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Extending the Domain with Application of Four-Step Nonlinear Scheme with Average Lipschitz Conditions

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Abstract: A novel local and semi-local convergence theorem for the four-step nonlinear scheme is presented. Earlier studies on local convergence were conducted without particular assumption on Lipschitz constant. In first part, the main local convergence theorems with a weak $\varkappa$-average (assuming it as a positively integrable function and dropping the essential property of ND) are obtained. In comparison to previous research, in another part, we employ majorizing sequences that are more accurate in their precision along with the certain form of $\varkappa$ average Lipschitz criteria. A finer local and semi-local convergence criteria, boosting its utility, by relaxing the assumptions is derived. Applications in engineering to a variety of specific cases, such as object motion governed by a system of differential equations, are illustrated.

Keywords: local convergence; nonlinear problem; convergence radius; Banach space; generalized Lipschitz conditions; $\varkappa$-average

MSC: 65H10; 65J15; 65G99; 47J25

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1. Introduction

Let the nonlinear operator T be a map in the domain D from χ to Y and taken as a Fréchet differentiable with T' its Fréchet derivative which maps χ a Banach space to another Banach space Y, $D \neq \emptyset$ convex open subset, which can be generated as

$$T(x) = 0. \tag{1}$$

Computational sciences have advanced significantly in mathematics, economic equilibrium theory, and engineering sciences. Iteration techniques are also used to solve optimization difficulties. In computer sciences, the discipline of numerical analysis for determining such solutions is fundamentally linked to versions of Newton's approach as

$$x_{n+1} \quad = \quad x_n - [T'(x_n)]^{-1}T(x_n),\ n \geq 0, \tag{2}$$

It is chosen despite its slow convergence speed. A survey on Newton's method [1] can be found in Kantorovich [2] and the references by Rall [3].

There is an extensive literature on the local convergence for Newton, Jarratt, Weerakoon schemes, etc., in the Banach space in the refs. [4–11]. Our objectives here are centered on the local convergence study of a four-step nonlinear scheme (FSS) under generalized/weak Lipschitz criteria which Wang [12] developed, where a non-decreasing positive integrable function (NDPIF) was incorporated rather than a Lipschitz constant. However, Wang with Li [13] discovered new conclusions on the convergence study of Newton's method (NM) in the Banach spaces where the T' meets the radius/center Lipschitz criteria but relaxing $\varkappa$-average. Shakhno [14] has explored local convergence for the Secant-type method [2] with a first-order non-differentiable operator satisfying the generalized/weak Lipschitz conditions.

We shall use the classical FSS [15] under the $\varkappa$-average condition to study the local convergence of FSS that is expressed as:

$$\begin{aligned} y_n &= x_n - [T'(x_n)]^{-1}T(x_n), \\ z_n &= y_n - [T'(x_n)]^{-1}T(y_n), \\ q_n &= z_n - [T'(x_n)]^{-1}T(z_n), \\ x_{n+1} &= q_n - [T'(x_n)]^{-1}T(q_n),\ n \geq 0. \end{aligned} \tag{3}$$

The method (3) is notable for being the simplest and most efficient fifth-order iterative procedure. We find, in the literature, a study using ω-continuity conditions on T'. While methods of greater R-order convergence are often not implemented regularly despite their great speed of convergence, this is due to the high operational expense. That being said, in stiff system challenges, the method of higher R-order convergence can be used cited by [2] where quick convergence is necessary.

We are extremely motivated from the captivating study [13] which gave us the possibility of relaxing the $\varkappa$-average Lipschitz condition and property of the ND of $\varkappa$ to be essential for the convergence of a fifth-order FSS scheme. In [16], we also illustrated the local convergence of a third-order Newton-like method under the same $\varkappa$-average Lipschitz condition taken above. Using such considerations, we derive a new local convergence study for the scheme (3), which enables us to enlarge the convergence ball by dropping out additional assumptions along-with an error/distance estimate. In addition, few corollaries with numerical examples are also stated.

In the literature, L.V. Kantorovich first investigated the semi-local convergence results in [2]. Many other scholars have since examined the enhancement of outcomes based on majorizing sequences and its variants [1,3,17–20], which is described as [21]:

Definition 1 (Majorized sequence)**.** *Let $\{a_n\}$ be a sequence in a Banach space X and $\{t_n\}$ be an increasing scalar sequence. We could say $\{a_n\}$ is majorized by $\{t_n\}$ if $\|a_{n+1} - a_n\| \leq t_{n+1} - t_n$, for each $n = 0, 1, 2, \cdots$.*

It is also important to provide a unified semi-local convergence analysis for the FSS (3) along-with the uniqueness of the solution. This analysis can improve existing results through specialization.

The structure of the presentation of the work is as follows. Section 2 comprises some conditions and preliminary lemma for $\varkappa$-average weak conditions. In Sections 3 and 4, we provide local convergence with its domain of uniqueness for FSS while relaxing the assumption that T' should satisfy radius/center Lipschitz criteria under weak $\varkappa$-average saying $\varkappa/\varkappa_0$ is assumed to be belonging to one of the families of PIF, which are not always ND for convergence-related theorems. This work unifies the semi-local analysis of FSS in Section 5 under majorizing sequences and more weak Lipschitz-type conditions than previously. Finally, applications and further corollaries are given in order to justify the significance of the findings.

2. Notions and Preliminary Results

Making the research as self-contained as one possibly can, we reintroduce some essential concepts and findings [12,13]. Let $M(\Sigma^*, \rho) = \{r : ||r - \Sigma^*|| < \rho\}$ be a ball where the radius is denoted by ρ and the center is denoted by Σ^*.

The notions about Lipschitz criteria are defined as follows.

Definition 2. *The operator T satisfies the radius Lipschitz criterion if*

$$||T'(r) - T'(s^\theta)|| \leq \varkappa(1-\theta)(||r - \Sigma^*|| + ||s - \Sigma^*||), \forall\, r,\, s \in M(\Sigma^*, \rho), \tag{4}$$

in which $s^\theta = \Sigma^* + \theta(s - \Sigma^*), 0 \le \theta \le 1$. *This definition is previously used by researchers with constant* $\varkappa$.

Definition 3. *The operator T satisfies the center Lipschitz criterion if*

$$||T'(r) - T'(\Sigma^*)|| \le 2\varkappa_0 ||r - \Sigma^*||, \forall\, r \in M(\Sigma^*, \rho), \tag{5}$$

with the constant $\varkappa_0$ *in which* $\varkappa_0 \le \varkappa$. *It turns out that substituting* $\varkappa$ *as* $\varkappa_0$ *when* $\varkappa_0 < \varkappa$ *[4,9,22] leads to:*

(i) *Larger convergence radius/domain.*
(ii) *At least as specific information on the solution's location* Σ^*.
(iii) *Closer error boundaries on distances* $||r_{n+1} - r_n||, ||r_n - \Sigma^*||$.

The novelty of our work is to see that $\varkappa$ used in the Lipschitz criteria is not required to be essentially constant; rather, it takes the form of an integrable positively function. In that case, condition (4) is substituted with:

Definition 4. *The operator T satisfies the* $\varkappa$*-average or generalized/weak Lipschitz criterion, if*

$$||T'(r) - T'(s^\theta)|| \le \int_{\theta(\varrho(r)+\varrho(s))}^{\varrho(r)+\varrho(s)} \varkappa(u)du, \forall\, r,\ s \in M(\Sigma^*, \rho), 0 \le \theta \le 1, \tag{6}$$

And condition (5), respectively, is substituted with:

Definition 5. *The operator T satisfies the center* $\varkappa$*-average criterion, if*

$$||T'(r) - T'(\Sigma^*)|| \le \int_0^{2\varrho(r)} \varkappa_0(u)du, \forall\, r \in M(\Sigma^*, \rho), \tag{7}$$

in which $\varrho(r) = ||r - \Sigma^*||$ *together with* $\varkappa_0(u) \le \varkappa(u)$.

As an illustration of motivation, assume that the motion of a three-dimensional object is regulated by a system of differential equations

$$\begin{aligned} f_1'(p) - f_1(p) - 1 &= 0, \\ f_2'(p) - (e-1)q - 1 &= 0, \\ f_3'(r) - 1 &= 0, \end{aligned}$$

Let $\chi = Y = \Re^3, \omega = \overline{M}(0,1)$ and the solution represented by $\Sigma^* = (0,0,0)^t$. Define the function T on ω for $o = (p,q,r)^t$ as

$$T(o) = (e^p - 1, \frac{e-1}{2}q^2 + q, r)^t.$$

We find the Fréchet derivative as

$$T'(o) = \begin{pmatrix} e^p & 0 & 0 \\ 0 & (e-1)q + 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{8}$$

Thus, $\varkappa = \frac{e}{2}, \varkappa_0 = \frac{e-1}{2}$ where $\varkappa_0 < \varkappa$ (as per Definitions 2 and 3). As a result, substituting $\varkappa$ with $\varkappa_0$ at the denominator enhances the convergence radius mentioned in example 1. When $\varkappa$, $\varkappa_0$ are not considered to be constants, then we can find $\varkappa_0(u) = \frac{(e-1)u}{2}$, $\varkappa(u) = \frac{eu}{2}$ *and* $\overline{\varkappa}(u) = \frac{e^{\frac{1}{(e-1)}}u}{2}$ (as per Definitions 4, 5 and Remark 1).

Next, we shall show in Lemma 1 the two major double integrals that will be used in the main results by solving through a change of variables.

Lemma 1. *Assuming T being continuously differentiable in* $M(\Sigma^*,\rho)$, $T(\Sigma^*) = 0$, $[T'(\Sigma^*)]^{-1}$ *exists*

(a) *If the center average Lipschitz condition under the* $\varkappa_0$*-average is satisfied by* $[T'(\Sigma^*)]^{-1}T'$:

$$||[T'(\Sigma^*)]^{-1}(T'(r^\theta) - T'(\Sigma^*))|| \leq \int_0^{2\theta\varrho(r)} \varkappa_0(u)du, \forall\, r \in M(\Sigma^*,\rho), 0 \leq \theta \leq 1, \tag{9}$$

in which $\varrho(r) = ||r - \Sigma^*||$ *and* $\varkappa_0$ *is (ND); therby, we see*

$$\int_0^1 ||[T'(\Sigma^*)]^{-1}(T'(r^\theta) - T'(\Sigma^*))||\varrho(r)d\theta \leq \int_0^{2\varrho(r)} \varkappa_0(u)\Big(\varrho(r) - \frac{u}{2}\Big)du. \tag{10}$$

(b) *If the radius average Lipschitz condition under the* $\varkappa$*-average is satisfied by* $[T'(\Sigma^*)]^{-1}T'$:

$$||[T'(\Sigma^*)]^{-1}(T'(r) - T'(s^\theta))|| \leq \int_{\theta(\varrho(r)+\varrho(s))}^{\varrho(r)+\varrho(s)} \varkappa(u)du, \forall\, r,\ s \in M(\Sigma^*,\rho), 0 \leq \theta \leq 1, \tag{11}$$

in which $s^\theta = \Sigma^* + \theta(s - \Sigma^*)$, $\varkappa$ *is positively integrable. Then,*

$$\int_0^1 ||[T'(\Sigma^*)]^{-1}(T'(r) - T'(s^\theta))||\varrho(s)d\theta \leq \int_0^{\varrho(r)+\varrho(s)} \varkappa(u)\frac{u}{\varrho(r)+\varrho(s)}\varrho(s)du. \tag{12}$$

Proof. The definition for average Lipschitz criteria (11) and (9), respectively, infers that

$$\begin{aligned}
\int_0^1 ||[T'(\Sigma^*)]^{-1}(T'(r) - T'(s^\theta))||\varrho(s)d\theta &\leq \int_0^1 \int_{\theta(\varrho(r)+\varrho(s))}^{\varrho(r)+\varrho(s)} \varkappa(u)du\varrho(s)d\theta \\
&= \int_0^{\varrho(r)+\varrho(s)} \varkappa(u)\frac{u}{\varrho(r)+\varrho(s)}\varrho(s)du, \\
\int_0^1 ||[T'(\Sigma^*)]^{-1}(T'(r^\theta) - T'(\Sigma^*))||\varrho(r)d\theta &\leq \int_0^1 \int_0^{2\theta\varrho(r)} \varkappa_0(u)du\varrho(r)d\theta \\
&= \int_0^{2\varrho(r)} \varkappa_0(u)\Big(\varrho(r) - \frac{u}{2}\Big)du.
\end{aligned}$$

where $r^\theta = \Sigma^* + \theta(r - \Sigma^*)$. □

3. Local Convergence Results for Four-Step Scheme (3)

Under this section, we present the key findings about local convergence as well as improved error estimates with distances.

Let the relation be satisfied by ρ as:

$$\int_0^{2\rho} \varkappa_0(u)du \leq 1 \text{ and } \frac{\int_0^{2\rho} \varkappa(u)udu}{2\rho(1 - \int_0^{2\rho} \varkappa_0(u)du)} \leq 1. \tag{13}$$

Lemma 2 ([13])**.** *Assume that* $\varkappa$ *is PIF along with the function* $\varkappa_\alpha$ *given by expression (46) to be ND for some* α *with* $0 \leq \alpha \leq 1$. *Then, the function* $\psi_{\beta,\alpha}$ *for each* $\beta \geq 0$ *takes the form*

$$\psi_{\beta,\alpha}(t) = \frac{1}{t^{\alpha+\beta}} \int_0^t u^\beta \varkappa(u)du, \tag{14}$$

is also ND.

Lemma 3. *Assume that* $\varkappa$ *is NDPIF. Then, the function defined by* $\frac{1}{t^2}\int_0^t \varkappa(u)udu$ *is also ND with respect to t.*

Proof. Obviously, since $\varkappa$ is monotone, we arrive at

$$\begin{aligned}\left(\frac{1}{t_2^2}\int_0^{t_2}-\frac{1}{t_1^2}\int_0^{t_1}\right)\varkappa(u)udu &= \left(\frac{1}{t_2^2}\int_{t_1}^{t_2}+\left(\frac{1}{t_2^2}-\frac{1}{t_1^2}\right)\int_0^{t_1}\right)\varkappa(u)udu \\ &\geq \varkappa(t_1)\left(\frac{1}{t_2^2}\int_{t_1}^{t_2}+\left(\frac{1}{t_2^2}-\frac{1}{t_1^2}\right)\int_0^{t_1}\right)udu \\ &= \varkappa(t_1)\left(\frac{1}{t_2^2}\int_0^{t_2}-\frac{1}{t_1^2}\int_0^{t_1}\right)udu = 0,\end{aligned}$$

for $0 < t_1 < t_2$. Thus, $\frac{1}{t^2}\int_0^t \varkappa(u)udu$ is ND with respect to t. □

Theorem 1. *Assuming T being continuously differentiable in $M(\Sigma^*,\rho)$, $T(\Sigma^*)=0$, $[T'(\Sigma^*)]^{-1}$ exists and Definitions 4 and 5 are satisfied by $[T'(\Sigma^*)]^{-1}T'$ along with $\varkappa$ and $\varkappa_0$ to be ND, ρ satisfies the relation (13). Then, the FSS (3) converges $\forall x_0 \in M(\Sigma^*,\rho)$ with*

$$||y_n-\Sigma^*|| \leq \frac{\int_0^{2\varrho(x_n)}\varkappa(u)udu}{2(1-\int_0^{2\varrho(x_n)}\varkappa_0(u)du)} \leq \frac{C_1}{\varrho(x_0)}\varrho(x_n)^2, \tag{15}$$

$$||z_n-\Sigma^*|| \leq \frac{\int_0^{\varrho(x_n)+\varrho(y_n)}\varkappa(u)udu}{(\varrho(x_n)+\varrho(y_n))(1-\int_0^{2\varrho(x_n)}\varkappa_0(u)du)}\varrho(y_n) \leq \frac{C_1^2}{\rho(x_0)^2}\rho(x_t)^3, \tag{16}$$

$$||q_n-\Sigma^*|| \leq \frac{\int_0^{\varrho(y_n)+\varrho(z_n)}\varkappa(u)udu}{(\varrho(y_n)+\varrho(z_n))(1-\int_0^{2\varrho(y_n)}\varkappa_0(u)du)}\varrho(z_n) \leq \frac{C_1^3}{\rho(x_0)^3}\rho(x_t)^4, \tag{17}$$

$$||x_{n+1}-\Sigma^*|| \leq \frac{\int_0^{\varrho(z_n)+\varrho(q_n)}\varkappa(u)udu}{(\varrho(z_n)+\varrho(q_n))(1-\int_0^{2\varrho(z_n)}\varkappa_0(u)du)}\varrho(q_n) \leq \frac{C_1^4}{\rho(x_0)^4}\rho(x_t)^5, \tag{18}$$

in which the quantities

$$\begin{aligned}C_1 &= \frac{\int_0^{2\varrho(x_0)}\varkappa(u)udu}{2\varrho(x_0)(1-\int_0^{2\varrho(x_0)}\varkappa_0(u)du)},\ C_2 = \frac{\int_0^{\varrho(x_0)+\varrho(y_0)}\varkappa(u)udu}{(\varrho(x_0)+\varrho(y_0))(1-\int_0^{2\varrho(x_0)}\varkappa_0(u)du)}, \\ C_3 &= \frac{\int_0^{\varrho(x_0)+\varrho(z_0)}\varkappa(u)udu}{(\varrho(x_0)+\varrho(z_0))(1-\int_0^{2\varrho(x_0)}\varkappa_0(u)du)},\ C_4 = \frac{\int_0^{\varrho(x_0)+\varrho(q_0)}\varkappa(u)udu}{(\varrho(x_0)+\varrho(q_0))(1-\int_0^{2\varrho(x_0)}\varkappa_0(u)du)},\end{aligned} \tag{19}$$

are found to be less than 1. In addition, with

$$||x_n-\Sigma^*|| \leq C_1^{5^n-1}||x_0-\Sigma^*||,\ n=1,2,\cdots.$$

Then, we propose a uniqueness theorem with a center-average Lipschitz condition for FSS, (3).

Theorem 2. *Assuming T being continuously differentiable in $M(\Sigma^*,\rho)$, $T(\Sigma^*)=0$, $[T'(\Sigma^*)]^{-1}$ exists and Definition 5 is satisfied by $[T'(\Sigma^*)]^{-1}T'$ and ρ satisfies the relation*

$$\frac{\int_0^{2\rho}\varkappa_0(u)(2\rho-u)du}{2\rho} \leq 1. \tag{20}$$

Then, $T(x)=0$ has a unique solution $\Sigma^* \in M(\Sigma^*,\rho)$.

Next, we provide proofs for the two core results.

Proof of Theorem 1. Clearly, if $x \in M(\Sigma^*, \rho)$, we have with the help of the center-average Lipschitz condition under $\varkappa$-average along with the assumption (13):

$$||[T'(\Sigma^*)]^{-1}[T'(x) - T'(\Sigma^*)]|| \leq \int_0^{2\varrho(x)} \varkappa_0(u)du \leq 1. \quad (21)$$

By the virtue of the Banach Lemma [2] and the above equation

$$||I - ([T'(\Sigma^*)]^{-1}T'(x) - I)||^{-1} = ||[T'(x)]^{-1}T'(\Sigma^*)||,$$

using the expression (21), we arrive at the following inequality:

$$||[T'(x)]^{-1}T'(\Sigma^*)|| \quad \leq \quad \frac{1}{1 - \int_0^{2\varrho(x)} \varkappa_0(u)du}. \quad (22)$$

WLOG picking $x_0 \in M(\Sigma^*, \rho)$, in which C_1, C_2, C_3 and C_4 are given as per the relation (19) and ρ fulfills the inequality (13), can be proved as:

$$\begin{aligned}
C_1 &= \frac{\int_0^{2\varrho(x_0)} \varkappa(u)udu}{2\varrho(x_0)^2(1 - \int_0^{2\varrho(x_0)} \varkappa_0(u)du)}\varrho(x_0) \\
&\leq \frac{\int_0^{2\rho} \varkappa(u)udu}{2\rho^2(1 - \int_0^{2\rho} \varkappa_0(u)du)}\varrho(x_0) \leq \frac{||x_0 - \Sigma^*||}{\rho} < 1, \\
C_2 &= \frac{\int_0^{\varrho(x_0)+\varrho(y_0)} \varkappa(u)udu}{(\varrho(x_0) + \varrho(y_0))^2(1 - \int_0^{2\varrho(x_0)} \varkappa_0(u)du)}(\varrho(x_0) + \varrho(y_0)) \\
&\leq \frac{\int_0^{2\rho} \varkappa(u)udu}{2\rho^2(1 - \int_0^{2\rho} \varkappa_0(u)du)}(\varrho(x_0) + \varrho(y_0)) \leq \frac{||x_0 - \Sigma^*|| + ||y_0 - \Sigma^*||}{2\rho} < 1 \\
C_3 &= \frac{\int_0^{\varrho(x_0)+\varrho(z_0)} \varkappa(u)udu}{(\varrho(x_0) + \varrho(z_0))^2(1 - \int_0^{2\varrho(x_0)} \varkappa_0(u)du)}(\varrho(x_0) + \varrho(z_0)) \\
&\leq \frac{\int_0^{2\rho} \varkappa(u)udu}{2\rho^2(1 - \int_0^{2\rho} \varkappa_0(u)du)}(\varrho(x_0) + \varrho(z_0)) \leq \frac{||x_0 - \Sigma^*|| + ||z_0 - \Sigma^*||}{2\rho} < 1 \\
C_4 &= \frac{\int_0^{\varrho(x_0)+\varrho(q_0)} \varkappa(u)udu}{(\varrho(x_0) + \varrho(q_0))^2(1 - \int_0^{2\varrho(x_0)} \varkappa_0(u)du)}(\varrho(x_0) + \varrho(q_0)) \\
&\leq \frac{\int_0^{2\rho} \varkappa(u)udu}{2\rho^2(1 - \int_0^{2\rho} \varkappa_0(u)du)}(\varrho(x_0) + \varrho(q_0)) \leq \frac{||x_0 - \Sigma^*|| + ||q_0 - \Sigma^*||}{2\rho} < 1.
\end{aligned}$$

In what follows, if $x_n \in M(\Sigma^*, \rho)$, then we have from the scheme (3)

$$\begin{aligned}
||y_n - \Sigma^*|| &= ||x_n - \Sigma^* - [T'(x_n)]^{-1}T(x_n)|| \\
&= ||[T'(x_n)]^{-1}[T'(x_n)(x_n - \Sigma^*) - T(x_n) + T(\Sigma^*)]||. \quad (23)
\end{aligned}$$

Through Taylor's expansion, we obtain from the expansion of $T(x_n)$ along Σ^*:

$$T(\Sigma^*) - T(x_n) + T'(x_n)(x_n - \Sigma^*) = T'(\Sigma^*) \int_0^1 [T'(\Sigma^*)]^{-1}[T'(x_n) - T'(x^\theta)]d\theta(x_n - \Sigma^*). \quad (24)$$

So, combining expressions (23) and (24) along with Definition 4,

$$\begin{aligned} ||y_n - \Sigma^*|| &\le ||[T'(x_n)]^{-1}T'(\Sigma^*)||.|| \int_0^1 [T'(\Sigma^*)]^{-1}[T'(x_n) - T'(x_n^\theta)]d\theta||.||(x_n - \Sigma^*)|| \\ &\le \frac{1}{1 - \int_0^{2\varrho(x_n)} \varkappa_0(u)du} \int_0^1 \int_{2\theta\varrho(x_n)}^{2\varrho(x_n)} \varkappa(u)du\rho(x_n)d\theta. \end{aligned} \tag{25}$$

That gives the first inequality of relation (15) with Lemma 1. With the method's second sub-step (3) and a parallel analogy, we see that

$$\begin{aligned} ||z_n - \Sigma_*|| &\le ||[T'(x_n)]^{-1}T'(\Sigma_*)||.|| \int_0^1 [T'(\Sigma_*)]^{-1}[T'(x_n) - T'(y_n^\theta)]d\theta||.||(y_n - \Sigma_*)|| \\ &\le \frac{1}{1 - \int_0^{2\varrho(x_n)} \varkappa_0(u)du} \int_0^1 \int_{\theta(\varrho(x_n)+\varrho(y_n))}^{\varrho(x_n)+\varrho(y_n)} \varkappa(u)du\varrho(y_n)d\theta. \end{aligned} \tag{26}$$

That gives the first inequality of relation (16) with Lemma 1. With the method's third sub-step (3) and in similar analogy, we obtain

$$\begin{aligned} ||q_n - \Sigma_*|| &\le ||[T'(x_n)]^{-1}T'(\Sigma_*)||.|| \int_0^1 [T'(\Sigma_*)]^{-1}[T'(x_n) - T'(z_n^\theta)]d\theta||.||(z_n - \Sigma_*)|| \\ &\le \frac{1}{1 - \int_0^{2\varrho(x_n)} \varkappa_0(u)du} \int_0^1 \int_{\theta(\varrho(x_n)+\varrho(z_n))}^{\varrho(x_n)+\varrho(z_n)} \varkappa(u)du\varrho(z_n)d\theta. \end{aligned} \tag{27}$$

Looking the Lemma 1, we obtain relation (17). At last, in the last sub-step of the scheme (3), we obtain

$$\begin{aligned} ||x_{n+1} - \Sigma^*|| &\le ||[T'(x_n)]^{-1}T'(\Sigma^*)||.|| \int_0^1 [T'(\Sigma^*)]^{-1}[T'(x_n) - T'(q_n^\theta)]d\theta||.||(q_n - \Sigma^*)|| \\ &\le \frac{1}{1 - \int_0^{2\varrho(x_n)} \varkappa_0(u)du} \int_0^1 \int_{\theta(\varrho(x_n)+\varrho(q_n))}^{\varrho(x_n)+\varrho(q_n)} \varkappa(u)du\varrho(q_n)d\theta. \end{aligned} \tag{28}$$

That gives the expression (18). Moreover, $\varrho(x_n)$, $\varrho(z_n)$, $\varrho(q_n)$ and $\varrho(y_n)$ are monotonically decreasing; hence, $\forall n = 0, 1, \ldots$, and we see

$$\begin{aligned} ||y_n - \Sigma^*|| &\le \frac{\int_0^{2\varrho(x_n)} \varkappa(u)udu}{2(1 - \int_0^{2\varrho(x_n)} \varkappa_0(u)du)} \\ &\le \frac{\int_0^{2\varrho(x_0)} \varkappa(u)udu}{2\varrho(x_0)^2(1 - \int_0^{2\varrho(x_n)} \varkappa_0(u)du)} 2\varrho(x_n)^2 \le \frac{C_1}{\varrho(x_0)}\varrho(x_n)^2. \end{aligned}$$

Setting $n = 0$ above gives us $\|y_0 - \Sigma^*\| \le C_1.\varrho(x_0) < \varrho(x_0)$. This result shows that $y_0 \in M(\Sigma^*, \rho)$, and it can therefore be repeated indefinitely as per Equation (3). Furthermore, all values of y_n will belong to $M(\Sigma^*, \rho)$ by a mathematical induction, and the value of $\varrho(y_n) = \|y_n - \Sigma^*\|$ will decrease monotonically.

By some computation in the first part of expression (15) and (16), we obtain

$$\begin{aligned} ||z_n - \Sigma_*|| &\le \frac{\int_0^{\varrho(x_n)+\varrho(y_n)} \varkappa(u)udu}{(\varrho(x_n) + \varrho(y_n))^2(1 - \int_0^{2\varrho(x_n)} \varkappa_0(u)du)} \varrho(y_n).[\varrho(x_n) + \varrho(y_n)] \\ &\le \frac{C_2}{\varrho(x_0) + \varrho(y_0)}[\varrho(x_n) + \varrho(y_n)]\varrho(y_n). \end{aligned} \tag{29}$$

By simplifying further,

$$\|z_n - \epsilon^*\| \le \frac{C_1^2}{\varrho(x_0)^2}\varrho(x_n)^3. \tag{30}$$

Setting $n = 0$ in inequality (29) gives us $\|z_0 - \Sigma^*\| \leq C_2.\varrho(y_0) < \varrho(x_0)$. This result shows that $z_0 \in M(\Sigma^*, \rho)$, and it can therefore be repeated indefinitely as per Equation (3). Furthermore, all values of z_n will belong to $M(\Sigma^*, \rho)$ by mathematical induction, and the value of $\varrho(z_n) = \|z_n - \Sigma^*\|$ will decrease monotonically.

$$\begin{aligned} ||q_n - \Sigma_*|| &\leq \frac{\int_0^{\varrho(x_n)+\varrho(z_n)} \varkappa(u)udu}{(\varrho(x_n)+\varrho(z_n))^2(1-\int_0^{2\varrho(x_n)} \varkappa_0(u)du)}\varrho(z_n).[\varrho(x_n)+\varrho(z_n)] \\ &\leq \frac{C_3}{\varrho(x_0)+\varrho(z_0)}[\varrho(x_n)+\varrho(z_n)]\varrho(z_n). \end{aligned} \tag{31}$$

By simplifying further,

$$\|q_n - \epsilon^*\| \leq \frac{C_1^3}{\varrho(x_0)^3}\varrho(x_n)^4. \tag{32}$$

Setting $n = 0$ in inequality (31) gives us $\|q_0 - \Sigma^*\| \leq C_3.\varrho(z_0) < \varrho(x_0)$. This result shows that $q_0 \in M(\Sigma^*, \rho)$, and it can therefore be repeated indefinitely as per Equation (3). Furthermore, all values of q_n will belong to $M(\Sigma^*, \rho)$ by mathematical induction, and the value of $\varrho(q_n) = \|q_n - \Sigma^*\|$ will decrease monotonically. In addition, the last expression (18) gives

$$\begin{aligned} ||x_{n+1} - \Sigma^*|| &\leq \frac{\int_0^{\varrho(x_n)+\varrho(q_n)} \varkappa(u)udu}{(\varrho(x_n)+\varrho(q_n))^2(1-\int_0^{2\varrho(x_n)} \varkappa_0(u)du)}\varrho(q_n).[\varrho(x_n)+\varrho(q_n)] \\ &\leq \frac{C_4}{\varrho(x_0)+\varrho(q_0)}[\varrho(x_n)+\varrho(q_n)]\varrho(q_n). \end{aligned} \tag{33}$$

By simplifying further,

$$\|x_{n+1} - \epsilon^*\| \leq \frac{C_1^4}{\varrho(x_0)^4}\varrho(x_n)^5. \tag{34}$$

That is all regarding inequalities (15)–(18). It remains to check (20); for that, we use mathematical induction. For $n = 0$, we have by the relation (16),

$$\|x_{n+1} - \epsilon^*\| \leq \frac{C_1^4}{\varrho(x_0)^4}\varrho(x_n)^5.$$

Subsequently, the aforementioned inequality can be transformed into an alternative form:

$$||x_1 - \epsilon^*|| \quad \leq \quad C_1^{(5-1)}\varrho(x_0).$$

That means the equality (20) is said to be true for $n = 1$. Next, assume the relation (20) holds for some integer $n > 1$. The below form is preserved by the aforementioned inequality:

$$\begin{aligned} ||x_{n+1} - \Sigma^*|| &\leq \left[\frac{C_1^{5^{t+1}-1}}{\varrho(x_0)^4}\right]\varrho(x_0)^5 \\ &\leq C_1^{(5^{t+1}-1)}\varrho(x_0). \end{aligned}$$

□

We are now prepared to demonstrate the uniqueness result.

Proof of Theorem 2. WLOG picking $\Sigma_1^* \in M(\Sigma^*, \rho)$, $\Sigma_1^* \neq \Sigma^*$ and considering the scheme, we obtain

$$\begin{aligned} ||\Sigma_1^* - \Sigma^*|| &= ||\Sigma_1^* - \Sigma^* - [t'(\Sigma^*)]^{-1} t(\Sigma_1^*)||. \\ &= ||[t'(\Sigma^*)]^{-1}[t'(\Sigma^*)(\Sigma_1^* - \Sigma^*) - t(\Sigma_1^*) + t(\Sigma^*)]||. \end{aligned} \quad (35)$$

Through Taylor's expansion, we obtain from expansion of $T(\Sigma_1^*)$ along Σ^*:

$$T(\Sigma^*) - T(\Sigma_1^*) + T'(\Sigma^*)(\Sigma_1^* - \Sigma^*) = \int_0^1 [T'(\Sigma^*)]^{-1}[T'(\Sigma_1^*)^\theta - T'(\Sigma^*)] d\theta (\Sigma_1^* - \Sigma^*). \quad (36)$$

So, combining expressions (35) and (36) along with Definition 5,

$$\begin{aligned} ||\Sigma_1^* - \Sigma^*|| &\leq ||[T'(\Sigma^*)]^{-1} T'(\Sigma^*)||.|| \int_0^1 [T'(\Sigma^*)]^{-1}[T'(\Sigma_1^*)^\theta - T'(\Sigma^*)] d\theta||.||(\Sigma_1^* - \Sigma^*)|| \\ &\leq \int_0^1 \int_0^{2\theta\varrho(\Sigma_1^*)} \varkappa_0(u) du \varrho(\Sigma_1^*) d\theta \cdot \end{aligned} \quad (37)$$

Looking at the relation (37) with Lemma 1, we have

$$\begin{aligned} ||\Sigma_1^* - \Sigma^*|| &\leq \frac{1}{2\varrho(\Sigma_1^*)} \int_0^{2\varrho(\Sigma_1^*)} \varkappa_0(u)[2\varrho(\Sigma_1^*) - u] du (\Sigma_1^* - \Sigma^*) \\ &\leq \frac{\int_0^{2\rho} \varkappa_0(u)(2\rho - u) du}{2\rho} \varrho(\Sigma_1^*) \leq ||\Sigma_1^* - \Sigma^*||. \end{aligned} \quad (38)$$

However, that is found to be a contradiction. Hence, we find that $\Sigma_1^* = \Sigma^*$. This gives the core result for this part. □

Specifically, assuming that $\varkappa$ and $\varkappa_0$ are constants, we can obtain the usual results on the Lipschitz condition.

4. Local Convergence with Weak $\varkappa$-Average

We shall provide local convergence results on re-assuming the hypotheses already presented in the first theorem by weakening it where $\varkappa$ is not taken as an ND function. This concept has already resulted in a more precise convergence study by decreasing the convergence order.

Theorem 3. *Assuming T being continuously differentiable in $M(\Sigma^*, \rho)$, $T(\Sigma^*) = 0$, $[T'(\Sigma^*)]^{-1}$ exists and Definitions 4 and 5 are satisfied by $[T'(\Sigma^*)]^{-1} T'$ along with $\varkappa$ and $\varkappa_0$ to be PIF, and ρ satisfies the relation*

$$\int_0^{2\rho} \varkappa_0(u) du \leq 1 \text{ and } \int_0^{2\rho} (\varkappa(u) + \varkappa_0(u)) du \leq 1. \quad (39)$$

Then, the FSS (3) converges $\forall x_0 \in M(\Sigma^*, \rho)$ with

$$||y_n - \Sigma^*|| \leq \frac{\int_0^{2\varrho(x_n)} \varkappa(u) u du}{2(1 - \int_0^{2\varrho(x_n)} \varkappa_0(u) du)} \leq M_1 \varrho(x_n), \quad (40)$$

$$||z_n - \Sigma^*|| \leq \frac{\int_0^{\varrho(x_n)+\varrho(y_n)} \varkappa(u) u du}{(\varrho(x_n) + \varrho(y_n))(1 - \int_0^{2\varrho(x_n)} \varkappa_0(u) du)} \varrho(y_n) \leq M_2 M_1 \varrho(x_n), \quad (41)$$

$$||q_n - \Sigma^*|| \leq \frac{\int_0^{\varrho(y_n)+\varrho(z_n)} \varkappa(u) u du}{(\varrho(y_n) + \varrho(z_n))(1 - \int_0^{2\varrho(y_n)} \varkappa_0(u) du)} \varrho(z_n) \leq M_3 M_2 M_1 \varrho(x_n), \quad (42)$$

$$||x_{n+1}-\Sigma^*|| \quad \leq \quad \frac{\int_0^{\varrho(z_n)+\varrho(q_n)}\varkappa(u)udu}{(\varrho(z_n)+\varrho(q_n))(1-\int_0^{2\varrho(z_n)}\varkappa_0(u)du)}\varrho(q_n)\leq M_4M_3M_2M_1\varrho(x_n), \tag{43}$$

in which the quantities

$$M_1=\frac{\int_0^{2\rho(x_0)}\varkappa(u)du}{1-\int_0^{2\rho(x_0)}\varkappa_0(u)du},\ M_2=\frac{\int_0^{\rho(x_0)+\rho(y_0)}\varkappa(u)du}{1-\int_0^{2\rho(x_0}\varkappa_0(u)du},$$
$$M_3=\frac{\int_0^{\rho(x_0)+\rho(z_0)}\varkappa(u)du}{1-\int_0^{2\rho(x_0}\varkappa_0(u)du},\ M_4=\frac{\int_0^{\rho(x_0)+\rho(q_0)}\varkappa(u)du}{1-\int_0^{2\rho(x_0}\varkappa_0(u)du}, \tag{44}$$

are found to be less than 1*. In addition,*

$$||x_n-\Sigma^*||\leq(M_1M_2M_3M_4)^n||x_0-\Sigma^*||,\ n=1,2,\ldots. \tag{45}$$

Moreover, assuming the function $\varkappa_\alpha$ *given as*

$$\varkappa_\alpha(t)=t^{1-\alpha}\varkappa(t), \tag{46}$$

is ND for some α *with* $0\leq\alpha\leq1$ *and* ρ *satisfies*

$$\frac{1}{2\rho}\int_0^{2\rho}(2\rho\varkappa_0(u)+u\varkappa(u))du\leq1. \tag{47}$$

Then, the FSS (3) *converges* $\forall x_0\in M(\Sigma^*,\rho)$ *with*

$$||x_n-\Sigma^*||\leq m_1^{\frac{(1+4\alpha)^n-1}{\alpha}}||x_0-\Sigma^*||,\ n=1,2,\cdots, \tag{48}$$

in which quantity m_1 *is the same as* C_1 *given in inequality* (19) *and less than* 1*.*

Proof. Clearly, if $x\in M(\Sigma^*,\rho)$, we have with the help of the center-average Lipschitz condition under a weak average along with the assumption (39):

$$||[T'(\Sigma^*)]^{-1}[T'(x)-T'(\Sigma^*)]||\leq\int_0^{2\varrho(x)}\varkappa_0(u)du\forall\ x\in M(\Sigma^*,\rho)\leq1. \tag{49}$$

Using the Banach Lemma [2] and the equation below

$$||I-([T'(\Sigma^*)]^{-1}T'(x)-I)||^{-1}=||[T'(x)]^{-1}T'(\Sigma^*)||,$$

using expression (49), we arrive at the following inequality:

$$||[T'(x)]^{-1}T'(\Sigma^*)|| \quad\leq\quad \frac{1}{1-\int_0^{2\varrho(x)}\varkappa_0(u)du}. \tag{50}$$

WLOG picking $x_0\in M(\Sigma^*,\rho)$, in which M_1,M_2,M_3 and M_4 are given as per the relation (44) and ρ fulfills the inequality (39), can be proved as:

$$
\begin{aligned}
M_1 &= \frac{\int_0^{2\varrho(x_0)} \varkappa(u)du}{(1-\int_0^{2\varrho(x_0)} \varkappa_0(u)du)} \leq \frac{\int_0^{\rho} \varkappa(u)du}{(1-\int_0^{2\rho} \varkappa_0(u)du)} < 1, \\
M_2 &= \frac{\int_0^{\varrho(x_0)+\varrho(y_0)} \varkappa(u)du}{(1-\int_0^{2\varrho(x_0)} \varkappa_0(u)du)} \leq \frac{\int_0^{2\rho} \varkappa(u)du}{(1-\int_0^{2\rho} \varkappa_0(u)du)} < 1 \\
M_3 &= \frac{\int_0^{\varrho(x_0)+\varrho(z_0)} \varkappa(u)du}{(1-\int_0^{2\varrho(x_0)} \varkappa_0(u)du)} \leq \frac{\int_0^{2\rho} \varkappa(u)du}{(1-\int_0^{2\rho} \varkappa_0(u)du)} < 1 \\
M_4 &= \frac{\int_0^{\varrho(x_0)+\varrho(q_0)} \varkappa(u)du}{(1-\int_0^{2\varrho(x_0)} \varkappa_0(u)du)} \leq \frac{\int_0^{2\rho} \varkappa(u)du}{(1-\int_0^{2\rho} \varkappa_0(u)du)} < 1.
\end{aligned} \tag{51}
$$

In what follows, if $x_n \in M(\Sigma^*, \rho)$, then from scheme (3), the first inequality of relation (40)–(43) is completely similar to Theorem 3. Additionally, $\varrho(x_n)$, $\varrho(z_n)$, $\varrho(q_n)$ and $\varrho(y_n)$ are monotonically decreasing, hence $\forall n = 0, 1, \ldots$, which leads to

$$
||y_n - \Sigma^*|| \leq \frac{\int_0^{2\varrho(x_n)} \varkappa(u)udu}{2(1-\int_0^{2\varrho(x_n)} \varkappa_0(u)du)} \leq \frac{\int_0^{2\varrho(x_0)} \varkappa(u)du}{(1-\int_0^{2\varrho(x_n)} \varkappa_0(u)du)}\varrho(x_n) \leq m_1\varrho(x_n).
$$

By some computation in first part of expression (41)–(43), we obtain

$$
\begin{aligned}
||z_n - \Sigma^*|| &\leq \frac{\int_0^{\varrho(x_n)+\varrho(y_n)} \varkappa(u)udu}{(\varrho(x_n)+\varrho(y_n))(1-\int_0^{2\varrho(x_n)} \varkappa_0(u)du)}\varrho(y_n) \\
&\leq \frac{\int_0^{\varrho(x_0)+\varrho(y_0)} \varkappa(u)du}{(1-\int_0^{2\varrho(x_0)} \varkappa_0(u)du)}\varrho(y_n) \leq m_2 m_1 \rho(x_n). \\
||q_n - \Sigma^*|| &\leq \frac{\int_0^{\varrho(x_n)+\varrho(z_n)} \varkappa(u)udu}{(\varrho(x_n)+\varrho(z_n))(1-\int_0^{2\varrho(x_n)} \varkappa_0(u)du)}\varrho(z_n) \\
&\leq \frac{\int_0^{\varrho(x_0)+\varrho(z_0)} \varkappa(u)du}{(1-\int_0^{2\varrho(x_0)} \varkappa_0(u)du)}\varrho(y_n) \leq m_3 m_2 m_1 \rho(x_n). \\
||x_{n+1} - \Sigma^*|| &\leq \frac{\int_0^{\varrho(x_n)+\varrho(q_n)} \varkappa(u)udu}{(\varrho(x_n)+\varrho(y_n))(1-\int_0^{2\varrho(x_n)} \varkappa_0(u)du)}\varrho(q_n) \\
&\leq \frac{\int_0^{\varrho(x_0)+\varrho(q_0)} \varkappa(u)du}{(1-\int_0^{2\varrho(x_0)} \varkappa_0(u)du)}\varrho(q_n) \leq m_4 m_3 m_2 m_1 \rho(x_n).
\end{aligned} \tag{52}
$$

We also can easily derive the inequality (45) through the aforementioned result. Assuming the function $\varkappa_\alpha$ given by the relation (46) is ND for some α with $0 \leq \alpha \leq 1$ and ρ is given by expression (47), in view of Lemma 2 and the first part of inequality of relation (40), we see

$$
\begin{aligned}
||y_n - \Sigma^*|| &\leq \frac{\psi_{1,\alpha}(2\varrho(x_n))2^\alpha}{(1-\int_0^{2\varrho(x_n)} \varkappa_0(u)du)}\varrho(x_n)^{\alpha+1} \\
&\leq \frac{\psi_{1,\alpha}(2\varrho(x_0))2^\alpha}{(1-\int_0^{2\varrho(x_n)} \varkappa_0(u)du)}\varrho(x_n)^{\alpha+1} = \frac{m_1}{\varrho(x_0)^\alpha}\varrho(x_n)^{\alpha+1}.
\end{aligned}
$$

Following that, when we see Lemma 2 and the initial part of inequality of (41)–(43), we find

$$\begin{aligned}
||z_n - \Sigma^*|| &\leq \frac{\psi_{1,\alpha}(\varrho(x_n) + \varrho(y_n))(\varrho(x_n) + \varrho(y_n))^\alpha}{(1 - \int_0^{2\varrho(x_n)} \varkappa_0(u)du)} \varrho(y_n), \\
&\leq \frac{\psi_{1,\alpha}(\varrho(x_0) + \varrho(y_0))(\varrho(x_n) + \varrho(y_n))^\alpha}{(1 - \int_0^{2\varrho(x_n)} \varkappa_0(u)du)} \varrho(y_n), \\
&\leq \frac{m_1}{(2\varrho(x_0))^\alpha}(\varrho(x_n) + \varrho(y_n))^\alpha \varrho(y_n), \\
&\leq \frac{m_1^2}{\varrho(x_0)^{2\alpha}} \varrho(x_n)^{2\alpha+1}. \\
||q_n - \Sigma^*|| &\leq \frac{\psi_{1,\alpha}(\varrho(x_n) + \varrho(z_n))(\varrho(x_n) + \varrho(z_n))^\alpha}{(1 - \int_0^{2\varrho(x_n)} \varkappa_0(u)du)} \varrho(z_n), \\
&\leq \frac{\psi_{1,\alpha}(\varrho(x_0) + \varrho(z_0))(\varrho(x_n) + \varrho(z_n))^\alpha}{(1 - \int_0^{2\varrho(x_n)} \varkappa_0(u)du)} \varrho(z_n), \\
&\leq \frac{m_1}{(2\varrho(x_0))^\alpha}(\varrho(x_n) + \varrho(z_n))^\alpha \varrho(z_n), \\
&\leq \frac{m_1^3}{\varrho(x_0)^{3\alpha}} \varrho(x_n)^{3\alpha+1}. \\
||x_{n+1} - \Sigma^*|| &\leq \frac{\psi_{1,\alpha}(\varrho(x_n) + \varrho(q_n))(\varrho(x_n) + \varrho(q_n))^\alpha}{(1 - \int_0^{2\varrho(x_n)} \varkappa_0(u)du)} \varrho(q_n), \\
&\leq \frac{\psi_{1,\alpha}(\varrho(x_0) + \varrho(q_0))(\varrho(x_n) + \varrho(q_n))^\alpha}{(1 - \int_0^{2\varrho(x_n)} \varkappa_0(u)du)} \varrho(q_n), \\
&\leq \frac{m_1}{(2\varrho(x_0))^\alpha}(\varrho(x_n) + \varrho(q_n))^\alpha \varrho(q_n), \\
&\leq \frac{m_1^4}{\varrho(x_0)^{4\alpha}} \varrho(x_n)^{4\alpha+1}.
\end{aligned}$$

in which (19) proves $m_1 < 1$. Moving forward the derivation of inequality (48), the method of mathematical induction will be employed. Initially, when $n = 0$, the inequality transforms into the following expression:

$$||x_1 - \Sigma^*|| \leq m_1^4 . \varrho(x_0).$$

Consequently, the expression (48) holds true for $n = 1$. To continue, let us assume that the inequality (48) is valid for an arbitrary integer $n > 1$. By utilizing the inequalities (48) for $n = n$, and rearranging its terms, the inequality retains its form:

$$\begin{aligned}
||x_{n+1} - \Sigma^*|| &\leq \left[m_1^4 . \frac{\varrho(x_k)^{(1+4\alpha)}}{\varrho(x_0)^{4\alpha}} \right] \\
&\leq m_1^{\frac{(1+4\alpha)^n - 1}{\alpha}} \varrho(x_0).
\end{aligned}$$

This demonstrates that the result holds true for $n = n + 1$ and is clearly concluding that x_n is convergent to Σ^*. Hence, the proof is said to be completed. □

Theorem 4. *Assuming T being continuously differentiable in $M(\Sigma^*, \rho)$, $T(\Sigma^*) = 0$, $[T'(\Sigma^*)]^{-1}$ exists and Definition 5 is satisfied by $[T'(\Sigma^*)]^{-1}T'$ along with $\varkappa_0$ to be PIF, and ρ satisfies the relation.*

$$\int_0^{2\rho} \varkappa_0(u)du \leq \frac{1}{3}. \tag{53}$$

Then, the FSS (3) *converges* $\forall x_0 \in M(\Sigma^*, \rho)$ *with*

$$||x_n - \Sigma^*|| \leq (\eta_1\eta_2\eta_3\eta_4)^n||x_0 - \Sigma^*||,\ n = 1, 2, ..., \tag{54}$$

holds for

$$\begin{aligned} \eta_1 &= \frac{2\int_0^{2\varrho(x_0)} \varkappa_0(u)du}{(1 - \int_0^{2\varrho(x_0)} \varkappa_0(u)du)}, \quad \eta_2 = \frac{\int_0^{2\varrho(x_0)} \varkappa_0(u)du + \int_0^{2\varrho(y_0)} \varkappa_0(u)du}{(1 - \int_0^{2\varrho(x_0)} \varkappa_0(u)du)}, \\ \eta_3 &= \frac{\int_0^{2\varrho(x_0)} \varkappa_0(u)du + \int_0^{2\varrho(z_0)} \varkappa_0(u)du}{(1 - \int_0^{2\varrho(x_0)} \varkappa_0(u)du)}, \eta_2 = \frac{\int_0^{2\varrho(x_0)} \varkappa_0(u)du + \int_0^{2\varrho(q_0)} \varkappa_0(u)du}{(1 - \int_0^{2\varrho(x_0)} \varkappa_0(u)du)}. \end{aligned} \tag{55}$$

Additionally, assuming the function $\varkappa_\alpha$ *given by the inequality* (46) *to be ND for some* α *when* $0 \leq \alpha \leq 1$, *we see*

$$||x_n - \Sigma^*|| \leq \eta_1^{\frac{(1+4\alpha)^n - 1}{\alpha}}||x_0 - \Sigma^*||,\ n = 1, 2, \cdots. \tag{56}$$

Proof. WLOG picking $x_0 \in M(\Sigma^*, \rho)$, in which η_1, η_2, η_3 and η_4 are given as per the relation (55) and ρ fulfills the inequality (53). In what follows, if $x_n \in M(\Sigma^*, \rho)$, then we have from the scheme (3), we are able to give the distance norms as in Theorem 3. Looking at Definition 5 and with relation (24), we obtain

$$\begin{aligned} ||y_n - \Sigma^*|| &\leq ||[T'(x_n)]^{-1}T'(\Sigma^*)||.||\int_0^1 [T'(\Sigma^*)]^{-1}[T'(x_n) - T'(\Sigma^*) + T'(\Sigma^*) - T'(x)^\theta]d\theta|| \\ &\quad .||(x_n - \Sigma^*)|| \\ &\leq \frac{1}{(1 - \int_0^{2\varrho(x_n)} \varkappa_0(u)du} \left(\int_0^1 \int_0^{2\theta\varrho(x_n)} \varkappa_0(u)du\varrho(x_n)d\theta\right) \\ &\quad + \left(\int_0^1 \int_0^{2\varrho(x_n)} \varkappa_0(u)du\varrho(x_n)d\theta\right). \end{aligned} \tag{57}$$

Looking at the Lemma 1, the aforementioned inequality becomes

$$\begin{aligned} ||y_n - \Sigma^*|| &\leq \frac{2\int_0^{2\varrho(x_n)} \varkappa_0(u)du\varrho(x_n) - \frac{1}{2}\int_0^{2\varrho(x_n)} \varkappa_0(u)udu}{1 - \int_0^{2\varrho(x_n)} \varkappa_0(u)du} \\ &\leq \frac{2\int_0^{2\varrho(x_n)} \varkappa_0(u)du}{1 - \int_0^{2\varrho(x_n)} \varkappa_0(u)du}\varrho(x_n) = \eta_1\varrho(x_n), \end{aligned}$$

The method's remaining sub-step (3) and a parallel analogy gives

$$\begin{aligned} ||z_n - \Sigma^*|| &\leq ||[T'(x_n)]^{-1}T'(\Sigma^*)||\left(||\int_0^1 [T'(\Sigma^*)]^{-1}[T'(x_n) - T'(\Sigma^*)]d\theta||.||(y_n - \Sigma^*)||\right) \\ &\quad + \left(||\int_0^1 [T'(\Sigma^*)]^{-1}[T'(\Sigma^*) - T'(y^\theta)]d\theta||.||(y_n - \Sigma^*)||\right) \\ &\leq \frac{1}{(1 - \int_0^{2\varrho(x_n)} \varkappa_0(u)du} \left(\int_0^1 \int_0^{2\theta\varrho(y_n)} \varkappa_0(u)du\varrho(y_n)d\theta\right) \\ &\quad + \left(\int_0^1 \int_0^{2\varrho(x_n)} \varkappa_0(u)du\varrho(y_n)d\theta\right). \end{aligned} \tag{58}$$

Looking at Lemma 1, the aforementioned inequality becomes

$$\begin{aligned} ||z_n - \Sigma^*|| &\leq \frac{\int_0^{2\varrho(x_n)} \varkappa_0(u)du\varrho(y_n) + \int_0^{2\varrho(y_n)} \varkappa_0(u)du\varrho(y_n) - \frac{1}{2}\int_0^{2\varrho(y_n)} \varkappa_0(u)udu}{1 - \int_0^{2\varrho(x_n)} \varkappa_0(u)du} \\ &\leq \frac{\int_0^{2\varrho(x_n)} \varkappa_0(u)du\varrho(y_n) + \int_0^{2\varrho(y_n)} \varkappa_0(u)du\varrho(y_n)}{1 - \int_0^{2\varrho(x_n)} \varkappa_0(u)du} \\ &= \eta_2\eta_1\varrho(x_n) \end{aligned}$$

$$
\begin{aligned}
||q_n - \Sigma^*|| &\leq \frac{\int_0^{2\varrho(x_n)} \varkappa_0(u)du\varrho(z_n) + \int_0^{2\varrho(z_n)} \varkappa_0(u)du\varrho(z_n) - \frac{1}{2}\int_0^{2\varrho(z_n)} \varkappa_0(u)udu}{1 - \int_0^{2\varrho(x_n)} \varkappa_0(u)du} \\
&\leq \frac{\int_0^{2\varrho(x_n)} \varkappa_0(u)du\varrho(z_n) + \int_0^{2\varrho(z_n)} \varkappa_0(u)du\varrho(z_n)}{1 - \int_0^{2\varrho(x_n)} \varkappa_0(u)du} \\
&= \eta_3\eta_2\eta_1\varrho(x_n)
\end{aligned}
$$

$$
\begin{aligned}
||x_{n+1} - \Sigma^*|| &\leq \frac{\int_0^{2\varrho(x_n)} \varkappa_0(u)du\varrho(q_n) + \int_0^{2\varrho(q_n)} \varkappa_0(u)du\varrho(q_n) - \frac{1}{2}\int_0^{2\varrho(q_n)} \varkappa_0(u)udu}{1 - \int_0^{2\varrho(x_n)} \varkappa_0(u)du} \\
&\leq \frac{\int_0^{2\varrho(x_n)} \varkappa_0(u)du\varrho(q_n) + \int_0^{2\varrho(q_n)} \varkappa_0(u)du\varrho(q_n)}{1 - \int_0^{2\varrho(x_n)} \varkappa_0(u)du} \\
&= \eta_4\eta_3\eta_2\eta_1\varrho(x_n),
\end{aligned}
$$

in which expression (55) gives $\eta_1 < 1, \eta_2, \eta_3$ and $\eta_4 < 1$. The relations proved above yield inequality (54) proving x_n is convergent to Σ^*. Assuming the function $\varkappa_\alpha$ given by the relation (46) is ND for some α with $0 \leq \alpha \leq 1$ and ρ is given by expression (53), in view of Lemma 2 and the aforementioned relations, we have

$$
\begin{aligned}
||y_n - \Sigma^*|| &\leq \frac{2\psi_{0,\alpha}(2\varrho(x_n))2^\alpha}{(1 - \int_0^{2\varrho(x_n)} \varkappa_0(u)du)}\varrho(x_n)^{\alpha+1}, \\
&\leq \frac{2\psi_{0,\alpha}(2\varrho(x_0))2^\alpha}{(1 - \int_0^{2\varrho(x_0)} \varkappa_0(u)du)}\varrho(x_n)^{\alpha+1} = \frac{\eta_1}{\varrho(x_0)^\alpha}\varrho(x_n)^{\alpha+1}.
\end{aligned}
$$

Following that, Lemma 2 gives

$$
\begin{aligned}
||z_n - \Sigma^*|| &\leq \frac{\psi_{0,\alpha}(2\varrho(x_n)) + \psi_{0,\alpha}(2\varrho(y_n)).(2\varrho(x_n))^\alpha.\varrho(y_n)}{(1 - \int_0^{2\varrho(x_n)} \varkappa_0(u)du)}\varrho(y_n), \\
&\leq \frac{\psi_{0,\alpha}(2\varrho(x_0)) + \psi_{0,\alpha}(2\varrho(y_0)).(2\varrho(x_n))^\alpha.\varrho(y_n)}{(1 - \int_0^{2\varrho(x_n)} \varkappa_0(u)du)}\varrho(y_n), \\
&\leq \frac{\eta_1}{(2\varrho(x_0))^\alpha}(\varrho(x_n) + \varrho(y_n))^\alpha\varrho(y_n), \\
&\leq \frac{\eta_1^2}{\varrho(x_0)^{2\alpha}}\varrho(x_n)^{2\alpha+1}. \\
||q_n - \Sigma^*|| &\leq \frac{\psi_{0,\alpha}(2\varrho(x_n)) + \psi_{0,\alpha}(2\varrho(z_n)).(2\varrho(x_n))^\alpha.\varrho(z_n)}{(1 - \int_0^{2\varrho(x_n)} \varkappa_0(u)du)}\varrho(z_n), \\
&\leq \frac{\psi_{0,\alpha}(2\varrho(x_0)) + \psi_{0,\alpha}(2\varrho(z_0)).(2\varrho(x_n))^\alpha.\varrho(z_n)}{(1 - \int_0^{2\varrho(x_n)} \varkappa_0(u)du)}\varrho(z_n), \\
&\leq \frac{\eta_1}{(2\varrho(x_0))^\alpha}(\varrho(x_n) + \varrho(z_n))^\alpha\varrho(z_n), \\
&\leq \frac{\eta_1^3}{\varrho(x_0)^{3\alpha}}\varrho(x_n)^{3\alpha+1}. \\
||x_{n+1} - \Sigma^*|| &\leq \frac{\psi_{0,\alpha}(2\varrho(x_n)) + \psi_{0,\alpha}(2\varrho(q_n)).(2\varrho(x_n))^\alpha.\varrho(q_n)}{(1 - \int_0^{2\varrho(x_n)} \varkappa_0(u)du)}\varrho(q_n), \\
&\leq \frac{\psi_{0,\alpha}(2\varrho(x_0)) + \psi_{0,\alpha}(2\varrho(q_0)).(2\varrho(x_n))^\alpha.\varrho(q_n)}{(1 - \int_0^{2\varrho(x_n)} \varkappa_0(u)du)}\varrho(q_n), \\
&\leq \frac{\eta_1}{(2\varrho(x_0))^\alpha}(\varrho(x_n) + \varrho(q_n))^\alpha\varrho(q_n), \\
&\leq \frac{\eta_1^4}{\varrho(x_0)^{4\alpha}}\varrho(x_n)^{4\alpha+1}.
\end{aligned}
$$

Continuing the derivation further results in the inequality (56) indicating that x_n is convergent to Σ^*. $\square$

Next, we will give special cases and the applications to our novel improved theorems with few specific functions on $\varkappa$, and the results through Theorems 3 and 4 are reconstructed.

Corollary 1. *Assuming T being continuously differentiable in $M(\Sigma^*,\rho)$, $T(\Sigma^*)=0$, $[T'(\Sigma^*)]^{-1}$ exists and Definitions 4 and 5 are satisfied by $[T'(\Sigma^*)]^{-1}T'$ along with $\varkappa(u)=\zeta a u^{a-1}$ and $\varkappa_0(u)=\zeta_0 a u^{a-1}$:*

$$||[T'(\Sigma^*)]^{-1}(T'(x)-T'(y^\theta))||\leq \zeta.(1-\theta^a)(||x-\Sigma^*||+||y-\Sigma^*||)^a \tag{59}$$

with

$$||[T'(\Sigma^*)]^{-1}(T'(x)-T'(\Sigma^*))||\leq \zeta_0 2^a||x-\Sigma^*||^a, \tag{60}$$

$\forall\ x,\ y\in M(\Sigma^*,\rho), 0\leq\theta\leq 1$, where $y^\theta=\Sigma^*+\theta(y-\Sigma^*)$, $0<a<1$, $\zeta>0$ and $\zeta_0>0$. ρ satisfies the relation

$$\rho \quad = \quad \left(\frac{a+1}{2^a(\zeta_0(a+1)+\zeta a)}\right)^{\frac{1}{a}}. \tag{61}$$

Then, the FSS, (3) converges $\forall x_0\in M(\Sigma^*,\rho)$ with

$$||y_n-\Sigma^*|| \quad\leq\quad \frac{\int_0^{2\varrho(x_n)}\varkappa(u)udu}{2(1-\int_0^{2\varrho(x_n)}\varkappa_0(u)du)}\leq M_1\varrho(x_n), \tag{62}$$

$$||z_n-\Sigma^*|| \quad\leq\quad \frac{\int_0^{\varrho(x_n)+\varrho(y_n)}\varkappa(u)udu}{(\varrho(x_n)+\varrho(y_n))(1-\int_0^{2\varrho(x_n)}\varkappa_0(u)du)}\varrho(y_n)\leq M_2M_1\varrho(x_n), \tag{63}$$

$$||q_n-\Sigma^*|| \quad\leq\quad \frac{\int_0^{\varrho(y_n)+\varrho(z_n)}\varkappa(u)udu}{(\varrho(y_n)+\varrho(z_n))(1-\int_0^{2\varrho(y_n)}\varkappa_0(u)du)}\varrho(z_n)\leq M_3M_2M_1\varrho(x_n), \tag{64}$$

$$||x_{n+1}-\Sigma^*|| \quad\leq\quad \frac{\int_0^{\varrho(z_n)+\varrho(q_n)}\varkappa(u)udu}{(\varrho(z_n)+\varrho(q_n))(1-\int_0^{2\varrho(z_n)}\varkappa_0(u)du)}\varrho(q_n)\leq M_4M_3M_2M_1\varrho(x_n), \tag{65}$$

holds for

$$\begin{aligned} M_1 &= \frac{\zeta a 2^a\varrho(x_0)^a}{(1+a)[1-2^a\zeta_0\varrho(x_0)^a]},\ M_2=\frac{\zeta a(\varrho(x_0)+\varrho(y_0))^a}{(a+1)(1-2^a\zeta_0\varrho(x_0)^a)},\\ M_3 &= \frac{\zeta a(\varrho(x_0)+\varrho(z_0))^a}{(a+1)(1-2^a\zeta_0\varrho(x_0)^a)},\ M_4=\frac{\zeta a(\varrho(x_0)+\varrho(q_0))^a}{(a+1)(1-2^a\zeta_0\varrho(x_0)^a)}, \end{aligned} \tag{66}$$

Hence,

$$||x_n-\Sigma^*||\leq(M_1M_2M_3M_4)^n||x_0-\Sigma^*||,\ n=1,2,\ldots. \tag{67}$$

Corollary 2. *Assuming T being continuously differentiable in $M(\Sigma^*,\rho)$, $T(\Sigma^*)=0$, $[T'(\Sigma^*)]^{-1}$ exists and Definition 5 is satisfied by $[T'(\Sigma^*)]^{-1}T'$ along with $\varkappa_0(u)=\zeta_0 a u^{a-1}$:*

$$||[T'(\Sigma^*)]^{-1}(T'(x)-T'(\Sigma^*))||\leq\zeta_0 2^a||x-\Sigma^*||^a,\forall\ x\in M(\Sigma^*,\rho), \tag{68}$$

in which $0 < a < 1$ *and* $\zeta_0 > 0.\rho$ *satisfies the relation*

$$\rho = \left(\frac{1}{3\zeta_0 2^a}\right)^{\frac{1}{a}}. \tag{69}$$

Then, the FSS (3) *converges* $\forall x_0 \in M(\Sigma^*, \rho)$ *with*

$$\eta_1 = \frac{\zeta_0 2^{a+1} \varrho(x_0)^a}{[1 - 2^a \zeta_0 \varrho(x_0)^a]}, \ \eta_2 = \frac{\zeta_0 2^a (\varrho(x_0)^a + \varrho(y_0)^a)}{(1 - 2^a \zeta_0 \varrho(x_0)^a)},$$
$$\eta_3 = \frac{\zeta_0 2^a (\varrho(x_0)^a + \varrho(z_0)^a)}{(1 - 2^a \zeta_0 \varrho(x_0)^a)}, \ \eta_2 = \frac{\zeta_0 2^a (\varrho(x_0)^a + \varrho(q_0)^a)}{(1 - 2^a \zeta_0 \varrho(x_0)^a)}, \tag{70}$$

$$||x_n - \Sigma^*|| \leq (\eta_1 \eta_2 \eta_3 \eta_4)^n ||x_0 - \Sigma^*||, \ n = 1, 2, ... \tag{71}$$

Remark 1. *(i) If* $\varkappa_0 = \varkappa$*, then our results narrow down to those proved by earlier researchers [5,8,12,13,23]. Thus, the results of the above condition mentioned above are special cases of our results. However, if* $\varkappa_0 < \varkappa$*, the wider convergence radius is achieved in our results due to the weakening of Lipschitz continuity conditions (the same as illustrated in the Examples 1 and 3) in the next section.*

(ii) The extension of scope of application of our results is described below. Assume equation $2\varkappa_0(u)u - 1 = 0$ *is said to have a minimal positive root* $\overline{\rho}$ *and* (5) *holds. Set* $\tilde{M} = M(\Sigma^*, \rho) \cap M(\Sigma^*, \overline{\rho})$*. Furthermore, set*

$$||T(x) - T(y^\theta)|| \leq \int_{\theta(\varrho(x)+\varrho(y))}^{\varrho(x)+\varrho(y)} \overline{\varkappa}(u) du, \tag{72}$$

in which $\forall\ x,\ y \in \tilde{M}, 0 \leq \theta \leq 1,$ *and* $\overline{\varkappa}$ *is as* $\varkappa$*. We see*

$$\overline{\varkappa}(u) \leq \varkappa(u)\ \forall\ u \in [0, min\{\rho, \overline{\rho}\}].$$

So, according to the above proofs, $\overline{\varkappa}$ *can replace* $\varkappa$ *in all the results under* $\varkappa$*. However, then if*

$$\overline{\varkappa}(u) < \varkappa(u)$$

the advantages mentioned in (i) *above can be extended even more. Thus, according to the motivational example, by setting a lower upper bound of* $\varkappa(u)$ *as* $\overline{\varkappa}$ *which will further enhance the convergence radius, we obtain*

$$\varkappa_0 < \overline{\varkappa} = \frac{e^{\frac{1}{(e-1)}}}{2} < \varkappa.$$

5. Semi-Local Convergence

This section follows semi-local convergence outcomes for highly comprehensive majorizing sequences of FSS (3). The study of iterative methods highly values majorizing sequences as they significantly contribute to the analysis of the given scheme. This is because majorizing sequences provide a way to bound the error of the iterative method, which is crucial in understanding the convergence properties of the method. By providing a tight upper bound on the error, majorizing sequences can be used to establish convergence results for iterative methods. We introduce an extensive majorizing sequence. Suppose that there exists a real function κ_0 defined on the interval $[0, +\infty)$ such that the equation $\kappa_0(n) - 1 = 0$ has a smallest positive solution ρ. Let also κ be a real function defined on the interval $[0, \rho)$. Let $a_0 = 0$ and b_0 be a non-negative parameter. Then, define the sequence $\{a_t\}$ by

$$\begin{aligned}
c_t &= b_t + \frac{\int_0^1 \kappa((1-\Theta)(b_t - a_t))d\Theta(b_t - a_t)}{1 - \kappa_0(a_t)}, \\
\alpha_t &= c_1 + \int_0^1 \kappa_0(b_t + \Theta(c_t - b_t))d\Theta(c_t - b_t) \\
&+ \int_0^1 \kappa((1-\Theta)(b_t - a_t))d\Theta(b_t - a_t), \\
d_t &= c_t + \frac{\alpha_t}{1 - \kappa_0(a_t)}, \\
l_t &= c_1 + \int_0^1 \kappa_0(b_t + \Theta(d_t - b_t))d\Theta(d_t - b_t) \\
&+ \int_0^1 \kappa((1-\Theta)(b_t - a_t))d\Theta(b_t - a_t), \\
a_{t+1} &= d_t + \frac{l_t}{1 - \kappa_0(a_t)}, \\
r_{t+1} &= \int_0^1 \kappa((1-\Theta)(a_{t+1} - a_t))d\Theta(a_{t+1} - a_t) + (1 - \kappa_0(a_t))(a_{t+1} - a_t), \qquad (73)
\end{aligned}$$

and

$$b_{t+1} = a_{t+1} + \frac{r_{t+1}}{1 - \kappa_0(a_{t+1})}.$$

The sequence a_t is shown to be majorizing for the sequence x_t in Theorem 4. Let us first develop convergence criteria for the sequence $\{a_t\}$.

Lemma 4. *Suppose there exists $\rho_0 \in [0, \rho)$ such that for each $t = 0, 1, 2, \cdots$*

$$\kappa_0(a_t) < 1 \text{ and } a_t \leq \rho_0. \qquad (74)$$

Then, the following assertions hold

$$0 \leq a_t \leq b_t \leq c_t \leq d_t \leq a_{t+1} \leq \rho_0, \qquad (75)$$

and there exists $a^* \in [0, \rho_0]$ such that $a_t \leq a^* \leq \rho_0$ and $\lim_{t \to \infty} a_t = a^*$.

Proof. By the definition of the sequence $\{a_t\}$ given by the formula (73) and the conditions (74), we see that the assertion (75) holds. Hence, the rest of the assertions also hold. □

Remark 2. *If the function κ_0 is strictly increasing, then set $\rho_0 = \kappa_0^{-1}(1)$. The functions κ_0, κ and the limit point a^* are associated with the method (3). Suppose:*

(A_1) *There exists a parameter $b_0 \geq 0$ and a point $x_0 \in \Omega$ such that the linear operator $T'(x_0)$ is invertible and $\|[T'(x_0)]^{-1}T(x_0)\| \leq b_0$.*
(A_2) $\|T'(x_0)^{-1}(T'(u) - T'(x_0))\| \leq \kappa_0(\|u - x_0\|)$ *for each $u \in \Omega$. Set $U_1 = U(x_0, \rho) \cap \Omega$*
(A_3) $\|T'(x_0)^{-1}(T'(u_2) - T'(u_1))\| \leq \kappa(\|u_2 - u_1\|)$ *for each $u_1, u_2 \in U_1$.*
(A_4) *The conditions in (74) hold and*
(A_5) $U[x_0, a^*] \subset \Omega$.

Remark 3. *(1) The parameter ρ can replace the limit point a^* in the condition (A_5).*
(2) Suppose that
(A_3') $\|T'(x_0)^{-1}T'(x)\| \leq \kappa_1(\|x - x_0\|)$ *for each $x \in U_1$, where κ_1 is a continuous and non-decreasing real function defined on the interval $[0, \rho]$.*

Then, under the conditions (A_2), we obtain in turn

$$\begin{aligned} \|T'(x_0)^{-1}(T'(x) - T'(x_0) + T'(x_0))\| &\leq 1 + \|T'(x_0)^{-1}(T'(x) - T'(x_0))\| \\ &\leq 1 + \kappa_0(\|x - x_0\|). \end{aligned}$$

That is, we can choose $\kappa_1(n) = 1 + \kappa_0(n)$. Then, the condition (A_3') holds for this choice. However, the function κ_1 can be smaller than the function $1 + \kappa_0(u)$ in some examples. As an example, define the real function $T(x) = \sin x$. Then, we obtain $\kappa_1(n) = n < 1 + \kappa_0(n)$. In practice, we shall be using the smaller of the functions κ_1 and $1 + \kappa_0(n)$. Moreover, if κ_1 is smaller, then (A_3') should be added in the conditions (A_1)–(A_5), since (A_2) implies (A_3') but not necessarily vice versa.

The main result for the FSS (3)' semi-local convergence is:

Theorem 5. *Suppose that the conditions (A_1)–(A_5) hold. Then, the sequence x_t generated by the method (3) is well defined in the ball $U(x_0, a^*)$, remains in $U(x_0, a^*)$ for each $t = 0, 1, 2, \cdots$ and is convergent to a solution $x^* \in U[x_0, a^*]$ of the equation $T(x) = 0$. Additionally, the following assertions hold for each $t = 0, 1, 2, \cdots$*

$$\|y_t - x_t\| \leq b_t - a_t, \tag{76}$$

$$\|z_t - y_t\| \leq c_t - b_t, \tag{77}$$

$$\|q_t - z_t\| \leq d_t - c_t, \tag{78}$$

$$\|x_{t+1} - q_t\| \leq a_{t+1} - d_t, \tag{79}$$

and

$$\|x_t - x^*\| \leq a^* - a_t. \tag{80}$$

Proof. Induction shall determine the assertions. The condition (A_1) and the method (3) for $t = 0$ give

$$\|y_0 - x_0\| = \|[T'(x_0)]^{-1}T(x_0)\| \leq b_0 = b_0 - a_0 < a^*.$$

Thus, the iterate $y_0 \in U(x_0, a^*)$ and the assertion (76) is established for $t = 0$. Let $u \in U(x_0, a^*)$. Then, by the condition (A_2), it follows

$$\|T'(x_0)^{-1}(T'(u) - T'(x_0))\| \leq \kappa_0(\|u - x_0\|) \leq \kappa_)(a^* < 1,$$

thus

$$\|[T'(u)]^{-1}T'(x_0)\| \leq \frac{1}{1 - \kappa_0(\|u - x_0\|)}. \tag{81}$$

We can write by the first sub-step

$$\begin{aligned} T(y_k) &= T(y_k) - T(x_k) + T(x_k) \\ &= T(y_k) - T(x_k) - T'(x_k)(y_k - x_k) \\ &= \int_0^1 [T'(x_k + \Theta(y_k - x_k))d\Theta - T'(x_k)](y_k - x_k). \end{aligned}$$

Hence, by (A_2),

$$\begin{aligned} \|[T'(x_0)]^{-1}T(y_k)\| &\leq \int_0^1 \kappa((1-\Theta)\|y_k-x_k\|)d\Theta\|y_k-x_k\|, \\ &\leq \int_0^1 \kappa((1-\Theta)\|y_k-x_k\|)d\Theta\|y_k-x_k\|, \\ &\leq \int_0^1 \kappa((1-\Theta)(b_k-a_k\|)d\Theta(b_k-a_k), \end{aligned}$$

and by the second sub-step

$$\begin{aligned} \|z_k-x_k\| &\leq \|[T'(x_k)]^{-1}T'(x_0)\|.\|[T'(x_0)]^{-1}T(y_k)\| \\ &\leq \frac{\int_0^1 \kappa((1-\Theta)(b_k-a_k\|)d\Theta(b_k-a_k)}{1-\kappa_0(a_k)} = c_k-b_k, \end{aligned}$$

and

$$\|z_k-x_0\| \leq \|z_k-y_k\|+\|y_k-x_0\| \leq c_k-b_k+b_k-a_0 = c_k < a^*,$$

where we also used (76) for $u=x_k$. Hence, the iterate $z_k \in U(x_0,a^*)$ and the assertion (77) holds. Then, we can write

$$\begin{aligned} T(z_k) &= T(z_k)-T(y_k)+T(y_k) \\ &= \int_0^1 [T'(y_k+\Theta(z_k-y_k))d\Theta](z_k-y_k)+T(y_k). \end{aligned}$$

Therefore,

$$\begin{aligned} \|[T'(x_0)]^{-1}T(z_k)\| &\leq (\int_0^1 \kappa_0(\|y_k-x_0\|+\Theta\|z_k-y_k\|)d\Theta\|z_k-y_k\|+\|[T'(x_0)]^{-1}T(y_k)\| \\ &\leq (\int_0^1 \kappa_0(b_k+\Theta(c_k-b_k))d\Theta)(c_k-b_k) \\ &+ (\int_0^1 \kappa((1-\Theta)(b_k-a_k))d\Theta)(b_k-a_k), \end{aligned}$$

Consequently,

$$\begin{aligned} \|q_k-z_k\| &\leq \|[T'(x_k)]^{-1}T'(x_0)\|.\|[T'(x_0)]^{-1}T(z_k)\| \\ &\leq \frac{\alpha_k}{1-\kappa_0(a_k)} = d_k-c_k, \end{aligned}$$

and

$$\|q_k-x_0\| \leq \|q_k-z_k\|+\|z_k-x_0\| \leq d_k-c_k+c_k-a_0 = d_k < a^*.$$

Hence, the iterate $q_k \in U(x_0,a^*)$ and the assertion (78) holds. Similarly, the last sub-step gives in turn

$$\begin{aligned} \|x_{k+1}-q_k\| &\leq \|[T'(x_k)]^{-1}T'(x_0)\|.\|[T'(x_0)]^{-1}(T(q_k)-T(y_k)+T(y_k)\| \\ &\leq \frac{(1+\int_0^1 \kappa_0(\|y_k-x_0\|+\Theta\|q_k-y_k\|)d\Theta)\|q_k-y_k\|+\|[T'(x_0)]^{-1}T(y_k)\|}{1-\kappa_0(a_k)} \\ &\leq \frac{l_k}{1-\kappa_0(a_k)} = a_{k+1}-d_k, \end{aligned}$$

and

$$\|x_{k+1}-x_0\| \leq \|x_{k+1}-q_k\|+\|q_k-x_0\| \leq a_{k+1}-d_k+d_k-a_0 = a_{k+1} < a^*.$$

Hence, the iterate $x_{k+1} \in U(x_0, a^*)$ and the assertion (79) holds. Moreover, we can write by the first sub-step

$$\begin{aligned} T(x_{k+1}) &= T(x_{k+1}) - T(x_k) + T(x_k) \\ &= T(x_{k+1}) - T(x_k) - T'(x_k)(x_{k+1} - x_k) \\ &+ T'(x_k)(x_{k+1} - x_k) - T'(x_k)(y_k - x_k) \\ &= T(x_{k+1}) - T(x_k) - T'(x_k)(x_{k+1} - x_k) + T'(x_k)(x_{k+1} - y_k), \end{aligned}$$

so

$$\begin{aligned} \|[T'(x_0)]^{-1}T(x_{k+1})\| &\leq \int_0^1 \kappa((1-\Theta)\|x_{k+1} - x_k\|)d\Theta\|x_{k+1} - x_k\| \\ &+ (1 + \kappa_0(\|x_k - x_0\|))\|x_{k+1} - y_k\| \\ &\leq (\int_0^1 \kappa((1-\Theta)(a_{k+1} - a_k))d\Theta(a_{k+1} - a_k) \\ &+ (1 + \kappa_0(a_k))(a_{k+1} - b_k) = r_{k+1}, \end{aligned} \tag{82}$$

Consequently,

$$\begin{aligned} \|y_{k+1} - x_{k+1}\| &\leq \|[T'(x_{k+1})]^{-1}T'(x_0)\|.\|[T'(x_0)]^{-1}T(x_{k+1})\| \\ &\leq \frac{r_{k+1}}{1 - \kappa_0(a_{k+1})} = b_{k+1} - a_{k+1} \end{aligned}$$

and

$$\|y_{k+1} - x_0\| \leq \|y_{k+1} - x_{k+1}\| + \|x_{k+1} - x_0\| \leq b_{k+1} - a_{k+1} + a_{k+1} - a_0 = b_{k+1} < a^*.$$

Hence, the iterate $y_{k+1} \in U(x_0, a^*)$ and the assertion (79) holds. The induction is terminated. Therefore, it is established that the sequence a_k majorizes the sequence x_k. Moreover, the sequence a_k is complete as convergent by the condition (A_4). Thus, the sequence x_k is also complete in Banach space . Hence, there exists $x^* \in U(x_0, a^*)$ such that $\lim_{k\to\infty} x_k = x^*$. Furthermore, if $k \to \infty$ in (82), then we conclude from $\lim_{k\to\infty} r_{k+1} = 0$, that $T(x^*) = 0$. Finally, let $i \to \infty$ in the estimate

$$\|x_{t+i} - x_t\| \leq a_{t+i} - a_t,$$

to obtain the assertion (80). □

The determination of the solution region's uniqueness follows.

Proposition 1. *Suppose there exists a solution $y^* \in U(x_0, \rho_1)$ of the equation $T(x) = 0$ for some $\rho_1 > 0$, the condition (A_2) holds in the ball $U(x_0, \rho_1)$ and there exists $\rho_2 \geq \rho_1$ such that*

$$\int_0^1 \kappa_0((1-\Theta)\rho_1 + \Theta\rho_2)d\Theta < 1. \tag{83}$$

Set $U_2 = U(x_0, \rho_2) \cap \Omega$. Subsequently, in the region U_2, the equation $T(x) = 0$ has to be uniquely solvable by y^*.

Proof. Let $z^* \in U_2$ be such that $T(z^*) = 0$. Then, by applying (A_2) and the condition (83), we obtain in turn for $M = \int_0^1 T'(y^* + \Theta(z^* - y^*))d\Theta$ that

$$\begin{aligned} \|[T'(x_0)]^{-1}(M - T'(x_0))\| &\leq \int_0^1 \kappa_0((1-\Theta)\|y^* - x_0\| + \Theta\|z^* - x_0\|)d\Theta \\ &\leq \int_0^1 \kappa_0((1-\Theta)\rho_1 + \Theta\rho_2)d\Theta < 1, \end{aligned}$$

Thus, the linear operator M is invertible. Hence, we obtain $z^* - y^* = M^{-1}(T(z^*) - T(y^*)) = M^{-1}(0) = 0$. Therefore, we conclude $z^* = y^*$. $\square$

Remark 4. *If all the conditions (A_1)–(A_5) hold in Proposition 1, then choose $y^* = x^*$ and $\rho_1 = a^*$.*

6. Applications

The applications are illustrated by the examples.

Example 1. *Looking back at the inspirational example, we are able to apply our hypothesis and see that all the assumptions are proved to be true. Applying (13) and $T'(\Sigma^*) = (1,1,1)^t$, we find the following:*

The old case we find for outcomes of past researchers [1,22] $\varkappa_0(u) = \varkappa(u) = \frac{e}{2}$ gives

$$\rho_0 = 0.245253.$$

Again, narrowing down $\varkappa_0(u)$ further, we have following two cases. Case $\varkappa_0(u) = \frac{e-1}{2}$ and $\varkappa(u) = \frac{e}{2}$ gives

$$\rho_1 = 0.324947$$

Case $\varkappa_0(u) = \frac{e-1}{2}$ and $\overline{\varkappa}(u) = \frac{e^{\frac{1}{(e-1)}}}{2}$ gives

$$\rho_2 = 0.382692$$

We clearly notice that

$$\rho_0 < \rho_1 < \rho_2.$$

Therefore, we are able to justify the advantages mentioned in the Remark 1 evaluation, i.e., it leads to the convergence domain of the proposed scheme of our study.

Example 2. *Let $\chi = Y = \Re$. We take*

$$T(u) = \int_0^u \left(1 + 2u \sin\frac{\pi}{u}\right) du, \ \forall u \in \Re.$$

So

$$T'(u) = \begin{cases} 1 + 2u\sin\frac{\pi}{u}, & u \neq 0, \\ 1, & u = 0, \end{cases}$$

Clearly, $\Sigma^* = 0$, a root for T. T' fulfills

$$||[T'(\Sigma^*)]^{-1}(T'(x) - T'(\Sigma^*))|| = \left|2u\sin\frac{\pi}{u}\right| \le 2|u - \Sigma^*|, \forall\, u \in \Re.$$

In the view of Theorem 4, for any $x_0 \in M(\Sigma^*, 1/6)$, we obtain an expression

$$||x_n - \Sigma^*|| \le C^{5^n - 1}||x_0 - \Sigma^*||, \ n = 1, 2, \cdots, C = \left(\frac{(8|x_0|^4).(|x_0| + |y_0|).(|x_0| + |z_0|)}{(1 - 2|x_0|)^3.|y_0|.|z_0|.|q_0|}\right).$$

Meanwhile, there seems to be no PIF $\varkappa$ that satisfies the inequality (6). Take note of the fact that since

$$||[T'(\Sigma^*)]^{-1}(T'(u) - T'(v^\theta))|| = \left|2u\sin\frac{\pi}{u} - 2v\theta\sin\frac{\pi}{v\theta}\right| = \frac{4}{2i+1},$$

for $u = 1/i, v = 1/i, \theta = \frac{2i}{2i+1}$ *and* $i = 1, 2, \cdots$ *Hence, we can say that possibly if there was a positively integrable function* $\varkappa$ *s.t., the relation* (6) *follows on* $M(\Sigma^*, \rho)$*; for some* $\rho > 0$*, it follows that* $\exists$ *some* $i_0 > 1$ *s.t.*

$$\int_0^{2\rho} \varkappa(u)du \geq \sum_{i=i_0}^{+\infty} \int_{\frac{4}{2i+1}}^{\frac{2}{i}} \varkappa(u)du \geq \sum_{i=i_0}^{+\infty} \frac{4}{2i+1} = +\infty,$$

that contradicts the above results. This mentioned example illustrates that Theorem 4 *is the consequence of Theorem* 3 *as critical enhancement if the convergence radius is to be ignored.*

Example 3 ([15])**.** *Choosing* $\chi = C[0,1]$ *and* $Y = C[0,1], \mathrm{Y} = \overline{M}(0,1)$, $\Sigma^* = 0$*. So, set* T *on the* Y*:*

$$T(p)(u) = p(u) - 5\int_0^1 u\theta p(\theta)^3 d\theta.$$

Then,

$$T'(p(s))(u) = s(u) - 15\int_0^1 u\theta p(\theta)^2 s(\theta)d\theta \; for \; all \; s \in \mathrm{Y}.$$

Therefore, we arrive at

$$\varkappa_0(u) = \frac{15}{2}u < \varkappa(u) = \overline{\varkappa}(u) = 15u.$$

Thereby, this leads to the same advantages as in Example 1 *by solving* (13) *and hence, it extends the scope of application of the scheme. In addition, over the previous work described in* [15]*, we have expanded the convergence domain, making our findings more beneficial.*

7. Conclusions

To estimate a locally unique solution, a local convergence criteria is successfully proposed for FSS using this new idea of weak $\varkappa$-average on a high-order scheme and the combination of weak/average radius Lipschitz/center Lipschitz criteria. In comparison to previous work in [15], our analysis is more beneficial in terms of the following advantages: sufficient weaker convergence criteria and a broader convergence domain. However, the scheme considered here is without a coefficient, which is a limitation, and this issue can be addressed by modifying the assumptions on the radius which the authors intend to take up in the future. This work has further scope of enhancement in condition for the scheme considered in this theory to make it applicable for semi-local and global domains. The proposed convergence criteria is superior over the existing convergence criteria for the FSS scheme of fifth order. By providing semi-local convergence results for incredibly broad majoring sequences, emphasis is on the broad applicability of results and their potential significance in the study of iterative schemes. In all, it is a contribution of new research directions in computational methods and numerical functional analysis.

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