac{1}{4}(M-N)\tau c_1 z + \left(\frac{1}{4}(M-N)\tau\left(c_2 - \frac{1}{2}c_1^2\right) + \frac{(M-N)(5-N)\tau^2 c_1^2}{16}\right)z^2 + \ldots \end{aligned} \tag{18}$$

Since $g \in \mathcal{R}(M, N, \lambda, \gamma)$, then

$$1 + \frac{1}{\gamma}\left(\frac{zD'(g(z)}{D(g(z)} - 1\right) = 1 + \frac{1}{\gamma}a_2 z + \frac{1}{\gamma}\left(2a_3 - a_2^2\right)z^2 + \ldots. \quad (19)$$

It is simple to show that by utilizing (14) and comparing the coefficients from (18) and (19), we obtain

$$a_2 = \frac{\gamma(M - N)\tau c_1}{4} \quad (20)$$

or, using (13), we obtain

$$\rho_2 = \frac{\gamma(M - N)\tau c_1}{4(1 + \lambda)}.$$

Applying the modulus on both side, we have

$$|\rho_2| \leq \frac{|\gamma|(M - N)\tau}{2(1 + \lambda)}.$$

Now, comparing the coefficients from (18) and (19) again, we have

$$\frac{2}{\gamma}a_3 = \frac{1}{4}(M - N)\tau\left(c_2 - \frac{1}{2}c_1^2\right) + \frac{(M - N)(5 - N)\tau^2 c_1^2}{16} + \frac{1}{\gamma}a_2^2,$$

$$\begin{aligned} a_3 &= \frac{\gamma(M - N)\tau c_2}{8} - \frac{\gamma(M - N)\tau}{8}\frac{c_1^2}{2} + \frac{\gamma(M - N)\tau^2}{32}\left\{\frac{\gamma M}{(1 + \lambda)^2} - \frac{(\gamma + 1)N}{(1 + \lambda)^2} + 5\right\} \\ &= \frac{\gamma(M - N)\tau}{8}\left\{c_2 - \frac{v}{2}c_1^2\right\}. \quad (21) \end{aligned}$$

By using (13), we obtain

$$\rho_3 = \frac{\gamma(M - N)\tau}{8(1 + 2\lambda)}\left\{c_2 - \frac{v}{2}c_1^2\right\},$$

where

$$v = 1 - \frac{\tau}{2}\left\{\frac{\gamma M}{(1 + \lambda)^2} - \left(\frac{\gamma}{(1 + \lambda)^2} + 1\right)N + 5\right\}.$$

This shows that $v > 2$ and is satisfied by the relation $M > N$. Hence, by applying Lemma 2, we obtain the required result.

The result is sharp for

$$1 + \frac{1}{\gamma}\left(\frac{zD'(g(z))}{D(g(z))} - 1\right) = 1 + \frac{(M - N)\tau}{2}z + \frac{(M - N)(5 - N)\tau^2}{4}z^2 + \ldots.$$

□

Taking the special values in Theorem 1, we have the following example.

Example 1. *Let* $\gamma = 1$, $M = 0.6$, $N = 0.4$, $\tau = \frac{1-\sqrt{5}}{2}$, $\lambda = 0.7$, *and* $g \in \mathcal{R}(0.6, 0.4, 0.7, 1)$. *Then,*

$$\begin{aligned} |\rho_2| &\leq 0.095177 \\ |\rho_3| &\leq 0.134466. \end{aligned}$$

Taking $\gamma = 1$ and $\lambda = 0$ in Theorem 1, we obtain the known corollary proven in [35] for starlike functions connected to the cardioid domain.

Corollary 1 ([35])**.** *Let $g \in \mathcal{R}(M, N)$ be given by (1), $-1 \leq N < M \leq 1$. Then,*

$$\begin{aligned} |a_2| &\leq \frac{(M-N)|\tau|}{2}, \\ |a_3| &\leq \frac{(M-N)|\tau|^2}{8}\{M - 2N + 5\}. \end{aligned}$$

Taking $\lambda = 1$ in Theorem 1, we obtain the new result for convex functions of complex order γ connected with the cardioid domain.

Theorem 2. *Let $g \in \mathcal{R}(M, N, \gamma)$ be given by (1), $-1 \leq N < M \leq 1$. Then,*

$$\begin{aligned} |a_2| &\leq \frac{|\gamma|(M-N)|\tau|}{4}, \\ |a_3| &\leq \frac{|\gamma|(M-N)|\tau|^2}{24}\left\{\frac{\gamma M}{4} - \frac{(\gamma+1)N}{4} + 5\right\}. \end{aligned}$$

Taking $\gamma = 1$ and $\lambda = 1$ in Theorem 1, we obtain the new result for a class of convex functions related with the cardioid domain.

Theorem 3. *Let $g \in \mathcal{C}(M, N)$ be given by (1), $-1 \leq N < M \leq 1$. Then,*

$$\begin{aligned} |a_2| &\leq \frac{(M-N)}{4}, \\ |a_3| &\leq \frac{(M-N)|\tau|^2}{24}\left\{\frac{M}{4} - \frac{N}{2} + 5\right\}. \end{aligned}$$

Theorem 4. *Let $g \in \mathcal{R}(M, N, \lambda, \gamma)$ and be of the form (1). Then,*

$$\begin{aligned} &\left|\rho_3 - \mu\rho_2^2\right| \\ &\leq \frac{|\gamma|(M-N)|\tau|}{8(1+2\lambda)} \max\left\{2, \left|\tau\left(-(M - 2N + 5) + \frac{2\gamma(1+2\lambda)(M-N)\mu}{(1+\lambda)^2}\right)\right|\right\}. \end{aligned}$$

This result is sharp.

Proof. Since $g \in \mathcal{R}(M, N, \lambda, \gamma)$, we have

$$1 + \frac{1}{\gamma}\left(\frac{zD'(g(z))}{D(g(z)} - 1\right) = \widetilde{p}(M, N; u(z)),\ z \in \mathcal{U},$$

where u is a Schwarz function such that $u(0)$ and $|u(z)| < 1$ in $\mathcal{U}$. Therefore,

$$z + 2a_2z^2 + 3a_3z^3 + ... = \gamma\left\{z + a_2z^2 + a_3z^3 + ...\right\}\left\{1 + p_1z + p_2z^2 + \dots\right\}.$$

Comparing the coefficients of both sides, we obtain

$$a_2 = \gamma p_1, \quad 2a_3 = \gamma(p_1a_2 + p_2).$$

Using (13), we obtain

$$a_2 = (1+\lambda)\rho_2 = \gamma p_1, \quad 2a_3 = 2(1+2\lambda)\rho_3 = \gamma\left(\gamma p_1^2 + p_2\right),$$

or

$$\rho_2 = \frac{\gamma p_1}{(1+\lambda)}, \quad \rho_3 = \frac{\gamma}{2(1+2\lambda)}\left(\gamma p_1^2 + p_2\right).$$

This implies that

$$\begin{aligned}\left|\rho_3 - \mu\rho_2^2\right| &= \frac{\gamma}{2(1+2\lambda)}\left|p_2 + \left(1 - \mu\frac{2(1+2\lambda)}{(1+\lambda)^2}\right)\gamma p_1^2\right| \\ &= \frac{\gamma}{2(1+2\lambda)}\left|p_2 - v p_1^2\right|,\end{aligned}$$

where

$$v = \left(\mu\frac{2(1+2\lambda)}{(1+\lambda)^2} - 1\right)\gamma.$$

By using (iv) of Lemma 1 for $v = \left(\mu\frac{2(1+2\lambda)}{(1+\lambda)^2} - 1\right)\gamma$, we obtain the required result. The equality

$$\left|\rho_3 - \mu\rho_2^2\right| = \frac{|\gamma|(M-N)|\tau|^2}{8(1+2\lambda)}\left|M - 2N + 5 - \frac{2\gamma(1+2\lambda)(M-N)\mu}{(1+\lambda)^2}\right|$$

holds for

$$g_*(z) = z + \frac{\tau}{2}(M-N)z^2 + \frac{\tau^2}{8}(M-N)(M-2N+5)z^3 + \ldots$$

Now consider that the function $g_0 : \mathcal{U} \to C$ is defined as:

$$g_0(z) = z\exp\int_0^z \frac{\widetilde{p}(M,N;t^2) - 1}{t}dt = z + \frac{\tau}{2}(M-N)z^3 + \ldots, \tag{22}$$

where $\widetilde{p}(M,N;z)$ is defined in (6). Hence, it is obvious that $g_0(0) = 0$ and $g_0'(0) = 1$ and

$$1 + \frac{1}{\gamma}\left(\frac{zD'(g_0(z))}{D(g_0(z))} - 1\right) = \widetilde{p}\left(M,N;t^2\right).$$

This demonstrates $g_0 \in \mathcal{R}(M,N,\lambda,\gamma)$. Hence, the equality

$$\left|\rho_3 - \mu\rho_2^2\right| = \frac{|\gamma|(M-N)|\tau|}{2(1+2\lambda)}$$

holds for the function g_0 given in (22). □

Taking $\lambda = 1$ in Theorem 4, we obtain the new result for convex functions of complex order γ associated with the cardioid domain.

Theorem 5. *Let $g \in \mathcal{R}(M,N,\gamma)$ and be of the form (1). Then,*

$$\left|a_3 - \mu a_2^2\right| \leq \frac{|\gamma|(M-N)|\tau|}{24}\max\left\{2, \left|\tau\left(-(M-2N+5) + \frac{3\gamma(M-N)\mu}{2}\right)\right|\right\}.$$

This result is sharp.

Taking $\gamma = 1$ and $\lambda = 1$ in Theorem 1, we obtain the new result for a class of convex functions associated with the cardioid domain.

Theorem 6. *Let $g \in C(M,N)$ and be of the form (1). Then,*

$$\left|a_3 - \mu a_2^2\right| \leq \frac{(M-N)|\tau|}{24} \max\left\{2, \left|\tau\left(-(M-2N+5) + \frac{3(M-N)\mu}{2}\right)\right|\right\}.$$

This result is sharp.

Taking $\gamma = 1$ and $\lambda = 0$ in Theorem 1, we obtain the known corollary proven in [35] for starlike functions associated with the cardioid domain.

Corollary 2 ([35]). *Let $g \in \mathcal{R}(M,N)$ and be of the form (1). Then,*

$$\left|a_3 - \mu a_2^2\right| \leq \frac{(M-N)|\tau|}{8} \max\{2, |\tau(-(M-2N+5) + 2(M-N)\mu)|\}.$$

This result is sharp.

Coefficient inequality for the class $\mathcal{R}(M,N,\lambda,\gamma)$.

Theorem 7. *Function $g \in \mathcal{A}$ is given by (1). If $g \in \mathcal{R}(M,N,\lambda,\gamma)$, then*

$$|\rho_n| \leq \frac{\prod_{k=2}^{n}\left(k - 2 + \frac{\left|\gamma\left((L-N)\frac{\tau}{2}\right)\right|}{1+\lambda(k-2)}\right)}{(1+\lambda(n-1))(n-1)!}, \ (n \in \mathbb{N}).$$

Proof. Suppose $g \in \mathcal{R}(M,N,\lambda,\gamma)$ and the function $q(z)$ is defined by

$$\begin{aligned} q(z) &= 1 + \frac{1}{\gamma}\left(\frac{zg'(z) + \lambda z^2 g''(z)}{(1-\lambda)g(z) + \lambda z g'(z)} - 1\right) \\ &= 1 + \frac{1}{\gamma}\left(\frac{zD'(g(z))}{D(g(z))} - 1\right) \end{aligned} \tag{23}$$

and

$$a_n = (1 + \lambda(n-1))\rho_n \tag{24}$$

for

$$D(g(z)) = z + \sum_{n=2}^{\infty} a_n z^n.$$

Then, by Definition 2, we have

$$q(z) \prec \widetilde{p}(L,N;z),$$

where $\gamma \in \mathbb{C}\backslash\{0\}$ and $\widetilde{p}(L,N;z)$ is given by (6). Hence, applying the Lemma 3, we obtain

$$\left|\frac{q^{(m)}(0)}{m!}\right| = |c_m| \leq \left|\overline{Q}_1\right|, \quad m \in \mathbb{N}, \tag{25}$$

where

$$q(z) = 1 + c_1 z + c_2 z^2 + \ldots$$

and by (8), we have

$$D(g(z)) = z + \sum_{n=2}^{\infty} a_n z^n.$$

$$|\widetilde{p}_1| = \left|(M-N)\frac{\tau}{2}\right|. \tag{26}$$

Also from (23), we find

$$zD^{'}(g(z)) = \{\gamma[q(\varsigma) - 1] + 1\}D(g(z)). \tag{27}$$

Since $a_1 = 1$, in view of (27), we obtain

$$(n-1)a_n = \gamma\{c_{n-1} + c_{n-2}a_2 + ... + c_1 a_{n-1}\} = \gamma \sum_{i=1}^{n-1} c_i a_{n-i}. \tag{28}$$

Applying (25) into (28), we obtain

$$(n-1)|a_n| \leq |\gamma||\widetilde{p}_1| \sum_{i=1}^{n-1} |a_{n-i}|, \quad n \in \mathbb{N}. \tag{29}$$

Using (24) in (29), we have

$$(n-1)(1+\lambda(n-1))\rho_n \leq |\gamma||\widetilde{p}_1| \sum_{i=1}^{n} |a_{n-i}|, \quad n \in \mathbb{N}.$$

For $n = 2, 3, 4$, we have

$$\begin{aligned} |\rho_2| &\leq \frac{|\gamma\widetilde{p}_1|}{1+\lambda}, \\ |\rho_3| &\leq \frac{|\gamma\widetilde{p}_1|}{2(1+2\lambda)}(1+|a_2|) \\ &\leq \frac{|\gamma\widetilde{p}_1|}{2(1+2\lambda)}\left(1+\frac{|\gamma\widetilde{p}_1|}{1+\lambda}\right) \end{aligned}$$

and

$$\begin{aligned} |\rho_4| &\leq \frac{|\gamma\widetilde{p}_1|}{3(1+3\lambda)}(1+|\rho_2|+|\rho_3|) \\ &\leq \frac{|\gamma\widetilde{p}_1|}{3(1+3\lambda)}\left(1+\frac{|\gamma\widetilde{p}_1|}{1+\lambda}+\frac{|\gamma\widetilde{p}_1|}{2(1+2\lambda)(1+\lambda)}\left(1+\frac{|\gamma\widetilde{p}_1|}{1+\lambda}\right)\right) \\ &= \frac{|\gamma\widetilde{p}_1|}{6(1+3\lambda)}\left(\left(2+\frac{|\gamma\widetilde{p}_1|}{1+2\lambda}\right)\left(1+\frac{|\gamma\widetilde{p}_1|}{1+\lambda}\right)\right). \end{aligned}$$

Applying the Equality (26) and the mathematical induction principle, we arrive at

$$\begin{aligned} |\rho_n| &\leq \frac{\prod_{k=2}^{n}\left((k-2)+\frac{|\gamma\widetilde{p}_1|}{1+\lambda(k-2)}\right)}{(1+\lambda(n-1))(n-1)!} \\ &= \frac{\prod_{k=2}^{n}\left(k-2+\frac{\left|\gamma\left((L-N)\frac{\tau}{2}\right)\right|}{1+\lambda(k-2)}\right)}{(1+\lambda(n-1))(n-1)!}. \end{aligned}$$

This completes the proof of Theorem 7. $\square$

3.1. Inverse Coefficients

Theorem 8. *Let $g \in \mathcal{R}(M, N, \lambda, \gamma)$ be given by (1), and let g^{-1} have the coefficients of the form (2) $-1 \leq N < M \leq 1$. Then,*

$$|A_2| \leq \frac{|\gamma|(M-N)|\tau|}{2(1+\lambda)} \tag{30}$$

and

$$|A_3| \leq \frac{|\gamma|(M-N)|\tau|}{8(1+2\lambda)} \max\left\{2, \tau\left|\left(3\gamma - \frac{\gamma}{(1+\lambda)^2}\right)M - \left(2\gamma - \frac{1}{(1+\lambda)^2}\right)N - 5\right|\right\}. \quad (31)$$

The result is sharp.

Proof. Let $g \in \mathcal{R}(M, N, \lambda, \gamma)$, which is of the form (1). Then, using (20) and (21), we have

$$a_2 = \frac{(1+\lambda)}{\gamma}\rho_2 = \frac{(M-N)\tau c_1}{4} \quad (32)$$

and

$$a_3 = \frac{\gamma(M-N)\tau c_2}{8} - \frac{\gamma(M-N)\tau}{8}\frac{c_1^2}{2} + \frac{\gamma(M-N)\tau^2}{32} \times\left\{\frac{\gamma M}{(1+\lambda)^2} - \left(\frac{\gamma}{(1+\lambda)^2} + 1\right)N + 5\right\}. \quad (33)$$

Since $g(g^{-1})(z) = w$, it is simple to show that using (2),

$$A_2 = -a_2. \quad (34)$$

By solving (32) and (34), we have

$$|A_2| \leq \frac{|\gamma|(M-N)|\tau|}{2(1+\lambda)}$$

and from (3), we have

$$A_3 = 2a_2^2 - a_3. \quad (35)$$

Putting (32) and (33) in (35), we obtain

$$|A_3| = \frac{|\gamma|(M-N)|\tau|}{8(1+2\lambda)}\left|c_2 - \frac{1}{2}V_0 c_1^2\right|,$$

where

$$V_0 = 1 - \frac{\tau}{2}\left(5 + \left(\frac{\gamma}{(1+\lambda)^2} - 3\gamma\right)M + \left(2\gamma - \frac{1}{(1+\lambda)^2}\right)N\right).$$

Hence, by using Lemma 2, we have

$$|A_3| \leq \frac{|\gamma|(M-N)|\tau|}{8(1+2\lambda)} \max\left\{2, \tau\left|\left(3\gamma - \frac{\gamma}{(1+\lambda)^2}\right)M - \left(2\gamma - \frac{1}{(1+\lambda)^2}\right)N - 5\right|\right\}.$$

Hence, the required result is proved.
The results (30) and (31) are sharp for the functions

$$g_*(z) = z + \frac{\tau}{2}(M-N)z^2 + \frac{\tau^2}{8}(M-N)(M-2N+5)z^3 + \ldots$$

The result

$$|A_3| \leq \frac{|\gamma|(M-N)|\tau|}{4(1+2\lambda)}$$

is sharp for the function given in (22). □

Taking $\gamma = 1$ and $\lambda = 0$ in Theorem 8, we obtain the known corollary proven in [35] for starlike functions associated with the cardioid domain.

Corollary 3 ([35]). *Let $g \in \mathcal{R}(M, N, 0, 1)$ be given by (1), and let g^{-1} have the coefficients of the form (2), $-1 \le N < M \le 1$. Then,*

$$|A_2| \le \frac{(M-N)|\tau|}{2}$$

and

$$|A_3| \le \frac{(M-N)|\tau|}{8} \max\{2, \tau|2M - 2N - 5|\}.$$

Taking $\lambda = 1$ in Theorem 8, we obtain the new result for convex functions of complex order γ associated with the cardioid domain.

Theorem 9. *Let $g \in \mathcal{R}(M, N, \gamma)$ be given by (1), and let g^{-1} have the coefficients of the form (2), $-1 \le N < M \le 1$. Then,*

$$|A_2| \le \frac{|\gamma|(M-N)|\tau|}{4}$$

and

$$|A_3| \le \frac{|\gamma|(M-N)|\tau|}{24} \max\left\{2, \tau\left|\left(3\gamma - \frac{\gamma}{4}\right)M - \left(2\gamma - \frac{1}{4}\right)N - 5\right|\right\}.$$

Taking $\gamma = 1$ and $\lambda = 1$ in Theorem 8, we obtain the new result for a class of convex order associated with the cardioid domain.

Theorem 10. *Let $g \in \mathcal{C}(M, N,)$ be given by (1), and let g^{-1} have the coefficients of the form (2), $-1 \le N < M \le 1$. Then,*

$$|A_2| \le \frac{(M-N)|\tau|}{4}$$

and

$$|A_3| \le \frac{(M-N)|\tau|}{24} \max\left\{2, \tau\left|\frac{11}{4}M - \frac{7}{4}N - 5\right|\right\}.$$

Theorem 11. *Let $g \in \mathcal{R}(M, N, \lambda, \gamma)$ and be of the form (1), and let g^{-1} have the coefficients of the form (2) $-1 \le N < M \le 1$. Then, for complex numbers μ and $|z| < \tau^2$:*

$$\begin{aligned}
&\left|A_3 - \mu A_2^2\right| \\
\le\ & \frac{|\gamma|(M-N)|\tau|}{8(1+2\lambda)} \max\left\{2, \left|\tau\left(\frac{\gamma}{1+\lambda}\left(\frac{(4-2\mu)(1+2\lambda)}{(1+\lambda)} - 1\right)(M-N) + N - 5\right)\right|\right\}.
\end{aligned}$$

The result is sharp.

Proof. Let $g \in \mathcal{R}(M, N, \lambda, \gamma)$. Then, using (20) and (21), we have

$$a_2 = \frac{(1+\lambda)}{\gamma}\rho_2 = \frac{(M-N)\tau c_1}{4} \tag{36}$$

and

$$\begin{aligned}
a_3 \quad = \quad & \frac{\gamma(M-N)\tau c_2}{8} - \frac{\gamma(M-N)\tau}{8}\frac{c_1^2}{2} + \frac{\gamma(M-N)\tau^2}{32} \\
& \times\left\{\frac{\gamma M}{(1+\lambda)^2} - \left(\frac{\gamma}{(1+\lambda)^2} + 1\right)N + 5\right\}.
\end{aligned} \tag{37}$$

Since $g(g^{-1})(z) = w$, it is simple to show that using (2),

$$A_2 = -a_2. \tag{38}$$

By solving (36) and (38), we have

$$A_2 = -\frac{\gamma(M-N)\tau c_1}{4(1+\lambda)} \tag{39}$$

and from (3), we have

$$A_3 = 2a_2^2 - a_3. \tag{40}$$

Therefore, by using $a_2 = \gamma p_1$ and $2a_3 = \gamma(p_1 a_2 + p_2)$, one can write

$$\left|A_3 - \mu A_2^2\right| = \frac{|\gamma|}{2(1+2\lambda)}\left|p_2 - \frac{\gamma}{1+\lambda}\left(\frac{(4-2\mu)(1+2\lambda)}{(1+\lambda)} - 1\right)p_1^2\right|.$$

Hence, by applying Lemma 1, part (iv), for

$$v = \frac{\gamma}{1+\lambda}\left(\frac{(4-2\mu)(1+2\lambda)}{(1+\lambda)} - 1\right),$$

we obtain the required result

$$\left|A_3 - \mu A_2^2\right| \leq \frac{|\gamma|(M-N)|\tau|}{8(1+2\lambda)}\max\left\{2, \left|\tau\left(\begin{array}{c}\frac{\gamma}{1+\lambda}\left(\frac{(4-2\mu)(1+2\lambda)}{(1+\lambda)} - 1\right)\\ \times(M-N) + N - 5\end{array}\right)\right|\right\}.$$

The Theorem 11 is sharp for the functions

$$g_*(z) = z + \frac{\tau}{2}(M-N)z^2 + \frac{\tau^2}{8}(M-N)(M-2N+5)z^3 + \ldots$$

and for the function given in (22). □

Taking $\gamma = 1$ and $\lambda = 0$ in Theorem 11, we obtain the known corollary proven in [35] for starlike functions associated with the cardioid domain.

Theorem 12 ([35])**.** *Let $g \in \mathcal{R}(M, N, \lambda, \gamma)$ and be of the form (1), and let g^{-1} have the coefficients of the form (2) $-1 \leq N < M \leq 1$. Then, for complex numbers μ and $|z| < \tau^2$.*

$$\left|A_3 - \mu A_2^2\right| \leq \frac{(M-N)|\tau|}{8}\max\{2, |\tau(3M-2N-5) - 2\mu(M-N)|\}.$$

Taking $\lambda = 1$ in Theorem 11, we obtain the new result for convex functions of complex order γ associated with the cardioid domain.

Theorem 13. *Let $g \in \mathcal{R}(M, N, \gamma)$ and be of the form (1), and let g^{-1} have the coefficients of the form (2) $-1 \leq N < M \leq 1$. Then, for complex number μ and $|z| < \tau^2$:*

$$\left|A_3 - \mu A_2^2\right| \leq \frac{|\gamma|(M-N)|\tau|}{24}\max\left\{2, \left|\tau\left(\frac{\gamma}{2}\left(\frac{(4-2\mu)3}{2} - 1\right)(M-N) + N - 5\right)\right|\right\}.$$

Taking $\gamma = 1$ and $\lambda = 1$ in Theorem 11, we obtain the new result for a class of convex functions associated with the cardioid domain.

Theorem 14. *Let $g \in \mathcal{C}(M, N,)$ and of the form (1), and let g^{-1} have the coefficients of the form (2) $-1 \leq N < M \leq 1$. Then, for complex number μ and $|z| < \tau^2$:*

$$\left|A_3 - \mu A_2^2\right| \leq \frac{(M-N)|\tau|}{24}\max\left\{2, \left|\tau\left(\frac{1}{2}\left(\frac{3(4-2\mu)}{2} - 1\right)(M-N) + N - 5\right)\right|\right\}.$$

This result is sharp.

3.2. Logarithmic Coefficients

Theorem 15. *Let $g \in \mathcal{R}(M, N, \lambda, \gamma)$ be given by (1), and let the coefficients of $\log \frac{g(z)}{z}$ be given in (4) $-1 \le N < M \le 1$. Then,*

$$|T_1| \le \frac{|\gamma|(M-N)|\tau|}{4(1+\lambda)}$$

and

$$|T_2| \le \frac{|\gamma|(M-N)|\tau|}{16(1+2\lambda)} \max\left\{2, \tau\left|\left(\frac{\gamma(1+2\lambda)}{(1+\lambda)^2} - 1\right)(M-N) + N - 5\right|\right\}.$$

Proof. Differentiating (4) and comparing coefficients gives

$$T_1 = \frac{1}{2}a_2 = \frac{1}{2}\left(\frac{1+\lambda}{\gamma}\right)\rho_2$$

and

$$\begin{aligned} T_2 &= \frac{1}{2}\left(a_3 - \frac{1}{2}a_2^2\right) = \frac{1}{2}\left\{\frac{\gamma p_1^2}{2(1+2\lambda)} + \frac{\gamma}{2(1+2\lambda)}p_2 - \frac{\gamma^2 p_1^2}{2(1+\lambda)^2}\right\} \\ &= \frac{\gamma}{4(1+2\lambda)}\left|p_2 - \mu p_1^2\right|, \end{aligned}$$

where

$$\mu = \frac{\gamma(1+2\lambda)}{(1+\lambda)^2} - 1.$$

Hence, by using Lemma 1, part (iv), we obtain the required result.
Equality holds for

$$1 + \frac{1}{\gamma}\left(\frac{zD'(g(z))}{D(g(z))} - 1\right) = 1 + \frac{(M-N)\tau}{2}z + \frac{(M-N)(5-N)\tau^2}{4}z^2 + \dots$$

□

Theorem 16. *Let $g \in \mathcal{R}(M, N, \lambda, \gamma)$ and be of the form (1), andlet the coefficients of log$\frac{g(z)}{z}$, $-1 \le N < M \le 1$. Then, for complex number μ and $|z| < \tau^2$:*

$$\begin{aligned} &\left|T_2 - \mu T_1^2\right| \\ \le\ &\frac{|\gamma|(M-N)|\tau|}{16(1+2\lambda)} \max\left\{2, \left|\tau\left(\frac{\gamma}{1+\lambda}\left(\frac{(1+\mu)(1+2\lambda)}{(1+\lambda)} - 1\right)(M-N) + N - 5\right)\right|\right\}. \end{aligned}$$

This result is sharp.

Proof. Since $T_1 = \frac{\gamma}{2}a_2$ and $T_2 = \frac{\gamma}{2}(2a_2^2 - a_3)$, by using $a_2 = p_1$, and $2a_3 = \gamma(p_1 a_2 + p_2)$, one can write

$$\left|T_2 - \mu T_1^2\right| = \frac{|\gamma|}{4(1+2\lambda)}\left|p_2 - \frac{\gamma}{1+\lambda}\left(\frac{(4-2\mu)(1+2\lambda)}{(1+\lambda)} - 1\right)p_1^2\right|.$$

Hence, by using Lemma 1, part (iv), for

$$v = \frac{\gamma}{1+\lambda}\left(\frac{(1+\mu)(1+2\lambda)}{(1+\lambda)} - 1\right).$$

we obtain

$$\begin{aligned}
&\left|T_2-\mu T_1^2\right| \\
\leq\ & \frac{|\gamma|(M-N)|\tau|}{16(1+2\lambda)} \\
&\times \max\left\{2, \left|\tau\left(\frac{\gamma}{1+\lambda}\left(\frac{(1+\mu)(1+2\lambda)}{(1+\lambda)}-1\right)(M-N)+N-5\right)\right|\right\}. \qquad (41)
\end{aligned}$$

Thus, inequality (41) is our required result.

The result is sharp for the function

$$g_*(z) = z + \frac{\tau}{2}(M-N)z^2 + \frac{\tau^2}{8}(M-N)(M-2N+5)z^3 + \ldots$$

and for the function given in (22). □

4. Conclusions

In the present article, three new subclasses of analytic functions are defined in relation to the concepts of subordination and cardioid domain. We have investigated a number of interesting problems for functions that belong to these classes of analytic functions, including bounds for the first two Taylor–Maclaurin coefficients, estimates for the Fekete–Szegö-type functional, and coefficient inequalities. It has been demonstrated that all bounds that we have examined in this article is sharp. The same type of sharp results were also investigated for the inverse and $\log\left(\frac{g(z)}{z}\right)$ functions. Some known consequences of our main results are also highlighted in our study.

Based on our current investigation, future research might take the well-known quantum or basic (or q-) calculus as in, for example, the relevant recent publications [36–40]. We hope that our work will provide a foundation for further studies investigating several other classes of analytic functions associated with the cardioid domain, and for these classes, a number of geometrical properties such as coefficient estimates, sufficiency criteria, radii of starlikeness, convexity, and close to convexity, extreme points, and distortion bounds can be investigated.

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