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A Full-Body Relative Orbital Motion of Spacecraft Using Dual Tensor Algebra and Dual Quaternions

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Abstract: This paper proposes a new non-linear differential equation for the six degrees of freedom (6-DOF) relative rigid bodies motion. A representation theorem is provided for the 6-DOF differential equation of motion in the arbitrary non-inertial reference frame. The problem of the 6-DOF relative motion of two spacecraft in the specific case of Keplerian confocal orbits is proposed. The result is an analytical method without secular terms and singularities. Tensors dual algebra and dual quaternions play a fundamental role, with the solution representation being the relative problem. Furthermore, the representation theorems for the rotation and translation parts of the 6-DOF relative orbital motion problems are obtained.

Keywords: non-inertial frame; full body problem; relative orbital motion; dual algebra

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1. Introduction

With the development of space technology, space activities have gradually become normal, and the diversity of space missions has increased. In recent years, in-orbit service, formation flying of spacecraft, rendezvous, and docking, refueling, and other short-range operations have attracted more and more attention from researchers. Traditional control methods assume that spacecraft translation and rotation are decoupled, and spacecraft control adopts the serial control mode of alternate attitude and orbit control. The approach implies considering the translation and attitude dynamics, using a mathematical formalism of real Euclidean vectors and tensors. However, for short-range missions with high accuracy, the coupling effect between attitude and orbit must be considered, and attitude and orbit must be controlled simultaneously, which requires establishing a dynamic coupled attitude-orbit model. The relative movement between the primary spacecraft denoted Chief, and the second spacecraft denoted Deputy, is a 6-DOF movement. In astrodynamics, this problem is called the full body relative orbital motion. In recent years, the 6-DOF motion of spacecraft has attracted particular attention [1–5], such as the control of the relative pose of satellite formations flying, which has become a crucial research topic [6–13]. This paper uses dual number algebra and dual quaternions to obtain the exact solution of the 6-DOF laws of relative orbital motion for the case of two Keplerian spacecrafts in the same gravitational center of attraction. Orthogonal dual tensors and dual quaternions are crucial for describing the complete onboard solution of 6-DOF relative orbital motion problems. The solution proves one must know only the Chief spacecraft's motion and the Deputy spacecraft's initial conditions in the Local Vertical-Local Horizontal (LVLH) non-inertial reference frame. This paper proves a novel general theorem for the 6-DOF motion in an arbitrary non-inertial frame. A decoupling problem for the attitude and translation components of the relative motion is also obtained. The relative motion of the Chief and Deputy spacecrafts' mass centers is an exact closed-form, coordinate-free solution. The novel result is achieved by analytical methods in the general topic, without implying any secular terms and is singularity-free.

In contrast to other approaches, the proposed solution is onboard; all calculations are carried out in the reference frame of the Chief spacecraft. The paper is structured as follows. Section 2 presents the dual algebra, vectors, and tensor for rigid body displacement and motion parameterization. The structural invariant on vector and tensor of rigid body displacement is proposed, and the exponential Rodrigues-like formula in dual algebra is demonstrated. The Lie group of rigid body displacement and this Lie algebra is isomorphic with orthogonal dual tensors and, respectively, with skew-symmetric dual tensors. The rigid body motion is parameterized with a curve in the Lie algebra of orthogonal dual tensor. The velocity field of rigid body and kinematics equation problem in dual algebra is considered. The reconstruction of rigid body motion is an approach for a dual twist in space or in the body frame. Section 3 presents a fundamental non-vectorial parameterization of rigid body displacement and motion: dual quaternions. The Lie group of unit dual quaternions is homomorphic with orthogonal dual tensors. Dual vector and unit dual quaternions represent all the properties of rigid body motion. Additionally, the kinematic equations for dual quaternion are proven. Section 4 establishes the state equations for a rigid body in an arbitrary non-inertial reference frame. A representation theorem that decouples the inertial and non-inertial components of the last motion is presented. In Section 5, using the general results of Section 4, the representation theorem and the solution for the onboard full-body relative orbital motion problem are proven. Furthermore, the representation theorems for the attitude and translation parts of the 6-DOF relative orbital motion problems are obtained. Finally, a case study to give the exact closed-form, coordinate-free solution of the translation part of the relative orbital motion problem is resolved. The last section is designated to the conclusions and further work.

2. Dual Number Algebra Parameterization by Rigid Body Motion

The contemporary approach starts with the property of the Lie group of rigid body displacements, accompanied by its Lie algebra. In geometric terminology, the Lie group of rigid body displacement is the semidirect product of the rotation group with the translation group in three-dimensional Euclidean space. A previous paper proves an isomorphism of the Lie group with orthogonal dual tensors and, an isomorphism of the Lie algebra with the dual vectors set. Orthogonal dual tensor maps and dual quaternions are a complete instrument for studying rigid body displacement and motion. Further information on dual numbers, dual vectors, dual tensors, and dual quaternions can be found in references [2,14–24].

2.1. General Theorems and Isomorphism between Lie Groups and Lie Algebras for Rigid Body Displacements

Let the set of orthogonal dual tensor:

$$\underline{\mathbb{SO}}_3 = \left\{ \underline{\boldsymbol{R}} \in \mathbf{L}(\underline{\boldsymbol{V}}_3, \underline{\boldsymbol{V}}_3) \middle| \underline{\boldsymbol{R}}\underline{\boldsymbol{R}}^T = \underline{\boldsymbol{I}}, \det \underline{\boldsymbol{R}} = 1 \right\} \tag{1}$$

where $\underline{\boldsymbol{I}}$ is the unit orthogonal dual tensor.

The orthogonal dual tensor properties of $\underline{\mathbb{SO}}_3$ are the same results that were detailed in our previous studies [15,16,20].

Theorem 1 (Structure Theorem). *For any orthogonal dual tensor $\underline{\boldsymbol{R}} \in \underline{\mathbb{SO}}_3$ a decomposition: where $\boldsymbol{Q} \in \mathbb{SO}_3$ and $\boldsymbol{\rho} \in \boldsymbol{V}_3$ are called **structural invariants**, $\varepsilon^2 = 0$, $\varepsilon \neq 0$ is uniques.*

$$\underline{\boldsymbol{R}} = \left(\boldsymbol{I} + \varepsilon \widetilde{\boldsymbol{\rho}} \right) \boldsymbol{Q} \tag{2}$$

For the Lie group structure of **Theorem 1**, it can be concluded that any orthogonal dual tensor $\underline{\boldsymbol{R}} \in \underline{\mathbb{SO}}_3$ can globally parameterize any displacements of rigid bodies.

Theorem 2 (Representation Theorem). *For any orthogonal dual tensor $\underline{\boldsymbol{R}}$ defined as in (Equation (2), a dual number $\underline{\alpha} = \alpha + \varepsilon d$ and a dual unit vector $\underline{\mathbf{u}} = \mathbf{u} + \varepsilon \mathbf{u}_0$ can be computed to have the following equation* [15,16]:

$$\underline{\boldsymbol{R}}(\underline{\alpha}, \underline{\mathbf{u}}) = \underline{\boldsymbol{I}} + (\sin \underline{\alpha})\underline{\widetilde{\boldsymbol{u}}} + (1 - \cos\underline{\alpha})\underline{\widetilde{\boldsymbol{u}}}^2 = \exp\left(\underline{\alpha}\underline{\widetilde{\boldsymbol{u}}}\right) \tag{3}$$

Equation (3) is a Rodrigues-like formula for $\underline{\mathbb{SO}}_3$. Dual angle $\underline{\alpha}$ and unit dual vector $\underline{\mathbf{u}}$ are called the **natural invariants** of $\underline{\boldsymbol{R}}$. The unit dual vector $\underline{\mathbf{u}} = \mathbf{u} + \varepsilon \mathbf{u}_0$ gives the Plücker representation of the Mozzi–Chalses axis [14,20], while the dual angle $\underline{\alpha} = \alpha + \varepsilon d$ contains the rotation angle α and the translated distance d.

The Lie algebra of the Lie group $\underline{\mathbb{SO}}_3$ is the skew-symmetric dual tensor set denoted by $\underline{\mathbb{so}}_3 = \left\{ \underline{\widetilde{\boldsymbol{\alpha}}} \in \mathbf{L}(\underline{\boldsymbol{V}}_3, \underline{\boldsymbol{V}}_3) \middle| \underline{\widetilde{\boldsymbol{\alpha}}} = -\underline{\widetilde{\boldsymbol{\alpha}}}^{\mathrm{T}} \right\}$, where commutator is the internal operation: $\left\langle \underline{\widetilde{\boldsymbol{\alpha}}}_1, \underline{\widetilde{\boldsymbol{\alpha}}}_2 \right\rangle = \widetilde{\underline{\boldsymbol{\alpha}}_1 \widetilde{\boldsymbol{\alpha}}_2}$.

The link between the Lie algebra $\underline{\mathbb{so}}_3$, the Lie group $\underline{\mathbb{SO}}_3$, and the exponential map is given by the following.

Theorem 3. *The exponential mapping is well defined and surjective.*

$$\begin{gathered} \exp : \underline{\mathbb{so}}_3 \to \underline{\mathbb{SO}}_3, \\ \exp\left(\underline{\widetilde{\boldsymbol{\alpha}}}\right) = e^{\underline{\widetilde{\boldsymbol{\alpha}}}} = \sum_{\mathrm{k}=0}^{\infty} \frac{\underline{\widetilde{\boldsymbol{\alpha}}}^{\mathrm{k}}}{\mathrm{k}!} \end{gathered} \tag{4}$$

The screw axis that embeds a rigid body displacement (via the Mozzi–Chalses theorem) is bound to the problem of finding the logarithm of an orthogonal dual tensor, which is a multiple valued function:

$$\begin{gathered} \log : \underline{\mathbb{SO}}_3 \to \underline{\mathbb{so}}_3, \\ \log\underline{\boldsymbol{R}} = \left\{ \underline{\widetilde{\boldsymbol{\alpha}}} \in \underline{\mathbb{so}}_3 \middle| \exp\left(\underline{\widetilde{\boldsymbol{\alpha}}}\right) = \underline{\boldsymbol{R}} \right\} \end{gathered} \tag{5}$$

and is the inverse of (Equation (4)).

From **Theorem 2** for any orthogonal dual tensor $\underline{\boldsymbol{R}}$, a dual vector $\underline{\boldsymbol{\alpha}} = \underline{\alpha}\underline{\mathbf{u}} = \boldsymbol{\alpha} + \varepsilon \boldsymbol{\alpha}_0$ is computed, denoted by the **Euler dual vector** (that includes the screw axis and screw parameters in dual form). The form of dual vector $\underline{\boldsymbol{\alpha}}$ implies that $\underline{\widetilde{\boldsymbol{\alpha}}} \in \log\underline{\boldsymbol{R}}$. The types of rigid body displacements that are parameterized by the Euler dual vector $\underline{\boldsymbol{\alpha}}$ are as below:

- screw displacement if $\boldsymbol{\alpha} \neq 0, \boldsymbol{\alpha}_0 \neq 0$ and $\boldsymbol{\alpha} \cdot \boldsymbol{\alpha}_0 \neq 0 \Longleftrightarrow |\underline{\boldsymbol{\alpha}}| \in \underline{\mathbb{R}}$ and $|\underline{\boldsymbol{\alpha}}| \notin \varepsilon\mathbb{R}$;
- only translation displacement if $\boldsymbol{\alpha} = 0$ and $\boldsymbol{\alpha}_0 \neq 0 \Longleftrightarrow |\underline{\boldsymbol{\alpha}}| \in \varepsilon\mathbb{R}$;
- only rotation displacement if $\boldsymbol{\alpha} \neq 0$ and $\boldsymbol{\alpha} \cdot \boldsymbol{\alpha}_0 = 0 \Longleftrightarrow |\underline{\boldsymbol{\alpha}}| \in \mathbb{R}$.

In previous relations, we have denoted the dual norm of the dual vector $\underline{\boldsymbol{\alpha}} = \boldsymbol{\alpha} + \varepsilon \boldsymbol{\alpha}_0$ as:

$$|\underline{\boldsymbol{\alpha}}| = \begin{cases} \|\boldsymbol{\alpha}\| + \varepsilon \frac{\boldsymbol{\alpha}_0 \cdot \boldsymbol{\alpha}}{\|\boldsymbol{\alpha}\|}, Re(\underline{\boldsymbol{\alpha}}) \neq 0 \\ \varepsilon \|\boldsymbol{\alpha}_0\|, Re(\underline{\boldsymbol{\alpha}}) = 0 \end{cases} \tag{6}$$

The conversion between natural invariants and structural invariants of rigid body motion is given by:

Theorem 4. *The natural invariants $\underline{\alpha} = \alpha + \varepsilon d, \underline{\mathbf{u}} = \mathbf{u} + \varepsilon \mathbf{u}_0$ can be used to directly recover the structural invariants $\boldsymbol{Q}$ and $\boldsymbol{\rho}$ from (Equation (2)):*

$$\begin{gathered} \boldsymbol{Q} = \boldsymbol{I} + \sin\alpha\widetilde{\boldsymbol{u}} + (1 - \cos\alpha)\widetilde{\boldsymbol{u}}^2 \\ \boldsymbol{\rho} = d\mathbf{u} + \sin\alpha\mathbf{u}_0 + (1 - \cos\alpha)\mathbf{u} \times \mathbf{u}_0 \end{gathered} \tag{7}$$

Theorem 5 (Isomorphism Theorem). *The special Euclidean Lie groups* ($\mathbb{SE}_3$) *and* $\underline{\mathbb{SO}}_3$) *are isomorphic:*

$$\begin{array}{c}\Phi : \mathbb{SE}_3 \to \underline{\mathbb{SO}}_3, \\ \Phi(\mathrm{g}) = \left(\boldsymbol{I} + \varepsilon\widetilde{\boldsymbol{\rho}}\right)\boldsymbol{Q},\end{array} \tag{8}$$

where $\mathrm{g} = \begin{bmatrix} \boldsymbol{Q} & \boldsymbol{\rho} \\ \mathbf{0} & 1 \end{bmatrix}, \boldsymbol{Q} \in \mathbb{SO}_3, \boldsymbol{\rho} \in \boldsymbol{V}_3$.

The Lie algebras $\mathfrak{se}(3)$ *and* $\underline{\boldsymbol{V}}_3$ *is isomorphic*:

$$\begin{array}{c}\varphi : \mathfrak{se}(3) \to \underline{\boldsymbol{V}}_3, \\ \varphi(\xi) = \boldsymbol{\omega} + \varepsilon\mathbf{v},\end{array} \tag{9}$$

where $\widehat{\xi} = \begin{bmatrix} \widetilde{\boldsymbol{\omega}} & \mathbf{v} \\ \mathbf{0} & 0 \end{bmatrix}, \widetilde{\boldsymbol{\omega}} \in \mathfrak{so}_3, \mathbf{v} \in \boldsymbol{V}_3$.

Proof. For any $\mathrm{g}_1, \mathrm{g}_2 \in \mathbb{SE}_3$, the map defined in Equation (7) yields

$$\Phi(\mathrm{g}_1 \cdot \mathrm{g}_2) = \Phi(\mathrm{g}_1) \cdot \Phi(\mathrm{g}_2). \tag{10}$$

Let $\underline{\boldsymbol{R}} \in \underline{\mathbb{SO}}_3$. Based on **Theorem 4**, which ensures a unique decomposition, we can conclude that the only choice for g, such that $\Phi(\mathrm{g}) = \underline{\boldsymbol{R}}$ is $\mathrm{g} = \begin{bmatrix} \boldsymbol{Q} & \boldsymbol{\rho} \\ \mathbf{0} & 1 \end{bmatrix}$. This underlines that Φ is a bijection and keeps all the internal operations, where $\boldsymbol{Q}$ and $\boldsymbol{\rho}$ are denoted as structural invariant of the orthogonal tensor $\boldsymbol{Q}$.

For any $\hat{\xi}_1, \hat{\xi}_2 \in \mathrm{se}(3)$, the mapping defined by Equation (7) verifies the identity

$$\Phi\left(\left[\hat{\xi}_1, \hat{\xi}_2\right]\right) = \Phi\left(\hat{\xi}_1\right) \times \Phi\left(\hat{\xi}_2\right) \tag{11}$$

Additionally, for any $\underline{\boldsymbol{\omega}} \in \boldsymbol{V}_3$, $\underline{\boldsymbol{\omega}} = \boldsymbol{\omega} + \varepsilon\mathbf{v}$, there is only determined $\hat{\xi} = \begin{bmatrix} \widetilde{\boldsymbol{\omega}} & \mathbf{v} \\ \mathbf{0} & 0 \end{bmatrix}$ such that $\varphi(\xi) = \underline{\boldsymbol{\omega}}$. Thus, φ is a bijective mapping. □

Remark 1. *The inverse of isomorphisms* Φ *and respectively* φ *is*:

$$\Phi^{-1} : \underline{\mathbb{SO}}_3 \longleftrightarrow \mathbb{SE}_3; \Phi^{-1}(\underline{\boldsymbol{R}}) = \begin{bmatrix} \boldsymbol{Q} & \boldsymbol{\rho} \\ \mathbf{0} & 1 \end{bmatrix} \tag{12}$$

where $\boldsymbol{Q} = Re(\underline{\boldsymbol{R}}), \boldsymbol{\rho} = vect\left(Du(\underline{\boldsymbol{R}}) \cdot \boldsymbol{Q}^{\mathrm{T}}\right)$.

$$\varphi^{-1} : \underline{\boldsymbol{V}}_3 \longleftrightarrow \mathfrak{se}(3); \varphi^{-1}(\underline{\boldsymbol{\omega}}) = \varphi^{-1}(\boldsymbol{\omega} + \varepsilon\mathbf{v}) = \begin{bmatrix} \widetilde{\boldsymbol{\omega}} & \mathbf{v} \\ \mathbf{0} & 0 \end{bmatrix} \tag{13}$$

where $\widetilde{\boldsymbol{\omega}} = skew(Re(\underline{\boldsymbol{\omega}})), \mathbf{v} = Du(\underline{\boldsymbol{\omega}})$.

Theorem 5 connects two distinct ways of rigid body displacement: the displacement of its points or the displacement of the oriented lines attached to the rigid body.

2.2. The Velocity Field and Kinematic Equation in Dual Number Algebra

Let the parametric equations of a rigid body motion: $\boldsymbol{Q} = \boldsymbol{Q}(t) \in \mathbb{SO}_3^{\mathbb{R}}$ and $\boldsymbol{\rho} = \boldsymbol{\rho}(t) \in \boldsymbol{V}_3^{\mathbb{R}}$, where all the functions are time differentiable. The rigid body motion can be parameterized by a curve in the Lie group of orthogonal dual tensors $\underline{\mathbb{SO}}_3$ by $\underline{\boldsymbol{R}}(t) = \left(\boldsymbol{I} + \varepsilon\widetilde{\boldsymbol{\rho}}(t)\right)\boldsymbol{Q}(t)$, where $t \in \mathbb{R}$ is time variable (see **Theorem 1**). Let $\underline{\mathbf{u}}_0$ the unit dual vector embed the oriented line feature at $t = t_0$. At a time t the oriented line is transformed into:

$$\underline{\mathbf{u}}(t) = \underline{\boldsymbol{R}}(t)\underline{\mathbf{u}}_0 \tag{14}$$

Theorem 6. *In a general rigid body motion, the dual velocity tensor function $\underline{\widetilde{\omega}}$, defined as:*

$$\dot{\underline{\mathbf{u}}} = \underline{\widetilde{\omega}\mathbf{u}}, \forall \underline{\mathbf{u}} \in \underline{\boldsymbol{V}}_3^{\mathbb{R}} \tag{15}$$

is given by

$$\underline{\widetilde{\omega}} = \underline{\dot{\boldsymbol{R}}\boldsymbol{R}}^{\mathrm{T}}, \tag{16}$$

$\underline{\widetilde{\omega}}$ *is a skew-symmetric dual tensor*: $\underline{\widetilde{\omega}} = -\underline{\widetilde{\omega}}^{\mathbf{T}}, \underline{\widetilde{\omega}} \in \underline{\mathbb{so}}_3^{\mathbb{R}}$.

The dual vector $\underline{\boldsymbol{\omega}} = vect\underline{\dot{\boldsymbol{R}}\boldsymbol{R}}^{\mathrm{T}}$ is the dual angular velocity of the rigid body motion and has the form:

$$\underline{\boldsymbol{\omega}} = \boldsymbol{\omega} + \varepsilon \mathbf{v}. \tag{17}$$

In Equation (15), $\boldsymbol{\omega}$ is the instantaneous angular velocity and $\mathbf{v} = \dot{\boldsymbol{\rho}} - \boldsymbol{\omega} \times \boldsymbol{\rho}$. The pair $(\boldsymbol{\omega}, \mathbf{v})$ is the **space twist** of the rigid body. All the information for the velocity field of rigid body motion is given by dual angular velocity and is also named a **dual twist**. Knowing the initial pose and dual twist $\underline{\boldsymbol{\omega}} = \boldsymbol{\omega} + \varepsilon \mathbf{v}$, rigid body motion reconstruction is always possible [5,15]:

Theorem 7. *For any continuous function $\underline{\boldsymbol{\omega}} \in \underline{\boldsymbol{V}}_3^{\mathbb{R}}$, the differential equation:*

$$\begin{array}{c} \dot{\underline{\boldsymbol{R}}} = \underline{\widetilde{\omega}\boldsymbol{R}} \\ \underline{\boldsymbol{R}}(t_0) = \underline{\boldsymbol{R}}_0, \underline{\boldsymbol{R}}_0 \in \underline{\mathbb{SO}}_3. \end{array} \tag{18}$$

has a unique solution, orthogonal dual tensor $\underline{\boldsymbol{R}} \in \underline{\mathbb{SO}}_3^{\mathbb{R}}$.

Proof. Let $\underline{\boldsymbol{R}} = \left(\boldsymbol{I} + \varepsilon\widetilde{\boldsymbol{\rho}}\right)\boldsymbol{Q}$. By differentiating Equation (16), result:

$$\dot{\boldsymbol{Q}} + \varepsilon\left(\dot{\widetilde{\boldsymbol{\rho}}}\boldsymbol{Q} + \widetilde{\boldsymbol{\rho}}\dot{\boldsymbol{Q}}\right) = \left(\widetilde{\omega} + \varepsilon\widetilde{\mathbf{v}}\right)\left(\boldsymbol{Q} + \varepsilon\widetilde{\boldsymbol{\rho}}\boldsymbol{Q}\right) = \widetilde{\omega}\boldsymbol{Q} + \varepsilon\left(\widetilde{\mathbf{v}}\boldsymbol{Q} + \widetilde{\omega}\widetilde{\boldsymbol{\rho}}\boldsymbol{Q}\right) \tag{19}$$

For Equation (17), separating the real parts, obtained differential equation:

$$\begin{cases} \dot{\boldsymbol{Q}} = \widetilde{\omega}\boldsymbol{Q} \\ \boldsymbol{Q}(t_0) = \boldsymbol{Q}_0 \in \mathbb{SO}_3 \end{cases} \tag{20}$$

Instantaneous angular velocity $\widetilde{\omega} = \widetilde{\omega}(t)$ is a continuous function, and Problem (20) admits a unique solution. We will prove that this solution is an orthogonal tensor.

Denote $\boldsymbol{Q}^{\mathrm{T}}$ the transpose of $\boldsymbol{Q}$. By differentiating:

$$\frac{d}{dt}\left(\boldsymbol{Q}\boldsymbol{Q}^{\mathrm{T}}\right) = \dot{\boldsymbol{Q}}\boldsymbol{Q}^{\mathrm{T}} + \boldsymbol{Q}\dot{\boldsymbol{Q}}^{\mathrm{T}} = \widetilde{\omega}\boldsymbol{Q}\boldsymbol{Q}^{\mathrm{T}} - \boldsymbol{Q}\boldsymbol{Q}^{\mathrm{T}}\widetilde{\omega} = 0 \tag{21}$$

results

$$\boldsymbol{Q}\boldsymbol{Q}^{\mathrm{T}} = \boldsymbol{Q}\boldsymbol{Q}^{\mathrm{T}}(t_0) = \boldsymbol{I} \tag{22}$$

From (22), $\det(\boldsymbol{Q}) \in \{-1, 1\}$. Since $\det(\boldsymbol{Q}(t_0)) = \det\boldsymbol{Q}_0 = 1$, it follows that:

$$\begin{cases} \boldsymbol{Q}\boldsymbol{Q}^{T} = I \\ det(\boldsymbol{Q}) = 1 \end{cases} \tag{23}$$

Therefore, $\boldsymbol{Q} \in \mathbb{SO}_3^{\mathbb{R}}$ is a rotation tensor map.

From Equation (17), separating dual parts gives:

$$\dot{\tilde{\boldsymbol{\rho}}} + \tilde{\boldsymbol{\rho}}\tilde{\boldsymbol{\omega}} = \tilde{\mathbf{v}} + \tilde{\boldsymbol{\omega}}\tilde{\boldsymbol{\rho}} \tag{24}$$

which, taking a step further implies that

$$\dot{\tilde{\boldsymbol{\rho}}} + \tilde{\boldsymbol{\rho}}\tilde{\boldsymbol{\omega}} - \tilde{\boldsymbol{\omega}}\tilde{\boldsymbol{\rho}} = \tilde{\mathbf{v}} \tag{25}$$

Using the identity: $\widetilde{\tilde{\boldsymbol{\omega}}\boldsymbol{\rho}} = \tilde{\boldsymbol{\omega}}\tilde{\boldsymbol{\rho}} - \tilde{\boldsymbol{\rho}}\tilde{\boldsymbol{\omega}}$, and Equation (23) results in the differential equation:

$$\begin{cases} \dot{\boldsymbol{\rho}} - \boldsymbol{\omega} \times \boldsymbol{\rho} = \mathbf{v} \\ \boldsymbol{\rho}(\mathrm{t}_0) = \boldsymbol{\rho}_0 \end{cases} \tag{26}$$

The solution of Problem (26) is:

$$\boldsymbol{\rho} = \boldsymbol{Q}(t)\left[\boldsymbol{\rho}_0 + \int_{t_0}^{t} \boldsymbol{Q}^{\mathrm{T}}(x)\mathbf{v}(x)dx\right] \tag{27}$$

where $\boldsymbol{Q}$ is the solution of Equation (20). □

Differential equation problem of **Theorem 7** is called a dual kinematics equation.

If denoted by $\underline{\boldsymbol{\omega}}^B$ the dual angular velocity in the body frame: $\underline{\boldsymbol{\omega}}^B = \underline{\boldsymbol{R}}^T\underline{\boldsymbol{\omega}}$, and $\tilde{\underline{\boldsymbol{\omega}}}^B = \underline{\boldsymbol{R}}^T\dot{\underline{\boldsymbol{R}}}$.

Remark 2. *The kinematic equation by dual angular velocity in the body frame is:*

$$\begin{cases} \dot{\underline{\boldsymbol{R}}} = \underline{\boldsymbol{R}}\tilde{\underline{\boldsymbol{\omega}}}^B \\ \underline{\boldsymbol{R}}(t_0) = \underline{\boldsymbol{R}}_0, \underline{\boldsymbol{R}}_0 \in \underline{\mathbb{SO}}_3 \end{cases} \tag{28}$$

Equations (18) and (28) also represent the Poisson–Darboux problem in dual algebra [5,25,26].

3. Dual Quaternions Parameterizations of the Rigid Body Displacement and Motion

The Lie group $\underline{\mathbb{SO}}_3$ admits multiple parameterizations. The Lie group of dual unit quaternions, which is one non-vectorial parameterizations for the 6-DOF rigid body displacement and motion [3].

A dual quaternion set, denotes $\underline{\boldsymbol{Q}} = \underline{\mathbb{R}} \times \underline{\boldsymbol{V}}_3$, is a pair of a dual scalar and dual vector:

$$\hat{\underline{\mathbf{q}}} = \left(\underline{\mathrm{q}}, \underline{\mathbf{q}}\right), \underline{\mathrm{q}} \in \underline{\mathbb{R}}, \underline{\mathbf{q}} \in \underline{\boldsymbol{V}}_3. \tag{29}$$

Dual quaternion set $\underline{\boldsymbol{Q}}$, by addition and multiplication with dual numbers, are a $\underline{\mathbb{R}}$-module of rank 4 over the ring of dual number $\underline{\mathbb{R}}$.

The product of two dual quaternions $\hat{\underline{\mathbf{q}}}_1 = \left(\underline{\mathrm{q}}_1, \underline{\mathbf{q}}_1\right)$ and $\hat{\underline{\mathbf{q}}}_2 = \left(\underline{\mathrm{q}}_2, \underline{\mathbf{q}}_2\right)$ is definite by:

$$\hat{\underline{\mathbf{q}}}_1\hat{\underline{\mathbf{q}}}_2 = \left(\underline{\mathrm{q}}_1 \cdot \underline{\mathrm{q}}_2 - \underline{\mathbf{q}}_1 \cdot \underline{\mathbf{q}}_2, \underline{\mathrm{q}}_1\underline{\mathbf{q}}_2 + \underline{\mathrm{q}}_2\underline{\mathbf{q}}_1 + \underline{\mathbf{q}}_1 \times \underline{\mathbf{q}}_2\right) \tag{30}$$

For dual quaternion by Equation (29), denoted by $\hat{\underline{\mathbf{q}}}^* = \left(\underline{\mathrm{q}}, -\underline{\mathbf{q}}\right)$ the conjugate dual quaternion. If $\hat{\underline{\mathbf{q}}}\hat{\underline{\mathbf{q}}}^* = 1$, a dual quaternion is denoted unit dual quaternion. A free $\underline{\mathbb{R}}$-module $\underline{\boldsymbol{Q}}$ contains two sub-modules: $\underline{\boldsymbol{Q}}_{\underline{\mathbb{R}}} = \left(\underline{\mathrm{q}}, \underline{\mathbf{0}}\right), \underline{\mathrm{q}} \in \underline{\mathbb{R}}$ and $\underline{\boldsymbol{Q}}_{\underline{\boldsymbol{V}}_3} = \left(\underline{0}, \underline{\mathbf{q}}\right), \underline{\mathbf{q}} \in \underline{\boldsymbol{V}}_3$. $\underline{\boldsymbol{Q}}_{\underline{\mathbb{R}}}$ is isomorphic with dual number set $\underline{\mathbb{R}}$, and $\underline{\boldsymbol{Q}}_{\underline{\boldsymbol{V}}_3}$ is isomorphic with dual vector set $\underline{\boldsymbol{V}}_3$.

A dual quaternion can be decomposition as $\underline{\widehat{\mathbf{q}}} = \underline{q} + \underline{\mathbf{q}}$, where $\underline{q} = \left(\underline{q}, \underline{\mathbf{0}}\right)$ is dual number and $\underline{\mathbf{q}} = \left(\underline{0}, \underline{\mathbf{q}}\right)$ is dual vector, or as $\underline{\widehat{\mathbf{q}}} = \widehat{\mathbf{q}} + \varepsilon\widehat{\mathbf{q}}_0$, where $\widehat{\mathbf{q}}$, $\widehat{\mathbf{q}}_0$ are real quaternions.

Let $\underline{\mathbb{U}}$ and $\mathbb{U}$ denote the set of unit dual quaternions and, respectively, the set of unit real quaternions. For any $\underline{\widehat{\mathbf{q}}} \in \underline{\mathbb{U}}$, the unique decomposition is valid [20]:

$$\underline{\widehat{\mathbf{q}}} = \left(1 + \varepsilon\frac{1}{2}\widehat{\rho}\right)\widehat{\mathbf{q}}, \tag{31}$$

where $\boldsymbol{\rho} \in \boldsymbol{V}_3$ and $\widehat{\boldsymbol{q}} \in \mathbb{U}$. Additionally, a dual number $\underline{\alpha}$ and a unit dual vector $\underline{\mathbf{u}}$ exist for the exponential formula [16]:

$$\underline{\widehat{\mathbf{q}}} = \cos\frac{\underline{\alpha}}{2} + \underline{\mathbf{u}}sin\frac{\underline{\alpha}}{2} = \exp\left(\frac{\underline{\alpha}}{2}\underline{\mathbf{u}}\right), \tag{32}$$

where $\underline{\alpha}$ and $\underline{\mathbf{u}}$ are the natural invariants of the rigid body displacement (see **Theorem 2**).

Remark 3. *The exponential mapping: exp: $\underline{\boldsymbol{V}}_3 \to \underline{\mathbb{U}}_3$, $\underline{\widehat{\mathbf{q}}} = \exp\left(\frac{1}{2}\underline{\boldsymbol{\alpha}}\right)$ is well defined and surjective.*

Remark 4. *The unit dual quaternions $\underline{\mathbb{U}}$ with multiplication is a Lie group. This Lie algebra is dual vectors set $\underline{\boldsymbol{V}}_3$ (as the internal operation with the cross product of dual vectors). The dual unit quaternions set which can be used to global parameterize all rigid body displacements. The rigid body motion can be parameterized by a curve in Lie group of unit dual quaternions $\underline{\widehat{\mathbf{q}}} = \underline{\widehat{\mathbf{q}}}(t), t \in \mathbb{R}$.*

The following theorem gives the connection between unit dual quaternions and orthogonal dual tensors:

Theorem 8. *The Lie group of unit dual quaternions $\underline{\mathbb{U}}$ and Lie group of orthogonal dual tensors $\underline{\mathbb{SO}}_3$ are linked by a surjective homomorphism:*

$$\Theta : \underline{\mathbb{U}} \to \underline{\mathbb{SO}}_3, \Theta\left(\underline{\widehat{\mathbf{q}}}\right) = \underline{\boldsymbol{I}} + 2\underline{q}\underline{\widetilde{\mathbf{q}}} + 2\underline{\widetilde{\mathbf{q}}}^2 ; \underline{\widehat{\mathbf{q}}} = \underline{q} + \underline{\mathbf{q}}. \tag{33}$$

Proof. Considering that any $\underline{\widehat{\mathbf{q}}} \in \underline{\mathbb{U}}$ can be decomposed as in (32), results that $\Theta\left(\underline{\widehat{\mathbf{q}}}\right) = \exp\left(\underline{\alpha}\underline{\widetilde{\boldsymbol{u}}}\right) \in \underline{\mathbb{SO}}_3$. This shows that the mapping given by (33) is well defined and surjective. Using direct calculus, we can also acknowledge that $\Theta\left(\underline{\widehat{\mathbf{q}}}_2\underline{\widehat{\mathbf{q}}}_1\right) = \Theta\left(\underline{\widehat{\mathbf{q}}}_2\right)\Theta\left(\underline{\widehat{\mathbf{q}}}_1\right)$.

Regarding surjectivity, any orthogonal dual tensor $\underline{\boldsymbol{R}} \in \underline{\mathbb{SO}}_3$ can be represented as in Theorem 3, $\underline{\boldsymbol{R}} = \exp\left(\underline{\alpha}\underline{\widetilde{\boldsymbol{u}}}\right)$. Thus, we can find a dual quaternion $\underline{\widehat{\mathbf{q}}} = \exp\left(\frac{\underline{\alpha}}{2}\underline{\mathbf{u}}\right)$ to have $\Theta\left(\underline{\widehat{\mathbf{q}}}\right) = \underline{\boldsymbol{R}}$, which proves that Θ is a surjective homomorphism.

One of the most important properties is $\Theta\left(\underline{\widehat{\mathbf{q}}}\right) = \Theta\left(-\underline{\widehat{\mathbf{q}}}\right)$ which shows that Lie group $\underline{\mathbb{U}}$ is a double cover for $\underline{\mathbb{SO}}_3$.

Let $\underline{\widehat{\mathbf{q}}} \in \underline{\mathbb{U}}^{\mathbb{R}}$ such that $\Theta\left(\underline{\widehat{\mathbf{q}}}\right) = \underline{\boldsymbol{R}}$. According to Equation (33), the kinematic equation from Equations (18) and (28) are equivalent to:

$$\begin{cases} \dot{\underline{\widehat{\mathbf{q}}}} = \frac{1}{2}\underline{\boldsymbol{\omega}}\underline{\widehat{\mathbf{q}}} \\ \underline{\widehat{\mathbf{q}}}(t_0) = \underline{\widehat{\mathbf{q}}}_0 \end{cases} \tag{34}$$

and respectively:

$$\begin{cases} \dot{\underline{\widehat{\mathbf{q}}}} = \frac{1}{2}\underline{\widehat{\mathbf{q}}}\underline{\boldsymbol{\omega}}^B \\ \underline{\widehat{\mathbf{q}}}(t_0) = \underline{\widehat{\mathbf{q}}}_0 \end{cases}, \tag{35}$$

where $\Theta\left(\underline{\hat{\mathbf{q}}}_0\right) = \underline{\boldsymbol{R}}_0$. □

Remark 5. *If* $Re\underline{\boldsymbol{\omega}} = \boldsymbol{\omega}$ *is an instantaneous angular velocity vector function with a fixed direction, then differential equations are given from Equations (18), (28), (34), and (35) have the closed-form solution as in [1,12,27].*

4. Rigid Body Motion Equations in Arbitrary Non-Inertial Frame Revised

In this section, we proposed a novel dual tensors-based model for the motion of the rigid body with respect to an arbitrary non-inertial frame.

Let $\underline{\boldsymbol{R}}_D$ and $\underline{\boldsymbol{R}}_C$ be the dual orthogonal tensors which describe the motion of two rigid bodies, denoted D and, respectively, C, relative to a given inertial reference frame.

Let $\underline{\boldsymbol{R}}$ the orthogonal dual tensor which models the 6-DOF relative motion of rigid body D relative to the reference frame originating from rigid body C, then:

$$\underline{\boldsymbol{R}} = \underline{\boldsymbol{R}}_C^T \underline{\boldsymbol{R}}_D \tag{36}$$

Let $\underline{\boldsymbol{\omega}}_C$ the dual angular velocity of the rigid body C in the body frame of C, and $\underline{\boldsymbol{\omega}}_D$ the dual angular velocity of the rigid body D, resolved in the body frame of C. If $\underline{\boldsymbol{\omega}}$ is the dual angular velocity of the rigid body D relative to reference frame originated from rigid body C, resolved in the body frame of C, with Equation (36) result:

$$\underline{\boldsymbol{\omega}} = \underline{\boldsymbol{\omega}}_D - \underline{\boldsymbol{\omega}}_C \tag{37}$$

The motion of the rigid body C is considered known. Let $\underline{\boldsymbol{\omega}}_D^B$ being the dual angular velocity vector of the rigid body D in this body frame. If the body frame of rigid body D is centered in the mass center, the dual equation of motion given in [28] is:

$$\underline{\boldsymbol{J}}\dot{\underline{\boldsymbol{\omega}}}_D^B + \underline{\boldsymbol{\omega}}_D^B \times \underline{\boldsymbol{J}}\underline{\boldsymbol{\omega}}_D^B = \underline{\boldsymbol{\tau}}^B \tag{38}$$

In Equation (38), $\underline{\boldsymbol{\tau}}^B = \mathbf{F}^B + \varepsilon\boldsymbol{\tau}^B$, where $\mathbf{F}^B$ the resulting force, and $\boldsymbol{\tau}^B$ is the resulting torque in the mass center of rigid body D. Additionally, in Equation (38), $\underline{\boldsymbol{J}}$ denote the inertia dual operator: $\underline{\boldsymbol{J}} = m_D \frac{d}{d\varepsilon}\boldsymbol{I} + \varepsilon\boldsymbol{J}$, where $\boldsymbol{J}$ is the inertia tensor of the rigid body D related to its mass centre and m_D is the mass of the rigid body D. By equation: $\underline{\boldsymbol{J}}^{-1} = \boldsymbol{J}^{-1}\frac{d}{d\varepsilon} + \varepsilon\frac{1}{m_D}\boldsymbol{I}$ with Equation (38) results:

$$\dot{\underline{\boldsymbol{\omega}}}_D^B + \underline{\boldsymbol{J}}^{-1}\left(\underline{\boldsymbol{\omega}}_D^B \times \underline{\boldsymbol{J}}\underline{\boldsymbol{\omega}}_D^B\right) = \underline{\boldsymbol{J}}^{-1}\underline{\boldsymbol{\tau}}^B. \tag{39}$$

Let $\underline{\boldsymbol{\omega}}_D = \underline{\boldsymbol{R}}\underline{\boldsymbol{\omega}}_D^B$, the dual angular velocity vector $\underline{\boldsymbol{\omega}}$ can be computed from (see Equation (37):

$$\underline{\boldsymbol{\omega}} = \underline{\boldsymbol{R}}\underline{\boldsymbol{\omega}}_D^B - \underline{\boldsymbol{\omega}}_C \tag{40}$$

After time differentiation of Equation (40), results:

$$\dot{\underline{\boldsymbol{\omega}}} + \dot{\underline{\boldsymbol{\omega}}}_C = \dot{\underline{\boldsymbol{R}}}\underline{\boldsymbol{\omega}}_D^B + \underline{\boldsymbol{R}}\dot{\underline{\boldsymbol{\omega}}}_D^B. \tag{41}$$

Multiplied by $\underline{\boldsymbol{R}}^T$ Equation (41) to obtain:

$$\underline{\boldsymbol{R}}^T\left(\dot{\underline{\boldsymbol{\omega}}} + \dot{\underline{\boldsymbol{\omega}}}_C\right) = \underline{\boldsymbol{R}}^T\dot{\underline{\boldsymbol{R}}}\underline{\boldsymbol{\omega}}_D^B + \dot{\underline{\boldsymbol{\omega}}}_D^B \tag{42}$$

with $\dot{\underline{\boldsymbol{R}}} = \tilde{\underline{\boldsymbol{\omega}}}\underline{\boldsymbol{R}}$, by Equation (42) results:

$$\underline{\boldsymbol{R}}^T\left(\dot{\underline{\boldsymbol{\omega}}} + \dot{\underline{\boldsymbol{\omega}}}_C\right) = \underline{\boldsymbol{R}}^T\tilde{\underline{\boldsymbol{\omega}}}\underline{\boldsymbol{R}}\underline{\boldsymbol{\omega}}_D^B + \dot{\underline{\boldsymbol{\omega}}}_D^B \tag{43}$$

After some algebra, Equation (43) proves:

$$\underline{\dot{\boldsymbol{\omega}}} + \underline{\dot{\boldsymbol{\omega}}}_C = \underline{\boldsymbol{R}}\underline{\dot{\boldsymbol{\omega}}}_D^B + \underline{\boldsymbol{\omega}} \times \underline{\boldsymbol{\omega}}_C \tag{44}$$

With Equations (39) and (43), from Equation (44) obtained:

$$\underline{\dot{\boldsymbol{\omega}}} + \underline{\dot{\boldsymbol{\omega}}}_C = \underline{\boldsymbol{R}}\underline{\boldsymbol{J}}^{-1}\underline{\boldsymbol{\tau}}^B - \underline{\boldsymbol{R}}\underline{\boldsymbol{J}}^{-1}\left(\underline{\boldsymbol{\omega}}_D^B \times \underline{\boldsymbol{J}}\underline{\boldsymbol{\omega}}_D^B\right) + \underline{\boldsymbol{\omega}} \times \underline{\boldsymbol{\omega}}_C \tag{45}$$

Because $\underline{\boldsymbol{\omega}}_D^B = \underline{\boldsymbol{R}}^T(\underline{\boldsymbol{\omega}} + \underline{\boldsymbol{\omega}}_C)$, by Equation (45):

$$\underline{\dot{\boldsymbol{\omega}}} + \underline{\dot{\boldsymbol{\omega}}}_C = \underline{\boldsymbol{R}}\underline{\boldsymbol{J}}^{-1}\left[\underline{\boldsymbol{\tau}}^B - \underline{\boldsymbol{R}}^T(\underline{\boldsymbol{\omega}} + \underline{\boldsymbol{\omega}}_C) \times \underline{\boldsymbol{J}}\underline{\boldsymbol{R}}^T(\underline{\boldsymbol{\omega}} + \underline{\boldsymbol{\omega}}_C)\right] + \underline{\boldsymbol{\omega}} \times \underline{\boldsymbol{\omega}}_C \tag{46}$$

The first order differential equation:

$$\begin{cases} \underline{\dot{\boldsymbol{R}}} = \underline{\widetilde{\boldsymbol{\omega}}}\underline{\boldsymbol{R}} \\ \underline{\dot{\boldsymbol{\omega}}} + \underline{\dot{\boldsymbol{\omega}}}_C = \underline{\boldsymbol{R}}\underline{\boldsymbol{J}}^{-1}[\underline{\boldsymbol{R}}^{\boldsymbol{T}}\underline{\boldsymbol{\tau}} - \underline{\boldsymbol{R}}^T(\underline{\boldsymbol{\omega}} + \underline{\boldsymbol{\omega}}_C) \times \underline{\boldsymbol{J}}\underline{\boldsymbol{R}}^T(\underline{\boldsymbol{\omega}} + \underline{\boldsymbol{\omega}}_C)] + \underline{\boldsymbol{\omega}} \times \underline{\boldsymbol{\omega}}_C \\ \underline{\boldsymbol{\omega}}(\mathrm{t}_0) = \underline{\boldsymbol{\omega}}_0, \underline{\boldsymbol{\omega}}_0 \in \underline{\boldsymbol{V}}_3 \\ \underline{\boldsymbol{R}}(t_0) = \underline{\boldsymbol{R}}_0, \underline{\boldsymbol{R}}_0 \in \underline{\mathrm{S}\mathbb{O}}_3 \end{cases} \tag{47}$$

is a compact differential equation to the state (pose and velocity field) of the rigid body in relation to the non-inertial reference frame attached to rigid body C. This equation of the dual vector and dual tensor is coupled and models the 6-DOF relative motion problem. The pose of rigid body D is given by the orthogonal dual tensor $\underline{\boldsymbol{R}}$, and velocity field of the dual angular velocities $\underline{\boldsymbol{\omega}}$.

In Equation (47), the proposed solution is onboard in the reference frame of rigid body C.

Using parametrization of dual quaternion by 6-DOF motion in the non-inertial reference frame, the differential Equation (47) is expressed by:

$$\begin{cases} \underline{\dot{\widehat{\mathbf{q}}}} = \frac{1}{2}\underline{\boldsymbol{\omega}}\underline{\widehat{\mathbf{q}}} \\ \underline{\dot{\boldsymbol{\omega}}} + \underline{\dot{\boldsymbol{\omega}}}_C = \mathbf{Ad}_{\underline{\widehat{\mathbf{q}}}}\underline{\boldsymbol{J}}^{-1}\left[\mathbf{Ad}_{\underline{\widehat{\mathbf{q}}}}{}^{-1}\underline{\boldsymbol{\tau}} - \mathbf{Ad}_{\underline{\widehat{\mathbf{q}}}}{}^{-1}(\underline{\boldsymbol{\omega}} + \underline{\boldsymbol{\omega}}_C) \times \underline{\boldsymbol{J}}\mathbf{Ad}_{\underline{\widehat{\mathbf{q}}}}{}^{-1}(\underline{\boldsymbol{\omega}} + \underline{\boldsymbol{\omega}}_C)\right] + \underline{\boldsymbol{\omega}} \times \underline{\boldsymbol{\omega}}_C \\ \underline{\boldsymbol{\omega}}(\mathrm{t}_0) = \underline{\boldsymbol{\omega}}_0, \underline{\boldsymbol{\omega}}_0 \in \underline{\boldsymbol{V}}_3 \\ \underline{\widehat{\mathbf{q}}}(t_0) = \underline{\widehat{\mathbf{q}}}_0, \underline{\widehat{\mathbf{q}}}_0 \in \underline{\mathbb{U}} \end{cases} \tag{48}$$

The previous equation denotes the adjoint dual quaternion application by:

$$\begin{array}{l} \mathbf{Ad}_{\underline{\widehat{\mathbf{q}}}} : \underline{\boldsymbol{V}}_3 \to \underline{\boldsymbol{V}}_3, \\ \mathbf{Ad}_{\underline{\widehat{\mathbf{q}}}}(\ ) = \underline{\widehat{\mathbf{q}}}\ (\ )\underline{\widehat{\mathbf{q}}}^*, \\ \mathbf{Ad}_{\underline{\widehat{\mathbf{q}}}}{}^{-1}() = \underline{\widehat{\mathbf{q}}}^*(\ )\underline{\widehat{\mathbf{q}}}\ ,\ \underline{\widehat{\mathbf{q}}} \in \underline{\mathbb{U}} \end{array} \tag{49}$$

Next, we will propose a representation theorem that decouples the inertial and non-inertial components of the unique solution of the differential equation of Equation (47).

In Equation (47), let the following substitution:

$$\underline{\boldsymbol{\omega}}_* = \underline{\boldsymbol{R}}^T(\underline{\boldsymbol{\omega}} + \underline{\boldsymbol{\omega}}_C) \tag{50}$$

Equation (50) leads to $\underline{\dot{\boldsymbol{\omega}}}_* = \underline{\dot{\boldsymbol{R}}}^T(\underline{\boldsymbol{\omega}} + \underline{\boldsymbol{\omega}}_C) + \underline{\boldsymbol{R}}^T(\underline{\dot{\boldsymbol{\omega}}} + \underline{\dot{\boldsymbol{\omega}}}_C) = -\underline{\boldsymbol{R}}^T\underline{\widetilde{\boldsymbol{\omega}}}(\underline{\boldsymbol{\omega}} + \underline{\boldsymbol{\omega}}_C) + \underline{\boldsymbol{R}}^T(\underline{\dot{\boldsymbol{\omega}}} + \underline{\dot{\boldsymbol{\omega}}}_C)$. The result is equivalent with $\underline{\dot{\boldsymbol{\omega}}}_* = \underline{\boldsymbol{R}}^T(\underline{\boldsymbol{\omega}}_C \times \underline{\boldsymbol{\omega}} + \underline{\dot{\boldsymbol{\omega}}} + \underline{\dot{\boldsymbol{\omega}}}_C)$ or

$$\underline{\boldsymbol{\omega}}_C \times \underline{\boldsymbol{\omega}} + \underline{\dot{\boldsymbol{\omega}}} + \underline{\dot{\boldsymbol{\omega}}}_C = \underline{\boldsymbol{R}}\underline{\dot{\boldsymbol{\omega}}}_*. \tag{51}$$

By Equations (47) and (51), results:

$$\begin{cases} \underline{J}\dot{\underline{\boldsymbol{\omega}}}_* + \underline{\boldsymbol{\omega}}_* \times \underline{J}\underline{\boldsymbol{\omega}}_* = \underline{\boldsymbol{\tau}}_* \\ \underline{\boldsymbol{\omega}}_*(t_0) = \underline{\boldsymbol{\omega}}_*^0 \end{cases} \tag{52}$$

In Equation (50), $\underline{\boldsymbol{\tau}}_* = \underline{\boldsymbol{R}}^T \underline{\boldsymbol{\tau}}$ is dual torque related to the mass center of the body D resolved in this body frame, and dual vector $\underline{\boldsymbol{\omega}}_*^0 = \underline{\boldsymbol{R}}_0^T(\underline{\boldsymbol{\omega}}_0 + \underline{\boldsymbol{\omega}}_C(t_0))$. Equation (52) represents, in dual form, the equation of motion on the rigid body in the inertial reference frame.

For $\underline{\boldsymbol{R}} \in \underline{\mathbb{SO}}_3^{\mathbb{R}}$, the solution of Equation (47) has the differential kinematic equation:

$$\begin{cases} \dot{\underline{\boldsymbol{R}}} = \widetilde{\underline{\boldsymbol{\omega}}}\underline{\boldsymbol{R}} \\ \underline{\boldsymbol{R}}(t_0) = \underline{\boldsymbol{R}}_0 \end{cases} \tag{53}$$

Using Equation (50), results that $\underline{\boldsymbol{R}}\underline{\boldsymbol{\omega}}_* = \underline{\boldsymbol{\omega}} + \underline{\boldsymbol{\omega}}_C$, transformed into $\widetilde{\underline{\boldsymbol{R}}\underline{\boldsymbol{\omega}}_*} = \widetilde{\underline{\boldsymbol{\omega}}} + \widetilde{\underline{\boldsymbol{\omega}}}_C \Leftrightarrow \underline{\boldsymbol{R}}\widetilde{\underline{\boldsymbol{\omega}}}_*\underline{\boldsymbol{R}}^T = \dot{\underline{\boldsymbol{R}}}\underline{\boldsymbol{R}}^T + \widetilde{\underline{\boldsymbol{\omega}}}_C$. Multiplying the last equation by dual orthogonal tensor $\underline{\boldsymbol{R}}$, we obtain the differential equation:

$$\begin{cases} \dot{\underline{\boldsymbol{R}}} = \underline{\boldsymbol{R}}\widetilde{\underline{\boldsymbol{\omega}}}_* - \widetilde{\underline{\boldsymbol{\omega}}}_C\underline{\boldsymbol{R}} \\ \underline{\boldsymbol{R}}(t_0) = \underline{\boldsymbol{R}}_0 \end{cases} \tag{54}$$

Let $\underline{\boldsymbol{R}}_{-\underline{\omega}_C} \in \underline{\mathbb{SO}}_3^{\mathbb{R}}$ solution of the following differential equation:

$$\begin{cases} \dot{\underline{\boldsymbol{R}}} + \widetilde{\underline{\boldsymbol{\omega}}}_C\underline{\boldsymbol{R}} = 0 \\ \underline{\boldsymbol{R}}(t_0) = I - \varepsilon\widetilde{\boldsymbol{r}}_C(t_0) \end{cases}. \tag{55}$$

Considering $\underline{\boldsymbol{R}} = \underline{\boldsymbol{R}}_{-\underline{\omega}_C}\underline{\boldsymbol{R}}_*$, obtain an exact solution of Equation (47). Previous consideration proves the following theorem:

Theorem 9 (Relative Motion Representation Theorem). *The solution of Equation* (47) *results applying the tensor* $\underline{\boldsymbol{R}}_{-\underline{\omega}_C}$ *(solution from Equation (55) to the solution of differential equation:*

$$\begin{cases} \dot{\underline{\boldsymbol{R}}}_* = \underline{\boldsymbol{R}}_*\widetilde{\underline{\boldsymbol{\omega}}}_* \\ \underline{J}\dot{\underline{\boldsymbol{\omega}}}_* + \underline{\boldsymbol{\omega}}_* \times \underline{J}\underline{\boldsymbol{\omega}}_* = \underline{\boldsymbol{\tau}}_* \\ \underline{\boldsymbol{\omega}}_*(t_0) = \underline{\boldsymbol{\omega}}_{*0} \\ \underline{\boldsymbol{R}}_*(t_0) = \underline{\boldsymbol{R}}_{*0} \end{cases} \tag{56}$$

where $\underline{\boldsymbol{\omega}}_{*0} = \underline{\boldsymbol{R}}_0^T(\underline{\boldsymbol{\omega}}_0 + \underline{\boldsymbol{\omega}}_C(t_0)), \underline{\boldsymbol{R}}_{*0} = \left(\boldsymbol{I} + \varepsilon\widetilde{\boldsymbol{r}}_C(t_0)\right)\underline{\boldsymbol{R}}_0$, $\underline{\boldsymbol{\tau}}_* = \underline{\boldsymbol{R}}^T\underline{\boldsymbol{\tau}}$.

Remark 6. *Using dual quaternions, a version of the* ***Theorem 9*** *result:*

Theorem 10. *The solution of differential Equation (47) is orthogonal dual tensor* $\underline{\boldsymbol{R}} = \Theta\left(\widehat{\underline{\mathbf{q}}}_{-\underline{\omega}_C}\widehat{\underline{\mathbf{q}}}_*\right)$ *where* $\widehat{\underline{\mathbf{q}}}_{-\underline{\omega}_C}$ *is the unique solution of differential equation:*

$$\begin{cases} \dot{\widehat{\underline{\mathbf{q}}}} + \frac{\underline{\boldsymbol{\omega}}_C}{2}\widehat{\underline{\mathbf{q}}} = \widehat{\underline{0}} \\ \widehat{\underline{\mathbf{q}}}(t_0) = \widehat{\underline{1}} \end{cases}, \tag{57}$$

with $\widehat{\underline{\mathbf{q}}}_*$ *being the solution of the differential equation problem below*:

$$\begin{cases} \dot{\widehat{\underline{\mathbf{q}}}}_* = \frac{1}{2}\widehat{\underline{\mathbf{q}}}_*\underline{\boldsymbol{\omega}}_* \\ \underline{J}\dot{\underline{\boldsymbol{\omega}}}_* + \underline{\boldsymbol{\omega}}_* \times \underline{J}\underline{\boldsymbol{\omega}}_* = \underline{\boldsymbol{\tau}}_* \\ \underline{\boldsymbol{\omega}}_*(t_0) = \underline{\boldsymbol{\omega}}_{*0} \\ \widehat{\underline{\mathbf{q}}}_*(t_0) = \widehat{\underline{\mathbf{q}}}_{*0} \end{cases}. \tag{58}$$

Theorems 9 and 10 give significant insight into the motion of any rigid body in a non-inertial reference frame. A simple method to approach its motion is revealed as follows:

1. The problem is solved in an inertial frame; our non-inertial frame is "frozen" at the initial moment t_0;
2. The solution to the non-inertial problem is obtained by applying tensor $\underline{\boldsymbol{R}}_{-\underline{\omega}_C}$, or the dual quaterninton $\underline{\widehat{\mathbf{q}}}_{-\underline{\omega}_C}$, to the solution obtained in the previous step.

5. Dual Algebra Solution of the Full-Body Relative Orbital Motion Problem

The relative orbital motion of spacecraft is a fundamental problem in Astrodynamics considering its numerous applications: rendezvous operations, distributed spacecraft missions, and formation flight of spacecraft [3,4,6–12]. Some formation flying spacecraft applications are space-based radar, ground-based terrestrial laser communication systems, Earth surveillance, remote sensing, stellar imaging, and astrometry.

The relative orbital motion model shows two spacecrafts flying in Keplerian orbits due to the same gravitational attraction center. The main problem is determining the relative motion of the Deputy spacecraft concerning a LVLH non-inertial frame originating from the center of mass of the Chief spacecraft. The relative 6-DOF motion of the Deputy concerning the LVLH frame is present in this section, using the general results of Section 4. The Chief and Deputy spacecrafts can be considered modeled by rigid bodies.

The vector that gives the instantaneous angular velocity of the LVLH in this reference frame is:

$$\boldsymbol{\omega}_C = \dot{f}_C \frac{\mathbf{h}_C}{\mathrm{h}_C} = \frac{1}{\mathrm{r}_C^2}\mathbf{h}_C = \left[\frac{1 + \mathrm{e}_C \cos f_C(\mathrm{t})}{\mathrm{p}_C}\right]^2 \mathbf{h}_C \tag{59}$$

The position vector of Chief mass center, $\mathbf{r}_C$, originating in the gravitational attracting center, is expressed in LVLH:

$$\mathbf{r}_C = \frac{p_C}{1 + e_C \cos f_C(t)} \frac{\mathbf{r}_C^0}{\mathrm{r}_C^0} \tag{60}$$

In Equations (59) and (60), $\mathbf{h}_C$ is the angular momentum, p_C is the conic parameter, $f_C(t)$ being the true anomaly and e_C is the eccentricity of the Chief spacecraft.

Furthermore, the time derivative of $\mathbf{r}_C$ in LVLH frame is:

$$\dot{\mathbf{r}}_C = \frac{e_C |\mathbf{h}_C| \sin f_C(\mathrm{t})}{\mathrm{p}_C} \frac{\mathbf{r}_C^0}{\mathrm{r}_C^0} \tag{61}$$

For $t = t_0$ they will be used in the following denotations:

$$\boldsymbol{\omega}_C^0 = \left[\frac{1 + e_C \cos f_C(\mathrm{t}_0)}{\mathrm{p}_C}\right]^2 \mathbf{h}_C \tag{62}$$

$$\dot{\mathbf{r}}_C^0 = \frac{e_C |\mathbf{h}_C| \sin f_C(\mathrm{t}_0)}{\mathrm{p}_C} \frac{\mathbf{r}_C^0}{\mathrm{r}_C^0} \tag{63}$$

where $\mathbf{u}_X = \frac{\mathbf{r}_C^0}{\mathrm{r}_C^0}$ is the unit vector of the X-axis, $\mathbf{u}_Z = \frac{\mathbf{h}_C}{|\mathbf{h}_C|}$ is the unit vector of the Z-axis, and $\mathbf{u}_Y = \mathbf{u}_Z \times \mathbf{u}_X$ is the unit vector of the Y-axis from LVLH.

The 6-DOF relative orbital motion is described by the Equation (47). In this specific case the dual angular velocity of the Chief spacecraft in the LVLH reference frame is given by equation:

$$\underline{\boldsymbol{\omega}}_C = \boldsymbol{\omega}_C + \varepsilon\left(\dot{\mathbf{r}}_C + \boldsymbol{\omega}_C \times \mathbf{r}_C\right) \tag{64}$$

The dual torque, related to the mass center of the Deputy spacecraft is:

$$\underline{\boldsymbol{\tau}} = -\frac{\mu}{|\mathbf{r}_c + \mathbf{r}|^3}(\mathbf{r}_c + \mathbf{r}) + \varepsilon\boldsymbol{\tau}. \tag{65}$$

Theorem 9 is applied using the Equations (59)–(63). The instantaneous angular velocity $\boldsymbol{\omega}_C$ has a fixed direction (see Equation (59)), and the solution to the differential Equation (55) is in a closed form, coordinate-free:

$$\underline{\boldsymbol{R}}_{-\underline{\omega}_C} = \left(\boldsymbol{I} - \varepsilon\widetilde{\boldsymbol{r}}_C(t)\right)\left(\boldsymbol{I} - \sin f_c^0 \frac{\widetilde{\mathbf{h}}_C}{\mathrm{h}_c} + \left(1 - \cos f_c^0\right)\frac{\widetilde{\mathbf{h}}_C^{\,2}}{\mathrm{h}_c^{\,2}}\right). \tag{66}$$

In Equation (66) denotes $\mathrm{h}_c = \|\mathbf{h}_c\|$ and $f_c^0 = f_c(t) - f_c(t_0)$.

Theorem 11. *The solution of (Equation (47)) results from the application of the tensor $\underline{\boldsymbol{R}}_{-\omega_C}$ (Equation (66)) to the solution of the inertial Problem (56), with dual angular velocity $\underline{\boldsymbol{\omega}}_C$ and dual torque $\underline{\boldsymbol{\tau}}$ given by Equation (64) and, respectively, Equation (65).*

The Attitude and Translation Equations of the 6-DOF Relative Orbital Motion

The attitude and translation parts of solution to the problem of motion Deputy spacecraft concerning to the LVLH frame will be obtained in the next **Theorem**.

Consider the real part of (Equation (47)) results in a first order differential equation:

$$\begin{cases} \dot{\boldsymbol{Q}} = \widetilde{\boldsymbol{\omega}}\boldsymbol{Q} \\ \dot{\boldsymbol{\omega}} + \dot{\boldsymbol{\omega}}_c = \boldsymbol{Q}\boldsymbol{J}^{-1}\left[\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{\tau} - \boldsymbol{Q}^{\mathrm{T}}(\boldsymbol{\omega} + \boldsymbol{\omega}_c) \times \boldsymbol{J}\boldsymbol{Q}^{\mathrm{T}}(\boldsymbol{\omega} + \boldsymbol{\omega}_c)\right] + \boldsymbol{\omega} \times \boldsymbol{\omega}_c \\ \boldsymbol{\omega}(\mathrm{t}_0) = \boldsymbol{\omega}_0, \boldsymbol{\omega}_0 \in \boldsymbol{V}_3 \\ \boldsymbol{Q}(\mathrm{t}_0) = \boldsymbol{Q}_0, \boldsymbol{Q}_0 \in \mathrm{S}\mathbb{O}_3 \end{cases} \tag{67}$$

which has the solution $\boldsymbol{Q} = \boldsymbol{Q}(\mathrm{t}) \in \mathrm{S}\mathbb{O}_3^{\mathbb{R}}$. The real tensor $\boldsymbol{Q}$ being the attitude motion of the Deputy in LVLH.

Consider now the dual part of Equation (47). By decomposition $\underline{\boldsymbol{R}} = \left(\boldsymbol{I} + \varepsilon\widetilde{\boldsymbol{r}}\right)\boldsymbol{Q}$, which Equation (2) gives, after some algebra, we obtain a second-order differential vector equation that models the translation of the Deputy spacecraft mass center expressed into the LVLH:

$$\begin{cases} \ddot{\mathbf{r}} + 2\boldsymbol{\omega}_c \times \dot{\mathbf{r}} + \boldsymbol{\omega}_c \times (\boldsymbol{\omega}_c \times \mathbf{r}) + \dot{\boldsymbol{\omega}}_c \times \mathbf{r} + \frac{\mu}{|\mathbf{r}_c + \mathbf{r}|^3}(\mathbf{r}_c + \mathbf{r}) - \frac{\mu}{r_c^3}\mathbf{r}_c = 0 \\ \mathbf{r}(t_0) = \mathbf{r}_0, \dot{\mathbf{r}}(t_0) = \mathbf{v}_0 \end{cases} \tag{68}$$

where $\mu > 0$ is the gravitational parameter and $\mathbf{r}$ represent the relative position vector of the mass centre of the Deputy spacecraft concerning to LVLH frame.

Based on the general **Theorem 9**, the next theorem results.

Theorem 12. *The solutions of problems (Equations (67) and (68)) are given by*

$$\begin{array}{c} \boldsymbol{Q} = \boldsymbol{R}_{-\omega_C}\boldsymbol{Q}_* \\ \mathbf{r} = \boldsymbol{R}_{-\omega_C}\mathbf{r}_* - \mathbf{r}_c \end{array} \tag{69}$$

where $\boldsymbol{Q}_*$ and $\mathbf{r}_*$ are the solutions of the Euler problem and, respectively, Kepler's problem:

$$\begin{cases} \dot{\boldsymbol{Q}}_* = \boldsymbol{Q}_*\widetilde{\boldsymbol{\omega}}_* \\ J\dot{\boldsymbol{\omega}}_* + \boldsymbol{\omega}_* \times J\boldsymbol{\omega}_* = \boldsymbol{\tau}_* \\ \boldsymbol{\omega}_*(t_0) = \boldsymbol{Q}_0^{\mathrm{T}}\left(\boldsymbol{\omega}_0 + \boldsymbol{\omega}_C^0\right) \\ \boldsymbol{Q}_*(t_0) = \boldsymbol{Q}_0 \end{cases} \tag{70}$$

and

$$\begin{cases} \ddot{\mathbf{r}}_* + \frac{\mu}{r_*^3}\mathbf{r}_* = 0; \\ \mathbf{r}_*(t_0) = \mathbf{r}_c^0 + \mathbf{r}_0; \\ \dot{\mathbf{r}}_*(t_0) = \dot{\mathbf{r}}_C^0 + \mathbf{v}_0 + \boldsymbol{\omega}_C^0 \times \left(\mathbf{r}_C^0 + \mathbf{r}_0\right) \end{cases} \quad (71)$$

were

$$\boldsymbol{R}_{-\omega_C} = \boldsymbol{I} - \sin f_c^0 \frac{\widetilde{\mathbf{h}}_C}{|\mathbf{h}_c|} + \left(1 - \cos f_c^0\right) \frac{\widetilde{\mathbf{h}}_C^2}{|\mathbf{h}_c|^2} \quad (72)$$

and $\mathbf{r}_c$ *is given by (Equation (60)).*

Remark 7. *The* ***Theorem 12*** *reduces the complex problem of the full body relative orbital motion into two classical problems in an inertial reference frame: the Euler fixed point problem (Equation (68)) and the Kepler problem (Equation (69)). However, the problems are coupled because, in the general case, the resulting torque depends on the position vector of the Deputy mass center relative to the attractive gravitational center. Even in these conditions, simplification is essential to the problem approach. Moreover, this approach can generate new analytical or semi-analytical solutions and the control theory of 6-DOF motion.*

This result shows an interesting property of the translational part of the relative orbital motion Problem (66); this problem by reducing to the super-integrable Kepler problem [11,12,29].

Next, we present a case study for the exact closed-form, a coordinate-free solution of the translation part of the relative orbital motion problem. The solution work for any reference spacecraft Chief motion mass center (elliptic, parabolic, hyperbolic inertial trajectories) and any Deputy motion mass center (elliptic, parabolic, hyperbolic, rectilinear inertial trajectories). From **Theorem 12**, the law of motion of the translational part is given by:

$$\mathbf{r} = \boldsymbol{R}_{-\omega_C}\mathbf{r}_* - \frac{p_C}{1 + e_C \cos f_C(t)} \frac{\mathbf{r}_C^0}{\mathbf{r}_{C\,c}^0} \quad (73)$$

$\boldsymbol{R}_{-\omega_C}$ is expressed by Equation (70) and $\mathbf{r}_*$ is the solution of the Kepler Problem (69).
Let the prime integrals of the Kepler problem [26–34]:
Specific energy:

$$\xi = \frac{1}{2}||\dot{\mathbf{r}}_C^0 + \mathbf{v}_0 + \boldsymbol{\omega}_C^0 \times (\mathbf{r}_C^0 + \mathbf{r}_0)||^2 - \frac{\mu}{\left\|\mathbf{r}_c^0 + \mathbf{r}_0\right\|} \quad (74)$$

Specific angular momentum:

$$\mathbf{h} = \left(\mathbf{r}_c^0 + \mathbf{r}_0\right) \times \left[\dot{\mathbf{r}}_C^0 + \mathbf{v}_0 + \boldsymbol{\omega}_C^0 \times \left(\mathbf{r}_C^0 + \mathbf{r}_0\right)\right] \quad (75)$$

Eccentricity vector:

$$\mathbf{e} = \frac{1}{\mu}\left[\left(\dot{\mathbf{r}}_C^0 + \mathbf{v}_0 + \boldsymbol{\omega}_C^0 \times \left(\mathbf{r}_C^0 + \mathbf{r}_0\right)\right] \times \mathbf{h} - \frac{\mathbf{r}_c^0 + \mathbf{r}_0}{\left\|\mathbf{r}_c^0 + \mathbf{r}_0\right\|} \quad (76)$$

If it denotes $n = \frac{\mu}{(2|\xi|)^{\frac{3}{2}}}$, the vectorial orbital elements of Kepler problem [34] are:

$$\mathbf{a} = \begin{cases} \frac{\mu}{2e|\xi|}\boldsymbol{e}, \boldsymbol{e} \neq 0 \\ \mathbf{r}_c^0 + \mathbf{r}_0, \boldsymbol{e} = 0' \end{cases} \quad (77)$$

$$\mathbf{b} = \begin{cases} \frac{1}{e\sqrt{2|\xi|}}\mathrm{h} \times \boldsymbol{e}, \boldsymbol{e} \neq 0 \\ \frac{1}{n}\left[\dot{\mathbf{r}}_C^0 + \mathbf{v}_0 + \boldsymbol{\omega}_C^0 \times \left(\mathbf{r}_C^0 + \mathbf{r}_0\right)\right], \boldsymbol{e} = 0' \end{cases} \quad (78)$$

If the Deputy spacecraft is an elliptic inertial motion, $\xi < 0$, the vector solution of the Kepler problem is:

$$\mathbf{r}_{*} = [\cos E(t) - e]\mathbf{a} + [\sin E(t)]\mathbf{b} \tag{79}$$

The eccentric anomaly $E(t)$ is given by Kepler equation:

$$E(t) - e\sin[E(t)] = n\left(t - t_p\right) \tag{80}$$

$$t_p = t_0 - \frac{1}{n}\left(E_o - e\sin E_o\right) \tag{81}$$

$$E_o = \operatorname{atan2}\left[n\frac{\mathbf{v}_0\cdot\left(\mathbf{r}_c^0 + \mathbf{r}_0\right)}{2|\xi|}\left(1 - \frac{\boldsymbol{\omega}_C^0\cdot\mathbf{h}}{\mu}||\mathbf{r}_c^0 + \mathbf{r}_0||\right), 1 - n\frac{\left\|\mathbf{r}_c^0 + \mathbf{r}_0\right\|}{\sqrt{2|\xi|}}\right. \tag{82}$$

The law of motion of the translational part by Equation (79) and **Theorem 12**, is:

$$\mathbf{r} = [\cos E(t) - e]\boldsymbol{R}_{-\omega_C}\mathbf{a} + [\sin E(t)]\boldsymbol{R}_{-\omega_C}\mathbf{b} - \frac{p_C}{1 + e_C\cos f_C(t)}\frac{\mathbf{r}_C^0}{\mathrm{r}_{C\,c}^0} \tag{83}$$

By Equation (83), after some calculus results:

$$\begin{aligned}\mathbf{r} = {} & [\cos E(t) - e]\left(\mathbf{a} - \sin f_c^0\frac{\widetilde{\mathbf{h}}_C\mathbf{a}}{|\mathbf{h}_c|} + \left(1 - \cos f_c^0\right)\frac{\widetilde{\mathbf{h}}_C^2\mathbf{a}}{|\mathbf{h}_c|^2}\right) \\ & + [\sin E(t)]\left(\mathbf{b} - \sin f_c^0\frac{\widetilde{\mathbf{h}}_C\mathbf{b}}{|\mathbf{h}_c|} + \left(1 - \cos f_c^0\right)\frac{\widetilde{\mathbf{h}}_C^2\mathbf{b}}{|\mathbf{h}_c|^2}\right) - \frac{p_C}{1+e_C\cos f_C(t)}\frac{\mathbf{r}_C^0}{\mathrm{r}_{C\,c}^0}\end{aligned} \tag{84}$$

This equation is coordinate-free and closed-form. Therefore, Equation (84) represent an exact solution, an alternative to the Tschauner–Hempel and Lawden linearized solution [35,36].

6. Conclusions

The present paper develops new methods for recovering a solution to the full-body relative orbital motion problem in the specific case of Keplerian confocal orbits. First, the coupled state equations for a rigid body in an arbitrary non-inertial reference frame are presented using orthogonal dual tensor or dual quaternion. Then, a representation theorem that decouples the inertial and non-inertial components of the last motion is presented. The core result of the paper offers meaningful insight and a natural geometric interpretation of the motion, namely that it is derived from the motion in a well-defined inertial frame, which is seen through a transformation that depends solely on an orthogonal dual tensor (or unit dual quaternion) that models the behavior of the non-inertial frame. The representation theorems for the rotation and translation parts of the 6-DOF relative orbital motion problems are obtained. A case study is present for the exact closed-form, a coordinate-free solution of the translation part of the problem of relative orbital motion. The results interest spacecraft formation flying, rendezvous orbital dynamics and control, advances in rendezvous trajectory safety, and robust analysis.

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Nomenclature

a	real number
$\underline{a}$	dual number
$\mathbf{a}$	real vector
$\underline{\mathbf{a}}$	dual vector
$\boldsymbol{A}$	real tensor
$\underline{\boldsymbol{A}}$	dual tensor
$\boldsymbol{V}_3$	real vectors set
$\underline{\boldsymbol{V}}_3$	dual vectors set
$\boldsymbol{V}_3^{\mathbb{R}}$	time depending real vector functions
$\underline{\boldsymbol{V}}_3^{\mathbb{R}}$	time depending dual vector functions
$\underline{\widetilde{\boldsymbol{a}}}$	skew-symmetric dual tensor corresponding to the dual vector $\underline{\mathbf{a}} = vect\underline{\widetilde{\boldsymbol{a}}}$
$\text{skew}\left(\underline{\widetilde{\boldsymbol{a}}}\ \right)$	dual vector corresponding a skew-symmetric dual tensor $\underline{\widetilde{\boldsymbol{a}}}$
$\boldsymbol{L}(\underline{\boldsymbol{V}}_3, \underline{\boldsymbol{V}}_3)$	Euclidean dual tensor set
$\mathbb{R}$	real numbers set
$\underline{\mathbb{R}}$	dual numbers set
$\mathbb{so}_3$	skew-symmetric Euclidean real tensor set
$\mathbb{SO}_3$	proper orthogonal Euclidean real tensor set
$\underline{\mathbb{so}}_3$	skew-symmetric Euclidean dual tensor set
$\underline{\mathbb{SO}}_3$	orthogonal Euclidean orthogonal dual tensor set
$\mathbb{SO}_3^{\mathbb{R}}$	time depending Euclidean real orthogonal tensor functions
$\underline{\mathbb{SO}}_3^{\mathbb{R}}$	time depending Euclidean dual orthogonal tensor functions
$\mathbb{U}$	unitreal quaternionsset
$\underline{\mathbb{U}}$	unit dual quaternions set
f_c	true anomaly of Keplerian motion
p_c	conic parameter
e_C	eccentricity of the Chief spacecraft
$\mathbf{h}_c$	specific angular momentum of the Chief spacecraft

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