mathematics

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Oscillation Criteria for Advanced Half-Linear Differential Equations of Second Order

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Abstract: In this paper, we find new oscillation criteria for second-order advanced functional half-linear differential equations. Our results extend and improve recent criteria for the same equations established previously by several authors and cover the existing classical criteria for related ordinary differential equations. We give some examples to illustrate the significance of the obtained results.

Keywords: oscillation; second order; half-linear; advanced differential equations

MSC: 34K11; 34C10; 34K25

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Citation: Hassan, T.S.; Kong, Q.; El-Matary, B.M. Oscillation Criteria for Advanced Half-Linear Differential Equations of Second Order. *Mathematics* **2023**, *11*, 1385. https://doi.org/10.3390/math11061385

Academic Editors: Carmen Chicone and Luigi Rodino

Received: 14 January 2023
Revised: 2 March 2023
Accepted: 10 March 2023
Published: 13 March 2023

1. Introduction

Differential equations with deviating arguments are indispensable in simulating the numerous processes in all areas of science. It is well known that the rate of change of a process described by a delay differential equation depends on how the process has changed in the past. In such a model, the prediction for the future time is logically accurate and dependable, which leads to simultaneous descriptions of a variety of qualitative phenomena such as periodicity, oscillation, and stability; see [1,2].

On the other hand, advanced differential equations have been derived from a variety of practical areas where the rates of evolution depends on both the present and the future. In order to reflect the influence of potential future factors in the decision-making process, we must include an advanced term in the equation. For instance, population dynamics, economic issues, or mechanical control engineering are typical fields where the dynamical growth is affected by future factors (see [1] for details).

Oscillation has been a problem for applied researchers which was rooted from mechanical vibrations and have been developed widely in the sciences and engineering. The oscillation models often contain delay or advanced terms to reflect the dependence of solutions on the past or future times. There has been extensive studies of oscillations for delay equations, see [3–16]; but studies of advanced oscillations are relatively few, see [17–20].

In this paper, we study the advanced oscillations, but focus on the half-linear case. As an extension of the Laplace equation, the half-linear differential equations have important applications in many areas such as non-Newtonian fluid theory, the turbulent flow of a polytrophic gas in a porous media, and mathematical biology; see, e.g., [21–32] for more details.

Now, we consider second-order half-linear advanced differential equations of the form

$$\left(r(t)\phi(x'(t))\right)' + p(t)\phi(x(\sigma(t))) = 0, \tag{1}$$

where $t \in [t_0, \infty)$ with $t_0 \geq 0$ is a constant, $\phi(u) := |u|^{\gamma-1}u$, $\gamma > 0$, p is a positive continuous function on $[t_0, \infty)$, σ is a continuous function satisfying $\sigma(t) \geq t$ for $t \in [t_0, \infty)$ and $\lim_{t\to\infty} \sigma(t) = \infty$, and r is a positive continuous function on $[t_0, \infty)$ such that

$$R(t) := \int_{t_0}^{t} \frac{\mathrm{d}\tau}{r^{1/\gamma}(\tau)} \to \infty \text{ as } t \to \infty. \tag{2}$$

By a solution of Equation (1) we mean a non-trivial real-valued function $x \in \mathrm{C}^1[T, \infty)$ with $T \in [t_0, \infty)$ such that $x', r(t)\phi(x'(t)) \in \mathrm{C}^1[T, \infty)$ and $x(t)$ satisfies Equation (1) on $[T, \infty)$. We shall not investigate solutions that vanish in the neighbourhood of infinity. A solution $x(t)$ of Equation (1) is said to be oscillatory if it is neither eventually positive nor eventually negative; otherwise, it is said to be non-oscillatory. Equation (1) is said to be oscillatory if all its solutions are oscillatory. We first review some existing oscillation results for differential equations that are related to Equation (1).

Fite [33] studied the oscillatory behaviour of solutions of the second-order linear ordinary differential equation

$$x''(t) + p(t)x(t) = 0, \tag{3}$$

and showed that if

$$\int_{t_0}^{\infty} p(\tau)\mathrm{d}\tau = \infty, \tag{4}$$

then Equation (3) is oscillatory. Note that if Equations (2) and (4) hold, then the Sturm–Liouville linear equation

$$\big(r(t)x'(t)\big)' + p(t)x(t) = 0 \tag{5}$$

is oscillatory by the Leighton–Wintner oscillation criterion, see [34]. Hille [35] improved Condition (4) and proved that if

$$t\int_{t}^{\infty} p(\tau)\mathrm{d}\tau \geq \beta > \frac{1}{4}, \tag{6}$$

then Equation (3) is oscillatory. For the case of Equation (2) and

$$\int_{t_0}^{\infty} p(\tau)\mathrm{d}\tau < \infty,$$

the Hille-type criterion for Equation (5) has been established and proven that if

$$\int_{t_0}^{t} \frac{\mathrm{d}\tau}{r(\tau)} \int_{t}^{\infty} p(\tau)\,\mathrm{d}\tau \geq \beta > \frac{1}{4}, \tag{7}$$

then Equation (5) is oscillatory, see, e.g., ([36], Chap. 2). These results has been extended to the half-linear ordinary differential equation

$$\big(r(t)\phi(x'(t))\big)' + p(t)\phi(x(t)) = 0, \tag{8}$$

and showed that if

$$R(t)\left(\int_{t}^{\infty} p(\tau)\,\mathrm{d}\tau\right)^{1/\gamma} \geq \beta > \frac{\gamma}{(1+\gamma)^{(1+\gamma)/\gamma}}, \tag{9}$$

then Equation (8) is oscillatory, see ([37], Section 3.1.1). Erbe [38] generalized the Hille-type Condition (6) to the delay differential equation

$$x''(t) + p(t)x(\sigma(t)) = 0, \tag{10}$$

where $\sigma(t) \le t$, and obtained if

$$t\int_t^\infty \frac{\sigma(\tau)}{\tau} p(\tau)\,\mathrm{d}\tau \ge \beta > \frac{1}{4}, \tag{11}$$

then Equation (10) is oscillatory. For oscillation of second-order advanced differential equations, Kusano [39] established comparison results and showed that oscillation of advanced differential equation

$$\left(r(t)x'(t)\right)' + p(t)x(\sigma(t)) = 0, \tag{12}$$

where $\sigma(t) \ge t$, follows from the oscillation of the ordinary differential equation

$$\left(r(t)x'(t)\right)' + p(t)x(t) = 0. \tag{13}$$

Furthermore, Džurina [40] presented new comparison results and showed that the oscillation of functional advanced differential Equation (12) follows from the oscillation of the ordinary differential equation

$$\left(r(t)x'(t)\right)' + \left(\frac{R(\sigma(t))}{R(t)}\right)^{\alpha_1} p(t)x(t) = 0$$

with $\alpha_1 > 0$ such that $R(t)\int_t^\infty p(\tau)\,\mathrm{d}\tau \ge \alpha_1$ and also proved that if

$$\frac{R(\sigma(t))}{R(t)} \ge \lambda > 1 \tag{14}$$

eventually and there exists a positive integer n such that $\alpha_i \le 1/4$ for $i = 1, 2, \ldots, n-1$ and $\alpha_n > 1/4$, where $\alpha_i = \lambda^{\alpha_{i-1}}\alpha_1$, $i = 2, 3, \ldots, n$, then Equation (12) is oscillatory. The following is a result for the oscillation of half-linear advanced differential Equation (1) obtained in [41].

Theorem 1. *Suppose there exists a constant β such that*

$$R(t)\left(\int_t^\infty p(\tau)\,\mathrm{d}\tau\right)^{1/\gamma} \ge \beta > \frac{\gamma}{(1+\gamma)^{(1+\gamma)/\gamma}}. \tag{15}$$

Then Equation (1) is oscillatory.

Since the advanced argument $\sigma(t)$ is not included in the aforementioned Condition (15), this criterion is more appropriate for the ordinary differential equation

$$\left(r(t)\phi(x'(t))\right)' + p(t)\phi(x(t)) = 0$$

and does not reveal the fact of how the oscillation depends on the advanced argument.

More specifically, if

$$R(t)\left(\int_t^\infty p(\tau)\,\mathrm{d}\tau\right)^{1/\gamma} \ge \beta \text{ with } \beta \le \frac{\gamma}{(1+\gamma)^{(1+\gamma)/\gamma}},$$

then Theorem 1 fails to work.

It should be noted that the research in this paper was strongly motivated by the contributions of [34–37,40,41]. The purpose of this paper is to modify Condition (15) to include the role of $\sigma(t)$ to obtain certain sharper conditions for the oscillation of Equation (1). We will show that our criteria cover the existing ones for ordinary differential equations, and give examples to show their significance. The reader is directed to papers concerning Hille-type criteria [42–48] as well as the sources listed therein.

2. Main Results

Without further mention, we assume that all the improper integrals involved are convergent in the following theorems. Otherwise, we find that Equation (1) is oscillatory, see [33]. We begin this section with two preliminary lemmas.

Lemma 1 (see [49])**.** *Suppose $x(t)$ is an eventually positive solution of Equation (1). Then*

$$x'(t) > 0 \quad \text{and} \quad \big(r(t)\phi(x'(t))\big)' < 0 \tag{16}$$

eventually.

Lemma 2. *Suppose $x(t)$ is a positive solution of Equation (1). Let $\beta_0 = 0$. Suppose there exist $n \in \mathbb{N}$ and $\beta_i > 0$, $i = 1, 2, \ldots, n$ such that*

$$R(t)\left(\int_t^\infty \left(\frac{R(\sigma(\tau))}{R(\tau)}\right)^{\gamma\beta_{i-1}} p(\tau)\,\mathrm{d}\tau\right)^{1/\gamma} \geq \beta_i \tag{17}$$

eventually, then

$$\left(\frac{x(t)}{R^{\beta_i}(t)}\right)' \geq 0 \tag{18}$$

eventually.

Proof. We show this by induction. Since $x'(t) > 0$ eventually, from Equation (1) we have that for large t,

$$r(t)\big(x'(t)\big)^\gamma \geq \int_t^\infty p(\tau)x^\gamma(\sigma(\tau))\,\mathrm{d}\tau \geq \int_t^\infty p(\tau)x^\gamma(\tau)\,\mathrm{d}\tau \geq x^\gamma(t)\int_t^\infty p(\tau)\,\mathrm{d}\tau.$$

Therefore,

$$\begin{aligned}
\left(\frac{x(t)}{R^{\beta_1}(t)}\right)' &= \frac{1}{R^{2\beta_1}(t)}\left[R^{\beta_1}(t)x'(t) - \beta_1\frac{R^{\beta_1-1}(t)}{r^{1/\gamma}(t)}x(t)\right] \\
&= \frac{1}{r^{1/\gamma}(t)R^{\beta_1+1}(t)}\left[R(t)r^{1/\gamma}(t)x'(t) - \beta_1 x(t)\right] \\
&\geq \frac{x(t)}{r^{1/\gamma}(t)R^{\beta_1+1}(t)}\left[R(t)\left(\int_t^\infty p(\tau)\,\mathrm{d}\tau\right)^{1/\gamma} - \beta_1\right] \geq 0.
\end{aligned}$$

Then Equation (18) holds for $i = 1$. Assume Equation (18) holds for $i = k \in \mathbb{N}$, i.e.,

$$\left(\frac{x(t)}{R^{\beta_k}(t)}\right)' \geq 0 \text{ eventually.}$$

This together with Equation (1) shows that

$$\begin{aligned}
r(t)\big(x'(t)\big)^\gamma &\geq \int_t^\infty p(\tau)x^\gamma(\sigma(\tau))\,\mathrm{d}\tau \geq \int_t^\infty \left(\frac{R(\sigma(\tau))}{R(\tau)}\right)^{\gamma\beta_k} x^\gamma(\tau)p(\tau)\,\mathrm{d}\tau \\
&\geq x^\gamma(t)\int_t^\infty \left(\frac{R(\sigma(\tau))}{R(\tau)}\right)^{\gamma\beta_k} p(\tau)\,\mathrm{d}\tau.
\end{aligned}$$

Therefore,

$$\left(\frac{x(t)}{R^{\beta_{k+1}}(t)}\right)' = \frac{1}{R^{2\beta_{k+1}}(t)}\left[R^{\beta_{k+1}}(t)x'(t) - \beta_n\frac{R^{\beta_{k+1}-1}(t)}{r^{1/\gamma}(t)}x(t)\right]$$

$$
\begin{aligned}
&= \frac{1}{r^{1/\gamma}(t)R^{\beta_{k+1}+1}(t)}\left[R(t)r^{1/\gamma}(t)x'(t)-\beta_n x(t)\right] \\
&\geq \frac{x(t)}{r^{1/\gamma}(t)R^{\beta_{k+1}+1}(t)}\left[R(t)\left(\int_t^\infty\left(\frac{R(\sigma(\tau))}{R(\tau)}\right)^{\gamma\beta_k}p(\tau)\,\mathrm{d}\tau\right)^{1/\gamma}-\beta_n\right]\geq 0.
\end{aligned}
$$

This demonstrates that Equation (18) holds when $i = k+1$. Therefore, Equation (18) holds for all $i = 1, \ldots, n$. $\square$

Theorem 2. *Let $\beta_0 = 0$. Suppose there exist $n \in \mathbb{N}$ and $\beta_i > 0$, $i = 1, 2, \ldots, n$ such that Equation (17) holds. If one of the following ordinary differential equations*

$$
(r(t)\phi(x'(t)))' + \left(\frac{R(\sigma(t))}{R(t)}\right)^{\gamma\beta_i} p(t)\phi(x(t)) = 0,\ i = 1, 2, \ldots, n, \tag{19}
$$

is oscillatory, then Equation (1) is oscillatory.

Proof. Assume x is a non-oscillatory solution of Equation (1) on $[t_0, \infty)$. Then, without the loss of generality let $x(t) > 0$ on $[t_0, \infty)$. By virtue of $\left(\frac{x(t)}{R^{\beta_i}(t)}\right)' \geq 0$, we deduce that

$$
x(\sigma(t)) \geq \left(\frac{R(\sigma(t))}{R(t)}\right)^{\beta_i} x(t).
$$

Therefore, from Equation (1), $x(t)$ satisfies

$$
\left(r(t)(x'(t))^\gamma\right)' + \left(\frac{R(\sigma(t))}{R(t)}\right)^{\gamma\beta_i} p(t)x^\gamma(t) \leq 0. \tag{20}
$$

Integrating Equation (20) from t to $v \geq t$ and letting $v \to \infty$ and noting that $x'(t) > 0$, we obtain

$$
x'(t) \geq \left(\frac{1}{r(t)}\int_t^\infty \left(\frac{R(\sigma(\tau))}{R(\tau)}\right)^{\gamma\beta_i} p(\tau)x^\gamma(\tau)\,\mathrm{d}\tau\right)^{1/\gamma}. \tag{21}
$$

Integrating Equation (21) from t_0 to t, we obtain

$$
x(t) \geq x(t_0) + \int_{t_0}^t \left(\frac{1}{r(\tau)}\int_\tau^\infty \left(\frac{R(\sigma(\tau_1))}{R(\tau_1)}\right)^{\gamma\beta_i} p(\tau_1)x^\gamma(\tau_1)\,\mathrm{d}\tau_1\right)^{1/\gamma} \mathrm{d}\tau.
$$

Next, we define a sequence $\{\omega_m(t)\}_{m\in\mathbb{N}_0}$ by

$$
\begin{aligned}
&\omega_0(t) = x(t) \\
&\omega_{m+1}(t) = x(t_0) + \textstyle\int_{t_0}^t \left(\frac{1}{r(\tau)}\int_\tau^\infty \left(\frac{R(\sigma(\tau_1))}{R(\tau_1)}\right)^{\gamma\beta_i} p(\tau_1)\omega_m^\gamma(\tau_1)\,\mathrm{d}\tau_1\right)^{1/\gamma} \mathrm{d}\tau,\ m \in \mathbb{N}_0.
\end{aligned}
$$

It is easy to check by induction that $\{\omega_m(t)\}$ is a well-defined decreasing sequence satisfying

$$
x(t_0) \leq \omega_m(t) \leq x(t) \quad \text{for } t \geq t_0 \text{ and } m \in \mathbb{N}_0.
$$

Thus, there exists a function ω on $[t_0, \infty)$ such that

$$
\lim_{m\to\infty} \omega_m(t) = \omega(t) \text{ and } x(t_0) \leq \omega_m(t) \leq x(t).
$$

By Lebesgue's dominated convergence theorem, it follows that

$$\omega(t) = x(t_0) + \int_{t_0}^{t} \left(\frac{1}{r(\tau)} \int_{\tau}^{\infty} \left(\frac{R(\sigma(\tau_1))}{R(\tau_1)} \right)^{\gamma\beta_i} p(\tau_1)\omega^{\gamma}(\tau_1)\, \mathrm{d}\tau_1 \right)^{1/\gamma} \mathrm{d}\tau. \tag{22}$$

Differentiating Equation (22) twice, we conclude that ω is a positive solution of Equation (19). This contradicts the assumption that Equation (19) is oscillatory and hence completes the proof. □

Theorem 3. *Let $\beta_0 = 0$. Suppose there exist $n \in \mathbb{N}$ and $\beta_i > 0$, $i = 1, 2, \ldots, n$ such that Equation (17) holds with*

$$\beta_n > \frac{\gamma}{(1+\gamma)^{(1+\gamma)/\gamma}}. \tag{23}$$

Then Equation (1) is oscillatory.

Proof. Without the loss of generality we assume that $n \in \mathbb{N}$ is the least number such that Equation (23) holds. Otherwise, we must replace it by the smallest one satisfying Equation (23). Then from Equations (17) and (23), we have

$$R(t) \left(\int_{t}^{\infty} \left(\frac{R(\sigma(t))}{R(t)} \right)^{\gamma\beta_{n-1}} p(\tau)\, \mathrm{d}\tau \right)^{1/\gamma} \geq \beta_n \in \left(\frac{\gamma}{(1+\gamma)^{(1+\gamma)/\gamma}}, \infty \right).$$

Applying Theorem 1 with $p(t)$ replaced by $\left(\frac{R(\sigma(t))}{R(t)} \right)^{\gamma\beta_i} p(t)$ to Equation (19), we see that Equation (19) is oscillatory with $i = n$. Therefore, by Theorem 2, Equation (1) is oscillatory. □

Remark 1. *Theorem 3 not only improves but also extends the result in Theorem 1. In particular, if Equation (23) holds with $n \geq 2$ and*

$$0 < \beta_i \leq \frac{\gamma}{(1+\gamma)^{(1+\gamma)/\gamma}}, \; i = 1, 2, \ldots, n-1 \quad \text{and} \quad \beta_n > \frac{\gamma}{(1+\gamma)^{(1+\gamma)/\gamma}},$$

then we know that Equation (1) is oscillatory by Theorem 3, but Theorem 1 fails to apply.

Example 1. *Consider second-order half-linear advanced differential equations*

$$\left(\frac{\gamma^{\gamma}}{(1+\gamma)^{1+\gamma}} \frac{\phi(x'(t))}{t} \right)' + \frac{\delta}{t^{\gamma+2}} \phi(x(\eta t)) = 0, \tag{24}$$

where $\delta > 0$ and $\eta \geq 1$. Now

$$\int_{t_0}^{\infty} \frac{\mathrm{d}\tau}{r^{1/\gamma}(\tau)} = \frac{(1+\gamma)^{1+1/\gamma}}{\gamma} \int_{t_0}^{\infty} \tau^{1/\gamma}\, \mathrm{d}\tau = \infty$$

and for $n \in \mathbb{N}$,

$$\begin{aligned} & R(t)\left(\int_t^\infty \left(\frac{R(\sigma(\tau))}{R(\tau)}\right)^{\gamma\beta_{i-1}} p(\tau)\,\mathrm{d}\tau\right)^{1/\gamma} \\ \geq\ & \delta^{1/\gamma}(1+\gamma)^{1/\gamma}\left(t^{1+1/\gamma} - t_0^{1+1/\gamma}\right)\left(\int_t^\infty \left(\eta^{1+1/\gamma} - \left(\frac{t_0}{\tau}\right)^{1+1/\gamma}\right)^{\gamma\beta_{i-1}} \frac{\mathrm{d}\tau}{\tau^{\gamma+2}}\right)^{1/\gamma} \\ =\ & \delta^{1/\gamma}\eta^{\beta_{i-1}(1+1/\gamma)}(1+\gamma)^{1/\gamma} t^{1+1/\gamma}\left(\int_t^\infty \frac{\mathrm{d}\tau}{\tau^{\gamma+2}}\right)^{1/\gamma}(1-\circ(1)), \end{aligned}$$

as $t \to \infty$. *Therefore, Condition* (17) *is satisfied for a large* t *provided there exist* $\beta_i > 0$, $i = 1, 2, \ldots, n$ *such that*

$$\delta^{1/\gamma}\eta^{\beta_{i-1}(1+1/\gamma)}(1+\gamma)^{1/\gamma} t^{1+1/\gamma}\left(\int_t^\infty \frac{\mathrm{d}\tau}{\tau^{\gamma+2}}\right)^{1/\gamma} = \delta^{1/\gamma}\eta^{\beta_{i-1}(1+1/\gamma)} > \beta_i,\ i = 1, 2, \ldots, n.$$

Hence, we may choose

$$\beta_i < \delta^{1/\gamma}\eta^{\beta_{i-1}(1+1/\gamma)},\ i = 1, 2, \ldots, n. \tag{25}$$

1. *For* $\delta = \gamma = 0.4$ *and* $\eta = 1.6$, *we have* $\gamma/(1+\gamma)^{(1+\gamma)/\gamma} = 0.1232$ *, and with Equation* (25) *we choose*

$$\begin{aligned} \beta_1 &= 0.10118; \\ \beta_2 &= 0.11951; \\ \beta_3 &= 0.12317; \\ \beta_4 &= 0.12391. \end{aligned}$$

   So, $\beta_i \leq \gamma/(1+\gamma)^{(1+\gamma)/\gamma}$, $i = 1, 2, 3$ *and* $\beta_4 > \gamma/(1+\gamma)^{(1+\gamma)/\gamma}$. *Then, by Theorem 3, Equation* (24) *is oscillatory.*

2. *For* $\delta = 0.2$, $\gamma = 1$, *and* $\eta = 1.7$, *we have* $\gamma/(1+\gamma)^{(1+\gamma)/\gamma} = 0.25$ *and*

$$\begin{aligned} \beta_1 &= 0.19999; \\ \beta_2 &= 0.24728; \\ \beta_3 &= 0.26001. \end{aligned}$$

   So, $\beta_i \leq \gamma/(1+\gamma)^{(1+\gamma)/\gamma}$, $i = 1, 2$ *and* $\beta_3 > \gamma/(1+\gamma)^{(1+\gamma)/\gamma}$. *Then, by Theorem 3, Equation* (24) *is oscillatory.*

3. *For* $\delta = 0.13$, $\gamma = 1.4$, *and* $\eta = 1.9$, *we have* $\gamma/(1+\gamma)^{(1+\gamma)/\gamma} = 0.31213$ *and*

$$\begin{aligned} \beta_1 &= 0.23285; \\ \beta_2 &= 0.30086; \\ \beta_3 &= 0.32423. \end{aligned}$$

   So, $\beta_i \leq \gamma/(1+\gamma)^{(1+\gamma)/\gamma}$, $i = 1, 2$ *and* $\beta_3 > \gamma/(1+\gamma)^{(1+\gamma)/\gamma}$. *Then, by Theorem 3, Equation* (24) *is oscillatory. Obviously, Theorem 1 fails to apply to these equations.*

3. Discussion and Conclusions

In this paper, our results extend and improve related contributions to the second-order differential equations with deviating arguments and cover the existing classical criteria for ordinary differential equations in the literature; see the following details
Theorems 2 and 3 are for the cases $\sigma(t) \geq t$ and $\gamma > 0$.

(I) When $\gamma = 1$, Equation (1) becomes the advanced differential Equation (12).

(i) The results in Theorem 2 improve those given in [39] due to:
- The oscillation of Equation (13) implies oscillation of Equation (12) (see [39]);
- The oscillation of Equation (19) implies oscillation of Equation (12) (Theorem 2);
- The oscillation of Equation (13) implies oscillation of Equation (19) since
$$\left(\frac{R(\sigma(t))}{R(t)}\right)^{\beta_i} \geq 1, \; i = 1, 2, \ldots, n$$
and using Sturm's comparison theorem.

(ii) If Equation (14) holds, β_i in Theorem 3 reduces to α_i of the results of [40].

(II) If $\sigma(t) \equiv t$, Equation (1) becomes the ordinary half-linear Equation (8), which includes the linear case Equation (5) ($\gamma = 1$) and Equation (3) ($\gamma = 1$, $r(t) \equiv 1$). Now we show that Theorem 3 covers the existing results for the above equations as seen in Section 1. Let $\sigma(t) \equiv t$. We note that all β_i in Equation (17) can be chosen to be the same. This can be denoted by β.

(i) In general, $\gamma > 0$ and $r(t) > 0$, Equation (17) clearly reduces to Equation (9). Thus, Theorem 3 guarantees the oscillation of Equation (8).

(ii) When $\gamma = 1$ and $r(t) > 0$, Equation (17) clearly reduces to Equation (7). Thus, Theorem 3 guarantees the oscillation of Equation (5).

(iii) When $\gamma = 1$ and $r(t) \equiv 1$, Equation (17) reduces to
$$(t - t_0) \int_t^\infty p(\tau) \, \mathrm{d}\tau \geq \beta > 0.$$
Note that $\int_t^\infty p(\tau) \, \mathrm{d}\tau$ is convergent. It is equivalent to
$$t \int_t^\infty p(\tau) \, \mathrm{d}\tau \geq \beta^*, \text{ for some } \beta^* > 0.$$
Thus, Theorem 3 guarantees the oscillation of Equation (3).

Author Contributions: Writing—original draft, T.S.H.; writing—review editing, T.S.H., Q.K. and B.M.E.-M.; supervision, T.S.H., Q.K. and B.M.E.-M.; validation, T.S.H., Q.K. and B.M.E.-M.; conceptualization, T.S.H.; project administration, T.S.H.; formal analysis, Q.K. and B.M.E.-M. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Acknowledgments: The researchers would like to thank the Deanship of Scientific Research of Qassim University for funding the publication of this project.

Conflicts of Interest: The authors declare no conflict of interest.

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