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Nonlinear Stability of the Monotone Traveling Wave for the Isothermal Fluid Equations with Viscous and Capillary Terms

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Abstract: We prove the existence of the monotone traveling wave for the isothermal fluid equations with viscous and capillary terms by the planar dynamical system method. We obtain that the monotone traveling wave is asymptotically stable under the suitable perturbation. In the process of establishing the uniform a priori estimate, we dispose the capillary term reasonably according to the feature of the equations, and find the appropriate weighted function to overcome the difficulty caused by the non-convex pressure function.

Keywords: nonlinear stability; monotone traveling wave; capillary term; non-convex pressure; weighted energy estimate

MSC: 35B40; 35A24

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1. Introduction

We consider the nonlinear stability of the monotone traveling wave for the following isothermal fluid equations with the viscous and capillary terms [1] in the Lagrangian coordinates

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v)_x - \varepsilon u_{xx} + \delta v_{xxx} = 0. \end{cases} \tag{1}$$

In (1), $t > 0$ is the time, and $x \in R$ is the coordinate. u and v represent the velocity and specific volume, respectively. εu_{xx} is the viscous term, and δv_{xxx} is the capillary term, where the coefficients ε and δ are constants, satisfying $\varepsilon > 0$ and $\delta > 0$. The capillarity was first proposed by Korteweg [2], so the system with the capillary term is also called a Korteweg type [3–7]. The van del Waals pressure $p(v)$ is in the form of

$$p(v) = \frac{a}{v-b} - \frac{c}{v^2}. \tag{2}$$

(1) with (2) could be treated as a simple model to describe the liquid–gas phase transition [8–10]. M. Affouf and R.E. Caflisch [1] used its simplified form

$$p(v) = v - (v-2)^3 \tag{3}$$

to analyze the stability for rarefaction waves, shock waves, and phase jump of Equation (1) by a numerical method.

The energy estimate method [11,12] is usually used to theoretically analyze the asymptotic stability of the traveling wave. Applying this method, the stability of the rarefaction wave and shock wave for a one-dimensional compressible model with viscous gas

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v)_x = \mu(\frac{u_x}{v})_x, \end{cases} \tag{4}$$

has been studied in [13–15]. Z. Chen et al. [6,7] studied the large time behavior of the rarefaction wave and traveling wave for the following fluid equations

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v)_x = \mu(\frac{u_x}{v})_x + \kappa\frac{1}{v}(\frac{1}{v}(\frac{1}{v}(\frac{1}{v})_x)_x)_x, \end{cases} \tag{5}$$

where $\kappa\frac{1}{v}(\frac{1}{v}(\frac{1}{v}(\frac{1}{v})_x)_x)_x$ is the capillary term. W. Zhang et al. [16] studied the existence and asymptotic stability of the monotone traveling waves of Equations (1) with (3). In the above studies, the pressure $p(v)$ is supposed to satisfy

$$p'(v) < 0, \tag{6}$$

$$p''(v) > 0. \tag{7}$$

This paper mainly focuses on the case of Equations (1) with (2), in which the pressure $p(v)$ is more complex. We assume that its first-order derivative $p'(v)$ satisfies (6), while the second-order derivative $p''(v)$ will change the signal on (v_-, v_+), where the constants $v_\pm$ are the asymptotic values of the traveling wave throughout, i.e.,

$$p''(v)\begin{cases} > 0, & when \;\; v < v*, \\ < 0, & when \;\; v > v* \end{cases} \quad v* \in (v_-, v_+). \tag{8}$$

This means the pressure $p(v)$ is not strictly convex. The case with a non-convex pressure function has been investigated by some researchers [17–22], but the models they studied had no capillarity. In light of [6,7,17–22], we studied the nonlinear stability of the monotone traveling wave, when the system has a capillary effect and the pressure $p(v)$, chosen as (2), satisfies (6) and (8).

In Section 2, we qualitatively analyze the existence of monotone traveling waves by the planar dynamical system method. In Section 3, in order to settle the difficulty caused by the non-convex pressure $p(v)$, we find the appropriate weight function to establish the uniformly prior estimate. In this process, we dispose the capillary term reasonably by the structure of Equation (1) itself. The uniformly prior estimate can be used to explain the asymptotic behavior under the suitable perturbation.

In this paper, we use Young's inequality and the differential mean value theorem. To enhance readability, we list them as follows:

Young's Inequality Suppose $u, v, \eta > 0$, $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$uv \leq \frac{\eta}{p}u^p + \frac{\eta^{-q/p}}{q}v^q.$$

Differential Mean Value Theorem If function $f(x)$ meets the following conditions:

(1) $f(x)$ is continuous on close interval $[a, b]$;

(2) $f(x)$ is derivable on open interval (a, b).

Then, at least there is one point $\xi \in (a, b)$ that can make the equation $f(b) - f(a) = f'(\xi)(b - a)$ true.

Notations. $L^2(R)$ denotes the space of measurable functions on R which are square integrable, with the norm $||f||^2_{L^2} = ||f||^2 = \int_R f^2(x)dx$. $H^l(l \geq 0)$ denoting the Sobolev space, with the norm $||f||^2_{H^l} = ||f||^2_l = \sum\limits_{j=0}^{l} ||\partial^j_x f||^2$.

2. Traveling Wave and Main Results

Suppose that Equations (1) with (2) have the traveling wave $(V,U)(\xi) = (V,U)(x - ct) = (v,u)(t,x)$, where c is the wave speed satisfying

$$\begin{cases} -cV_\xi - U_\xi = 0, \\ -cU_\xi + p(V)_\xi = \varepsilon U_{\xi\xi} - \delta V_{\xi\xi\xi}. \end{cases} \tag{9}$$

Integrating the above formula on $(-\infty,\xi)$ and $(\xi,+\infty)$, respectively, yields

$$\begin{cases} -cV - U = a_1, \\ -cU + p(V) = \varepsilon U_\xi - \delta V_{\xi\xi} + a_2, \end{cases} \tag{10}$$

where $a_1 = -cv_\pm - u_\pm$, $a_2 = -p(v_\pm) - cu_\pm$. Then c satisfies

$$\begin{cases} -c(v_+ - v_-) = u_+ - u_-, \\ -c(u_+ - u_-) = p(v_+) - p(v_-), \end{cases} \tag{11}$$

which is the Rankine–Hugoniot condition.

When the viscous coefficient $\varepsilon = 0$ and capillary coefficient $\delta = 0$, the system (1) has two eigenvalues $\lambda_{1,2} = \pm\sqrt{-p'(v)}$. The wave speed c satisfies the Lax shock condition

$$\lambda_i^+ < c < \lambda_i^-, i = 1,2. \tag{12}$$

In this paper, we only discuss the case of $c > 0$, i.e.,

$$\sqrt{-p'(v_+)} < c < \sqrt{-p'(v_-)}. \tag{13}$$

The case of $c < 0$ can be discussed similarly.

Theorem 1 (Existence of the traveling wave)**.** *When $p'(v) < 0$, there exists a monotone traveling wave $(V,U)(\xi)$ in Equations (1) and (2), with $V_\xi > 0$ and $U_\xi < 0$, satisfying the Rankine–Hugoniot condition (11) and the Lax shock condition (13).*

Proof. We prove Theorem 1 with the planar dynamical system method [23].

From (9), the traveling wave $V(\xi)$ satisfies

$$\delta V_{\xi\xi\xi} + \varepsilon c V_{\xi\xi} + (c^2 V + p(V))_\xi = 0. \tag{14}$$

Integrating above on $(\xi,+\infty)$, we have

$$\delta V_{\xi\xi} + \varepsilon c V_\xi + (c^2 V + p(V) - g) = 0, \tag{15}$$

where g is an integral constant.

Letting $x = V(\xi)$ and $y = V_\xi(\xi)$, then (15) is equivalent to the planar dynamical system

$$\begin{cases} \dfrac{dx}{d\xi} = y, \\ \dfrac{dy}{d\xi} = -\dfrac{c\varepsilon}{\delta} y - \dfrac{c^2 x + p(x) - g}{\delta}. \end{cases} \tag{16}$$

We want to find the monotone traveling wave $(V,U)(\xi)$ of Equation (1), satisfying $V_\xi > 0$ and $U_\xi < 0$, with the asymptotic value $(v_\pm, u_\pm)$, as long as we find the bounded orbit connecting $v_\pm$, where $v_\pm$ are the real roots of $c^2 x + p(x) - g = 0$ satisfying $v_- < v_+$.

On the phase plane (x, y), we denote the singular points to be $P_1(v_-, 0)$ and $P_2(v_+, 0)$, at which the Jocabi matrix is

$$J(v_\pm, 0) = \begin{pmatrix} 0 & 1 \\ -\dfrac{1}{\delta}(c^2 + p'(v_\pm)) & -\frac{c\varepsilon}{\delta} \end{pmatrix}. \tag{17}$$

The characteristic polynomial of $J(v_\pm, 0)$ is $\epsilon^2 + \frac{c\varepsilon}{\delta}\epsilon + \frac{1}{\delta}(c^2 + p'(v_\pm))$. From the Lax shock condition (13), we have

$$\begin{cases} c^2 + p'(v_+) > 0, \\ c^2 + p'(v_-) < 0. \end{cases} \tag{18}$$

From the planar dynamical system theory, we know that $P_1(v_-, 0)$ is a saddle point, since $J(v_-, 0)$ has two real eigenvalues with opposite signs; $P_2(v_+, 0)$ is a stable node point, since $J(v_+, 0)$ has two real negative eigenvalues.

In the next step, we give the tendency of the separatrix at the right side of the saddle point P_1. For this purpose, we establish a triangle region, surrounded by the straight lines $P_1A:\ x = v_-$, $P_1P_2:\ y = 0$, and $P_2A:\ y = k(x - v_+)$, where $k < 0$ will be determined later. The triangle region is generalized non-tangential. Since the tangent slope of the orbits of system (16) at P_1A, P_1P_2 and P_2A are

$$\begin{aligned} &\frac{dy}{dx}\Big|_{(x,y)\in P_1A} = -\frac{c\varepsilon}{\delta} < 0, \\ &\frac{dy}{dx}\Big|_{(x,y)\in P_1P_2} = \infty, \\ &\frac{dy}{dx}\Big|_{(x,y)\in P_2A} = -\frac{c\varepsilon}{\delta} - \frac{c^2}{\delta k}\frac{c^2 v + p(v) - g}{v - v_+} \leq k. \end{aligned} \tag{19}$$

The third formula of (19) holds as long as k satisfies

$$-\frac{c\varepsilon}{2\delta} - \frac{1}{2}\sqrt{\frac{c^2\varepsilon^2}{\delta^2} - \frac{4(c^2 v + p(v) - g)}{\delta(v - v_+)}} \leq k \leq -\frac{c\varepsilon}{2\delta} + \frac{1}{2}\sqrt{\frac{c^2\varepsilon^2}{\delta^2} - \frac{4(c^2 v + p(v) - g)}{\delta(v - v_+)}}. \tag{20}$$

See details in Figure 1.

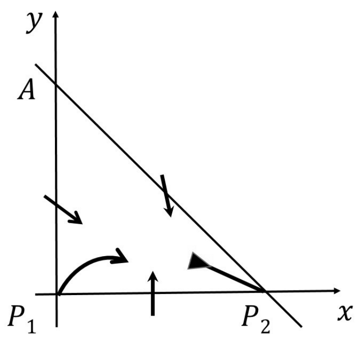

Figure 1. The tangent slopes of the orbit at P_1A, P_1P_2 and P_2A.

From the direction of the vector field described in Figure 1, we know that the separatrix line of the saddle point P_1 will not pass through the triangle region $\triangle P_1P_2A$. Note that P_2 is a stable node point, so the separatrix coming from the saddle point P_1 must trend to P_2. Hence, there must be a bounded orbit connecting the points P_1 and P_2, which corresponds to the monotone increasing traveling wave since $y = V_\xi > 0$. $\square$

Theorem 2 (Property of the traveling wave)**.** *There exists a constant C, independent of t, such that the monotone traveling wave $(V,U)(\xi)$ of Equations (1) and (2) obtained in Theorem 1 satisfies*

$$|V_\xi| \le C|v_+ - v_-|. \tag{21}$$

(21) is very important in the energy estimate, which can be obtained from (9) by direct calculations, so we omit the proof. See details in [16].

We discuss the traveling wave solution of Equations (1) and (2) with the initial condition

$$(v,u)(0,x) = (v_0,u_0)(x), \tag{22}$$

where v_0, u_0 are measurable functions, satisfying $v_0 \to v_\pm$ and $u_0 \to u_\pm$ as $x \to \pm\infty$. $v_\pm$ and $u_\pm$ are constants.

Let $(v,u)(t,x) = (V,U)(\xi) + (\Phi_\xi,\Psi_\xi)(t,\xi)$, i.e.,

$$\Phi(t,\xi) = \int_{-\infty}^{\xi}(v_0 - V)(x)dx,\ \Psi(t.\xi) = \int_{-\infty}^{\xi}(u_0 - U)(x)dx, \tag{23}$$

and then (Φ,Ψ) satisfy

$$\begin{cases} \Phi_{\xi_t} - c\Phi_{\xi\xi} - \Psi_{\xi\xi} = 0 \\ \Psi_{\xi_t} - c\Psi_{\xi\xi} + (p(V+\Phi_\xi) - p(V))_\xi = \varepsilon\Psi_{\xi\xi\xi} - \delta\Phi_{\xi\xi\xi\xi}. \end{cases} \tag{24}$$

We consider the system (24) with the initial condition

$$(\Phi_\xi,\Psi_\xi)(0,\xi) \triangleq (\Phi_{0\xi},\Psi_{0\xi})(\xi) = (v_0 - V, u_0 - U)(\xi), \tag{25}$$

where

$$\begin{matrix} \Phi(0,\xi) \triangleq \Phi_0(\xi) = \int_{-\infty}^{\xi}(v_0 - V)(x)dx, \\ \Psi(0,\xi) \triangleq \Psi_0(x) = \int_{-\infty}^{\xi}(u_0 - U)(x)dx, \end{matrix} \quad \forall\xi \in R \tag{26}$$

Suppose that

$$\int_{-\infty}^{+\infty}(v_0 - V)(x)dx = 0,\ \int_{-\infty}^{+\infty}(u_0 - U)(x)dx = 0. \tag{27}$$

Theorem 3 (Nonlinear stability)**.** *$(V,U)(\xi)$ is the monotone traveling wave obtained in Theorem 1. Then there exist constant $\lambda \ll 1$ and $\eta \ll 1$, such that Equation (1) has a unique global solution $(v,u)(t,x)$ with the initial value $(v_0,\ u_0)$, satisfying*

$$\begin{cases} v - V \in L^\infty(0,\infty;H^2) \cap L^2(0,\infty;H^2) \\ u - U \in L^\infty(0,\infty;H^1) \cap L^2(0,\infty;H^2), \end{cases} \tag{28}$$

if

$$|v_+ - v_-| \le \eta, \tag{29}$$

and

$$\begin{cases} N_0 = ||\Phi_0||_1 + ||\Psi_0|| \le \lambda, \\ ||\Phi_{0\xi\xi}||_1 + ||\Psi_{0\xi}||_1 < \infty. \end{cases} \tag{30}$$

Furthermore, the asymptotic behaviors of the global solution are shown in the form of

$$\lim_{t\to+\infty}\{||(v-V)(t)||_{W^{2,+\infty}(R)} + ||(u-U)(t)||_{W^{1,+\infty}(R)}\} = 0. \tag{31}$$

3. Proof to Theorem 3 on Nonlinear Stability

Integrate (24), and make the integration constant to be zero. Then we get

$$\begin{cases} \Phi_t - c\Phi_\xi - \Psi_\xi = 0 \\ \Psi_t - c\Psi_\xi + (p(V+\Phi_\xi) - p(V)) = \varepsilon\Psi_{\xi\xi} - \delta\Phi_{\xi\xi\xi}. \end{cases} \tag{32}$$

Linearizing (32) yields

$$\begin{cases} \Phi_t - c\Phi_\xi - \Psi_\xi = 0 \\ \Psi_t - c\Psi_\xi + p'(V)\Phi_\xi = \varepsilon\Psi_{\xi\xi} - \delta\Phi_{\xi\xi\xi} + F(V,\Phi_\xi), \end{cases} \tag{33}$$

among which $F(V,\Phi_\xi) = -p(V+\Phi_\xi) + p(V) + p'(V)\Phi_\xi$.

We consider the initial value problem (33) with

$$(\Phi,\Psi)(0,\xi) = (\Phi_0,\Psi_0)(\xi) \tag{34}$$

under the solution space

$$M(0,T) = \{(\Phi,\Psi) \in L^\infty(0,T;H^3\times H^2), \Phi_\xi \in L^2(0,T;H^2), \Psi_\xi \in L^2(0,T;H^2)\}.$$

Theorem 4. *Assume $(\Phi_0,\Psi_0) \in H^3 \times H^2$ and the condition in Theorem 3 holds. There exists a positive constant C, independent of t, s.t. (33) and (34) has a unique global solution (Φ,Ψ) in $M(0,\infty)$ under the condition (29) and (30). Furthermore, $\forall t \in [0,\infty)$,*

$$||\Phi(t)||_1^2 + ||\Psi(t)||^2 + \int_0^t (||\sqrt{V_\xi}\Phi||^2 + ||\Phi_\xi||_1^2 + ||\sqrt{V_\xi}\Psi||^2 + ||\Psi_\xi||^2)d\tau \le C(||\Phi_0||_1^2 + ||\Psi_0||^2). \tag{35}$$

$$||\Phi_\xi(t)||_2^2 + ||\Psi_\xi(t)||_1^2 + \int_0^t ||\Phi_\xi||_2^2 + ||\Psi_\xi||_2^2 d\tau \le C(||\Phi_0||_3^2 + ||\Psi_0||_2^2). \tag{36}$$

Theorem 3 could be proved by Theorem 4 directly. Actually, from the uniqueness of the global solution, its existence can be obtained by the global solution (Φ,Ψ) of the initial problem (33) and (34). Meanwhile, from (35) and (36), we know that

$$\int_0^{+\infty} \{||\Phi_\xi(t)||_2^2 + ||\Psi_\xi(t)||_2^2 + \frac{d}{dt}(||\Phi_\xi(t)||_2^2 + ||\Psi_\xi(t)||_1^2)\}dt < \infty \tag{37}$$

Furthermore, from the Sobolev inequality, then (31) in Theorem 3 holds.

Theorem 4 can be proved by two parts: the local existence and the a priori estimate. The first part can be arrived at in the standard way, so we omit it. We only give the proof for the a priori estimate. Combining the two parts, we can give the global existence by continuations.

Proposition 1 (Local existence)**.** *Suppose $||\Phi_0||_3 + ||\Psi_0||_2 \le \lambda_0$. There is a positive constant T_0 only depending on λ_0, such that the problem (33) and (34) has a unique small solution $(\Phi,\Psi) \in M(0,T_0)$.*

Proposition 2 (A priori estimate)**.** *$(\Phi,\Psi) \in M(0,T)$ is the solution of the initial problem (33) and (34). Denote*

$$N(t) = \sup_{0\le\tau\le t}\{\|(\Phi,\Psi)(\tau)\|_{H^1\times L^2}\}, t \in [0,T]. \tag{38}$$

Then there exists a positive constant $\lambda'(\le 2\lambda_0)$, independent of t, when $N(t) \le \lambda' << 1$, s.t. (35) and (36) hold.

Proof. We first give the weight energy estimate of (Φ,Ψ) on $L^\infty(H^1 \times L^2)$.

We multiply $a(V)\Phi$ and $w(V)\Psi$ with the first formula and second formula in (33), respectively, where $a(V) = -w(V)p'(V)$. Summing the results, then we have

$$\begin{aligned}&(\tfrac{a(V)}{2}\Phi^2+\tfrac{w(V)}{2}\Psi^2)_t+(-\tfrac{a(V)}{2}\Phi^2-\tfrac{w(V)}{2}\Psi^2-a(V)\Phi\Psi)_\xi+\tfrac{ca(V)_\xi}{2}\Phi^2+\tfrac{cw(V)_\xi}{2}\Psi^2+\varepsilon w(V)\Psi_\xi^2\\&=-a(V)_\xi\Phi\Psi-\delta w(V)\Phi_{\xi\xi\xi}\Psi-\varepsilon w(V)_\xi\Psi_\xi\Psi+w(V)F\Psi\end{aligned}\tag{39}$$

To simplify the term $-\delta w(V)\Phi_{\xi\xi\xi}\Psi$ in (39), we differentiate the first formula in (33) with respect to ξ, and multiply $\delta w(V)\Phi_\xi$. Summing (39), then we get

$$\begin{aligned}&(\tfrac{a(V)}{2}\Phi^2+\tfrac{w(V)}{2}\Psi^2+\tfrac{\delta w(V)}{2}\Phi_\xi^2)_t+\tfrac{ca(V)_\xi}{2}\Phi^2+\tfrac{c\delta w(V)_\xi}{2}\Phi_\xi^2+\tfrac{cw(V)_\xi}{2}\Psi^2+\varepsilon w(V)\Psi_\xi^2\\&+(-\tfrac{a(V)}{2}\Phi^2-\tfrac{w(V)}{2}\Psi^2-a(V)\Phi\Psi-\tfrac{c\delta w(V)}{2}\Phi_\xi^2+\delta w(V)\Phi_{\xi\xi}\Psi-\delta w(V)\Phi_\xi\Psi_\xi)_\xi\\&=-a(V)_\xi\Phi\Psi-\varepsilon w(V)_\xi\Psi_\xi\Psi+\delta w(V)_\xi\Phi_\xi\Psi_\xi-\delta w(V)_\xi\Phi_{\xi\xi}\Psi_\xi+w(V)F\Psi\end{aligned}\tag{40}$$

Note that $(-\frac{a(V)}{2}\Phi^2-\frac{w(V)}{2}\Psi^2-a(V)\Phi\Psi-\frac{c\delta w(V)}{2}\Phi_\xi^2+\delta w(V)\Phi_{\xi\xi}\Psi-\delta w(V)\Phi_\xi\Psi_\xi)_\xi$ will disappear after integrating, so we write it as $\{\ldots\}_\xi$ for short in the following.

In order to find the appropriate weight function $w(V)$, we rewrite (40) as

$$\begin{aligned}&(\tfrac{a(V)}{2}\Phi^2+\tfrac{w(V)}{2}\Psi^2+\tfrac{\delta w(V)}{2}\Phi_\xi^2)_t+\{\ldots\}_\xi+\tfrac{a(V)_\xi}{2c}(c\Phi+\Psi)^2+\tfrac{\varepsilon w(V)}{4}(\Psi_\xi+\tfrac{w(V)_\xi}{w(V)}\Psi)^2\\&\quad+\tfrac{c\delta w(V)_\xi}{2}\Phi_\xi^2+\tfrac{\varepsilon w(V)}{2}\Psi_\xi^2+(\tfrac{cw(V)_\xi}{2}-\tfrac{a(V)_\xi}{2c}-\tfrac{\varepsilon w(V)}{2}(\tfrac{w(V)_\xi}{w(v)})^2)\Psi^2\\&=\delta w(V)_\xi\Phi_\xi\Psi_\xi-\delta w(V)_\xi\Phi_{\xi\xi}\Psi_\xi+w(V)F\Psi\end{aligned}\tag{41}$$

To obtain the lower order a priori estimate, the weight function $w(V)$ should be chosen to make

$$\frac{cw(V)_\xi}{2}-\frac{1}{2c}a(V)_\xi-\frac{\varepsilon w(V)}{2}(\frac{w(V)_\xi}{w(V)})^2>0$$

hold. Further,

$$\begin{aligned}&\tfrac{cw(V)_\xi}{2}-\tfrac{1}{2c}a(V)_\xi-\tfrac{\varepsilon w(V)}{2}(\tfrac{w(V)_\xi}{w(V)})^2\\&=\tfrac{w(V)}{2c}[-c\varepsilon(\tfrac{w(V)_\xi}{w(V)})^2+c^2\tfrac{w(V)_\xi}{w(V)}-\tfrac{a(V)_\xi}{w(V)}]\\&=\tfrac{w(V)}{2c}V_\xi[-c\varepsilon V_\xi(\tfrac{w'(V)}{w(V)})^2+(c^2+p'(V))\tfrac{w'(V)}{w(v)}+p''(V)]>0.\end{aligned}\tag{42}$$

We could choose

$$w(V)=\begin{cases}1, & p''(V)>0\\ \dfrac{1}{(c^2+p'(V))^2}, & p''(V)<0\end{cases}\tag{43}$$

Note that, under the selection of (43), the coefficient of all terms at the left of (41) is positive.

On the other hand, from the Schwartz inequality, the first term at the right of (41) satisfies

$$\delta w(V)_\xi\Phi_\xi\Psi_\xi\le\frac{c\delta\theta}{2}w(V)_\xi\Phi_\xi^2+\frac{\delta}{2c\theta}w(V)_\xi\Psi_\xi^2,\theta<1.\tag{44}$$

Integrating (41) on $(0,t)\times R$ and applying (43) and (44), we obtain

$$\begin{aligned}&||\Phi(t)||_1^2+||\Psi(t)||^2+\textstyle\int_0^t(||\sqrt{V_\xi}\Phi||_1^2+||\sqrt{V_\xi}\Psi||^2+||\Psi_\xi||^2)d\tau\\&\le C(||\Phi_0||_1^2+||\Psi_0||^2+\textstyle\int_0^t||\sqrt{V_\xi}\Psi_\xi||^2d\tau+\int_0^t\int_R(|w(V)_\xi\Phi_{\xi\xi}\Psi_\xi|+|F\Psi|)d\xi d\tau).\end{aligned}\tag{45}$$

From Theorem 2, $V_\xi\le c|v_+-v_-|$, if $|v_+-v_-|\le\eta<<1$, and then we obtain

$$\begin{aligned}&||\Phi(t)||_1^2+||\Psi(t)||^2+\textstyle\int_0^t(||\sqrt{V_\xi}\Phi||_1^2+||\sqrt{V_\xi}\Psi||^2+||\Psi_\xi||^2)d\tau\\&\le C(||\Phi_0||_1^2+||\Psi_0||^2+\textstyle\int_0^t\int_R(|w(V)_\xi\Phi_{\xi\xi}\Psi_\xi|+|F\Psi|)d\xi d\tau).\end{aligned}\tag{46}$$

To control $\int_0^t \int_R |w(V)_\xi \Phi_{\xi\xi} \Psi_\xi| d\xi d\tau$ in (46), we give the estimate $\Phi_{\xi\xi}$ on L^2. Multiply Φ_ξ with $(33)_2$, and then we get

$$(\frac{\varepsilon}{2}\Phi_\xi^2)_t - p'(V)\Phi_\xi^2 + \delta\Phi_{\xi\xi}^2 + \{\dots\}_\xi = (\Psi\Phi_\xi)_t + \Psi_\xi^2 + F\Phi_\xi. \tag{47}$$

Integrating (47) on $(0,t) \times R$, and combining (46), we have

$$||\Phi_\xi(t)||^2 + \int_0^t (||\Phi_\xi||^2 + ||\Phi_{\xi\xi}||^2) d\tau \le C(||\Phi_0||_1^2 + ||\Psi_0||^2 + \int_0^t \int_R |F\Phi_\xi| d\xi d\tau). \tag{48}$$

Moreover,

$$\begin{aligned} &||\Phi(t)||_1^2 + ||\Psi(t)||^2 + \int_0^t (||\sqrt{V_\xi}\Phi||^2 + ||\Phi_\xi||_1^2 + ||\sqrt{V_\xi}\Psi||^2 + ||\Psi_\xi||^2) d\tau \\ &\le C(||\Phi_0||_1^2 + ||\Psi_0||^2 + \int_0^t \int_R (|w(V)_\xi \Phi_{\xi\xi} \Psi_\xi| + |F\Psi| + |F\Phi_\xi|) d\xi d\tau). \end{aligned} \tag{49}$$

By Young's inequality,

$$\int_0^t \int_R |w(V)_\xi \Phi_{\xi\xi} \Psi_\xi| d\xi d\tau) \le \mu \int_0^t \int_R \Psi_\xi^2 d\xi d\tau + C(\mu) \int_0^t \int_R (\sqrt{V_\xi}\Phi_{\xi\xi})^2 d\xi d\tau, \tag{50}$$

where $0 < \mu << 1$.

From the Taylor expansion,

$$F(V, \Phi_\xi) = -p(V + \Phi_\xi) + p(V) + p'(V)\Phi_\xi \sim O(\Phi_\xi^2), |\Phi_\xi| \to 0,$$

which means $|\Phi_\xi|^2$ is the dominant term, since it is much bigger that $|\Phi_\xi|^3$, $|\Phi_\xi|^4, \dots$, as $|\Phi_\xi| \to 0$. Since $N(t) = \sup\limits_{0 \le \tau \le t} \{\|(\Phi, \Psi)(\tau)\|_{H^1 \times L^2}\}, t \in [0, T]$ in (38), then

$$\int_0^t \int_R (|F\Psi| + |F\Phi_\xi|) d\xi d\tau \le CN(t) \int_0^t ||\Phi_\xi||^2 d\tau \tag{51}$$

From the smallness of μ, V_ξ and $N(t)$, we obtain the lower-order weight energy estimate (35).

Next, we give the weight energy estimate of (Φ_ξ, Ψ_ξ) on $L^\infty(H^1 \times L^2)$.

Differentiating the two formulas in (33) with respect to ξ,

$$\begin{cases} \Phi_{\xi t} - c\Phi_{\xi\xi} - \Psi_{\xi\xi} = 0, \\ \Psi_{\xi t} - c\Psi_{\xi\xi} + p'(V)\Phi_{\xi\xi} + p'(V)_\xi \Phi_\xi = \varepsilon\Psi_{\xi\xi\xi} - \delta\Phi_{\xi\xi\xi\xi} + F_\xi(V, \Phi_\xi), \end{cases} \tag{52}$$

and multiplying $a(V)\Phi_\xi$ and $w(V)\Psi_\xi$ with the first formula and second formula in (52), respectively. Summing the results and noting $\Phi_{\xi\xi\xi}\Psi_{\xi\xi} = \Phi_{\xi\xi\xi}(\Phi_{\xi t} - c\Phi_{\xi\xi})$, we can obtain

$$\begin{aligned} &(\tfrac{a(V)}{2}\Phi_\xi^2 + \tfrac{w(V)}{2}\Psi_\xi^2 + \tfrac{\delta w(V)}{2}\Phi_{\xi\xi}^2)_t + \tfrac{ca(V)_\xi}{2}\Phi_\xi^2 + \tfrac{cw(V)_\xi}{2}\Psi_\xi^2 + \varepsilon w(V)\Psi_{\xi\xi}^2 + \tfrac{c\delta w(V)_\xi}{2}\Phi_{\xi\xi}^2 + \{\dots\}_\xi \\ &= -a(V)_\xi \Phi_\xi \Psi_\xi - \varepsilon w(V)_\xi \Psi_{\xi\xi} \Psi_\xi - \delta w(V)_\xi \Phi_{\xi\xi} \Psi_{\xi\xi} - w(V) p'(V)_\xi \Phi_\xi \Psi_\xi \\ &\quad + \delta w(V)_\xi \Phi_{\xi\xi\xi} \Psi_{\xi\xi} + w(V) F_\xi \Psi \end{aligned} \tag{53}$$

Simplify (53), similarly with (41), and $w(V)$ is also chosen as (43). Integrating (53) on $(0,t) \times R$, and the terms $-a(V)_\xi \Phi_\xi \Psi_\xi$ and $-\varepsilon w(V)_\xi \Psi_{\xi\xi} \Psi_\xi$ can be controlled by the left. We could have

$$\begin{aligned} &||\Phi_\xi(t)||_1^2 + ||\Psi_\xi(t)||^2 + \int_0^t (||\sqrt{V_\xi}\Phi_\xi||_1^2 + ||\sqrt{V_\xi}\Psi_\xi||^2 + ||\Psi_{\xi\xi}||^2) d\tau \\ &\le C(||\Phi_{0\xi}||_1^2 + ||\Psi_{0\xi}||^2 + \int_0^t \int_R (|p'(v)_\xi \Phi_\xi \Psi_\xi| + |w(V)_\xi \Phi_{\xi\xi\xi} \Psi_{\xi\xi}| + |F_\xi \Psi|) d\xi d\tau. \end{aligned} \tag{54}$$

To obtain the estimate of $\Phi_{\xi\xi\xi}$ on L^2, we multiply $\Phi_{\xi\xi}$ on the second formula in (52). Then we have

$$(\frac{\varepsilon}{2}\Phi_{\xi\xi}^2)_t - p'(V)\Phi_{\xi\xi}^2 + \delta\Phi_{\xi\xi\xi}^2 + \{\dots\}_\xi = (\Psi_\xi\Phi_{\xi\xi})_t + \Psi_{\xi\xi}^2 + p'(V)_\xi \Phi_{\xi\xi}\Phi_\xi + F_\xi\Phi_{\xi\xi}. \tag{55}$$

Integrating (55) on $(0,t)\times R$, and combining the lower-order estimate (35), we have

$$\begin{aligned}&||\Phi_{\xi\xi}(t)||^2+\int_0^t(||\Phi_\xi||^2+||\Phi_{\xi\xi}||^2)d\tau\\&\le C(||\Phi_{0\xi}||_1^2+||\Psi_{0\xi}||^2+\int_0^t\int_R(|p'(V)_\xi\Phi_{\xi\xi}\Phi_\xi|+|F_\xi\Phi_{\xi\xi}|)d\xi d\tau.\end{aligned}\tag{56}$$

Similarly, by Young's inequality and the smallness of V_ξ, then we have

$$\begin{aligned}&||\Phi_\xi(t)||_1^2+||\Psi_\xi(t)||^2+\int_0^t(||\sqrt{V_\xi}\Phi_\xi||^2+||\Phi_{\xi\xi}||_1^2+||\sqrt{V_\xi}\Psi_\xi||^2+||\Psi_{\xi\xi}||^2)d\tau\\&\le C(||\Phi_{0\xi}||_1^2+||\Psi_{0\xi}||^2+\int_0^t\int_R(|F_\xi\Psi|+|F_\xi\Phi_{\xi\xi}|)d\xi d\tau).\end{aligned}\tag{57}$$

To give the estimate of $(\Phi_{\xi\xi\xi},\Psi_{\xi\xi})$ on $L^\infty(L^2\times L^2)$, we multiply $-\Psi_{\xi\xi\xi}$ on the second formula in (52), and note that $\Phi_{\xi\xi}\Psi_{\xi\xi\xi}=\Phi_{\xi\xi}(\Phi_{\xi\xi t}-c\Phi_{\xi\xi\xi})$. Then we obtain

$$(\frac{1}{2}\Psi_{\xi\xi}^2-\frac{p'(V)}{2}\Phi_{\xi\xi}^2+\frac{\delta}{2}\Phi_{\xi\xi\xi}^2)_t+\varepsilon\Psi_{\xi\xi\xi}^2-\frac{cp'(V)_\xi}{2}\Phi_{\xi\xi}^2+\{\ldots\}_\xi=p'(V)_\xi\Phi_\xi\Psi_{\xi\xi\xi}-F_\xi\Psi_{\xi\xi\xi}.\tag{58}$$

Integrating (58) on $(0,t)\times R$, then combining (57) and Young's inequality, the smallness of V_ξ, we have

$$\begin{aligned}&||\Phi_\xi(t)||_2^2+||\Psi_\xi(t)||_1^2+\int_0^t(||\sqrt{V_\xi}\Phi_\xi||^2+||\Phi_{\xi\xi}||_1^2+||\sqrt{V_\xi}\Psi_\xi||^2+||\Psi_{\xi\xi}||^2+||\Psi_{\xi\xi\xi}||^2)d\tau\\&\le C(||\Phi_{0\xi}||_2^2+||\Psi_{0\xi}||_1^2+\int_0^t\int_R|F_\xi\Psi_{\xi\xi\xi}|)d\xi d\tau).\end{aligned}\tag{59}$$

From (35), (57) and (59), we can obtain

$$\begin{aligned}&||\Phi(t)||_3^2+||\Psi(t)||_2^2+\int_0^t||\sqrt{V_\xi}\Phi||^2+||\sqrt{V_\xi}\Psi||^2+||\Phi_\xi||_2^2+||\Psi_\xi||_2^2d\tau\\&\le C(||\Phi_0||_3^2+||\Psi_0||_2^2+\int_0^t\int_R(|F_\xi\Psi|+|F_\xi\Phi_{\xi\xi}|+|F_\xi\Psi_{\xi\xi\xi}|)d\xi d\tau).\end{aligned}\tag{60}$$

Applying the differential mean value theorem,

$$|F_\xi|=|\theta_1p'''(V+\theta_3\Phi_\xi)V_\xi\Phi_\xi^2+p''(V+\theta_2\Phi_\xi)\Phi_\xi\Phi_{\xi\xi}|\le C(V_\xi\Phi_\xi^2+|\Phi_\xi\Phi_{\xi\xi}|),$$

where $0<\theta_1,\theta_2,\theta_3<1$, by Young's inequality,

$$\begin{aligned}&\int_0^t\int_R|F_\xi\Psi|d\xi d\tau\le C\int_0^t\int_R(V_\xi\Phi_\xi^2|\Psi|+|\Phi_\xi\Phi_{\xi\xi}||\Psi|)d\xi d\tau\\&\le CN(t)\int_0^t||\sqrt{V_\xi}\Phi_\xi(\tau)||^2d\tau+CN(t)\int_0^t\int_R(|\Phi_\xi\Phi_{\xi\xi}|)d\xi d\tau\\&\le CN(t)\int_0^t||\sqrt{V_\xi}\Phi_\xi(\tau)||^2d\tau+C\mu\int_0^t||\Phi_\xi||^2d\tau+C(\mu)N(t)\int_0^t||\Phi_{\xi\xi}||^2d\tau,\end{aligned}\tag{61}$$

$$\int_0^t\int_R|F_\xi\Phi_{\xi\xi}|d\xi d\tau\le C\mu\int_0^t||\sqrt{V_\xi}\Phi_\xi||^2d\tau+C(\mu)N(t)\int_0^t||\Phi_{\xi\xi}||^2d\tau+CN(t)\int_0^t||\Phi_{\xi\xi}||^2d\tau\tag{62}$$

$$\int_0^t\int_R|F_\xi\Psi_{\xi\xi\xi}|d\xi d\tau\le C\mu\int_0^t||\sqrt{V_\xi}\Phi_\xi||^2d\tau+C(\mu)N(t)\int_0^t||\Psi_{\xi\xi\xi}||^2d\tau+\mu\int_0^t||\Phi_{\xi\xi}||^2d\tau\tag{63}$$

From the smallness of μ and $N(t)$, (36) holds naturally. □

4. Discussion

In this paper, we investigated the nonlinear stability of the monotone traveling wave for the isothermal fluid equations with viscous and capillary terms under the suitable perturbation. It should be pointed out that, in the proof of Proposition 2, we only use the smallness of $\sup_{0\le\tau\le t}\{\|(\Phi,\Psi)(\tau)\|_{H^1\times L^2}\}$, which can be controlled by $||(\Phi_0,\Psi_0)||_{H^1\times L^2}$ from (35). So in the condition of Theorem 3, we assume $||(\Phi_0,\Psi_0)||_{H^1\times L^2}<\lambda<<1$. However, for the higher-order derivative of the perturbation $||(\Phi_{0\xi\xi},\Psi_{0\xi})||_{H^1\times H^1}$, we only need to assume it to be bounded. See details in (61)–(63). This is enough to ensure the prior estimate holds.

The condition in Theorem 3 is only a sufficient condition. In future research, we want to find the optimal condition which the initial perturbation satisfies to make the traveling wave stable, and will search some counter-examples by numerical simulation. We are also interested in the linear stability [24–26], the blowup phenomenon [27], and the problem of control for nonlinear systems [28–30] in the future.

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