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Li–Yau-Type Gradient Estimate along Geometric Flow

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Abstract: In this article we derive a Li–Yau-type gradient estimate for a generalized weighted parabolic heat equation with potential on a weighted Riemannian manifold evolving by a geometric flow. As an application, a Harnack-type inequality is also derived in the end.

Keywords: gradient estimate; weighted Laplacian; parabolic equation; geometric flow

MSC: 53C21; 53E20; 35B45

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1. Introduction

The gradient estimation for both elliptic and parabolic equations plays a significant role in geometric analysis. Harnack estimation is also one of the powerful tools in heat kernel analysis. The local and global behavior of positive solutions of nonlinear elliptic equations on $\mathbb{R}^n$ $(n > 2)$ near an isolated singularity were studied by Gidas and Spruck [1]. In [2], Hamilton proved a Harnack estimate on the Riemannian manifold for Ricci flow with a weakly positive curvature operator, which was later used in solving the Poincaré conjecture. Li and Yau [3] established parabolic gradient estimates on solutions to the linear heat equation

$$(\Delta - \partial_t)u = q(x,t)u \tag{1}$$

on Riemannian manifold having Ricci curvature bounded from below, where $q(x,t)$ is C^2 in first variable x and C^1 in second variable t, where C^2 and C^1 denote the space of all twice differentiable and one-time differentiable functions, respectively. After a remarkable work by Perelman [4–6] in Ricci flow, this topic gained massive importance. Thus, this topic becomes one of the important tools in geometric analysis and modern PDE theory. In [7], Jiyau Li considered the heat-type equation

$$(\Delta - \partial_t)u(x,t) + h(x,t)u^{\alpha}(x,t) = 0 \tag{2}$$

on $M \times [0,\infty)$, where $h(x,t)$, is a function on $M \times [0,\infty)$, which is C^2 in the first variable and C^1 in the second variable, $\alpha \in \mathbb{R}$ and derived the gradient estimates and Harnack inequalities for a positive solution to the above nonlinear parabolic equation. This equation represents a simple ecological model for population dynamics, where $u(x,t)$ is the population density at time t.

Wu [8] studied gradient estimates for the nonlinear parabolic equation

$$(\Delta_{\phi} - \partial_t)u + \mu(x,t)u + p(x,t)u^{\alpha} + q(x,t)u^{\beta} = 0, \tag{3}$$

where Δ_ϕ is the weighted Laplacian, $p(x,t)$, $q(x,t)$ are C^2 in x and C^1 in t. Abolarinwa et al. [9–12] studied gradient and Harnack estimates for various nonlinear parabolic equations. In [13], Dung et al. studied various gradient estimations for solutions of the following f-heat type equations

$$u_t = \Delta_f u + au\log u + bu + Cu^p + Du^{-q} \tag{4}$$
$$\text{and } u_t = \Delta_f u + Ce^{pu} + De^{-pu} + E, \tag{5}$$

where $a, b \in \mathbb{R}$ and C, D, E are smooth functions, on a complete smooth metric measure space $(M, g, e^{-f}dv)$ with Bakry–Émery Ricci curvature bounded from below. In [14], Azami studied gradient estimates for a weighted parabolic equation

$$(\Delta_\phi - \partial_t)u(x,t) = q(x,t)u^{a+1}(x,t) + p(x,t)A(u(x,t)) \tag{6}$$

evolving under the geometric flow, where $p(x,t), q(x,t), A(u(x,t))$ are C^2 in x and C^1 in t. Thereafter many authors studied the geometric aspect of analysis on the Riemannian manifold, see [15–23] and the references therein. Recently, Hui et al. studied Hamilton-Souplet-Zhang type gradient estimation for nonlinear weighted parabolic equation in [24], the same estimation for a system of equations in [25] and Saha et al. [26] studied first eigenvalue of weighted p-Laplacian along the Cotton flow.

Motivated by the above works in this paper we consider a generalized non-linear parabolic equation with potential by

$$\Delta_\phi u = \frac{\partial u}{\partial t} + A(u)p(x,t) + B(u)q(x,t) + \xi(x,t)u(x,t), \tag{7}$$

where $p(x,t), q(x,t)$ and $\xi(x,t)$ are C^2 functions of x, t. We derive a Li–Yau-type gradient estimate for a positive solution of (7) on a weighted Riemannian manifold which evolves under an abstract geometric flow.

In particular, if we consider $A(u) = u^\alpha,\ B(u) = u^\beta,\ \xi = \mu(x,t)$ then (7) reduces to (3), which was studied by Wu [8]. If we take $A(u) = u\log u,\ B(u) = u,\ \xi = Cu^p + Du^q$ then (7) reduces to (4) and if $A(u) = Ce^{pu},\ B(u) = De^{-pu},\ \xi = \frac{E}{u}$ then (7) reduces to (5), both of which were studied by Dung et al.[13]. The generalized Lichnerowicz type equation studied by Dung [13] comes from our Equation (7) by considering $A(u) = u^\alpha \log u,\ B(u) = u^\beta$ and p, q, ξ are suitable constants. Finally for $B(u) = u^{a+1}$ and $\xi = 0$ we have (6), which was studied by Azami [14]. Thus, our Equation (7) generalizes all the cases.

2. Preliminaries

Let us consider an n-dimensional closed weighted Riemannian manifold $(M^n, g, e^{-\phi}d\mu)$, where $e^{-\phi}d\mu$ is the weighted volume measure, g is Riemannian metric and $\phi \in C^2(M)$. Choose $\{e_1, e_2, \cdots, e_n\}$ as an orthonormal frame on M. Let $g(t)$ be a one-parameter family of Riemannian metrics evolving along the following abstract geometric flow

$$\frac{\partial}{\partial t} g_{ij}(t) = 2S_{ij}(t), \tag{8}$$

where $S_{ij}(t) := \mathcal{S}(e_i, e_j)(t)$ is smooth symmetric (0, 2)-type tensor on $(M, g(t))$. Let us define one parameter family of functions $S(t) = trace(\mathcal{S})(t) = g^{ij}(t)S_{ij}(t)$ on M. The weighted Laplacian operator is defined by

$$\Delta_\phi = \Delta - \nabla\phi\nabla,$$

where Δ is the Laplace operator and ∇ is the gradient operator. Let $u = e^f$ be a positive solution of (7), then Equation (7) transforms to

$$\Delta_\phi f = \partial_t f - |\nabla f|^2 + \hat{A}(f)p + \hat{B}(f)q + \xi, \tag{9}$$

where $\hat{A}(f) = \frac{A(u)}{u}$, $\hat{B}(f) = \frac{B(u)}{u}$. We define

$$\hat{A}_f = A'(u) - \frac{A(u)}{u}, \quad \hat{A}_{ff} = uA''(u) - A'(u) + \frac{A(u)}{u}. \tag{10}$$

Example 1. *Let $u = e^f$ and $A(u) = |u|^{\alpha-1}u$. Therefore $\hat{A}(f) = \frac{A(u)}{u} = e^{(\alpha-1)f}$, which gives*

1. $\hat{A}_f = (\alpha-1)e^{(\alpha-1)f}$
2. $\hat{A}_{ff} = (\alpha-1)^2 e^{(\alpha-1)f}$
3. $\nabla\hat{A} = (\alpha-1)e^{(\alpha-1)f}\nabla f = \hat{A}_f \nabla f$
4. $\Delta\hat{A} = (\alpha-1)^2 e^{(\alpha-1)f}|\nabla f|^2 + (\alpha-1)e^{(\alpha-1)f}\Delta f = \hat{A}_{ff}|\nabla f|^2 + \hat{A}_f \Delta f.$

Let $\bar{f} = \hat{A}p + \hat{B}q + \xi$ so that Equation (9) reduces to

$$\Delta_\phi f = -|\nabla f|^2 + f_t + \bar{f}. \tag{11}$$

Definition 1 ([27] Bakry–Émery Ricci tensor)**.** *For any integer $m > n$, an $(m-n)-$ Bakry–Émery tensor is defined by*

$$Ric_\phi^{m-n} := Ric + Hess\ \phi - \frac{\nabla\phi \otimes \nabla\phi}{m-n},$$

where Hess is the Hessian operator. The case when $m = n$ occurs if and only if ϕ is a constant function. Furthermore, when $m \to \infty$ the $\infty-$Bakry–Émery Ricci tensor is defined by

$$Ric_\phi := Ric + Hess\ \phi.$$

Lemma 1 ([14] Weighted Bochner Formula)**.** *For any smooth function u on a weighted Riemannian manifold $(M, g, e^{-\phi}d\mu)$, we have the weighted version of Bochner formula*

$$\frac{1}{2}\Delta_\phi|\nabla u|^2 = |Hess\ u|^2 + \langle\nabla\Delta_\phi u, \nabla u\rangle + Ric_\phi(\nabla u, \nabla u),$$

where $\langle\cdot,\cdot\rangle$ is the induced inner product by the Riemannian metric g.

Lemma 2 ([14])**.** *Under the geometric flow Equation (8) and for any smooth function u on a weighted Riemannian manifold $(M, g, e^{-\phi}d\mu)$ we have the following evolution formulas*

1. $\frac{\partial}{\partial t}|\nabla u|^2 = -2\mathcal{S}(\nabla u, \nabla u) + 2\langle\nabla u, \nabla u_t\rangle,$
2. $\frac{\partial}{\partial t}(\Delta_\phi u) = \Delta_\phi u_t - 2S^{ij}\nabla_i\nabla_j u - \langle 2div\ \mathcal{S} - \nabla S, \nabla u\rangle + 2\mathcal{S}(\nabla\phi, \nabla u) - \langle\nabla u, \nabla\phi_t\rangle$, *where div $\mathcal{S}$ denotes the divergence of $\mathcal{S}$ and $S^{ij} = g^{ik}g^{jl}S_{kl}$.*

Let $T > 0$ be any real number. For any two points $x, y \in M$ and for any $t \in [0, T]$, the quantity $d(x, y, t)$ denotes the geodesic distance between x and y under the metric $g(t)$. For any fixed $x_0 \in M$ and $R > 0$ we define a compact set

$$Q_{2R,T} = \{(x,t) : d(x, x_0, t) \le 2R, 0 \le t \le T\} \subset M^n \times (-\infty, +\infty). \tag{12}$$

Now for $u > 0$ we define some non-negative real numbers

$$\begin{array}{lll}
\lambda_1 := \sup\limits_{Q_{2R,T}} |\hat{A}| & \lambda_2 := \sup\limits_{Q_{2R,T}} |\hat{A}_f| & \lambda_3 := \sup\limits_{Q_{2R,T}} |\hat{A}_{ff}| \\
\Lambda_1 := \sup\limits_{M\times[0,T]} |\hat{A}| & \Lambda_2 := \sup\limits_{M\times[0,T]} |\hat{A}_f| & \Lambda_3 := \sup\limits_{M\times[0,T]} |\hat{A}_{ff}| \\
b_1 := \sup\limits_{Q_{2R,T}} |\hat{B}| & b_2 := \sup\limits_{Q_{2R,T}} |\hat{B}_f| & b_3 := \sup\limits_{Q_{2R,T}} |\hat{B}_{ff}| \\
B_1 := \sup\limits_{M\times[0,T]} |\hat{B}| & B_2 := \sup\limits_{M\times[0,T]} |\hat{B}_f| & B_3 := \sup\limits_{M\times[0,T]} |\hat{B}_{ff}| \\
\sigma_1 := \sup\limits_{Q_{2R,T}} |q| & \sigma_2 := \sup\limits_{Q_{2R,T}} |\nabla q| & \sigma_3 := \sup\limits_{Q_{2R,T}} |\Delta_\phi q| \\
\Sigma_1 := \sup\limits_{M\times[0,T]} |q| & \Sigma_2 := \sup\limits_{M\times[0,T]} |\nabla q| & \Sigma_3 := \sup\limits_{M\times[0,T]} |\Delta_\phi q| \\
\gamma_1 := \sup\limits_{Q_{2R,T}} |p| & \gamma_2 := \sup\limits_{Q_{2R,T}} |\nabla p| & \gamma_3 := \sup\limits_{Q_{2R,T}} |\Delta_\phi p| \\
\Gamma_1 := \sup\limits_{M\times[0,T]} |p| & \Gamma_2 := \sup\limits_{M\times[0,T]} |\nabla p| & \Gamma_3 := \sup\limits_{M\times[0,T]} |\Delta_\phi p| \\
\theta_1 := \sup\limits_{Q_{2R,T}} |\nabla \phi| & \theta_2 := \sup\limits_{Q_{2R,T}} |\nabla \phi_t| & \Theta_1 := \sup\limits_{M\times[0,T]} |\nabla \phi| \\
\Theta_2 := \sup\limits_{M\times[0,T]} |\nabla \phi_t| & m_1 := \sup\limits_{Q_{2R,T}} |\nabla \xi| & m_2 := \sup\limits_{Q_{2R,T}} |\Delta_\phi \xi| \\
m_3 := \sup\limits_{Q_{2R,T}} |\xi| & M_1 := \sup\limits_{M\times[0,T]} |\nabla \xi| & M_2 := \sup\limits_{M\times[0,T]} |\Delta_\phi \xi| \\
M_3 := \sup\limits_{M\times[0,T]} |\xi| & &
\end{array}$$

Lemma 3 ([14]). *For any smooth function f on an n-dimensional Riemannian manifold $(M^n, g, e^{-\phi} d\mu)$ and $m > n$ we have the following relation connecting Hessian and weighted Laplacian*

$$|Hess\ f|^2 \geq \frac{(\Delta_\phi f)^2}{m} - \frac{1}{m-n}\langle \nabla f, \nabla \phi\rangle^2. \tag{13}$$

Proof. Let $m > n$. Then we see that

$$\begin{aligned}
0 &\leq \left(\sqrt{\frac{m-n}{mn}}\Delta f + \sqrt{\frac{n}{m(m-n)}}\langle \nabla f, \nabla \phi\rangle\right)^2 \\
&= (\frac{1}{n} - \frac{1}{m})(\Delta f)^2 + \frac{2}{m}\Delta f\langle \nabla f, \nabla \phi\rangle + (\frac{1}{m-n} - \frac{1}{m})\langle \nabla f, \nabla \phi\rangle^2 \\
&\leq |\text{Hess}\ f|^2 - \frac{1}{m}\left((\Delta f)^2 - 2\Delta f\langle \nabla f, \nabla \phi\rangle + \langle \nabla f, \nabla \phi\rangle^2\right) + \frac{1}{m-n}\langle \nabla f, \nabla \phi\rangle^2 \\
&= |\text{Hess}\ f|^2 - \frac{(\Delta_\phi f)^2}{m} + \frac{1}{m-n}\langle \nabla f, \nabla \phi\rangle^2.
\end{aligned}$$

Thus $|\text{Hess}\ f|^2 \geq \frac{(\Delta_\phi f)^2}{m} - \frac{1}{m-n}\langle \nabla f, \nabla \phi\rangle^2$. □

Lemma 4 ([28] Young's inequality). *If $a,\ b$ are nonnegative real numbers and $p > 1,\ q > 1$ are real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$ then*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

Let $\alpha > 0$ be any real number. Put $a = \alpha a$ and $b = \frac{b}{\alpha}$ in the above expression we get Peter-Paul type inequality

$$ab \leq \alpha^p \frac{a^p}{p} + \frac{b^q}{\alpha^q q}. \tag{14}$$

If we put $a = a\sqrt{2\alpha}$, $b = \frac{b}{\sqrt{2\alpha}}$, $p = q = 2$ in Young's inequality then we have the well known Peter-Paul inequality

$$ab \leq \alpha a^2 + \frac{b^2}{4\alpha}. \tag{15}$$

In this paper we use these inequalities with a suitable choice of α.

3. Li-iYau-Type Gradient Estimation

In this section, we are going to derive a bound for the quantity $\frac{|\nabla u|^2}{u^2}$ on a compact domain $Q_{2R,T}$ of M, where u satisfies (7). This estimation is known as local Li–Yau-type estimation. After that, we derive global Li–Yau-type estimation on the whole of M. This method enables us to find the heat ratio between two points on a manifold by deriving a Harnack-type inequality. For this, we fix a point $x_0 \in M$ and let $R > 0$ be a real number. Let u be a positive solution to (7) in $Q_{2R,T}$.

Theorem 1. *If k_1, k_2, k_3, k_4 are positive constants such that*

$$Ric_\phi^{m-n} \geq -(m-1)k_1 g, \ -k_2 g \leq \mathcal{S} \leq k_3 g, \ |\nabla S| \leq k_4$$

on $Q_{2R,T}$, then for any solution u of (7), any $\lambda > 1$ and $\delta \in (0,1)$ we have

$$\frac{|\nabla u|^2}{u^2} - \lambda\left(\frac{u_t}{u} + \frac{A(u)}{u}p + \frac{B(u)}{u}q + \xi\right) \leq \frac{m\lambda^2}{2t(1-\lambda\epsilon)} + \frac{m\lambda^2}{2(1-\lambda\epsilon)}\tilde{D_1} + \tilde{E_1}, \tag{16}$$

where

$$\begin{aligned}
\tilde{D_1} &= \frac{c_0}{R}(m-1)(\sqrt{k_1} + \frac{2}{R}) + \frac{3c_1}{R^2} + c_2 k_2 + \frac{m\lambda^2 c_1}{4(1-\lambda\epsilon)(\lambda-1)R^2} + \frac{1}{\lambda}, \\
\tilde{E_1} &= \left(\frac{m\lambda^2}{2(1-\lambda\epsilon)}E_1\right)^{\frac{1}{2}}, \\
E_1 &= \frac{m\lambda^2}{4(1-\lambda\epsilon)(1-\delta)(\lambda-1)^2}\bar{C_1}^2 + 2\lambda k_2 \epsilon \theta_1^2 + \frac{n\lambda}{2\epsilon}(k_2+k_3)^2 \\
&+ \frac{9}{8}n\lambda^2 k_4 + (\lambda_1\gamma_3 + b_1\sigma_3) + \frac{3}{4}\left(\frac{2m\lambda^2}{(1-\lambda\epsilon)(\lambda-1)\delta}\right)^{\frac{1}{3}}(2\lambda_2\gamma_2 \\
&+ 2b_2\sigma_2)^{\frac{4}{3}} + m_2 + \frac{3}{4}\left(\frac{m\lambda^2}{2(1-\lambda\epsilon)(1-\delta)(\lambda-1)^2}\right)^{\frac{1}{3}}\lambda^{\frac{4}{3}}\theta_2^{\frac{4}{3}} \\
&+ \frac{3}{4}\left(\frac{m\lambda^2}{2(1-\lambda\epsilon)(1-\delta)(\lambda-1)^2}\right)^{\frac{1}{3}}(\lambda_1\gamma_2 + b_1\sigma_2 + m_1)^{\frac{4}{3}}, \\
\bar{C_1} &= \frac{\lambda k_2}{2\epsilon} + 2k_4 + \lambda_3\gamma_1 + b_3\sigma_1 + 2(\lambda-1)k_3 + \frac{\lambda-1}{\lambda} + \gamma_1\lambda_2 + \sigma_1 b_2 \\
&\quad +2(1-\lambda\epsilon)(m-1)k_1.
\end{aligned}$$

To prove the theorem we need the following lemma.

Lemma 5. *If $u = e^f$ is a positive solution to (7) and $F := t(|\nabla f|^2 - \lambda(f_t + \bar{f}))$, where $\bar{f} = \hat{A}p + \hat{B}q + \xi$ then for any $\epsilon \in (0, \frac{1}{\lambda})$ and assuming conditions of Theorem 1 we have*

$$\begin{aligned}(\Delta_\phi - \partial_t)F &\geq 2t(1-\lambda\epsilon)\frac{(\Delta_\phi f)^2}{m} - \frac{\lambda t k_2}{2\epsilon}|\nabla f|^2 - 2\lambda t k_2 \epsilon |\nabla\phi|^2 \\ &- 2t(1-\lambda\epsilon)(m-1)k_1|\nabla f|^2 - 2\nabla F \nabla f - \frac{F}{t} - 2t(\lambda-1)k_3|\nabla f|^2 \\ &- \frac{n\lambda t}{2\epsilon}(k_2+k_3)^2 - 3\lambda t\sqrt{n}k_4|\nabla f|^2 + \mathcal{H},\end{aligned} \tag{17}$$

where $\mathcal{H} = -2t(\lambda-1)\nabla\bar{f}\nabla f - \lambda t \nabla f \nabla\phi_t - \lambda t \Delta_\phi \bar{f}$.

Proof. Let u be a solution of (7) and consider $F := t(|\nabla f|^2 - \lambda(f_t + \bar{f}))$, where $\bar{f} = \hat{A}p + \hat{B}q + \xi$. Hence

$$\frac{F}{t} = |\nabla f|^2 - \lambda(f_t + \bar{f}) \tag{18}$$

and applying Lemma 1 (Weighted Bochner formula) we have

$$\begin{aligned}\Delta_\phi F &= 2t|\text{Hess } f|^2 + 2t\langle \nabla\Delta_\phi f, \nabla f\rangle + 2tRic_\phi(\nabla f, \nabla f) - \lambda t \Delta_\phi f_t \\ &- \lambda t \Delta_\phi \bar{f}.\end{aligned} \tag{19}$$

Now $\Delta_\phi f = -\frac{F}{t} - (\lambda-1)(f_t + \bar{f})$, so $\nabla\Delta_\phi f = -\frac{\nabla F}{t} - (\lambda-1)(\nabla f_t + \nabla\bar{f})$.
Hence

$$\begin{aligned}\Delta_\phi F &= 2t|\text{Hess } f|^2 - 2\nabla F \nabla f - 2t(\lambda-1)(\nabla f_t + \nabla\bar{f})\nabla f + 2tRic_\phi(\nabla f, \nabla f) \\ &- \lambda t \Delta_\phi f_t - \lambda t \Delta_\phi \bar{f}.\end{aligned} \tag{20}$$

Furthermore,

$$\partial_t(\Delta_\phi f) = \frac{F}{t^2} - \frac{F_t}{t} - (\lambda-1)(f_{tt} + \bar{f}_t). \tag{21}$$

Using (21) on (20) we get

$$\begin{aligned}\Delta_\phi F &= 2t|\text{Hess } f|^2 - 2\nabla F\nabla f - 2t(\lambda-1)(\nabla f_t + \nabla\bar{f})\nabla f + 2tRic_\phi(\nabla f, \nabla f) \\ &- \frac{\lambda F}{t} + \lambda F_t + \lambda(\lambda-1)t(f_{tt} + \bar{f}_t) - 2\lambda t\langle \mathcal{S}, \text{Hess } f\rangle - 2\lambda t\langle \text{div}\mathcal{S} - \frac{1}{2}\nabla S, \nabla f\rangle \\ &+ 2\lambda t\mathcal{S}(\nabla\phi, \nabla f) - \lambda t\langle \nabla f, \nabla\phi_t\rangle - \lambda t\Delta_\phi \bar{f},\end{aligned} \tag{22}$$

and

$$\partial_t F = \frac{F}{t} + t(\partial_t|\nabla f|^2 - \lambda(f_{tt} + \bar{f}_t)). \tag{23}$$

From (22) and (23) we get

$$\begin{aligned}(\Delta_\phi - \partial_t)F &= 2t|\text{Hess } f|^2 - 2\nabla F\nabla f - 2t(\lambda-1)(\nabla f_t + \nabla\bar{f})\nabla f \\ &+ 2tRic_\phi(\nabla f, \nabla f) - 2t(\lambda-1)\mathcal{S}(\nabla f, \nabla f) \\ &+ 2t(\lambda-1)\nabla f\nabla f_t - 2\lambda t\langle \mathcal{S}, \text{Hess } f\rangle \\ &- 2\lambda t\langle \text{div } \mathcal{S} - \frac{1}{2}\nabla S, \nabla f\rangle + 2\lambda t\mathcal{S}(\nabla\phi, \nabla f) - \lambda t\nabla f\nabla\phi_t \\ &- \lambda t\Delta_\phi\bar{f} - \frac{F}{t}.\end{aligned} \tag{24}$$

$$\begin{aligned}\text{or, } (\Delta_\phi - \partial_t)F &= 2t|\text{Hess } f|^2 + 2tRic_\phi(\nabla f, \nabla f) - 2\nabla F\nabla f - \frac{F}{t} \\ &- 2t(\lambda-1)\mathcal{S}(\nabla f, \nabla f) + 2\lambda t\mathcal{S}(\nabla\phi, \nabla f) - 2\lambda t\langle \mathcal{S}, \text{Hess } f\rangle \\ &- 2\lambda t\langle \text{div } \mathcal{S} - \frac{1}{2}\nabla S, \nabla f\rangle + \mathcal{H},\end{aligned} \tag{25}$$

where $\mathcal{H} = -2t(\lambda-1)\nabla\bar{f}\nabla f - \lambda t\nabla f\nabla\phi_t - \lambda t\Delta_\phi\bar{f}$.
Given that

$$-(k_2+k_3)g_{ij} \leq S_{ij} \leq (k_2+k_3)g_{ij}, \tag{26}$$

which implies

$$|S|^2 \leq n(k_2+k_3)^2, \tag{27}$$

as S_{ij} is a symmetric tensor.

Following [14], for any $\epsilon \in (0, \frac{1}{\lambda})$ using Young's inequality, we have

$$\langle \mathcal{S}, \text{Hess } f\rangle \leq \epsilon|\text{Hess } f|^2 + \frac{n}{4\epsilon}(k_2+k_3)^2, \tag{28}$$

$$2\lambda t\mathcal{S}(\nabla\phi, \nabla f) \geq -\frac{\lambda t k_2}{2\epsilon}|\nabla f|^2 - 2\lambda t k_2\epsilon|\nabla\phi|^2. \tag{29}$$

Also

$$|\text{div } S_{ij} - \frac{1}{2}\nabla S| \leq \frac{3}{2}\sqrt{n}k_4. \tag{30}$$

Using Lemma 3, Equations (26)–(30) and bounds of Ric_ϕ^{m-n}, $\mathcal{S}$ in (25) we have (17). □

Proof of Theorem 1. Consider a C^2-function ψ on $[0,\infty)$,

$$\psi(s) = \begin{cases} 1, & s \in [0,1], \\ 0, & s \in [2,\infty), \end{cases}$$

and it satisfies $\psi(s) \in [0,1]$, $-c_0 \leq \psi'(s) \leq 0$, $\psi''(s) \geq -c_1$ and $\frac{|\psi''(s)|^2}{\psi(s)} \leq c_1$, where c_1 is a constant and for $R \geq 1$ we defined a function

$$\eta(x,t) = \psi\left(\frac{r(x,t)}{R}\right),$$

where $r(x,t) = d(x,x_0,t)$. Applying the same argument as in [3] we can apply a maximum principle and use Calabi's trick [29] to assume everywhere smoothness of $\eta(x,t)$, as $\psi(s)$ is Lipschitz.

By generalized Laplacian comparison theorem [14], we have

1. $\Delta_\phi r(x) \leq (m-1)\sqrt{k_1}\coth(\sqrt{k_1}r(x))$,
2. $\Delta_\phi\eta \geq -\frac{c_0}{R}(m-1)(\sqrt{k_1}+\frac{2}{R}) - \frac{c_1}{R^2}$,
3. $\frac{|\nabla\eta|^2}{\eta} \leq \frac{c_1}{R^2}$.

Let $G = \eta F$. Fix any $T_1 \in (0,T]$ and assume G achieves maximum at $(x_0,t_0) \in Q_{2R,T_1}$. If $G(x_0,t_0) \leq 0$ then the result is trivial and hence nothing to be proved, so assume that $G(x_0,t_0) \geq 0$.

Thus, at (x_0,t_0) we have

$$\nabla G = 0, \quad \Delta G \leq 0, \quad \partial_t G \geq 0.$$

Therefore

$$\nabla F = -\frac{F}{\eta}\nabla\eta \tag{31}$$

and

$$0 \geq (\Delta_\phi - \partial_t)G = F(\Delta_\phi - \partial_t)\eta + \eta(\Delta_\phi - \partial_t)F + 2\langle\nabla\eta, \nabla F\rangle. \tag{32}$$

By [16], there is a constant c_2 such that

$$-F\eta_t \geq -c_2k_2F. \tag{33}$$

Using (31) and (33) in (32) we get

$$0 \geq -\left(\frac{c_0}{R}(m-1)(\sqrt{k_1}+\frac{2}{R})+\frac{3c_1}{R^2}+c_2k_2\right)F+\eta(\Delta_\phi-\partial_t)F. \tag{34}$$

Following [14,20,23], we set

$$\xi = \frac{|\nabla f|^2}{F}\bigg|_{(x_0,t_0)} \geq 0,$$

then at (x_0, t_0) we have

$$\begin{aligned} |\nabla f| &= \sqrt{\xi F}, & (35)\\ (\xi - \frac{t_0\xi - 1}{\lambda t_0})F &= |\nabla f|^2 - (f_t + \bar{f}), & (36)\\ \eta\langle \nabla f, \nabla F\rangle &\leq \frac{\sqrt{c_1}}{R}\eta^{\frac{1}{2}}F|\nabla f|, & (37)\\ 3\lambda\sqrt{n}k_4|\nabla f| &\leq 2k_4|\nabla f|^2 + \frac{9}{8}n\lambda^2k_4. & (38) \end{aligned}$$

Using Lemma 5 in (34) we have

$$\begin{aligned} 0 \geq & -\left(\frac{c_0}{R}(m-1)(\sqrt{k_1}+\frac{2}{R})+\frac{3c_1}{R^2}+c_2k_2\right)F+\frac{2\eta t_0(1-\lambda\epsilon)}{m}(\Delta_\phi f)^2-\frac{\lambda\eta t_0k_2}{2\epsilon}|\nabla f^2| \\ & - 2\lambda t_0\eta k_2\epsilon|\nabla\phi|^2-2\eta t_0(1-\lambda\epsilon)(m-1)k_1|\nabla f|^2-2\eta\nabla F\nabla f-\frac{\eta F}{t_0} \\ & - 2t_0(\lambda-1)\eta k_3|\nabla f|^2-\frac{n\eta\lambda t_0}{2\epsilon}(k_2+k_3)^2-3\eta\lambda t_0\sqrt{n}k_4|\nabla f|+\eta\mathcal{H}. \end{aligned} \tag{39}$$

Multiplying (39) with ηt_0 and using results from (35)–(38) we get

$$\begin{aligned} 0 \geq & -\left(\frac{c_0}{R}(m-1)(\sqrt{k_1}+\frac{2}{R})+\frac{3c_1}{R^2}+c_2k_2\right)Gt_0+\frac{2\eta^2t_0^2(1-\lambda\epsilon)}{m}(\Delta_\phi f)^2 \\ & - \frac{\lambda\eta t_0^2k_2\xi}{2\epsilon}G-2\lambda\eta^2t_0^2k_2\epsilon|\nabla\phi|^2-2\eta\xi t_0^2(1-\lambda\epsilon)(m-1)k_1G-2t_0\frac{\sqrt{c_1}}{R}G^{\frac{3}{2}}\xi^{\frac{1}{2}} \\ & - \eta G-2\eta^2t_0^2(\lambda-1)k_3|\nabla f|^2-\frac{n\eta^2t_0^2\lambda}{2\epsilon}(k_2+k_3)^2-2k_4t_0^2\xi G\eta \\ & - \frac{9}{8}n\lambda^2\eta^2k_4t_0^2+\eta^2t_0\mathcal{H}. \end{aligned} \tag{40}$$

Now we use Young's inequality by choosing suitable values for a, b, α, p, q as in Lemma 4. Set $a = \frac{2\sqrt{c_1}}{R}G^{\frac{1}{2}}$, $b = G\xi^{\frac{1}{2}}$, $p = 2$, $q = 2$, $\alpha = \frac{m\lambda^2}{4(1-\lambda\epsilon)(\lambda-1)}$ and apply Lemma 4 (Young's inequality) we get

$$2t_0\frac{\sqrt{c_1}}{R}G^{\frac{3}{2}}\xi^{\frac{1}{2}} \leq \frac{4(1-\lambda\epsilon)}{m\lambda^2}(\lambda-1)\xi G^2t_0+\frac{m\lambda^2c_1t_0G}{4(1-\lambda\epsilon)(\lambda-1)R^2}. \tag{41}$$

Cauchy–Schwarz inequality gives

$$\begin{aligned} \eta^2\lambda\langle\nabla f,\nabla\phi_t\rangle &\leq \eta^2\lambda|\nabla f||\nabla\phi_t| \\ &\leq \lambda\theta_2G^{\frac{1}{2}}\xi^{\frac{1}{2}}. \end{aligned}$$

Set $a = \lambda\theta_2, b = \xi^{\frac{1}{2}}G^{\frac{1}{2}}, p = \frac{4}{3}, q = 4, \alpha = \left(\frac{m\lambda^2}{2(1-\lambda\epsilon)(1-\delta)(\lambda-1)^2}\right)^{\frac{1}{4}}$ and apply Lemma 4 we get

$$\begin{aligned}\eta^2\lambda\langle\nabla f, \nabla\phi_t\rangle \leq & \frac{(1-\lambda\epsilon)(1-\delta)}{2m\lambda^2}(\lambda-1)^2\xi^2G^2 + \frac{3}{4}\left(\frac{m\lambda^2}{2(1-\epsilon\lambda)(1-\delta)(\lambda-1)^2}\right)^{\frac{1}{3}}\lambda^{\frac{4}{3}}\theta_2^{\frac{4}{3}}, \\ & \text{for all } \delta\in(0,1).\end{aligned} \tag{42}$$

We have $\bar{f} = \hat{A}p + \hat{B}q + \xi$. Hence

$$\nabla\bar{f} = p\hat{A}_f\nabla f + q\hat{B}_f\nabla f + \hat{A}\nabla p + \hat{B}\nabla q + \nabla\xi,$$

$$\begin{aligned}\Delta\bar{f} &= \nabla\nabla\bar{f} \\ &= (\hat{A}\Delta p + \hat{B}\Delta q) + p(\hat{A}_{ff}|\nabla f|^2 + \hat{A}_f\Delta f) + q(\hat{B}_{ff}|\nabla f|^2 + \hat{B}_f\Delta f) \\ &+ 2(\hat{A}_f\langle\nabla f, \nabla p\rangle + \hat{B}_f\langle\nabla f, \nabla q\rangle) + \Delta\xi.\end{aligned}$$

Hence

$$\begin{aligned}\Delta_\phi\bar{f} &= \eta^2(\Delta\bar{f} - \nabla\phi\nabla\bar{f}) \\ &= (\hat{A}\Delta_\phi p + \hat{B}\Delta_\phi q) + (p\hat{A}_f + q\hat{B}_f)\Delta_\phi f + (p\hat{A}_{ff} + q\hat{B}_{ff})|\nabla f|^2 \\ &+ 2(\hat{A}_f\langle\nabla p, \nabla f\rangle + \hat{B}_f\langle\nabla q, \nabla f\rangle) + \Delta_\phi\xi.\end{aligned} \tag{43}$$

Again

$$\begin{aligned}2\eta^2\hat{A}_f\langle\nabla p, \nabla f\rangle &\leq 2\lambda_2\eta^2|\nabla p||\nabla f|, \text{ using Cauchy-Schwarz inequality} \\ &\leq 2\lambda_2\gamma_2\eta^2|\nabla f| \\ &\leq 2\lambda_2\gamma_2\xi^{\frac{1}{2}}G^{\frac{1}{2}}.\end{aligned} \tag{44}$$

Similarly

$$2\eta^2\hat{B}_f\langle\nabla q, \nabla f\rangle \leq 2b_2\sigma_2\xi^{\frac{1}{2}}G^{\frac{1}{2}}. \tag{45}$$

Adding (45) and (45) gives

$$2\eta^2(\hat{A}_f\langle\nabla p, \nabla f\rangle + \hat{B}_f\langle\nabla q, \nabla f\rangle) \leq 2(\gamma_2\lambda_2 + b_2\sigma_2)\xi^{\frac{1}{2}}G^{\frac{1}{2}}. \tag{46}$$

Using (46) in (43) and applying Young's inequality with $a = 2(\gamma_2\lambda_2 + b_2\sigma_2)$, $b = \xi^{\frac{1}{2}}G^{\frac{1}{2}}$, $p = \frac{4}{3}$, $q = 4$ and $\alpha = \left(\frac{2m\lambda^2}{(1-\lambda\epsilon)\delta(\lambda-1)}\right)^{\frac{1}{4}}$ we obtain

$$\begin{aligned}\eta^2\Delta_\phi\bar{f} &\leq (\lambda_1\gamma_3 + b_1\sigma_3) + (\gamma_1\lambda_2 + \sigma_1 b_2)\eta^2\Delta_\phi f \\ &+ (\lambda_3\gamma_1 + b_3\sigma_1)\xi G + \frac{2(1-\lambda\epsilon)\delta}{m\lambda^2}(\lambda-1)^2\xi^2G^2 \\ &+ \frac{3}{4}\left(\frac{2m\lambda^2}{(1-\lambda\epsilon)\delta(\lambda-1)}\right)^{\frac{1}{3}}(2\lambda_2\gamma_2 + 2b_2\sigma_2)^{\frac{4}{3}} + m_2.\end{aligned} \tag{47}$$

Similarly we get

$$\begin{aligned}\eta^2\langle\nabla\bar{f},\nabla f\rangle &\leq \frac{(1-\lambda\epsilon)(1-\delta)(\lambda-1)^2}{2m\lambda^2}\xi^2G^2 \\ &+ \frac{3}{4}\left(\frac{m\lambda^2}{2(1-\lambda\epsilon)(1-\delta)(\lambda-1)^2}\right)^{\frac{1}{3}}(\lambda_1\gamma_2+b_1\sigma_2+m_1)^{\frac{4}{3}} \\ &+ (\gamma_1\lambda_2+\sigma_1 b_2)\xi G.\end{aligned} \tag{48}$$

Equations (47) and (48) are the quantities that estimates $\mathcal{H}$.
From (36) we have

$$\begin{aligned}\Delta_\phi f &= |\nabla f|^2-(f_t+\bar{f}) \\ &= \left(\xi-\frac{t_0\xi-1}{\lambda t_0}\right)F.\end{aligned}$$

Thus

$$\begin{aligned}\frac{2\eta^2t_0^2(1-\lambda\epsilon)}{m}(\Delta_\phi f)^2 &= \frac{2(1-\lambda\epsilon)}{m\lambda^2}G^2-\frac{4\xi t_0(1-\lambda\epsilon)(\lambda-1)}{m\lambda^2}G^2 \\ &+ \frac{2(1-\lambda\epsilon)}{m\lambda^2}\xi^2t_0^2(\lambda-1)^2G^2\end{aligned} \tag{49}$$

and

$$\eta^2\Delta_\phi f = -\frac{1}{\lambda t_0}G-\frac{t_0(\lambda-1)}{\lambda t_0}\xi G. \tag{50}$$

Set

$$\bar{C_1} := \left\{\frac{\lambda k_2}{2\epsilon}+2k_4+\lambda_3\gamma_1+b_3\sigma_1+2(\lambda-1)k_3+\frac{\lambda-1}{\lambda}+\gamma_1\lambda_2+\sigma_1b_2+2(1-\lambda\epsilon)(m-1)k_1\right\}$$

and apply Peter-Paul inequality with $a=\xi G$, $b=\bar{C_1}$, $\alpha=\frac{m\lambda^2}{(1-\epsilon\lambda)(1-\delta)(\lambda-1)^2}$ we get

$$\bar{C_1}\xi G \leq \frac{(1-\epsilon\lambda)(1-\delta)(\lambda-1)^2}{m\lambda^2}\xi^2G^2+\frac{m\lambda^2}{4(1-\lambda\epsilon)(1-\delta)(\lambda-1)^2}\bar{C_1}^2. \tag{51}$$

Set

$$D_1 := 1+t_0\left(\frac{c_0}{R}(m-1)(\sqrt{k_1}+\frac{2}{R})+\frac{3c_1}{R^2}+c_2k_2+\frac{m\lambda^2c_1}{4(1-\lambda\epsilon)(\lambda-1)R^2}+\frac{1}{\lambda}\right), \tag{52}$$

$$\begin{aligned}E_1 &:= \frac{m\lambda^2}{4(1-\lambda\epsilon)(1-\delta)(\lambda-1)^2}\bar{C_1}^2+2\lambda k_2\epsilon\theta_1^2 \\ &+ \frac{n\lambda}{2\epsilon}(k_2+k_3)^2+\frac{9}{8}n\lambda^2k_4+(\lambda_1\gamma_3+b_1\sigma_3)+\frac{3}{4}\left(\frac{2m\lambda^2}{(1-\lambda\epsilon)(\lambda-1)\delta}\right)^{\frac{1}{3}}(2\lambda_2\gamma_2 \\ &+ 2b_2\sigma_2)^{\frac{4}{3}}+m_2+\frac{3}{4}\left(\frac{m\lambda^2}{2(1-\lambda\epsilon)(1-\delta)(\lambda-1)^2}\right)^{\frac{1}{3}}\lambda^{\frac{4}{3}}\theta_2^{\frac{4}{3}} \\ &+ \frac{3}{4}\left(\frac{m\lambda^2}{2(1-\lambda\epsilon)(1-\delta)(\lambda-1)^2}\right)^{\frac{1}{3}}(\lambda_1\gamma_2+b_1\sigma_2+m_1)^{\frac{4}{3}}.\end{aligned} \tag{53}$$

Using (41) to (52) in (40) we obtain

$$0 \geq \frac{2(1-\lambda\epsilon)}{m\lambda^2}G^2 - D_1 G - t_0^2 E_1. \tag{54}$$

For a positive number p and two non-negative numbers q, r, the quadratic inequality of the form $px^2 - qx - r \leq 0$ implies that $x \leq \frac{q}{p} + \sqrt{\frac{r}{p}}$.

So at (x_0, t_0) we have

$$G \leq D_1 \frac{m\lambda^2}{2(1-\lambda\epsilon)} + t_0\sqrt{\frac{m\lambda^2 E_1}{2(1-\lambda\epsilon)}}. \tag{55}$$

Since $\eta(x,t) = 1$ whenever $d(x, x_0, T_1) \leq R$, hence

$$\frac{F(x,T_1)}{T_1} = (|\nabla f|^2 - \lambda(f_t + \bar{f}))\Big|_{(x,T_1)} \leq \frac{G(x_0,t_0)}{T_1} \leq \frac{1}{T_1}\left(D_1\frac{m\lambda^2}{2(1-\lambda\epsilon)} + t_0\sqrt{\frac{m\lambda^2 E_1}{2(1-\lambda\epsilon)}}\right).$$

Since $t_0 \leq T_1$, so

$$\frac{1}{T_1}\left(D_1\frac{m\lambda^2}{2(1-\lambda\epsilon)} + t_0\sqrt{\frac{m\lambda^2 E_1}{2(1-\lambda\epsilon)}}\right) \leq \frac{m\lambda^2}{2T_1(1-\lambda\epsilon)} + \frac{\tilde{D}}{T_1}\frac{m\lambda^2}{2(1-\lambda\epsilon)} + \sqrt{\frac{m\lambda^2 E_1}{2(1-\lambda\epsilon)}},$$

where $\tilde{D} = t_0\tilde{D_1}$ and $\tilde{D_1} = \left(\frac{c_0}{R}(m-1)(\sqrt{k_1} + \frac{2}{R}) + \frac{3c_1}{R^2} + c_2 k_2\right) + \frac{m\lambda^2 c_1}{4(1-\lambda\epsilon)(\lambda-1)R^2} + \frac{1}{\lambda}$ satisfying $\frac{\tilde{D}}{T_1} \leq \tilde{D_1}$. Since T_1 is arbitrary so

$$|\nabla f|^2 - \lambda(f_t + \hat{A}p + \hat{B}q + \xi) \leq \frac{m\lambda^2}{2t(1-\lambda\epsilon)} + \frac{m\lambda^2}{2(1-\lambda\epsilon)}\tilde{D_1} + \tilde{E_1}, \tag{56}$$

where $\tilde{E_1} = \left(\frac{m\lambda^2}{2(1-\lambda\epsilon)}E_1\right)^{\frac{1}{2}}$.

Substituting $f = \log u$ on (56) and using the definition of $\hat{A}, \hat{B}$, we get (16). This completes the proof. □

Corollary 1. *If k_1, k_2, k_3, k_4 are positive constants such that*

$$Ric_\phi^{m-n} \geq -(m-1)k_1 g,\ -k_2 g \leq \mathcal{S} \leq k_3 g,\ |\nabla S| \leq k_4$$

on M, then for any $\lambda > 1$ and $\delta \in (0,1)$ we have

$$\frac{|\nabla u|^2}{u^2} - \lambda\left(\frac{u_t}{u} + \frac{A(u)}{u}p + \frac{B(u)}{u}q + \xi\right) \leq \frac{m\lambda^2}{2t(1-\lambda\epsilon)} + \frac{m\lambda^2}{2(1-\lambda\epsilon)}\tilde{D_2} + \tilde{E_2}, \tag{57}$$

where

$$
\begin{aligned}
\tilde{D_2} &= c_2k_2 + \frac{1}{\lambda}, \\
\tilde{E_2} &= \left(\frac{m\lambda^2}{2(1-\lambda\epsilon)}E_2\right)^{\frac{1}{2}}, \\
E_2 &= \frac{m\lambda^2}{4(1-\lambda\epsilon)(1-\delta)(\lambda-1)^2}\bar{C_2}^2 + 2\lambda k_2\epsilon\Theta_1^2 + \frac{n\lambda}{2\epsilon}(k_2+k_3)^2 \\
&+ \frac{9}{8}n\lambda^2k_4 + (\Lambda_1\Gamma_3 + B_1\Sigma_3) + \frac{3}{4}\left(\frac{2m\lambda^2}{(1-\lambda\epsilon)(\lambda-1)\delta}\right)^{\frac{1}{3}}(2\Lambda_2\Gamma_2 + 2B_2\Sigma_2)^{\frac{4}{3}} \\
&+ M_2 + \frac{3}{4}\left(\frac{m\lambda^2}{2(1-\lambda\epsilon)(1-\delta)(\lambda-1)^2}\right)^{\frac{1}{3}}\lambda^{\frac{4}{3}}\Theta_2^{\frac{4}{3}} \\
&+ \frac{3}{4}\left(\frac{m\lambda^2}{2(1-\lambda\epsilon)(1-\delta)(\lambda-1)^2}\right)^{\frac{1}{3}}(\Lambda_1\Gamma_2 + B_1\Sigma_2 + M_1)^{\frac{4}{3}}, \\
\bar{C_2} &= \left(\frac{\lambda k_2}{2\epsilon} + 2k_4 + \Lambda_3\Gamma_1 + B_3\Sigma_1 + 2(\lambda-1)k_3 + \frac{\lambda-1}{\lambda} + 2(1-\lambda\epsilon)(m-1)k_1\right).
\end{aligned}
$$

Proof. We know $g(t)$ is uniformly equivalent to the initial metric $g(0)$. For a fixed $\delta \in (0,1)$ if we let R tend to $+\infty$ then we obtain our result. □

Theorem 2. *If k_1, k_2, k_3, k_4 are positive constants such that*

$$Ric_\phi^{m-n} \geq -(m-1)k_1g,\ -k_2g \leq \mathcal{S} \leq k_3g,\ |\nabla S| \leq k_4$$

on M and let u be a positive solution to (7) under the flow (8) then we have the Harnack inequality

$$u(y_1,s_1) \leq u(y_2,s_2)\left(\frac{s_2}{s_1}\right)^{\frac{m\lambda}{2(1-\lambda\epsilon)}} \exp\left\{\frac{\lambda}{4}\mathcal{I}(s_1,s_2) + (s_2-s_1)(\Lambda_1\Gamma_1 + B_1\Sigma_1 + M_3 + \frac{1}{\lambda}\tilde{F_2})\right\}, \tag{58}$$

where $\mathcal{I}(s_1,s_2) = \inf_\zeta \int_{s_1}^{s_2} |\zeta'(t)|^2 dt$ and $\zeta : [s_1,s_2] \to M$ is a path joining the points (y_1,s_1), (y_2,s_2) in $M \times [0,T]$ and $\tilde{F_2} = \frac{m\lambda^2}{2(1-\lambda\epsilon)}\tilde{D_2} + \tilde{E_2}$.

Proof. Set $\tilde{F_2} = \frac{m\lambda^2}{2(1-\lambda\epsilon)}\tilde{D_2} + \tilde{E_2}$ then (57) becomes

$$\frac{|\nabla u|^2}{u^2} - \lambda\left(\frac{u_t}{u} + \frac{A(u)}{u}p + \frac{B(u)}{u}q + \xi\right) \leq \frac{m\lambda^2}{2t(1-\lambda\epsilon)} + \tilde{F_2}. \tag{59}$$

For $u = e^f$ we have

$$|\nabla f|^2 - \lambda\left(f_t + \hat{A}p + \hat{B}q + \xi\right) \leq \frac{m\lambda^2}{2t(1-\lambda\epsilon)} + \tilde{F_2}. \tag{60}$$

Let $(y_1,s_1),(y_2,s_2) \in M \times [0,T]$ be such that $s_1 < s_2$. Take a geodesic path $\zeta : [s_1,s_2] \to M$ satisfying $\zeta(s_1) = y_1,\ \zeta(s_2) = y_2$. Using (60) we obtain

$$\begin{aligned} f(y_1,s_1)-f(y_2,s_2) &= -\int_{s_1}^{s_2}\frac{d}{dt}f(\zeta(t),t)dt \\ &= -\int_{s_1}^{s_2}\partial_t f dt-\int_{s_1}^{s_2}\langle\nabla f,\zeta'(t)\rangle dt \\ &\le \frac{m\lambda}{2(1-\lambda\epsilon)}\ln(\frac{s_2}{s_1})+(s_2-s_1)(\Lambda_1\Gamma_1+B_1\Sigma_1+M_3+\frac{1}{\lambda}\tilde{F_2}) \\ &- \int_{s_1}^{s_2}\frac{1}{\lambda}|\nabla f|^2dt-\int_{s_1}^{s_2}\langle\nabla f,\zeta'(t)\rangle dt. \end{aligned} \tag{61}$$

Now using the relation $-ax^2-bx\le\frac{b^2}{4a}$, we set $x=\nabla f$, $a=\frac{1}{\lambda}$ and $b=\zeta'(t)$ we get

$$\begin{aligned} f(y_1,s_1)-f(y_2,s_2) &\le \frac{m\lambda}{2(1-\lambda\epsilon)}\ln(\frac{s_2}{s_1})-\int_{s_1}^{s_2}\frac{\lambda|\zeta'(t)|^2}{4}dt \\ &+ (s_2-s_1)(\Lambda_1\Gamma_1+B_1\Sigma_1+M_3+\frac{1}{\lambda}\tilde{F_2}). \end{aligned} \tag{62}$$

Take infimum of (62) over all possible curves ζ on M and put $f=\ln u$ to obtain (58). □

4. Conclusions

In this paper, we have established Li–Yau-type estimate for a positive solution of the equation

$$\Delta_\phi u=\frac{\partial u}{\partial t}+A(u)p(x,t)+B(u)q(x,t)+\xi(x,t)u(x,t),$$

along the flow $\partial_t g_{ij}=2S_{ij}$ and related Harnack type inequality. In particular if $\xi(x,t)=0$, $B(u)=u^{a+1}$ then the results are same as in Section 2 of [14]. Thus, our paper generalizes some results of [14].

Further $A(u)=B(u)=\xi(u)=0$ gives the classical Li–Yau-type estimate for positive solution of the weighted heat equation

$$\Delta_\phi u=\partial_t u \tag{63}$$

under the geometric flow $\partial_t g_{ij}=2S_{ij}$. To obtain this estimate we put

1. $A(u)=B(u)=\xi(x,t)=0$
2. $\lambda_1=\lambda_2=\lambda_3=0$
3. $b_1=b_2=b_3=0$
4. $p(x,t)=q(x,t)=0$

in (16) and get

$$\frac{|\nabla u|^2}{u^2}-\lambda\frac{u_t}{u} \le \frac{m\lambda^2}{2t(1-\lambda\epsilon)}+\frac{m\lambda^2}{2(1-\lambda\epsilon)}\tilde{D_3}+\tilde{E_3}, \tag{64}$$

where

$$\begin{aligned} \tilde{D_3} &= \frac{c_0}{R}(m-1)(\sqrt{k_1}+\frac{2}{R})+\frac{3c_1}{R^2}+c_2k_2+\frac{m\lambda^2c_1}{4(a-\lambda\epsilon)(\lambda-1)R^2}+\frac{1}{\lambda}, \\ \tilde{E_3} &= \left(\frac{m\lambda^2}{2(1-\lambda\epsilon)}E_3\right)^{\frac{1}{2}}, \end{aligned}$$

$$
\begin{aligned}
E_3 &= \frac{m\lambda^2}{4(1-\lambda\epsilon)(1-\delta)(\lambda-1)^2}\bar{C_3}^2 + 2\lambda k_2\epsilon\theta_1^2 + \frac{n\lambda}{2\epsilon}(k_2+k_3)^2 + \frac{9}{8}n\lambda^2 k_4 \\
&\quad + \frac{3}{4}\left(\frac{m\lambda^2}{2(1-\lambda\epsilon)(1-\delta)(\lambda-1)^2}\right)^{\frac{1}{3}}\lambda^{\frac{4}{3}}\theta_2^{\frac{4}{3}}, \\
\bar{C_3} &= \frac{\lambda k_2}{2\epsilon} + 2k_4 + 2(\lambda-1)k_3 + \frac{\lambda-1}{\lambda} + 2(1-\lambda\epsilon)(m-1)k_1.
\end{aligned}
$$

Here if we let $R \to +\infty$ then we get the classical Li–Yau-type global gradient estimate for (63) along the flow $\partial_t g_{ij} = 2S_{ij}$. The key ingredient in this estimation is the assumption of bounds for the weight function ϕ and its derivative $|\nabla\phi|$ (see Preliminaries section), it would be interesting if one can derive Li–Yau-type estimation for a positive solution u of (7) without assuming bounds for ϕ, $|\nabla\phi|$. One can consider this problem as a future work for this article.

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