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On the Construction of Pandiagonal Magic Cubes

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Abstract: This paper investigates the construction method of pandiagonal magic cube. First, we define a pandiagonal Latin cube. According to this definition, the cube can be constructed by simple methods. After designing a set of orthogonal pandiagonal Latin cubes, the corresponding order pandiagonal magic cube can be constructed. In addition, we give the algebraic conditions of the universal diagonal Latin cube orthogonality and the strict theoretical proof. Based on the proposed method, it can be shown that at least $6(n!)^3$ pandiagonal magic cubes of order n is formed through a pandiagonal Latin cube. Moreover, our method is easy to implement by computer program.

Keywords: magic cube; Latin cube; pandiagonal magic cube; pandiagonal Latin cube

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1. Introduction

A square matrix of order n, composed of continuous natural numbers from 1 to n^2, is called the magic square of order n if the sums of elements in each row, column and diagonal are the same. This sum is called magic sum, and it is equal to $\frac{n(1+n^2)}{2}$ [1]. Magic square is one of the important research objects of combinatorial design. It is applied in the fields of image processing, computer science, and cryptography [2–4]. The study of magic squares can be traced back to ancient China. The "Luoshu" discovered 4000 years ago is a magic square of order three [5]. Thereafter, there has been a lot of research on magic squares, and magic squares in these studies have different characteristics [6–8]. If the sums of the numbers on each pandiagonal are the same in a magic square, this magic square is called pandiagonal magic square. The so-called pandiagonal refers to the "broken diagonal" paralleled to the diagonal, and wrap round at the edges of the square [9,10]. For example, let a magic square be $(a_{ij})_{n\times n}$, then $a_{12}, a_{23}, \cdots, a_{n-1,n}, a_{n,1}$ are elements of a pandiagonal. There are also other pandiagonals. After extending a magic square to the three-dimensional, a magic cube is obtained. An integer magic cube of order n is a three-dimensional array $A = (a_{ijk})$ which consists of the numbers $1, 2, \cdots, n^3$, with each section and six diagonal planes being magic squares. Previous studies have given the construction methods of concrete magic cubes with various characteristics [11–13]. If every cross section, every diagonal and every pandiagonal of the magic cube is a pandiagonal magic square, it is called a pandiagonal magic cube. The relevant concepts are strictly defined below.

An integer cube of order n is a three-dimensional array $A = (a_{ijk})$ which consists of the numbers $1, 2, \cdots, n^3$. Matrices $(a_{ijk})_{i,j=1,2,\cdots,n}$, $(a_{ijk})_{i,k=1,2,\cdots,n}$, $(a_{ijk})_{j,k=1,2,\cdots,n}$ are called cross sections. In particular, $(a_{ij1})_{i,j=1,2,\cdots,n}$, $(a_{i1k})_{i,k=1,2,\cdots,n}$, $(a_{1jk})_{j,k=1,2,\cdots,n}$ are called surfaces. Similarly, matrices $(a_{ijj})_{i,j=1,2,\cdots,n}$, $(a_{i,n-j+1,j})_{i,j=1,2,\cdots,n}$, $(a_{iji})_{i,j=1,2,\cdots,n}$, $(a_{i,j,n-i+1})_{i,j=1,2,\cdots,n}$, $(a_{iij})_{i,j=1,2,\cdots,n}$, $(a_{i,n-i+1,j})_{i,j=1,2,\cdots,n}$ are called diagonal planes.

Definition 1. *An integer cube is called a pandiagonal magic cube, if all of its cross sections and diagonal planes are pandiagonal magic squares.*

The concept of pandiagonal magic square can be found in [6–8,14]. Because a line parallel to an edge of an integer cube is a row or a column of a cross section, a great diagonal is a diagonal of a diagonal plane, so that a pandiagonal magic cube is certainly a magic cube [15].

The studies of magic squares and magic cubes have made relatively rich achievements. Cammann and Andrews introduced the early research results of magic squares and magic cubes [12,16]. Abe put forward 23 questions about magic squares, involving basic magic squares, pandiagonal magic squares, sparse magic squares, anti-magic squares, etc., which promoted the study of magic squares [17]. Loly et al. [18], Lee et al. [19], Nordgren et al. [20], and Hou et al. [21] discussed the algebraic properties of magic squares. Trenkler proposed an algorithm for making magic cubes [22]. However, the study on pandiagonal magic cubes is rare.

In this paper, we will study the existence and construction of pandiagonal magic cubes of order n based on the pandiagonal magic square idea in [6]. The following is the outline of this paper. First, a three-dimensional auxiliary cube is defined and its properties are discussed. Then, based on the discussion of Latin cube and pandiagonal Latin cube, the design method of the pandiagonal Latin cube is given. At the same time, we give the necessary and sufficient conditions for the orthogonality of three pandiagonal Latin cubes. Finally, by using the orthogonal pandiagonal Latin cubes, the construction method of pandiagonal magic cubes is obtained. In addition, we give a method to find another two pandiagonal Latin cubes orthogonal to one known pandiagonal Latin cube.

2. Auxiliary Cube and Its Properties

In this section, the definition and properties of the auxiliary cube are shown.

Definition 2. *A cube $A = (a_{ijk})$ which consists of consecutive integers from 1 to n^3 is called an Auxiliary Cube (hereafter written as Aux Cube) if it satisfies the following conditions:*

(1) The top surface (a_{ij1}) is an auxiliary matrix [6], namely (a_{ij1}) satisfies

$$a_{i11} - a_{111} = a_{i21} - a_{121} = \cdots = a_{in1} - a_{1n1} \qquad (i = 1, 2, \cdots, n)$$

(2) The cross sections $(a_{ijk})_{i,j=1,2,\cdots,n}$ and $(a_{ij1})_{i,j=1,2,\cdots,n}$ satisfy

$$a_{ijk} - a_{ij1} = \beta_k \qquad (k = 1, 2, \cdots, n)$$

where β_k is independent of i and j.

The preceding conditions (1) and (2) are equivalent to the fact that each cross section is an auxiliary matrix [6].

Let

$$a_{ijk} = n(i-1) + j + n^2(k-1)$$

then $A = (a_{ijk})$ is called Natural Auxiliary Cube. Clearly, by interchanging two parallel cross sections of an Aux Cube A, we can form other Aux Cubes. A is an Aux Cube if, and only if, A can be obtained from Natural Aux Cube through finite interchanges of parallel cross sections in succession. Because there are $n!$ permutations of $1, 2, \cdots, n$, we can create $(n!)^3$ Aux Cubes from Natural Aux Cube by interchanging parallel cross sections.

Lemma 1. *Let A be an Aux Cube of order n, then the sum of any n entries in different rows, different columns and different levels is the same. Namely, if $A = (a_{ijk})$, $i_1, i_2, \cdots, i_n$, $j_1, j_2, \cdots, j_n$, $k_1, k_2, \cdots, k_n$ are all permutations of $1, 2, \cdots, n$, then*

$$\sum_{s=1}^{n} a_{i_s j_s k_s} = constant$$

Proof. Assume

$$a_{i11} - a_{111} = a_{i21} - a_{121} = \cdots = a_{in1} - a_{1n1} = \alpha_i$$

Then we have

$$a_{ijk} = a_{ij1} + \beta_k \qquad a_{ij1} = a_{1j1} + \alpha_i$$

Suppose that

$$a_{i_1 j_1 k_1}, \cdots, a_{i_n j_n k_n}$$

are n entries in different rows, different columns, and different levels of A. This is equivalent to the fact that "$i_1, i_2, \cdots, i_n$", "$j_1, j_2, \cdots, j_n$", "$k_1, k_2, \cdots, k_n$" are all permutations of $1, 2, \cdots, n$. The sum of these entries is given by

$$\begin{aligned}\sum_{s=1}^{n} a_{i_s j_s k_s} &= \sum_{s=1}^{n} (a_{1 j_s 1} + \alpha_{i_s} + \beta_{k_s}) \\ &= \sum_{s=1}^{n} a_{1 j_s 1} + \sum_{s=1}^{n} \alpha_{i_s} + \sum_{s=1}^{n} \beta_{k_s} \\ &= \sum_{s=1}^{n} a_{1 s 1} + \sum_{s=1}^{n} \alpha_s + \sum_{s=1}^{n} \beta_s\end{aligned}$$

The last three terms are independent of "$i_1, i_2, \cdots, i_n$", "$j_1, j_2, \cdots, j_n$", "$k_1, k_2, \cdots, k_n$". This completes the proof. □

Let A be an Aux Cube. From Lemma 1, a pandiagonal magic cube can be obtained by adjusting the entries of A so that its each row, each column and each pandiagonal of its each cross section and each diagonal plane consists of entries in distinct rows, distinct columns and distinct levels of A, respectively.

Definition 3. *The integer cube A of order n consisting of $1, 2, \cdots, n$ is said to be a pandiagonal Latin cube if all of its cross sections and diagonal planes are pandiagonal Latin squares [6].*

Definition 4. *A set of pandiagonal Latin cubes $L_1 = (l^1_{ijk})$, $L_2 = (l^2_{ijk})$, $L_3 = (l^3_{ijk})$ is called orthogonal if entries of the cube*

$$L = \left((l^1_{ijk}, l^2_{ijk}, l^3_{ijk}) \right)$$

run through all ordered three-tuples $(1, 1, 1)$ to (n, n, n) [8,23].

For an integer cube $A = (a_{ijk})$, its entries have three subscripts i, j and k. The set of first integers, i.e., i's, form another integer cube, and similarly for the second and third ones. It is not difficult to understand the following fact. If $A = (a(i, j, k))$ is an Aux Cube, $L_1 = (l^1_{ijk})$, $L_2 = (l^2_{ijk})$, $L_3 = (l^3_{ijk})$ are orthogonal pandiagonal Latin cubes. Define $B = (b_{suv})$ as

$$b_{suv} = a(l^1_{suv}, l^2_{suv}, l^3_{suv}) \;\; (s, u, v = 1, 2, \cdots, n)$$

Then B is a pandiagonal magic cube. Therefore, we can construct a pandiagonal magic cube as follows. First, construct a set of orthogonal pandiagonal Latin cubes, and join them into a cube of 3-tuples, and then change entries into entries of some auxiliary cube so that the subscripts are these three-tuples. This provides a pandiagonal magic cube.

If we use the first, second and third subscripts of each element in a pandiagonal magic cube to form a number cube in the original order, respectively, three orthogonal pandiagonal Latin cubes can be obtained.

In this paper, we will use a transformation T_s, whose definition can be found in [6]. Let $i_1, i_2, \cdots, i_n$ be a permutation of $1, 2, \cdots, n$. $T_s(i_1, i_2, \cdots, i_n)$ $(3 \le s \le n-1)$ is a square matrix such that the first row is

$$i_1, \cdots, i_{s-1}, i_s, i_{s+1}, \cdots, i_n$$

and the second row is

$$i_s, i_{s+1}, \cdots, i_n, i_1, \cdots, i_{s-1}$$

By using the same rule, we form third row, and so on, to obtain $T_s(i_1, i_2, \cdots, i_n)$, which is written as

$$\left.\begin{bmatrix} i_1 & i_2 & \cdots & i_s & i_{s+1} & \cdots & i_n \\ i_s & i_{s+1} & \cdots & \cdots & \cdots & \cdots & i_{s-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}\right\} n \text{ rows}$$

For example, $T_3(1,2,3,4,5)$ is given by

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 1 & 2 \\ 5 & 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 & 1 \\ 4 & 5 & 1 & 2 & 3 \end{bmatrix}$$

3. Special Case: A Pandiagonal Magic Cube of Order 11

In this section, an example of a pandiagonal magic cube of order 11 is revealed.

First, we construct a pandiagonal Latin cube of order 11. Let Figure 1 represent the integer cube. Let the first row of top surface, namely DC be

$$1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \quad 10 \quad 11$$

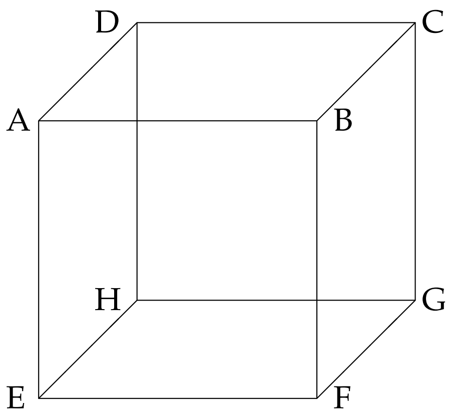

Figure 1. A Integer Cube.

From [6], in order that $DCGH$ is a pandiagonal Latin square, it suffices to select some k, $3 \leq k \leq n-1$, such that $DCGH = T_k(1, 2, \cdots, 11)$ (because $n = 11$ is a prime number). Let k = 7, i.e., let the back surface $DCGH$ be $T_3(1, 2, \cdots, 11)$:

1	2	3	4	5	6	7	8	9	10	11
3	4	5	6	7	8	9	10	11	1	2
5	6	7	8	9	10	11	1	2	3	4
7	8	9	10	11	1	2	3	4	5	6
9	10	11	1	2	3	4	5	6	7	8
11	1	2	3	4	5	6	7	8	9	10
2	3	4	5	6	7	8	9	10	11	1
4	5	6	7	8	9	10	11	1	2	3
6	7	8	9	10	11	1	2	3	4	5
8	9	10	11	1	2	3	4	5	6	7
10	11	1	2	3	4	5	6	7	8	9

Then take s, $3 \le s \le n-1$, $s \ne k$, for example, $s = 5$. Apply T_5 to each row of $DCGH$ to obtain 11 horizontal cross sections. They form the integer cube $ABCD$-$EFGH$, the top surface of which is $T_5(1, 2, \cdots, 11)$:

1	2	3	4	5	6	7	8	9	10	11
5	6	7	8	9	10	11	1	2	3	4
9	10	11	1	2	3	4	5	6	7	8
2	3	4	5	6	7	8	9	10	11	1
6	7	8	9	10	11	1	2	3	4	5
10	11	1	2	3	4	5	6	7	8	9
3	4	5	6	7	8	9	10	11	1	2
7	8	9	10	11	1	2	3	4	5	6
11	1	2	3	4	5	6	7	8	9	10
4	5	6	7	8	9	10	11	1	2	3
8	9	10	11	1	2	3	4	5	6	7

The second cross section under the top surface is $T_5(3, 4, 5, 6, 7, 8, 9, 10, 11, 1, 2)$:

3	4	5	6	7	8	9	10	11	1	2
7	8	9	10	11	1	2	3	4	5	6
11	1	2	3	4	5	6	7	8	9	10
4	5	6	7	8	9	10	11	1	2	3
8	9	10	11	1	2	3	4	5	6	7
1	2	3	4	5	6	7	8	9	10	11
5	6	7	8	9	10	11	1	2	3	4
9	10	11	1	2	3	4	5	6	7	8
2	3	4	5	6	7	8	9	10	11	1
6	7	8	9	10	11	1	2	3	4	5
10	11	1	2	3	4	5	6	7	8	9

etc. We denote the cube $ABCD$-$EFGH$ as $T_{3,5}(1, 2, \cdots, 11)$, since the above cube is constructed by T_3 and T_5.

Now, we turn to prove that the cube $T_{3,5}(1, 2, \cdots, 11)$ is a pandiagonal Latin cube of order 11. First, it should be noted that cross sections parallel to the top surface are all pandiagonal Latin squares. The back surface $DCGH$ is also a pandiagonal Latin square. It follows from symmetry of the top and back surfaces that cross sections parallel to back surface $DCGH$ are also pandiagonal Latin squares. We have to prove that cross sections parallel to $DAEH$ and the six diagonal planes are pandiagonal magic squares. The surface $DAEH$ is given by

1	5	9	2	6	10	3	7	11	4	8
3	7	11	4	8	1	5	9	2	6	10
5	9	2	6	10	3	7	11	4	8	1
7	11	4	8	1	5	9	2	6	10	3
9	2	6	10	3	7	11	4	8	1	5
11	4	8	1	5	9	2	6	10	3	7
2	6	10	3	7	11	4	8	1	5	9
4	8	1	5	9	2	6	10	3	7	11
6	10	3	7	11	4	8	1	5	9	2
8	1	5	9	2	6	10	3	7	11	4
10	3	7	11	4	8	1	5	9	2	6

This is just right $T_7(1,5,9,2,6,10,3,7,11,4,8)$. Similarly, we can prove that all cross sections parallel to $DAEH$ are created through applying T_7 to some column of top surface. Thus they are all pandiagonal Latin squares. The diagonal plane $DCFE$ is given by

1	2	3	4	5	6	7	8	9	10	11
7	8	9	10	11	1	2	3	4	5	6
2	3	4	5	6	7	8	9	10	11	1
8	9	10	11	1	2	3	4	5	6	7
3	4	5	6	7	8	9	10	11	1	2
9	10	11	1	2	3	4	5	6	7	8
4	5	6	7	8	9	10	11	1	2	3
10	11	1	2	3	4	5	6	7	8	9
5	6	7	8	9	10	11	1	2	3	4
11	1	2	3	4	5	6	7	8	9	10
6	7	8	9	10	11	1	2	3	4	5

which is $T_7(1,2,\cdots,11)$, a pandiagonal Latin square. The diagonal plane $ABGH$ is

8	9	10	11	1	2	3	4	5	6	7
6	7	8	9	10	11	1	2	3	4	5
4	5	6	7	8	9	10	11	1	2	3
2	3	4	5	6	7	8	9	10	11	1
11	1	2	3	4	5	6	7	8	9	10
9	10	11	1	2	3	4	5	6	7	8
7	8	9	10	11	1	2	3	4	5	6
5	6	7	8	9	10	11	1	2	3	4
3	4	5	6	7	8	9	10	11	1	2
1	2	3	4	5	6	7	8	9	10	11
10	11	1	2	3	4	5	6	7	8	9

We see that the above square is $T_{10}(8,9,10,1,2,3,4,5,6,7)$, a pandiagonal Latin square. The diagonal planes $ADGF$, $CBEH$, $DHFB$, and $CGEA$ are, respectively, given by

$$T_7(1,5,9,2,6,10,3,7,11,4,8)$$

$$T_4(11,4,8,1,5,9,2,6,10,3,7)$$

$$T_9(1,3,5,7,9,11,2,4,6,8,10)$$

$$T_8(11,2,4,6,8,10,1,3,5,7,9)$$

They are all pandiagonal Latin squares.

The preceding facts make clear that the cube $T_{3,5}(1,2,\cdots,11)$ is a pandiagonal Latin cube.

By the same method, we can construct $T_{6,9}(1,2,\cdots,11)$ and $T_{4,7}(1,2,\cdots,11)$. It can be shown that they are all pandiagonal Latin cubes, and that $T_{3,5}(1,2,\cdots,11)$, $T_{6,9}(1,2,\cdots,11)$, $T_{4,7}(1,2,\cdots,11)$ are orthogonal.

Now we construct cube of three-tuple $L = \left(l^1_{ijk}, l^2_{ijk}, l^3_{ijk}\right)_{i,j,k=1,2,\cdots,11}$, where $l^1_{ijk} \in T_{3,5}(1,2,\cdots,11)$, $l^2_{ijk} \in T_{6,9}(1,2,\cdots,11)$, $l^3_{ijk} \in T_{4,7}(1,2,\cdots,11)$. For example, the top surface of L is given by

$(1,1,1)$	$(2,2,2)$	$(3,3,3)$	$(4,4,4)$	$(5,5,5)$	$(6,6,6)$	$(7,7,7)$	$(8,8,8)$	$(9,9,9)$	$(10,10,10)$	$(11,11,11)$
$(5,9,7)$	$(6,10,8)$	$(7,11,9)$	$(8,1,10)$	$(9,2,11)$	$(10,3,1)$	$(11,4,2)$	$(1,5,3)$	$(2,6,4)$	$(3,7,5)$	$(4,8,6)$
$(9,6,2)$	$(10,7,3)$	$(11,8,4)$	$(1,9,5)$	$(2,10,6)$	$(3,11,7)$	$(4,1,8)$	$(5,2,9)$	$(6,3,10)$	$(7,4,11)$	$(8,5,1)$
$(2,3,8)$	$(3,4,9)$	$(4,5,10)$	$(5,6,11)$	$(6,7,1)$	$(7,8,2)$	$(8,9,3)$	$(9,10,4)$	$(10,11,5)$	$(11,1,6)$	$(1,2,7)$
$(6,11,3)$	$(7,1,4)$	$(8,2,5)$	$(9,3,6)$	$(10,4,7)$	$(11,5,8)$	$(1,6,9)$	$(2,7,10)$	$(3,8,11)$	$(4,9,1)$	$(5,10,2)$
$(10,8,9)$	$(11,9,10)$	$(1,10,11)$	$(2,11,1)$	$(3,1,2)$	$(4,2,3)$	$(5,3,4)$	$(6,4,5)$	$(7,5,6)$	$(8,6,7)$	$(9,7,8)$
$(3,5,4)$	$(4,6,5)$	$(5,7,6)$	$(6,8,7)$	$(7,9,8)$	$(8,10,9)$	$(9,11,10)$	$(10,1,11)$	$(11,2,1)$	$(1,3,2)$	$(2,4,3)$
$(7,2,10)$	$(8,3,11)$	$(9,4,1)$	$(10,5,2)$	$(11,6,3)$	$(1,7,4)$	$(2,8,5)$	$(3,9,6)$	$(4,10,7)$	$(5,11,8)$	$(6,1,9)$
$(11,10,5)$	$(1,11,6)$	$(2,1,7)$	$(3,2,8)$	$(4,3,9)$	$(5,4,10)$	$(6,5,11)$	$(7,6,1)$	$(8,7,2)$	$(9,8,3)$	$(10,9,4)$
$(4,7,11)$	$(5,8,1)$	$(6,9,2)$	$(7,10,3)$	$(8,11,4)$	$(9,1,5)$	$(10,2,6)$	$(11,3,7)$	$(1,4,8)$	$(2,5,9)$	$(3,6,10)$
$(8,4,6)$	$(9,5,7)$	$(10,6,8)$	$(11,7,9)$	$(1,8,10)$	$(2,9,11)$	$(3,10,1)$	$(4,11,2)$	$(5,1,3)$	$(6,2,4)$	$(7,3,5)$

By using this cube of 3-tuple L, we construct the integer cube $C = (c_{ijk})$ with

$$c_{ijk} = 11\left(l^1_{ijk} - 1\right) + l^2_{ijk} + 11^2\left(l^3_{ijk} - 1\right), \quad i,j,k = 1,2,\cdots,n.$$

Then C is a pandiagonal magic cube which is shown in Appendix A.

4. Main Results and Their Proofs

Our purpose is for any integer n to derive existence conditions of pandiagonal magic cubes of order n and to study a construction method by using the approach developed in Section 3. First, we construct three orthogonal pandiagonal Latin cubes, and then join them into a cube of ordered three-tuples. By replacing elements of the cube into elements of some auxiliary cube, we obtain a pandiagonal magic cube.

Let $i_1, i_2, \cdots, i_n$ be any permutation of $1,2,\cdots,n$. Without loss of generality let it just be $1,2,\cdots,n$. Otherwise, it suffices to prove the result for $1,2,\cdots,n$ first, then change j into i_j $(j = 1,2,\cdots,n)$. Since this transformation changes a permutation of $1,2,\cdots,n$ into another permutation of $1,2,\cdots,n$, the following discussion is valid in general.

Taking $1,2,\cdots,n$ as an edge, we construct an integer cube $ABCD$-$EFGH$ as follows (see Figure 1). Assume $3 \le k \le n-1$, $3 \le s \le n-1$. Let the back plane $DCGH$ be $T_k(1,2,\cdots,n)$. Apply T_s to each row of $DCGH$ to obtain n horizontal cross sections. They form the integer cube $ABCD$-$EFGH$. Denote it by $T_{k,s}(1,2,\cdots,n)$ or briefly $T_{k,s}$.

Now let us study sufficient conditions such that $T_{k,s}$ is a pandiagonal Latin cube. By using the same method as in Section 3, we easily follows that parallel cross sections of the cube $T_{k,s}$ have identical generating transformation T_t for some t. Sections parallel to pandiagonal sections of $T_{k,s}$ also have identical generating transformation $T_{t'}$ for some t'. Therefore it suffices to find conditions, such that $ABCD$, $DCGH$, $ADHE$, and six diagonal sections are pandiagonal Latin squares. For pandiagonal Latin squares, we have the following theorem.

Theorem 1. *Let n and k be integers with $3 \le k \le n-1$. Then $T_k(1,2,\cdots,n)$ is a pandiagonal Latin square of order n if, and only if,*

$$(k,n) = 1, \quad (k-1,n) = 1, \quad (k-2,n) = 1$$

Here, $(i,j) = 1$ means that i and j are mutually prime, namely, the maximum common factor of i and j is equal to 1.

When n is a prime number, the proof of this theorem can be found in [6]. The proof of general situation requires the following two preliminary facts.

Lemma 2. *If the first column of $T_k(1,2,\cdots,n)$ is a permutation of $1,2,\cdots,n$, so is every column of $T_k(1,2,\cdots,n)$.*

Proof. We regard an element of $T_k(1,2,\cdots,n)$ as itself modulo n, then $[i]=[jn+i]$ (j is an integer. We choose representative elements of congruence group of modulo n as $1,2,\cdots,n$. Hereafter suppose just so). Consider numbers in the bracket [ ]. Since

$$T_k(1,2,\cdots,n)=\begin{bmatrix} 1 & 2 & \cdots & n \\ [1+(k-1)] & [2+(k-1)] & \cdots & [n+(k-1)] \\ \cdots & \cdots & \cdots & \cdots \\ [1+(n-1)(k-1)] & [2+(n-1)(k-1)] & \cdots & [n+(n-1)(k-1)] \end{bmatrix}$$

it is clear that if 1, $[1+(k-1)]$, $\cdots$, $[1+(n-1)(k-1)]$ is a permutation of $1,2,\cdots,n$, so are i, $[i+(k-1)]$, $\cdots$, $[i+(n-1)(k-1)]$ ($i=1,2,\cdots,n$). This proves the Lemma. □

Lemma 3. *If some diagonal of $T_k(1,2,\cdots,n)$ is a permutation of $1,2,\cdots,n$, so is each pandiagonal which is parallel to the diagonal. If some pandiagonal of $T_k(1,2,\cdots,n)$ is a permutation of $1,2,\cdots,n$, so is each pandiagonal which is parallel to the pandiagonal.*

Proof. Since $[i]=[jn+i]$, we can substitute i for $[jn+i]$ for some j. The following are the first n rows of $T_k(1,2,\cdots,2n)$

$$\begin{bmatrix} [1] & [2] & \cdots & [n] & [n+1] & \cdots & [2n] \\ [2n+k] & [2n+k+1] & \cdots & [2n+k+(n-1)] & [2n+k+n] & \cdots & [2n+k+2n-1] \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ b_1 & b_2 & \cdots & b_n & b_{n+1} & \cdots & b_{2n} \end{bmatrix}$$

From the generating rule, we see that the first n columns just form $T_k(1,2,\cdots,n)$. Since $[i]=[n+i]$, $[2n+k+i]=[2n+k+i+n]$, the last n columns also form $T_k(1,2,\cdots,n)$. Hence the line from b_1 to $[n]$ is a diagonal. The line from b_2 to $[n+1]$ stands for the pandiagonal $[1]b_2\cdots[2n+k+(n-1)]$, and the line from b_n to $[2n-1]$ stands for the pandiagonal $[n-1][2n+k+n-2]\cdots b_n$. Since $b_1\cdots[n]$ is a diagonal, $(b_1+i)\cdots[n+i]$ ($i=1,\cdots,n-1$) are all pandiagonals which parallel to line $b_1\cdots[n]$. Thus, if one of them is a permutation of $1,2,\cdots,n$, then so are others. The proof for pandiagonals which are parallel to diagonal $[1]\cdots b_n$ is the same. This completes the proof of Lemma 3. □

Proof of Theorem 1. *Sufficiency.* Suppose the conditions hold. From Lemmas 2 and 3, it suffices to prove that the first column and two diagonals are permutations of $1,2,\cdots,n$. From the definition of $T_k(1,2,\cdots,n)$, its first column is

$$[1],[1+(k-1)],\cdots,[1+(n-1)(k-1)]$$

i.e., the difference of two successive elements is $k-1$. Since $(k-1,n)=1$, if $3\le k\le n-1$, the first column is a permutation of $1,2,\cdots,n$.

Similarly, the difference of two elements of the diagonal starting from [1] is $[(2n+k+1)-1]=[k]=k$. It follows at once from $(k,n)=1$ that the diagonal is a permutation of $1,2,\cdots,n$. Similar proof is applicable to another diagonal because $[(2n+k+2n-2)-2n]=k-2$ and $(k-2,n)=1$.

Necessity. Suppose one of the conditions $(k,n)=1,(k-1,n)=1,(k-2,n)=1$ does not hold. For example, let $(k-1,n)=d\neq 1$. So there exists integers p and q, such that $k-1=dq$, $n=dp$, $(p,q)=1$. Thus the first column of $T_k(1,2,\cdots,n)$ is $[1][1+(k-1)]\cdots[1+p(k-1)]\cdots[1+(n-1)(k-1)]$. Since $[1+p(k-1)]=[1+n/d\cdot dq]=[1+nq]=[1]$, the pth element is equal to the first element. This implies that the first column is not a permutation of $1,2,\cdots,n$, which is a contradiction. The other cases are treated similarly. This completes the proof of Theorem 1. □

The following is one of the main theorems of this paper.

Theorem 2. *Let $3\le k\le n-1$, $3\le s\le n-1$. Suppose that the following conditions are satisfied*

(1) $(k,n)=1$, $(k-1,n)=1$, $(k-2,n)=1$;

(2) $(s,n)=1$, $(s-1,n)=1$, $(s-2,n)=1$;
(3) $(k-s,n)=1$, $(s-k+1,n)=1$, $(k-s+1,n)=1$;
(4) $(k+s-1,n)=1$, $(k+s-2,n)=1$, $(k+s-3,n)=1$.

Then the preceding $T_{k,s}(i_1,i_2,\cdots,i_n)$ *is a pandiagonal Latin cube, where* $i_1,i_2,\cdots,i_n$ *is a permutation of* $1,2,\cdots,n$, *and vice versa.*

Proof. We show that the conditions (1)–(4) are necessary and sufficient so that each surface and each diagonal section is a pandiagonal Latin square.

Because the surface $DCGH$ is $T_k(1,2,\cdots,n)$, (1) is a necessary and sufficient condition such that $DCGH$ is a pandiagonal Latin square. Similarly, (2) is a necessary and sufficient condition such that $ABCD$ is a pandiagonal Latin square.

The left surface $ADHE$ is given by

$$\begin{bmatrix} 1 & n+s & 2n+(2s-1) & \cdots & (n-1)n+(n-1)s-(n-2) \\ n+k & n+k+(s-1) & n+k+2(s-1) & \cdots & n+k+(n-1)(s-1) \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

From Theorem 1, in order that this surface be a pandiagonal Latin square, necessary and sufficient conditions are

$$(s-1,n)=1, \quad (k-1,n)=1, \quad (k+s-2,n)=1, \quad (s-k,n)=1.$$

These are included in (1)–(4).

The diagonal section $DCFE$ is $T_{[k+s-1]}(1,2,\cdots,n)$. It is a pandiagonal Latin square if, and only if,

$$(k+s-1,n)=1, \quad (k+s-2,n)=1, \quad (k+s-3,n)=1.$$

These are exactly conditions in (4).

The diagonal section $ABGH$ is given by

$$\begin{bmatrix} (n-1)n+(n-1)s-(n-2) & (n-1)n+(n-1)s-(n-2)+1 & \cdots & n^2-(n-1)+(n-1)s \\ n+k+(n-2)(s-1) & n+k+(n-2)(s-1)+1 & \cdots & 2n-1+k+(n-2)(s-1) \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

i.e.,

$$\begin{bmatrix} n+2-s & n+3-s & \cdots & n+1-s \\ k-2s+2 & k-2s+3 & \cdots & k-2s+1 \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

It is a pandiagonal Latin square if, and only if,

$$((k-2s+2)-(n+2-s),n)=1,$$

$$((k-2s+2)-(n+3-s),n)=1,$$

$$((k-2s+3)-(n+2-s),n)=1,$$

i.e.,

$$(k-s,n)=1, \quad (s-k+1,n)=1, \quad (k-s+1,n)=1.$$

These conditions are also included in (1)–(4).

The diagonal section $BCHE$ is

$$\begin{bmatrix} n & s-1 & \cdots & (n-1)(s-1) \\ k-2 & k+s-3 & \cdots & k-2+(n-1)(s-1) \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

In order that this section be a pandiagonal Latin square, necessary and sufficient conditions are

$$(n-s+1,n)=1, \quad (n-k+2,n)=1, \quad (k-s-1,n)=1, \quad (n-k-s+3,n)=1.$$

i.e.,

$$(s-1,n)=1, \qquad (k-2,n)=1, \qquad (s-k+1,n)=1, \qquad (k+s-3,n)=1.$$

These conditions are included in (1)–(4).

The diagonal section $DHFB$ is

$$\begin{bmatrix} 1 & n+k & \cdots & (n-1)n+(n-1)k-(n-2) \\ n+s+1 & n+s+k & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

It is a pandiagonal Latin square if, and only if,

$$(n+s+1-1,n)=1, \quad (n+k-1,n)=1, \quad (s+1-k,n)=1, \quad (n+s+k-1,n)=1.$$

i.e.,

$$(s,n)=1, \qquad (k-1,n)=1, \qquad (s-k+1,n)=1, \qquad (s+k-1,n)=1.$$

These conditions are also included in (1)–(4).

The diagonal section $CGEA$ is

$$\begin{bmatrix} n & 2n-1+k & \cdots & n^2-(n-1)+(n-1)k \\ 2n-2+s & 2n-2+(s-1)+k & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

It is a pandiagonal Latin square if, and only if,

$$(k-1,n)=1, \qquad (s-2,n)=1, \qquad (k-s+1,n)=1, \qquad (s+k-3,n)=1.$$

These conditions are also included in (1)–(4).

The diagonal section $ABGH$ is

$$\begin{bmatrix} (n-1)n+(n-1)s-(n-2) & (n-1)n+(n-1)s-(n-2)+1 & \cdots & n^2-(n-1)+(n-1)s \\ n+k+(n-2)(s-1) & n+k+(n-2)(s-1)+1 & \cdots & 2n-1+k+(n-2)(s-1) \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

It is a pandiagonal Latin square if, and only if,

$$(k-s,n)=1, \qquad (s-k+1,n)=1, \qquad (k-s+1,n)=1.$$

This is the condition (3).

The diagonal section $ADGF$ is

$$\begin{bmatrix} 1 & 1+(s-1) & \cdots & 1+(n-1)(s-1) \\ k+1 & k+1+(s-1) & \cdots & k+1+(n-1)(s-1) \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

It is a pandiagonal Latin square if, and only if,

$$(s-1,n)=1, \qquad (k,n)=1, \qquad (k-s+1,n)=1, \qquad (k+s-1,n)=1.$$

These conditions are included in (1)–(4). The proof is completed. $\square$

Remark 1. *We observe that if* $T_{k,s}(i_1, i_2, \cdots, i_n)$ *is a pandiagonal Latin cube, so is* $T_{s,k}(i_1, i_2, \cdots, i_n)$. *Since conditions* (1)–(4) *of Theorem 2 are symmetric with respect to* s, k.

Corollary 1. *Let* n *be a prime number. Then if*

$$3 \leq k \leq n-1, \qquad 3 \leq s \leq n-1;$$

$$k \neq s, \quad k \neq s+1, \quad s \neq k+1;$$

$$k+s-1 \neq n, \quad k+s-2 \neq n, \quad k+s-3 \neq n,$$

are satisfied, $T_{k,s}(i_1, i_2, \cdots, i_n)$ *is a pandiagonal Latin cube.*

Proof. Immediate from Theorem 2. □

Now we give the second main theorem of this paper.

Theorem 3. *Suppose* $T_{k_i,s_i}(j_1, j_2, \cdots, j_n)$ $(i = 1, 2, 3)$ *are pandiagonal Latin cubes. They are orthogonal if, and only if, the determinant*

$$\Delta = \begin{vmatrix} s_1 & k_1 & 1 \\ s_2 & k_2 & 1 \\ s_3 & k_3 & 1 \end{vmatrix}$$

and n *are coprime, i.e.,* $(\Delta, n) = 1$.

Proof. Without loss of generality it suffices to prove that $(\Delta, n) = 1$ if, and only if, number 1, which is on the common vertex $(1, 1, 1)$ of $T_{k_i,s_i}(1, 2, \cdots, n)$ $(i = 1, 2, 3)$, cannot appear on another common place. Indeed, if 1 appears again on common place (i, j, r), then there are relations

$$1 + (i-1)(s_i - 1) + (r-1)(k_i - 1) + (j-1) \equiv 1 \qquad (mod\ n) \qquad (i = 1, 2, 3)$$

Thus there exist integers u_1, u_2, u_3, such that

$$(i-1)(s_i - 1) + (r-1)(k_i - 1) + (j-1) = u_i n \qquad (i = 1, 2, 3) \tag{1}$$

This is a system of linear equations for $(i-1), (j-1), (r-1)$. The determinant of coefficient matrix is

$$\begin{vmatrix} s_1 - 1 & k_1 - 1 & 1 \\ s_2 - 1 & k_2 - 1 & 1 \\ s_3 - 1 & k_3 - 1 & 1 \end{vmatrix} = \begin{vmatrix} s_1 & k_1 & 1 \\ s_2 & k_2 & 1 \\ s_3 & k_3 & 1 \end{vmatrix} = \Delta$$

From cyclic group theory [24], $(\Delta, n) = 1$ if, and only if, there is a unique solution of system (1) with $1 \leq i \leq n, 1 \leq j \leq n, 1 \leq r \leq n$. Since $(i, j, k) = (1, 1, 1)$ is a solution, there is no other solution. The proof is complete. □

From this theorem we obtain the following result.

Corollary 2. *Let* $T_{k_i,s_i}(1, 2, \cdots, n)$ $(i = 1, 2, 3)$ *be three pandiagonal Latin cubes of order* n. *In order that they are orthogonal, two necessary conditions are*

(i) *Not all of* $s_i + k_i$ $(i = 1, 2, 3)$ *are the same;*

(ii) *Not all of* $s_i - k_i$ $(i = 1, 2, 3)$ *are the same.*

We prove (i). If $s_1 + k_1 = s_2 + k_2 = s_3 + k_3$, then

$$\Delta = \begin{vmatrix} s_1 & k_1 & 1 \\ s_2 & k_2 & 1 \\ s_3 & k_3 & 1 \end{vmatrix} = 0$$

It is not coprime with n. The proof of (ii) is similar.

The following theorem give a method of generating orthogonal pandiagonal Latin cubes from a given pandiagonal Latin cube.

Theorem 4. *Suppose $T_{k,s}(i_1, i_2, \cdots, i_n)$ is a pandiagonal Latin cube. Choose integers α, β, $3 \le \alpha \le n-1$, $3 \le \beta \le n-1$, such that*

$$1 + \alpha(s-1) \equiv k \qquad (mod\ n) \tag{2}$$

$$1 + \beta(s-1) \equiv 2 \qquad (mod\ n) \tag{3}$$

Then $T_{k,s}(i_1, i_2, \cdots, i_n), T_{s,k}(j_1, j_2, \cdots, j_n), T_{\alpha+1,\beta+1}(k_1, k_2, \cdots, k_n)$ are orthogonal pandiagonal Latin cubes.

Proof. The proof is performed in five steps. Steps (a) to (d) show that $T_{\alpha+1,\beta+1}$ is a pandiagonal Latin cube. Step (e) shows that $T_{k,s}, T_{s,k}, T_{\alpha+1,\beta+1}$ are orthogonal. For convenience we denote $\alpha+1, \beta+1$ by α', β', respectively.

(a) We prove (1) of Theorem 2, i.e.,

$$(\alpha', n) = 1, \qquad (\alpha'-1, n) = 1, \qquad (\alpha'-2, n) = 1$$

1. If $(\alpha', n) \neq 1$, i.e., α' and n have a common factor d which is larger than 1:

$$\alpha + 1 = \alpha' = pd \tag{4}$$

$$n = qd \tag{5}$$

where p and q are positive integer, $(p, q) = 1$. Substituting (4) into (2) yields

$$k \equiv (pd-1)(s-1) + 1 \qquad (mod\ n)$$

i.e.,

$$k \equiv pd(s-1) - s + 2 \qquad (mod\ n)$$

This can be written as

$$k + s - 2 \equiv pd(s-1) \qquad (mod\ n)$$

This means d is a common factor of $k+s-2$ and n, i.e., $(k+s-2, n) \neq 1$. This contradicts assumptions. Thus, the relation $(\alpha', n) = 1$ is true.

2. If $(\alpha'-1, n) \neq 1$, then there is $d > 1$ which is a common factor of $\alpha'-1$ and n. So we obtain

$$\alpha = \alpha' - 1 = pd \tag{6}$$

$$n = qd \tag{7}$$

Substituting (6) into (2) we obtain

$$k \equiv pd(s-1) + 1 \qquad (mod\ n)$$

i.e.,

$$k - 1 \equiv pd(s-1) \qquad (mod\ n)$$

Therefore, $(k-1, n) \neq 1$. This contradicts assumptions, so that we have $(\alpha'-1, n) = 1$.

3. If $(\alpha'-2, n) \neq 1$, then there exists an integer $d > 1$, such that

$$\alpha - 1 = \alpha' - 2 = pd \tag{8}$$

$$n = qd \tag{9}$$

Substituting (8) into (2) yields

$$k \equiv (pd+1)(s-1)+1 \quad (mod\ n)$$

i.e.,

$$k-s \equiv pd(s-1) \quad (mod\ n)$$

So $(k-s,n) \neq 1$. This is a contradiction. Thus $(\alpha'-2,n)=1$.

(b) We prove (2) of Theorem 2, i.e.,

$$(\beta',n)=1, \quad (\beta'-1,n)=1, \quad (\beta'-2,n)=1$$

1. If $(\beta',n) \neq 1$, then there exists $d>1$ which is a common factor of β' and n:

$$\beta+1=\beta'=pd \tag{10}$$
$$n=qd \tag{11}$$

Substituting (10) into (3), we obtain

$$2 \equiv (pd-1)(s-1)+1 \quad (mod\ n)$$

i.e.,

$$s \equiv pd(s-1) \quad (mod\ n)$$

This means $(s,n) \neq 1$. This is a contradiction. So $(\beta',n)=1$.

2. If $(\beta'-1,n) \neq 1$, then there exists $d>1$ which is a common factor of $\beta'-1$ and n:

$$\beta=\beta'=pd \tag{12}$$
$$n=qd \tag{13}$$

Substituting (12) into (3) we obtain

$$2 \equiv pd(s-1)+1 \quad (mod\ n)$$

i.e.,

$$1 \equiv pd(s-1) \quad (mod\ n)$$

This means d is a factor of number 1, a contradiction. So $(\beta'-1,n)=1$.

3. If $(\beta'-2,n) \neq 1$, then $\beta'-2$ and n have a common factor $d>1$:

$$\beta-1=\beta'-2=pd \tag{14}$$
$$n=qd \tag{15}$$

Substituting (14) into (3) yields

$$2 \equiv (pd+1)(s-1)+1 \quad (mod\ n)$$

i.e.,

$$-s \equiv pd(s-1) \quad (mod\ n)$$

So $(s,n) \neq 1$. This is a contradiction. Thus $(\beta'-2,n)=1$.

(c) We prove (3) of Theorem 2, i.e.,

$$(\alpha'-\beta',n)=1, \quad (\alpha'-\beta'+1,n)=1, \quad (\beta'-\alpha'+1,n)=1$$

1. If $(\alpha'-\beta',n) \neq 1$, then there exists $d>1$ which is a common factor of $\alpha'-\beta'$ and n:

$$\alpha-\beta=\alpha'-\beta'=pd \tag{16}$$
$$n=qd \tag{17}$$

Substituting (16) into (2) we obtain

$$k \equiv (\beta + pd)(s-1) + 1 \qquad (mod\ n)$$

i.e.,

$$k \equiv [\beta(s-1) + 1] + pd(s-1) \qquad (mod\ n)$$

By applying (3), we obtain

$$k - 2 \equiv pd(s-1) \qquad (mod\ n)$$

This means $(k-2, n) \neq 1$. This is a contradiction. So $(\alpha' - \beta', n) = 1$.

2. If $(\alpha' - \beta' + 1, n) \neq 1$, then there exists $d > 1$ which is a common factor of $\alpha' - \beta' + 1$ and n:

$$\alpha - \beta + 1 = \alpha' - \beta' + 1 = pd \tag{18}$$

$$n = qd \tag{19}$$

Substituting (18) into (2) yields

$$k \equiv (\beta + pd - 1)(s-1) + 1 \qquad (mod\ n)$$

i.e.,

$$k \equiv [\beta(s-1) + 1] + pd(s-1) - (s-1) \qquad (mod\ n)$$

Because of (3), this can be written as

$$k \equiv 2 + pd(s-1) - (s-1) \qquad (mod\ n)$$

i.e.,

$$k + s - 3 \equiv pd(s-1) \qquad (mod\ n)$$

This means $(k+s-3, n) \neq 1$. This contradicts the assumption. So $(\alpha' - \beta' + 1, n) = 1$.

3. If $(\beta' - \alpha' + 1, n) \neq 1$, then there exists $d > 1$ which is a common factor of $\beta' - \alpha' + 1$ and n such that

$$\beta - \alpha + 1 = \beta' - \alpha' + 1 = pd \tag{20}$$

$$n = qd \tag{21}$$

Substituting (20) into (2) yields

$$k \equiv (\beta - pd + 1)(s-1) + 1 \qquad (mod\ n)$$

i.e.,

$$k \equiv [\beta(s-1) + 1] - pd(s-1) + (s-1) \qquad (mod\ n)$$

Because of (3), this can be written as

$$k \equiv 2 - pd(s-1) + (s-1) \qquad (mod\ n)$$

i.e.,

$$s - k + 1 \equiv pd(s-1) \qquad (mod\ n)$$

This means $(s-k+1,n)\neq 1$. This contradicts assumption. So $(\beta'-\alpha'+1,n)=1$.

(d) We prove (4) of Theorem 2, i.e.,

$$(\alpha'+\beta'-1,n)=1, \qquad (\alpha'+\beta'-2,n)=1, \qquad (\alpha'+\beta'-3,n)=1$$

1. If $(\alpha'+\beta'-1,n)\neq 1$, then $\alpha'+\beta'-1$ and n have a common factor $d>1$:

$$\alpha+\beta+1=\alpha'+\beta'-1=pd \tag{22}$$
$$n=qd \tag{23}$$

Substituting (22) into (2) yields

$$k\equiv (pd-\beta-1)(s-1)+1 \pmod n$$

i.e.,

$$k\equiv pd(s-1)-\beta(s-1)-(s-1)+1 \pmod n$$

From (3), we obtain

$$k+s-1\equiv pd(s-1) \pmod n$$

So $(k+s-1,n)\neq 1$. This is a contradiction. Thus $(\alpha'+\beta'-1,n)=1$.

2. If $(\alpha'+\beta'-2,n)\neq 1$, then $\alpha'+\beta'-2$ and n have a common factor $d>1$:

$$\alpha+\beta=\alpha'+\beta'-2=pd \tag{24}$$
$$n=qd \tag{25}$$

Substituting (24) into (2) yields

$$k\equiv (pd-\beta)(s-1)+1 \pmod n$$

i.e.,

$$k\equiv pd(s-1)-\beta(s-1)+1 \pmod n$$

Using (3), we obtain

$$k\equiv pd(s-1) \pmod n$$

So $(k,n)\neq 1$. This is a contradiction. Thus $(\alpha'+\beta'-2,n)=1$.

3. If $(\alpha'+\beta'-3,n)\neq 1$, then $\alpha'+\beta'-3$ and n have a common factor $d>1$:

$$\alpha+\beta-1=\alpha'+\beta'-3=pd \tag{26}$$
$$n=qd \tag{27}$$

Substituting (26) into (2) yields

$$k\equiv (pd-\beta+1)(s-1)+1 \pmod n$$

i.e.,

$$k\equiv pd(s-1)-\beta(s-1)+(s-1)+1 \pmod n$$

It follows from (3) that

$$k-s+1\equiv pd(s-1) \pmod n$$

So $(k-s+1,n)\neq 1$. This is a contradiction. Thus $(\alpha'+\beta'-3,n)=1$.

(a), (b), (c), and (d) above show that $\alpha+1, \beta+1$ satisfy the conditions of Theorem 2, so that $T_{\alpha+1,\beta+1}$ is a pandiagonal magic cube. Now we have three pandiagonal magic cubes $T_{k,s}, T_{s,k}, T_{\alpha+1,\beta+1}$.

(e) We now turn to prove the orthogonality of $T_{k,s}, T_{s,k}, T_{\alpha+1,\beta+1}$. First, construct three determinants:

$$\Delta = \begin{vmatrix} s & k & 1 \\ k & s & 1 \\ \alpha+1 & \beta+1 & 1 \end{vmatrix} = \begin{vmatrix} s-1 & k-1 & 1 \\ k-1 & s-1 & 1 \\ \alpha & \beta & 1 \end{vmatrix}$$

$$\Delta(s-1) = \begin{vmatrix} s-1 & k-1 & 1 \\ k-1 & s-1 & 1 \\ \alpha(s-1) & \beta(s-1) & s-1 \end{vmatrix}$$

$$\Delta' = \begin{vmatrix} s-1 & k-1 & 1 \\ k-1 & s-1 & 1 \\ k-1 & 1 & s-1 \end{vmatrix} = (s-k)[(s+k-2)(s-1)-k]$$

From assumptions of (2), (3), we have $(s-1,n)=1$. Therefore, we obtain

$$(\Delta, n) = 1 \Longleftrightarrow (\Delta(s-1), n) = 1 \Longleftrightarrow (\Delta', n) = 1$$

Note that $(s+k-2)(s-1)-k = (s-2)(k+s-1)$, so $((s+k-2)(s-1)-k, n) = 1$. Because $T_{k,s}(1,2,\cdots,n)$ is a pandiagonal Latin square (third condition (2) and first condition (4) in Theorem 2), $((s+k-2)(s-1)-k,n)=1$. Compounded with $(s-k,n)=1$, we have $(\Delta',n)=1$, so $(\Delta,n)=1$. From Theorem 3 it follows that $T_{k,s}, T_{s,k}, T_{\alpha+1,\beta+1}$ are orthogonal. This completes the proof. □

The following theorem give another method to construct orthogonal pandiagonal Latin cubes.

Theorem 5. *Suppose* $3 \le k \le n-1$, $3 \le s \le n-1$. *If the conditions*

(1) $(k+i,n)=1\ \ (i=-2,-1,0,1,2)$;

(2) $(s+i,n)=1\ \ (i=-2,-1,0,1,2)$;

(3) $(k-s,n)=1\ \ (s-k+1,n)=1$, $(k-s+1,n)=1$;

(4) $(k+s+i,n)=1\ \ (i=-3,-2,-1,0,1,2,3)$

are satisfied, then $T_{k,s}$, $T_{s,k}$, $T_{n-k,n-s}$, $T_{n-s,n-k}$ *are all pandiagonal Latin cubes and three of them are orthogonal.*

Proof. For $\{s,k\}$ or $\{n-k,n-s\}$, the conditions of Theorem 2 are satisfied. Then $T_{k,s}$, $T_{s,k}$, $T_{n-k,n-s}$, $T_{n-s,n-k}$ are all pandiagonal Latin cubes.

In addition,

$$\Delta = \begin{vmatrix} s & k & 1 \\ k & s & 1 \\ n-s & n-k & 1 \end{vmatrix} = 2n(k-s) - 2(k-s)(k+s),$$

and n are coprime if, and only if, $(2(k-s)(k+s),n)=1$. However, this is true because $(2,n)=1$, $(k-s,n)=1$, $(k+s,n)=1$.

Hence from Theorem 3, any three of them are orthogonal. This completes the proof. □

Now, we can construct a pandiagonal magic cube by using Natural Aux Cube. We obtain the following theorem.

Theorem 6. *Let $T_{k_i,s_i}(j_1^{(i)}, j_2^{(i)}, \cdots, j_n^{(i)}) = (l_{tuv}^{(i)})$ $(i = 1,2,3)$ be orthogonal pandiagonal Latin cubes. Construct integers cube $C = (c_{tuv})$, where*

$$c_{tuv} = n(l_{tuv}^{(1)} - 1) + l_{tuv}^{(2)} + n^2(l_{tuv}^{(3)} - 1) \quad (t, u, v = 1, 2, \cdots, n)$$

Then C is a pandiagonal magic cube.

Proof. Consider any section or diagonal S of C. According to the properties of pandiagonal Latin cubes, $l_{tuv}^{(1)}, l_{tuv}^{(2)}, l_{tuv}^{(3)}$ take the complete permutation of $1, 2, \cdots, n$ on any row, column or pandiagonal line of S. When adding the elements of C on such a row, column, or pandiagonal, there is

$$\begin{aligned}
\sum c_{tuv} &= n\left(\sum l_{tuv}^{(1)} - \sum 1\right) + \sum l_{tuv}^{(2)} + n^2\left(\sum l_{tuv}^{(3)} - \sum 1\right) \\
&= n\left(\sum_{i=1}^{n} i - n\right) + \sum_{i=1}^{n} i + n^2\left(\sum_{i=1}^{n} i - n\right) \\
&= \frac{n^2(n-1)}{2} + \frac{n(n+1)}{2} + \frac{n^3(n-1)}{2} \\
&= \frac{n(1+n^3)}{2}
\end{aligned}$$

They are all equal to magic sum, so C is a pandiagonal magic cube. □

If a set of orthogonal pandiagonal Latin cubes is given, we can construct $6(n!)^3$ pandiagonal magic squares. This is because there exist six arrangements of three pandiagonal Latin squares and there are $n!$ permutations of $1, 2, \cdots, n$.

Next, we give some examples of satisfying the Theorems and Corollaries.

Example 1. *For $n = 11$, computer studies show that there exist 23 pandiagonal Latin cubes which satisfy the conditions of Corollary 1, and there are 9870 orthogonal sets. For example, three orthogonal pandiagonal Latin cube sets are $\{T_{3,5}, T_{6,9}, T_{4,7}\}$, $\{T_{6,10}, T_{3,8}, T_{5,10}\}$, $\{T_{6,10}, T_{3,8}, T_{6,3}\}$.*

Example 2. *Suppose $n = 143$.*

Let $k_1 = 67$, $s_1 = 4$, $k_2 = 96$, $s_2 = 5$, $k_3 = 120$, $s_3 = 8$.

Then they satisfy the conditions of Theorem 3.

Example 3. *For $n = 121$, let*

$k = 29$, $s = 21$, $\alpha + 1 = 75$, $\beta + 1 = 116$.

$k = 21$, $s = 29$, $\alpha + 1 = 19$, $\beta + 1 = 14$

$k = 5$, $s = 109$, $\alpha + 1 = 28$, $\beta + 1 = 97$

These are three sets of parameters such that orthogonal pandiagonal Latin cubes satisfy the conditions of Theorem 4.

5. Further Property

The pandiagonal magic cube considered in the foregoing sections has further interesting properties.

Let $ABCD$-$EFGH$ in Figure 2 be an integer cube, and NM-KL be a pandiagonal line on the top surface $ABCD$. Using two planes which are parallel to the edge DH, we can cut $ABCD$-$EFGH$ into three parts. Let cross sections be $NMM'N'$ and $KLL'K'$. Through parallel translation, we obtain a square matrix $NMKLL'K'M'N'$ (or $KLNMM'N'L'K'$), which is called a pandiagonal plane. There are six sets of such planes, where each set has n pandiagonal planes which are parallel to each other.

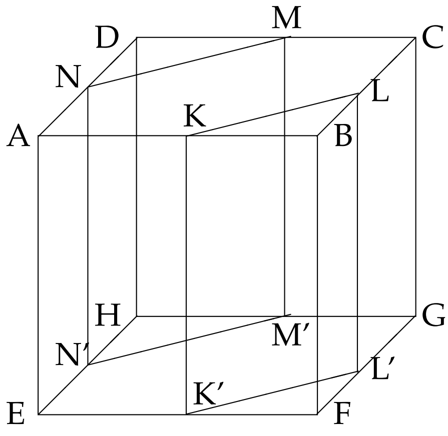

Figure 2. The Integer Cube-$ABCD$-$EFGH$.

A pandiagonal plane of a pandiagonal magic cube is also a pandiagonal magic square. In fact, let us begin with a pandiagonal Latin cube. Let $ABCD$-$EFGH$ be a pandiagonal Latin cube, then its each pandiagonal plane has the same generating transformation with the diagonal plane parallel to itself. For example, the diagonal plane $ACGE$ of $T_{3,5}(1, 2, \cdots, 11)$ of Section 3 is given by

8	5	2	10	7	4	1	9	6	3	11
10	7	4	1	9	6	3	11	8	5	2
1	9	6	3	11	8	5	2	10	7	4
3	11	8	5	2	10	7	4	1	9	6
5	2	10	7	4	1	9	6	3	11	8
7	4	1	9	6	3	11	8	5	2	10
9	6	3	11	8	5	2	10	7	4	1
11	8	5	2	10	7	4	1	9	6	3
2	10	7	4	1	9	6	3	11	8	5
4	1	9	6	3	11	8	5	2	10	7
6	3	11	8	5	2	10	7	4	1	9

A pandiagonal plane parallel to $ACGE$ is

6	3	11	8	5	2	10	7	4	1	9
8	5	2	10	7	4	1	9	6	3	11
10	7	4	1	9	6	3	11	8	5	2
1	9	6	3	11	8	5	2	10	7	4
3	11	8	5	2	10	7	4	1	9	6
5	2	10	7	4	1	9	6	3	11	8
7	4	1	9	6	3	11	8	5	2	10
9	6	3	11	8	5	2	10	7	4	1
11	8	5	2	10	7	4	1	9	6	3
2	10	7	4	1	9	6	3	11	8	5
4	1	9	6	3	11	8	5	2	10	7

They are all generated by T_4. This fact can be proved in general. Because of this and the fact that all of diagonal planes are pandiagonal magic squares, pandiagonal planes are all pandiagonal magic squares for a pandiagonal magic cube.

As an example, in Appendix B, we give a set of pandiagonal planes of the pandiagonal magic cube shown in Appendix A. They are cross sections of another pandiagonal magic cube. In other words, if $A = (a_{ijk})$ is a pandiagonal magic cube, then $B = (b_{ijk}) = (a_{j[k-j+1]i})$ (here [ ] expresses modulo n, we define that [0] = n) is also a pandiagonal magic cube. Similarly, $(b_{ijk}) = (a_{j[k+j-1]i})$, $(b_{ijk}) = (a_{[k-j+1]ij})$, $B = (b_{ijk}) = (a_{ij[k-j+1]})$,

$(b_{ijk}) = (a_{[k+j-1]ij})$, $(b_{ijk}) = (a_{ij[k+j-1]})$ are all pandiagonal magic cubes. Appendix B shows $B = (b_{ijk}) = (a_{j[k-j+1]i})$.

6. Conclusions

In this paper, we have defined a kind of magic cube called pandiagonal magic cube, whose cross sections and diagonal planes are pandiagonal magic squares. We have given a construction method for this pandiagonal magic cube. First, we construct three orthogonal pandiagonal Latin cubes, join them into a cube of three-tuples, and then replace entries by entries of some auxiliary cube so that the subscripts are these three-tuple. The last cube is a pandiagonal magic cube. If we have a pandiagonal Latin cube, Theorems 4 and 5 tell us how to construct a set of orthogonal pandiagonal Latin cubes. Some examples are also included. The construction method of pandiagonal magic cubes proposed in this paper can be extended to higher dimensions.

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Appendix A

The following are the sections of the pandiagonal magic cube generated by $T_{3,5}(1, 2, \cdots, 11)$, $T_{6,9}(1, 2, \cdots, 11)$ and $T_{4,7}(1, 2, \cdots, 11)$.

Section 1

1	134	267	400	533	666	799	932	1065	1198	1331
779	912	1045	1167	1300	102	235	247	380	513	646
215	348	481	493	626	759	881	1014	1147	1280	82
861	994	1127	1260	62	195	328	461	594	716	728
308	430	563	696	829	962	974	1107	1240	42	175
1075	1208	1220	22	144	277	410	543	676	809	942
390	523	656	789	922	1055	1188	1310	112	124	257
1157	1290	92	225	358	370	503	636	769	902	1024
604	616	738	871	1004	1137	1270	72	205	338	471
1250	52	185	318	451	573	706	839	851	984	1117
686	819	952	1085	1097	1230	32	165	287	420	553

Section 2

391	524	657	790	923	1056	1178	1311	113	125	258
1158	1291	93	226	359	371	504	637	770	892	1025
605	606	739	872	1005	1138	1271	73	206	339	472
1251	53	186	319	441	574	707	840	852	985	1118
687	820	953	1086	1098	1231	33	155	288	421	554
2	135	268	401	534	667	800	933	1066	1199	1321
780	913	1035	1168	1301	103	236	248	381	514	647
216	349	482	494	627	749	882	1015	1148	1281	83
862	995	1128	1261	63	196	329	462	584	717	729
298	431	564	697	830	963	975	1108	1241	43	176
1076	1209	1221	12	145	278	411	544	677	810	943

Section 3

781	903	1036	1169	1302	104	237	249	382	515	648
217	350	483	495	617	750	883	1016	1149	1282	84
863	996	1129	1262	64	197	330	452	585	718	730
299	432	565	698	831	964	976	1109	1242	44	166
1077	1210	1211	13	146	279	412	545	678	811	944
392	525	658	791	924	1046	1179	1312	114	126	259
1159	1292	94	227	360	372	505	638	760	893	1026
595	607	740	873	1006	1139	1272	74	207	340	473
1252	54	187	309	442	575	708	841	853	986	1119
688	821	954	1087	1099	1232	23	156	289	422	555
3	136	269	402	535	668	801	934	1067	1189	1322

Section 4

1160	1293	95	228	361	373	506	628	761	894	1027
596	608	741	874	1007	1140	1273	75	208	341	463
1253	55	177	310	443	576	709	842	854	987	1120
689	822	955	1088	1100	1222	24	157	290	423	556
4	137	270	403	536	669	802	935	1057	1190	1323
771	904	1037	1170	1303	105	238	250	383	516	649
218	351	484	485	618	751	884	1017	1150	1283	85
864	997	1130	1263	65	198	320	453	586	719	731
300	433	566	699	832	965	977	1110	1243	34	167
1078	1200	1212	14	147	280	413	546	679	812	945
393	526	659	792	914	1047	1180	1313	115	127	260

Section 5

219	352	474	486	619	752	885	1018	1151	1284	86
865	998	1131	1264	66	188	321	454	587	720	732
301	434	567	700	833	966	978	1111	1233	35	168
1068	1201	1213	15	148	281	414	547	680	813	946
394	527	660	782	915	1048	1181	1314	116	128	261
1161	1294	96	229	362	374	496	629	762	895	1028
597	609	742	875	1008	1141	1274	76	209	331	464
1254	45	178	311	444	577	710	843	855	988	1121
690	823	956	1089	1090	1223	25	158	291	424	557
5	138	271	404	537	670	803	925	1058	1191	1324
772	905	1038	1171	1304	106	239	251	384	517	639

Section 6

598	610	743	876	1009	1142	1275	77	199	332	465
1244	46	179	312	445	578	711	844	856	989	1122
691	824	957	1079	1091	1224	26	159	292	425	558
6	139	272	405	538	671	793	926	1059	1192	1325
773	906	1039	1172	1305	107	240	252	385	507	640
220	342	475	487	620	753	886	1019	1152	1285	87
866	999	1132	1265	56	189	322	455	588	721	733
302	435	568	701	834	967	979	1101	1234	36	169
1069	1202	1214	16	149	282	415	548	681	814	936
395	528	650	783	916	1049	1182	1315	117	129	262
1162	1295	97	230	363	364	497	630	763	896	1029

Section 7

867	1000	1133	1255	57	190	323	456	589	722	734
303	436	569	702	835	968	969	1102	1235	37	170
1070	1203	1215	17	150	283	416	549	682	804	937
396	518	651	784	917	1050	1183	1316	118	130	263
1163	1296	98	231	353	365	498	631	764	897	1030
599	611	744	877	1010	1143	1276	67	200	333	466
1245	47	180	313	446	579	712	845	857	990	1112
692	825	947	1080	1092	1225	27	160	293	426	559
7	140	273	406	539	661	794	927	1060	1193	1326
774	907	1040	1173	1306	108	241	253	375	508	641
210	343	476	488	621	754	887	1020	1153	1286	88

Section 8

1246	48	181	314	447	580	713	846	858	980	1113
693	815	948	1081	1093	1226	28	161	294	427	560
8	141	274	407	529	662	795	928	1061	1194	1327
775	908	1041	1174	1307	109	242	243	376	509	642
211	344	477	489	622	755	888	1021	1154	1287	78
868	1001	1123	1256	58	191	324	457	590	723	735
304	437	570	703	836	958	970	1103	1236	38	171
1071	1204	1216	18	151	284	417	550	672	805	938
386	519	652	785	918	1051	1184	1317	119	131	264
1164	1297	99	221	354	366	499	632	765	898	1031
600	612	745	878	1011	1144	1266	68	201	334	467

Section 9

305	438	571	704	826	959	971	1104	1237	39	172
1072	1205	1217	19	152	285	418	540	673	806	939
387	520	653	786	919	1052	1185	1318	120	132	254
1165	1298	89	222	355	367	500	633	766	899	1032
601	613	746	879	1012	1134	1267	69	202	335	468
1247	49	182	315	448	581	714	847	848	981	1114
683	816	949	1082	1094	1227	29	162	295	428	561
9	142	275	397	530	663	796	929	1062	1195	1328
776	909	1042	1175	1308	110	232	244	377	510	643
212	345	478	490	623	756	889	1022	1155	1277	79
869	991	1124	1257	59	192	325	458	591	724	736

Section 10

684	817	950	1083	1095	1228	30	163	296	429	551
10	143	265	398	531	664	797	930	1063	1196	1329
777	910	1043	1176	1309	100	233	245	378	511	644
213	346	479	491	624	757	890	1023	1145	1278	80
859	992	1125	1258	60	193	326	459	592	725	737
306	439	572	694	827	960	972	1105	1238	40	173
1073	1206	1218	20	153	286	408	541	674	807	940
388	521	654	787	920	1053	1186	1319	121	122	255
1166	1288	90	223	356	368	501	634	767	900	1033
602	614	747	880	1002	1135	1268	70	203	336	469
1248	50	183	316	449	582	715	837	849	982	1115

Section 11

1074	1207	1219	21	154	276	409	542	675	808	941
389	522	655	788	921	1054	1187	1320	111	123	256
1156	1289	91	224	357	369	502	635	768	901	1034
603	615	748	870	1003	1136	1269	71	204	337	470
1249	51	184	317	450	583	705	838	850	983	1116
685	818	951	1084	1096	1229	31	164	297	419	552
11	133	266	399	532	665	798	931	1064	1197	1330
778	911	1044	1177	1299	101	234	246	379	512	645
214	347	480	492	625	758	891	1013	1146	1279	81
860	993	1126	1259	61	194	327	460	593	726	727
307	440	562	695	828	961	973	1106	1239	41	174

Appendix B

The following is a set of pandiagonal planes of the pandiagonal magic cube in Appendix A. They form another pandiagonal magic cube.

Section 1

1	819	185	871	358	1055	410	1107	594	1280	646
779	134	952	318	1004	370	1188	543	1240	716	82
215	912	267	1085	451	1137	503	1310	676	42	728
861	348	1045	400	1097	573	1270	636	112	809	175
308	994	481	1167	533	1230	706	72	769	124	942
1075	430	1127	493	1300	666	32	839	205	902	257
390	1208	563	1260	626	102	799	165	851	338	1024
1157	523	1220	696	62	759	235	932	287	984	471
604	1290	656	22	829	195	881	247	1065	420	1117
1250	616	92	789	144	962	328	1014	380	1198	553
686	52	738	225	922	277	974	461	1147	513	1331

Section 2

391	1209	564	1261	627	103	800	155	852	339	1025
1158	524	1221	697	63	749	236	933	288	985	472
605	1291	657	12	830	196	882	248	1066	421	1118
1251	606	93	790	145	963	329	1015	381	1199	554
687	53	739	226	923	278	975	462	1148	514	1321
2	820	186	872	359	1056	411	1108	584	1281	647
780	135	953	319	1005	371	1178	544	1241	717	83
216	913	268	1086	441	1138	504	1311	677	43	729
862	349	1035	401	1098	574	1271	637	113	810	176
298	995	482	1168	534	1231	707	73	770	125	943
1076	431	1128	494	1301	667	33	840	206	892	258

Section 3

781	136	954	309	1006	372	1179	545	1242	718	84
217	903	269	1087	442	1139	505	1312	678	44	730
863	350	1036	402	1099	575	1272	638	114	811	166
299	996	483	1169	535	1232	708	74	760	126	944
1077	432	1129	495	1302	668	23	841	207	893	259
392	1210	565	1262	617	104	801	156	853	340	1026
1159	525	1211	698	64	750	237	934	289	986	473
595	1292	658	13	831	197	883	249	1067	422	1119
1252	607	94	791	146	964	330	1016	382	1189	555
688	54	740	227	924	279	976	452	1149	515	1322
3	821	187	873	360	1046	412	1109	585	1282	648

Section 4

1160	526	1212	699	65	751	238	935	290	987	463
596	1293	659	14	832	198	884	250	1057	423	1120
1253	608	95	792	147	965	320	1017	383	1190	556
689	55	741	228	914	280	977	453	1150	516	1323
4	822	177	874	361	1047	413	1110	586	1283	649
771	137	955	310	1007	373	1180	546	1243	719	85
218	904	270	1088	443	1140	506	1313	679	34	731
864	351	1037	403	1100	576	1273	628	115	812	167
300	997	484	1170	536	1222	709	75	761	127	945
1078	433	1130	485	1303	669	24	842	208	894	260
393	1200	566	1263	618	105	802	157	854	341	1027

Section 5

219	905	271	1089	444	1141	496	1314	680	35	732
865	352	1038	404	1090	577	1274	629	116	813	168
301	998	474	1171	537	1223	710	76	762	128	946
1068	434	1131	486	1304	670	25	843	209	895	261
394	1201	567	1264	619	106	803	158	855	331	1028
1161	527	1213	700	66	752	239	925	291	988	464
597	1294	660	15	833	188	885	251	1058	424	1121
1254	609	96	782	148	966	321	1018	384	1191	557
690	45	742	229	915	281	978	454	1151	517	1324
5	823	178	875	362	1048	414	1111	587	1284	639
772	138	956	311	1008	374	1181	547	1233	720	86

Section 6

598	1295	650	16	834	189	886	252	1059	425	1122
1244	610	97	783	149	967	322	1019	385	1192	558
691	46	743	230	916	282	979	455	1152	507	1325
6	824	179	876	363	1049	415	1101	588	1285	640
773	139	957	312	1009	364	1182	548	1234	721	87
220	906	272	1079	445	1142	497	1315	681	36	733
866	342	1039	405	1091	578	1275	630	117	814	169
302	999	475	1172	538	1224	711	77	763	129	936
1069	435	1132	487	1305	671	26	844	199	896	262
395	1202	568	1265	620	107	793	159	856	332	1029
1162	528	1214	701	56	753	240	926	292	989	465

Section 7

867	343	1040	406	1092	579	1276	631	118	804	170
303	1000	476	1173	539	1225	712	67	764	130	937
1070	436	1133	488	1306	661	27	845	200	897	263
396	1203	569	1255	621	108	794	160	857	333	1030
1163	518	1215	702	57	754	241	927	293	990	466
599	1296	651	17	835	190	887	253	1060	426	1112
1245	611	98	784	150	968	323	1020	375	1193	559
692	47	744	231	917	283	969	456	1153	508	1326
7	825	180	877	353	1050	416	1102	589	1286	641
774	140	947	313	1010	365	1183	549	1235	722	88
210	907	273	1080	446	1143	498	1316	682	37	734

Section 8

1246	612	99	785	151	958	324	1021	376	1194	560
693	48	745	221	918	284	970	457	1154	509	1327
8	815	181	878	354	1051	417	1103	590	1287	642
775	141	948	314	1011	366	1184	550	1236	723	78
211	908	274	1081	447	1144	499	1317	672	38	735
868	344	1041	407	1093	580	1266	632	119	805	171
304	1001	477	1174	529	1226	713	68	765	131	938
1071	437	1123	489	1307	662	28	846	201	898	264
386	1204	570	1256	622	109	795	161	858	334	1031
1164	519	1216	703	58	755	242	928	294	980	467
600	1297	652	18	836	191	888	243	1061	427	1113

Section 9

305	991	478	1175	530	1227	714	69	766	132	939
1072	438	1124	490	1308	663	29	847	202	899	254
387	1205	571	1257	623	110	796	162	848	335	1032
1165	520	1217	704	59	756	232	929	295	981	468
601	1298	653	19	826	192	889	244	1062	428	1114
1247	613	89	786	152	959	325	1022	377	1195	561
683	49	746	222	919	285	971	458	1155	510	1328
9	816	182	879	355	1052	418	1104	591	1277	643
776	142	949	315	1012	367	1185	540	1237	724	79
212	909	275	1082	448	1134	500	1318	673	39	736
869	345	1042	397	1094	581	1267	633	120	806	172

Section 10

684	50	747	223	920	286	972	459	1145	511	1329
10	817	183	880	356	1053	408	1105	592	1278	644
777	143	950	316	1002	368	1186	541	1238	725	80
213	910	265	1083	449	1135	501	1319	674	40	737
859	346	1043	398	1095	582	1268	634	121	807	173
306	992	479	1176	531	1228	715	70	767	122	940
1073	439	1125	491	1309	664	30	837	203	900	255
388	1206	572	1258	624	100	797	163	849	336	1033
1166	521	1218	694	60	757	233	930	296	982	469
602	1288	654	20	827	193	890	245	1063	429	1115
1248	614	90	787	153	960	326	1023	378	1196	551

Section 11

1074	440	1126	492	1299	665	31	838	204	901	256
389	1207	562	1259	625	101	798	164	850	337	1034
1156	522	1219	695	61	758	234	931	297	983	470
603	1289	655	21	828	194	891	246	1064	419	1116
1249	615	91	788	154	961	327	1013	379	1197	552
685	51	748	224	921	276	973	460	1146	512	1330
11	818	184	870	357	1054	409	1106	593	1279	645
778	133	951	317	1003	369	1187	542	1239	726	81
214	911	266	1084	450	1136	502	1320	675	41	727
860	347	1044	399	1096	583	1269	635	111	808	174
307	993	480	1177	532	1229	705	71	768	123	941

References

1. Sesiano, J. *Magic Squares*; Springer: Berlin/Heidelberg, Germany, 2019.
2. Ganapathy, G.; Mani, K. Add-on security model for public-key cryptosystem based on magic square implementation. In Proceedings of the World Congress on Engineering and Computer Science, San Francisco, CA, USA, 20–22 October 2009; Volume 1, pp. 22–27.
3. Loly, P.D. Franklin squares: A chapter in the scientific studies of magical squares. *Complex Syst. Champaign* **2007**, *17*, 143.
4. Ollerenshaw, K.; Bree, D. Most-perfect pandiagonal magic squares. *Math. Today Bull. Inst. Math. Its Appl.* **1998**, *34*, 139–143.
5. Swetz, F.J.; Luo, R. Legacy of the Luoshu. *Aust. Math. Soc.* **2008**, *184*.
6. Fucheng, L. Construction of Pandiagonal Magic Squarey of Odd Order. *Chin. J. Eng.* **1992**, *14*, 594–598.
7. Fucheng, L. Pandiagonal Magic Squares. *Chin. J. Eng.* **2001**, *23*, 6–8.
8. Keedwell, A.D.; Denes, J. *Latin Squares and Their Applications*; Elsevier: Amsterdam, The Netherlands, 2015.

9. Xu, C.X.; Lu, Z.W. Pandiagonal magic squares. *Lect. Notes Comput. Sci.* **1995**, *959*, 388–391.
10. Li, W.; Wu, D.; Pan, F. A construction for doubly pandiagonal magic squares. *Discret. Math.* **2012**, *312*, 479–485. [CrossRef]
11. Benson, W.H.; Jacoby, O. *Magic Cubes: New Recreations*; Courier Corporation: Chelmsford, MA, USA, 1981.
12. Andrews, W.S. *Magic Squares and Cubes*; Open Court Publishing Company: Chicago, IL, USA, 1917.
13. Trenkler, M. A construction of magic cubes. *Math. Gaz.* **2000**, *84*, 36–41. [CrossRef]
14. Candy, A.L. *Construction, Classification and Census of Magic Squares of Order Five*; Edwards Brothers: Ann Arbor, MI, USA, 1939.
15. Adler, A.; Li, S.Y.R. Magic cubes and Prouhet sequences. *Am. Math. Mon.* **1977**, *84*, 618–627. [CrossRef]
16. Cammann, S. The evolution of magic squares in China. *J. Am. Orient. Soc.* **1960**, *80*, 116–124. [CrossRef]
17. Abe, G. Unsolved problems on magic squares. *Discret. Math.* **1994**, *127*, 3–13. [CrossRef]
18. Loly, P.; Cameron, I.; Trump, W.; Schindel, D. Magic square spectra. *Linear Algebra Appl.* **2009**, *430*, 2659–2680. [CrossRef]
19. Lee, M.Z.; Love, E.; Narayan, S.K.; Wascher, E.; Webster, J.D. On nonsingular regular magic squares of odd order. *Linear Algebra Appl.* **2012**, *437*, 1346–1355. [CrossRef]
20. Nordgren, R.P. On properties of special magic square matrices. *Linear Algebra Appl.* **2012**, *437*, 2009–2025. [CrossRef]
21. Houa, X.D.; Lecuonab, A.G.; Mullenc, G.L.; Sellersd, J.A. On the dimension of the space of magic squares over a field. *Linear Algebra Appl.* **2013**, *438*, 3463–3475. [CrossRef]
22. Trenkler, M. An algorithm for making magic cubes. *π ME J.* **2005**, *12*, 105–106.
23. Arkin, J.; Hoggatt, V.E.; Straus, E.G. Systems of magic Latin k-cubes. *Can. J. Math.* **1976**, *28*, 1153–1161. [CrossRef]
24. Hungerford, T.W. *Algebra*; Springer Science & Business Media: Berlin/Heidelberg, Germany, 2012.