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Canonical F-Planar Mappings of Spaces with Affine Connection onto m-Symmetric Spaces

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Abstract: In this paper, we consider canonical F-planar mappings of spaces with affine connection onto m-symmetric spaces. We obtained the fundamental equations of these mappings in the form of a closed system of Chauchy-type equations in covariant derivatives. Furthermore, we established the number of essential parameters on which its general solution depends.

Keywords: F-planar mapping; space with affine connection; symmetric space; 2-symmetric space; m-symmetric space

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1. Introduction

In this paper, we further investigate F-planar mappings of spaces with affine connection. Ideologically, the theory concerning these mappings goes back to T. Levi-Civita's work [1], where he posed a problem of finding Riemannian spaces with common geodesic. He solved this problem in the special coordinate system. This problem is closely related to another topic, which is the study of the equations of mechanical system dynamics.

Many authors contributed to the development of the theory of geodesic mappings, including T. Thomas, H. Weyl, P.A. Shirokov, A.S. Solodovnikov, N.S. Sinyukov, A.V. Aminova, J. Mikeš, and others. The study of geodesic mappings raised questions many authors addressed and developed, i.e., V.F. Kagan, G. Vrănceanu, Ya.L. Shapiro, D.V. Vedenyapin, and others. The listed authors found special classes of $(n-2)$-projective spaces.

A. Z. Petrov [2] introduced the concept of quasi-geodesic mappings. Special quasi-geodesic mappings, in particular, are holomorphically projective mappings of Kaehler spaces, considered by T. Otsuki, Ya. Tashiro, M. Prvanović, and J. Mikeš et al.

The study continued with a natural generalization of these classes of mappings called almost geodesic mappings. N.S. Sinyukov introduced almost geodesic mappings [3]. He also determined three types of almost geodesic mappings, namely, π_1, π_2, and π_3.

As the broadest generalization of geodesic, quasi-geodesic, and holomorphic-procjective mappings, the F-planar mappings were introduced into consideration by J. Mikeš and N.S. Sinyukov [4]. At the same time, almost geodesic mappings of the second type π_2 are special F-planar mappings. Substantial refinements of the fundamental concepts of F-planar mappings are in the articles by I. Hinterleitner, J. Mikeš, and P. Peška [5–7].

The above results are presented in a developed form in monographs and researchers, e.g., [8–16].

The theory of F-planar mappings is developed in many works, for example, [17–25]. Our work is devoted to the study of F-planar mappings onto m-symmetric spaces. In this case, we find the fundamental equations in a new form. Analogous results were found for simpler cases in the theory of geodesic and almost geodesic mappings, for example, [24–33].

In conclusion, we emphasize that the mappings mentioned above were found as diffeomorphisms preserving special curves: geodesic, holomorphically projective, and F-planar. The work of [7] shows the possibility of formulating the definitions as diffeomorphisms that map all geodesic curves onto the indicated types of curves. Therefore, we can use them to model the physical processes associated with these curves, which are implicitly described in the already mentioned works by Levi-Civita [1], Petrov [2], and Bejan, Kowalski [18]. These curves are highly important in physics, especially theoretical mechanics and physics. The meaning of geodesics is widely known.

The study of the physical properties of special F-planar curves is described in the work of Petrov [2] (quasi-geodesics) and also currently in the works of Bejan and Druţă-Romaniuc [34] (magnetic curves). These curves are trajectories of the particles on which forces perpendicular to the direction of motion act. As a consequence, an operator F can be used to model magnetic forces.

2. Basic Concepts of the Theory of *F*-Planar Mappings of Spaces with Affine Connection

The following definitions and theorems for F-planar mappings are described in detail in the monograph [15,16] and the review article [6]. The research is conducted locally, in a class of sufficiently smooth functions.

Consider the n-dimensional space A_n with torsion-free affine connection ∇, assigned to the local coordinate system $x^1, x^2, \dots, x^n$, in which the affinor structure F (i.e., a tensor field of type $(1,1)$) is defined, for which in coordinates $F_i^h \neq a \cdot \delta_i^h$, where δ_i^h is the Kronecker symbol, a is some function.

Definition 1. *A curve ℓ defined by the equation $\ell = \ell(t)$ is called F-planar if its tangent vector $\lambda(t) = d\ell(t)/dt (\neq 0)$ remains, under parallel translation along the curve ℓ, in the distribution generated by the vector functions λ and $F\lambda$ along ℓ.*

According to this definition, a curve ℓ is F-planar if and only if the following condition holds:

$$\nabla_{\lambda(t)}\lambda(t) = \rho_1(t)\,\lambda(t) + \rho_2(t)\,F\lambda(t),$$

where $\rho_1(t)$ and $\rho_2(t)$ are some functions of the parameter t.

The class of F-planar curves is wide enough. It includes geodesic (if $F = \rho\,\mathrm{Id}$, where ρ is a function and Id is the identity operator, or a function $\rho_2 \equiv 0$), quasi-geodesic, planar, and analytically planar curves.

Let A_n and $\overline{A}_n$ be two spaces with torsion-free affine connections ∇ and $\overline{\nabla}$, respectively. Let F and $\overline{F}$ be affine structures defined on A_n and $\overline{A}_n$, respectively.

Definition 2. *The mapping π: $A_n \to \overline{A}_n$ is called F-planar if any F-planar curve of space A_n is mapped onto an $\overline{F}$-planar curve of space $\overline{A}_n$.*

Let us recall what a deformation tensor is, see [9,15,35]. Consider the affine connection spaces A_n and $\overline{A}_n$ in a common F-planar coordinate system $x^1, x^2, \dots, x^n$. The tensor

$$P_{ij}^h(x) = \overline{\Gamma}_{ij}^h(x) - \Gamma_{ij}^h(x), \tag{1}$$

is called a *tensor of the deformation of connections*. Here, $\Gamma_{ij}^h(x)$ and $\overline{\Gamma}_{ij}^h(x)$ are components of affine connections ∇ and $\overline{\nabla}$, respectively.

From Theorems 1 and 2 of [4], and more precisely [6,7], see [16] (Chapter 14), it actually follows that the mapping π: $A_n \to \overline{A}_n$ $(n > 2)$ will be F-planar if and only if, for the deformation tensor P in the coordinate system $x^1, x^2, \dots, x^n$, the following equality holds:

$$P^h_{ij} = \delta^h_{(i}\psi_{j)} + F^h_{(i}\varphi_{j)},$$

where $\psi_i(x)$ and $\varphi_i(x)$ are some covectors, and the brackets mean symmetrization by the specified indices without division.

F-planar mapping is called *canonical* if ψ_i vanishes. Each F-planar mapping can be represented as a composition of a canonical F-planar mapping and a geodesic mapping. The latter can be considered a trivial F-planar.

Thus, canonical F-planar mappings in the common coordinate system $x^1, x^2, \dots, x^n$ are characterized by the equations

$$P^h_{ij} = F^h_{(i}\varphi_{j)}. \tag{2}$$

Suppose that the affinor F defines in the space A_n an e-structure [9] (p. 177), which satisfies the condition $F^2 = e\,\mathrm{Id}$, $e = \pm 1$, in coordinates:

$$F^h_\alpha F^\alpha_i = e\delta^h_i. \tag{3}$$

In this case, F-planar mapping will be denoted $\pi(e)$.

3. Properties of Vector $\boldsymbol{\varphi_i}$

It is known [9] that there is a dependence between the Riemann tensors of spaces A_n and $\overline{A}_n$

$$\overline{R}^h_{ijk} = R^h_{ijk} + P^h_{ik,j} - P^h_{ij,k} + P^\alpha_{ik}P^h_{\alpha j} - P^\alpha_{ij}P^h_{\alpha k}. \tag{4}$$

Given that the deformation tensor of connections (1) has the structure (2), from the Formula (4) after transformations we obtain

$$\varphi_{i,j}F^h_k + \varphi_{k,j}F^h_i - \varphi_{i,k}F^h_j - \varphi_{j,k}F^h_i = B^h_{ijk}, \tag{5}$$

where

$$\begin{aligned} B^h_{ijk} = \overline{R}^h_{ijk} - R^h_{ijk} - \varphi_i\big(F^h_{k,j} - F^h_{j,k} + e\delta^h_k\varphi_j + \varphi_\alpha F^\alpha_k F^h_j - e\delta^h_j\varphi_k - \varphi_\alpha F^\alpha_j F^h_k\big) \\ -\varphi_k\big(F^h_{i,j} + \varphi_\alpha F^\alpha_i F^h_j\big) + \varphi_j\big(F^h_{i,k} + \varphi_\alpha F^\alpha_i F^h_k\big). \end{aligned} \tag{6}$$

Note, that the right hand side of the Equation (5) does not depend on the derivatives of φ_i. Contracting (5) with the affinor F^m_ρ with respect to the indices ρ and h, we obtain

$$\delta^m_k\varphi_{i,j} + \delta^m_i\varphi_{k,j} - \delta^m_j\varphi_{i,k} - \delta^m_i\varphi_{j,k} = eB^\alpha_{ijk}F^m_\alpha. \tag{7}$$

Next, we contract (7) with respect to the indices m and i. As a result, we find

$$\varphi_{k,j} - \varphi_{j,k} = \frac{e}{n+1}B^\alpha_{\beta jk}F^\beta_\alpha. \tag{8}$$

After contraction of (7) with respect to the indices m and k, we obtain

$$n\varphi_{i,j} - \varphi_{j,i} = eB^\alpha_{ij\beta}F^\beta_\alpha. \tag{9}$$

The Equations (9) after taking into account (8) can be written as

$$\varphi_{i,j} = \frac{e}{n-1}\big(B^\alpha_{ij\beta} - \frac{1}{n+1}B^\alpha_{\beta ji}\big)F^\beta_\alpha. \tag{10}$$

Note that the Formula (10) is obtained for the general case of canonical F-planar mappings $\pi(e)$ $(e = \pm 1)$.

Therefore, we proved the following theorem.

Theorem 1. *The vector φ_i, participating in the Equations (2) of canonical F-planar mappings $\pi(e)$, $e = \pm 1$ satisfies the conditions (10), where the tensor B^h_{ijk} is defined by the Formulas (6).*

The right part of Equation (10) depends on the unknown tensor $\overline{R}^h_{ijk}$, the unknown vector φ_i, and the known affinor F^h_i and its covariant derivative $F^h_{i,k}$ in A_n.

4. Canonical *F*-Planar Mappings $\pi(e)$ $(e = \pm 1)$ of Spaces with Affine Connection onto 2-Symmetric Spaces

Space $\overline{A}_n$ with affine connection is called *(locally) symmetric* if the Riemann tensor in it is absolutely parallel (P. A. Shirokov [36], É. Cartan [37], S. Helgason [38]). That is, symmetric spaces are characterized by the condition

$$\overline{R}^h_{ijk;m} = 0,$$

where $\overline{R}^h_{ijk}$ is the Riemann tensor of the space $\overline{A}_n$; the sign " ; " denotes the covariant derivative with respect to the connection $\overline{\nabla}$ of the space $\overline{A}_n$.

Space $\overline{A}_n$ is called *2-symmetric* [27,39] if the conditions are met for the Riemann tensor $\overline{R}^h_{ijk}$

$$\overline{R}^h_{ijk;m\rho_1} = 0. \tag{11}$$

Naturally, symmetric spaces are 2-symmetric spaces.

Consider canonical F-planar mappings $\pi(e)$ $(e = \pm 1)$ of spaces with an affine connection onto *2-symmetric spaces $\overline{A}_n$, which are characterized by the Equations (2), and the affinor F^h_i satisfying the conditions (3) is defined in the space A_n. We assume that the spaces A_n and $\overline{A}_n$ are related to the common coordinate system $x^1, x^2, \dots, x^n$.

Because

$$\overline{R}^h_{ijk;m} = \frac{\partial \overline{R}^h_{ijk}}{\partial x^m} + \overline{\Gamma}^h_{m\alpha}\overline{R}^\alpha_{ijk} - \overline{\Gamma}^\alpha_{mi}\overline{R}^h_{\alpha jk} - \overline{\Gamma}^\alpha_{mj}\overline{R}^h_{i\alpha k} - \overline{\Gamma}^\alpha_{mk}\overline{R}^h_{ij\alpha},$$

then, given the Formula (1), we can write

$$\overline{R}^h_{ijk;m} = \overline{R}^h_{ijk,m} + P^h_{m\alpha}\overline{R}^\alpha_{ijk} - P^\alpha_{mi}\overline{R}^h_{\alpha jk} - P^\alpha_{mj}\overline{R}^h_{i\alpha k} - P^\alpha_{mk}\overline{R}^h_{ij\alpha}. \tag{12}$$

Based on the definition of the covariant derivative

$$\begin{aligned}\left(\overline{R}^h_{ijk;m}\right)_{,\rho_1} = \frac{\partial \overline{R}^h_{ijk;m}}{\partial x^{\rho_1}} + \Gamma^h_{\alpha\rho_1}\overline{R}^\alpha_{ijk;m} - \Gamma^\alpha_{i\rho_1}\overline{R}^h_{\alpha jk;m} - \Gamma^\alpha_{j\rho_1}\overline{R}^h_{i\alpha k;m}\\ -\Gamma^\alpha_{k\rho_1}\overline{R}^h_{ij\alpha;m} - \Gamma^\alpha_{m\rho_1}\overline{R}^h_{ijk;\alpha},\end{aligned}$$

and taking into account the Formula (1), we have

$$\begin{aligned}\left(\overline{R}^h_{ijk;m}\right)_{,\rho_1} = \overline{R}^h_{ijk;m\rho_1} - P^h_{\alpha\rho_1}\overline{R}^\alpha_{ijk;m} + P^\alpha_{i\rho_1}\overline{R}^h_{\alpha jk;m} + P^\alpha_{j\rho_1}\overline{R}^h_{i\alpha k;m}\\ +P^\alpha_{k\rho_1}\overline{R}^h_{ij\alpha;m} + P^\alpha_{m\rho_1}\overline{R}^h_{ijk;\alpha}.\end{aligned} \tag{13}$$

Differentiate (12) by x^{ρ_1} in the space A_n. We obtain

$$\begin{aligned}\left(\overline{R}^h_{ijk;m}\right)_{,\rho_1} = \overline{R}^h_{ijk,m\rho_1} + P^h_{m\alpha,\rho_1}\overline{R}^\alpha_{ijk} + P^h_{m\alpha}\overline{R}^\alpha_{ijk,\rho_1} - P^\alpha_{mi,\rho_1}\overline{R}^h_{\alpha jk} - P^\alpha_{mi}\overline{R}^h_{\alpha jk,\rho_1}\\ -P^\alpha_{mj,\rho_1}\overline{R}^h_{i\alpha k} - P^\alpha_{mj}\overline{R}^h_{i\alpha k,\rho_1} - P^\alpha_{mk,\rho_1}\overline{R}^h_{ij\alpha} - P^\alpha_{mk}\overline{R}^h_{ij\alpha,\rho_1}.\end{aligned} \tag{14}$$

Comparing Equations (13) and (14), we have

$$\begin{aligned}\overline{R}^h_{ijk,m\rho_1} = \overline{R}^h_{ijk;m\rho_1} - P^h_{\alpha\rho_1}\overline{R}^\alpha_{ijk;m} + P^\alpha_{i\rho_1}\overline{R}^h_{\alpha jk;m} + P^\alpha_{j\rho_1}\overline{R}^h_{i\alpha k;m} + P^\alpha_{k\rho_1}\overline{R}^h_{ij\alpha;m} \\ +P^\alpha_{m\rho_1}\overline{R}^h_{ijk;\alpha} - P^h_{m\alpha,\rho_1}\overline{R}^\alpha_{ijk} - P^h_{m\alpha}\overline{R}^\alpha_{ijk,\rho_1} + P^\alpha_{mi,\rho_1}\overline{R}^h_{\alpha jk} + P^\alpha_{mi}\overline{R}^h_{\alpha jk,\rho_1} \\ +P^\alpha_{mj,\rho_1}\overline{R}^h_{i\alpha k} + P^\alpha_{mj}\overline{R}^h_{i\alpha k,\rho_1} + P^\alpha_{mk,\rho_1}\overline{R}^h_{ij\alpha} + P^\alpha_{mk}\overline{R}^h_{ij\alpha,\rho_1}.\end{aligned} \tag{15}$$

Taking account of (2) and (12), we might write (15) in the form

$$\overline{R}^h_{ijk,m\rho_1} = \overline{R}^h_{ijk;m\rho_1} + \Theta^h_{ijkm\rho_1}, \tag{16}$$

where

$$\begin{aligned}\Theta^h_{ijkm\rho_1} = -F^h_{(\alpha}\varphi_{\rho_1)}\big(\overline{R}^\alpha_{ijk,m} + \Theta^\alpha_{ijkm}\big) + F^\alpha_{(i}\varphi_{\rho_1)}\big(\overline{R}^h_{\alpha jk,m} + \Theta^h_{\alpha jkm}\big) \\ +F^\alpha_{(j}\varphi_{\rho_1)}\big(\overline{R}^h_{i\alpha k,m} + \Theta^h_{i\alpha km}\big) + F^\alpha_{(k}\varphi_{\rho_1)}\big(\overline{R}^h_{ij\alpha,m} + \Theta^h_{ij\alpha m}\big) \\ +F^\alpha_{(m}\varphi_{\rho_1)}\big(\overline{R}^h_{ijk,\alpha} + \Theta^h_{ijk\alpha}\big) - F^h_{(m}\varphi_{\alpha),\rho_1}\overline{R}^\alpha_{ijk} - F^h_{(m}\varphi_{\alpha)}\overline{R}^\alpha_{ijk,\rho_1} \\ +F^\alpha_{(m}\varphi_{i),\rho_1}\overline{R}^h_{\alpha jk} + F^\alpha_{(m}\varphi_{i)}\overline{R}^h_{\alpha jk,\rho_1} + F^\alpha_{(m}\varphi_{j),\rho_1}\overline{R}^h_{i\alpha k} \\ +F^\alpha_{(m}\varphi_{j)}\overline{R}^h_{i\alpha k,\rho_1} + F^\alpha_{(m}\varphi_{k),\rho_1}\overline{R}^h_{ij\alpha} + F^\alpha_{(m}\varphi_{k)}\overline{R}^h_{ij\alpha,\rho_1},\end{aligned} \tag{17}$$

$$\Theta^h_{ijkm} = F^h_{(m}\varphi_{\alpha)}\overline{R}^\alpha_{ijk} - F^\alpha_{(m}\varphi_{i)}\overline{R}^h_{\alpha jk} - F^\alpha_{(m}\varphi_{j)}\overline{R}^h_{i\alpha k} - F^\alpha_{(m}\varphi_{k)}\overline{R}^h_{ij\alpha}. \tag{18}$$

Given the structure of the tensor Θ^h_{ijkm} defined by the Formula (18), it is easy to see that the tensor $\Theta^h_{ijkm\rho_1}$ defined by the Formula (17) depends on the tensors F^h_k, $\overline{R}^h_{ijk}$, φ_k, as well as on covariant derivatives of the specified tensors by the connection ∇ of the space A_n. In this case, the tensor F^h_k is considered to be given, and the conditions (3) are met for this tensor.

Let us introduce the tensor $\overline{R}^h_{ijkm}$ in the following way:

$$\overline{R}^h_{ijk,m} = \overline{R}^h_{ijkm}, \tag{19}$$

Assume that the space $\overline{A}_n$ is 2-symmetric. Then, for the Riemann tensor $\overline{R}^h_{ijk}$ of this space, the conditions (11) are met. Taking into account (19) from (16), we have

$$\overline{R}^h_{ijkm,\rho_1} = \Theta^h_{ijkm\rho_1}, \tag{20}$$

where the tensor $\Theta^h_{ijkm\rho_1}$ is defined by the Formulas (17).

We assume that in (20) the tensors $\varphi_{i,j}$, $\overline{R}^h_{ijk,m}$ are expressed in accordance with (10) and (19).

Obviously, Equations (10), (19) and (20) in this space A_n represent a system of equations in covariant derivatives of the Cauchy type with respect to functions $\varphi_i(x)$, $\overline{R}^h_{ijk}(x)$, $\overline{R}^h_{ijkm}(x)$.

The functions $\overline{R}^h_{ijk}(x)$ and $\overline{R}^h_{ijkl}(x)$ must satisfy algebraic conditions that follow from the properties of the Riemannian tensor of $\overline{A}_n$:

$$\overline{R}^h_{i(jk)} = 0, \quad \overline{R}^h_{(ijk)} = 0, \quad \overline{R}^h_{i(jk)l} = 0, \quad \overline{R}^h_{(ijk)l} = 0. \tag{21}$$

Thus, we proved the following Theorem.

Theorem 2. *In order that an affine connection space A_n admits a canonical F-planar mapping $\pi(e)$ $(e = \pm 1)$ onto a 2-symmetric space $\overline{A}_n$, it is necessary and sufficient that in the space A_n*

a solution exists of a closed mixed system of Cauchy type equations in covariant derivatives (10), (19)–(21) *with respect to functions* $\varphi_i(x)$, $\overline{R}^h_{ijk}(x)$ *and* $\overline{R}^h_{ijkm}(x)$.

Obviously, the general solution of the closed mixed system of Cauchy-type equations in covariant derivatives (10), (19)–(21) depends on no more than

$$1/3\, n^2\ (n^3 + n^2 - n - 1) + n$$

essential parameters.

The proof of Theorem 2 was actually done by us in the work [25] but in a different form.

5. Canonical *F*-Planar Mappings $\pi(e)$ $(e = \pm 1)$ of Spaces with Affine Connection onto *m*-Symmetric Spaces

The space of affine connection $\overline{A}_n$ is called *m-symmetric* if the Riemann tensor $\overline{R}^h_{ijk}$ of this space satisfies the conditions

$$\overline{R}^h_{ijk;\rho_1\rho_2\ldots\rho_m} = 0. \tag{22}$$

The m-symmetric spaces are a natural generalization of symmetric and 2-symmetric spaces [39].

Based on the definition of the covariant derivative

$$\begin{aligned}(\overline{R}^h_{ijk;m\rho_1})_{,\rho_2} = \frac{\partial \overline{R}^h_{ijk;m\rho_1}}{\partial x^{\rho_2}} + \Gamma^h_{\alpha\rho_2}\overline{R}^\alpha_{ijk;m\rho_1} - \Gamma^\alpha_{i\rho_2}\overline{R}^h_{\alpha jk;m\rho_1} - \Gamma^\alpha_{j\rho_2}\overline{R}^h_{i\alpha k;m\rho_1}\\ -\Gamma^\alpha_{k\rho_2}\overline{R}^h_{ij\alpha;m\rho_1} - \Gamma^\alpha_{m\rho_2}\overline{R}^h_{ijk;\alpha\rho_1} - \Gamma^\alpha_{\rho_1\rho_2}\overline{R}^h_{ijk;m\alpha},\end{aligned}$$

and taking into account the Formula (1), we have

$$\begin{aligned}(\overline{R}^h_{ijk;m\rho_1})_{,\rho_2} = \overline{R}^h_{ijk;m\rho_1\rho_2} - P^h_{\alpha\rho_2}\overline{R}^\alpha_{ijk;m\rho_1} + P^\alpha_{i\rho_2}\overline{R}^h_{\alpha jk;m\rho_1} + P^\alpha_{j\rho_2}\overline{R}^h_{i\alpha k;m\rho_1}\\ +P^\alpha_{k\rho_2}\overline{R}^h_{ij\alpha;m\rho_1} + P^\alpha_{m\rho_2}\overline{R}^h_{ijk;\alpha\rho_1} + P^\alpha_{\rho_1\rho_2}\overline{R}^h_{ijk;m\alpha}.\end{aligned} \tag{23}$$

From the Formula (23) based on the Formulas (2) and (16), we obtain

$$\begin{aligned}(\overline{R}^h_{ijk;m\rho_1})_{,\rho_2} = \overline{R}^h_{ijk;m\rho_1\rho_2} - F^h_{(\alpha}\varphi_{\rho_2)}(\overline{R}^\alpha_{ijk,m\rho_1} - \Theta^\alpha_{ijkm\rho_1}) + F^\alpha_{(i}\varphi_{\rho_2)}(\overline{R}^h_{\alpha jk,m\rho_1} - \Theta^h_{\alpha jkm\rho_1})\\ +F^\alpha_{(j}\varphi_{\rho_2)}(\overline{R}^h_{i\alpha k,m\rho_1} - \Theta^h_{i\alpha km\rho_1}) + F^\alpha_{(k}\varphi_{\rho_2)}(\overline{R}^h_{ij\alpha,m\rho_1} - \Theta^h_{ij\alpha m\rho_1})\\ +F^\alpha_{(m}\varphi_{\rho_2)}(\overline{R}^h_{ijk,\alpha\rho_1} - \Theta^h_{ijk\alpha\rho_1}) + F^\alpha_{(\rho_1}\varphi_{\rho_2)}(\overline{R}^h_{ijk,m\alpha} - \Theta^h_{ijkm\alpha}).\end{aligned} \tag{24}$$

Differentiate (16) by x^{ρ_2} in the space A_n. Taking into account the Formulas (24), we have

$$\begin{aligned}\overline{R}^h_{ijk,m\rho_1\rho_2} = \overline{R}^h_{ijk;m\rho_1\rho_2} - F^h_{(\alpha}\varphi_{\rho_2)}(\overline{R}^\alpha_{ijk,m\rho_1} - \Theta^\alpha_{ijkm\rho_1}) + F^\alpha_{(i}\varphi_{\rho_2)}(\overline{R}^h_{\alpha jk,m\rho_1} - \Theta^h_{\alpha jkm\rho_1})\\ +F^\alpha_{(j}\varphi_{\rho_2)}(\overline{R}^h_{i\alpha k,m\rho_1} - \Theta^h_{i\alpha km\rho_1}) + F^\alpha_{(k}\varphi_{\rho_2)}(\overline{R}^h_{ij\alpha,m\rho_1} - \Theta^h_{ij\alpha m\rho_1})\\ +F^\alpha_{(m}\varphi_{\rho_2)}(\overline{R}^h_{ijk,\alpha\rho_1} - \Theta^h_{ijk\alpha\rho_1}) + F^\alpha_{(\rho_1}\varphi_{\rho_2)}(\overline{R}^h_{ijk,m\alpha} - \Theta^h_{ijkm\alpha}) + \Theta^h_{ijkm\rho_1,\rho_2}.\end{aligned} \tag{25}$$

We introduce the tensors $\overline{R}^h_{ijkm\rho_1}$ and $\Theta^h_{ijkm\rho_1\rho_2}$ and assume

$$\overline{R}^h_{ijkm,\rho_1} = \overline{R}^h_{ijkm\rho_1}, \tag{26}$$

$$\begin{aligned}\Theta^h_{ijkm\rho_1\rho_2} = {} & -F^h_{(\alpha}\varphi_{\rho_2)}\big(\overline{R}^\alpha_{ijk,m\rho_1} - \Theta^\alpha_{ijkm\rho_1}\big) + F^\alpha_{(i}\varphi_{\rho_2)}\big(\overline{R}^h_{\alpha jk,m\rho_1} - \Theta^h_{\alpha jkm\rho_1}\big) \\ & + F^\alpha_{(j}\varphi_{\rho_2)}\big(\overline{R}^h_{i\alpha k,m\rho_1} - \Theta^h_{i\alpha km\rho_1}\big) + F^\alpha_{(k}\varphi_{\rho_2)}\big(\overline{R}^h_{ij\alpha,m\rho_1} - \Theta^h_{ij\alpha m\rho_1}\big) \\ & + F^\alpha_{(m}\varphi_{\rho_2)}\big(\overline{R}^h_{ijk,\alpha\rho_1} - \Theta^h_{ijk\alpha\rho_1}\big) + F^\alpha_{(\rho_1}\varphi_{\rho_2)}\big(\overline{R}^h_{ijk,m\alpha} - \Theta^h_{ijkm\alpha}\big) + \Theta^h_{ijkm\rho_1,\rho_2}.\end{aligned} \tag{27}$$

Taking into account (26) and (27) from (25), we have

$$\overline{R}^h_{ijkm\rho_1,\rho_2} = \overline{R}^h_{ijk;m\rho_1\rho_2} + \Theta^h_{ijkm\rho_1\rho_2}. \tag{28}$$

Let us introduce tensors $\overline{R}^h_{ijk\rho_1\rho_2\rho_3}, \ldots, \overline{R}^h_{ijk\rho_1\rho_2\rho_3\ldots\rho_{m-2}\rho_{m-1}}$, and let us put

$$\begin{aligned}\overline{R}^h_{ijk\rho_1\rho_2,\rho_3} &= \overline{R}^h_{ijk\rho_1\rho_2\rho_3}, \\ & \ldots \\ \overline{R}^h_{ijk\rho_1\rho_2\rho_3\ldots\rho_{m-2},\rho_{m-1}} &= \overline{R}^h_{ijk\rho_1\rho_2\rho_3\ldots\rho_{m-2}\rho_{m-1}}.\end{aligned} \tag{29}$$

Using the Equation (28), we covariantly differentiate $(m-2)$ times with respect to the connection of the space A_n, and in the left part we proceed to the covariant derivative with respect to the connection of the space $\overline{A}_n$ using the formula

$$\begin{aligned}\big(\overline{R}^h_{ijk;\rho_1\ldots\rho_{\tau-2}\rho_{\tau-1}}\big)_{,\rho_\tau} = {} & \overline{R}^h_{ijk;\rho_1\ldots\rho_{\tau-2}\rho_{\tau-1}\rho_\tau} - P^h_{\alpha\rho_\tau}\overline{R}^\alpha_{ijk;\rho_1\ldots\rho_{\tau-2}\rho_{\tau-1}} + P^\alpha_{i\rho_\tau}\overline{R}^h_{\alpha jk;\rho_1\ldots\rho_{\tau-2}\rho_{\tau-1}} \\ & + P^\alpha_{j\rho_\tau}\overline{R}^h_{i\alpha k;\rho_1\ldots\rho_{\tau-2}\rho_{\tau-1}} + P^\alpha_{k\rho_\tau}\overline{R}^h_{ij\alpha;\rho_1\ldots\rho_{\tau-2}\rho_{\tau-1}} + P^\alpha_{\rho_1\rho_\tau}\overline{R}^h_{ijk;\alpha\ldots\rho_{\tau-2}\rho_{\tau-1}} \\ & + \cdots + P^\alpha_{\rho_{\tau-1}\rho_\tau}\overline{R}^h_{ijk;\rho_1\ldots\rho_{\tau-2}\alpha}.\end{aligned} \tag{30}$$

The Formula (30) is derived from (1).

Suppose that the space $\overline{A}_n$ is m-symmetric $(m > 2)$. Then, taking into account (22) and (29) from the equation obtained in this way after substitutions and transformations, we have

$$\overline{R}^h_{ijk\rho_1\ldots\rho_{m-2}\rho_{m-1},\rho_m} = \Theta^h_{ijk\rho_1\ldots\rho_{m-1}\rho_m}, \tag{31}$$

where $\Theta^h_{ijk\rho_1\ldots\rho_{m-1}\rho_m}$ is some tensor depending on unknown tensors φ_i, $\overline{R}^h_{ijk}$, $\overline{R}^h_{ijk\rho_1}$, …, $\overline{R}^h_{ijk\rho_1\ldots\rho_{m-1}}$, as well as on some well-known tensors.

Obviously, Equations (10), (19), (20), (26), (29) and (31) form a closed system of Cauchy type equations with respect to functions $\varphi_i(x)$, $\overline{R}^h_{ijk}(x)$, $\overline{R}^h_{ijk\rho_1}(x)$, …, $\overline{R}^h_{ijk\rho_1\ldots\rho_{m-1}}(x)$; moreover, the conditions of an algebraic nature (21) must be fulfilled and

$$\overline{R}^h_{i(jk)l_1\ldots l_\rho} = 0 \quad \text{and} \quad \overline{R}^h_{(ijk)l_1\ldots l_\rho} = 0, \quad \rho = 2, 3, \ldots, m-1. \tag{32}$$

Thus, we proved the following theorem.

Theorem 3. *In order that an affine connection space A_n admits a canonical F-planar mapping $\pi(e)$ $(e = \pm 1)$ onto an m-symmetric space $\overline{A}_n$, it is necessary and sufficient that a solution of a closed mixed system of Cauchy-type equations in covariant derivatives* (10), (19)–(21), (26), (29), (31), (32) *exists with respect to unknown functions $\varphi_i(x)$, $\overline{R}^h_{ijk}(x)$, $\overline{R}^h_{ijk\rho_1}(x)$, …, $\overline{R}^h_{ijk\rho_1\ldots\rho_{m-1}}(x)$.*

Obviously, the general solution of the closed mixed system of the above mentioned equations depends on no more than

$$1/3\, n^2\, (n^2-1)\, (1 + n + n^2 + \cdots + n^{m-1}) + n$$

essential parameters.

6. Conclusions

The paper deals with F-planar mappings of affine connection spaces with affinor e-structure onto m-symmetric spaces. We have found the fundamental equations of the considered mapping, which are in Cauchy form. Therefore, the general solution to this problem depends on the finite number of real parameters. The number of these parameters was also calculated.

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