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Submanifolds of Almost-Complex Metallic Manifolds

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Abstract: Our paper aims to study the geometry of submanifolds of an almost-complex metallic manifold. We give an example of this type of manifold, and reveal the fundamental properties of structure induced on submanifolds. We establish subsets of submanifolds in almost-complex metallic manifolds, such as invariant, anti-invariant, and slant submanifolds.

Keywords: complex metallic means; almost-complex metallic structure; almost-complex metallic manifold; submanifold

MSC: 53C15; 53C40; 53C42

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1. Introduction

In differential geometry, manifolds equipped with different geometric structures, such as almost-complex (briefly AC) structures, almost-product structures, and almost-contact structures, have been investigated extensively. Spidanel defined metallic mean in [1]. Inspired by metallic mean, golden Riemannian manifolds and metallic Riemannian (briefly MR) manifolds were introduced in [2–4]. Golden Riemannian manifolds, MR manifolds, and their submanifolds have been studied widely by many authors. Various types of submanifolds of golden and MR manifolds—such as invariant, anti-invariant, slant, and lightlike—were examined in [5–15]. Recently, different types of manifolds, called the almost-poly-Norden manifold and the almost-bronze manifold, were introduced in [16] and [17], respectively. See [18–20] for several studies about poly-Norden manifolds.

Consider the equation $x^2 - ax + \frac{3}{2}b = 0$, where a and b are the real numbers satisfying $-\sqrt{6b} < a < \sqrt{6b}$ and $b \geq 0$. The equation has complex roots, as $C_{a,b} = \frac{a \pm \sqrt{a^2-6b}}{2}$. The complex numbers $C_{a,b} = \frac{a+\sqrt{a^2-6b}}{2}$ were named complex metallic means family in [21]. In particular, if $a = 1$ and $b = 1$, then the complex metallic means family $C_{a,b} = \frac{a+\sqrt{a^2-6b}}{2}$ reduces to the complex golden mean: $C_{1,1} = \frac{1+\sqrt{5}i}{2}$, $i^2 = -1$, which is a complex analog of the well-known golden mean [2]. Inspired by the complex metallic means family, the almost-complex metallic (briefly ACM) structure and the ACM manifold were introduced in [21].

In this paper, we instigate a study of submanifolds in ACM manifolds. We present fundamental properties of structure induced on submanifolds, and investigate the geometry of some subsets of submanifolds in ACM manifolds, such as invariant, anti-invariant, and slant submanifolds.

2. Preliminaries

The positive solution of

$$x^2 - ax + \frac{3}{2}b = 0$$

is called complex mean [21], which is described by

$$C_{a,b} = \frac{a + \sqrt{a^2 - 6b}}{2}, \tag{1}$$

where a and b are the real numbers satisfying $b \geq 0$ and $-\sqrt{6b} < a < \sqrt{6b}$. In [21], by using the complex mean given in (1), a new type of structure on a Riemannian manifold was defined by the authors. Let $\mathcal{M}$ be a Riemannian manifold. An ACM structure is a $(1,1)$ tensor field, $\tilde{J}_{\mathcal{M}}$ on $\mathcal{M}$, which satisfies the relation

$$\tilde{J}^2_{\mathcal{M}} - a\tilde{J}_{\mathcal{M}} + \frac{3}{2}bI = 0, \tag{2}$$

where I is the identity $(1,1)$ tensor field on $\mathcal{M}$. In this case, we say $\mathcal{M}$ is an ACM manifold equipped with an ACM structure $\tilde{J}_{\mathcal{M}}$.

Note that if we take $a = m$ and $b = \frac{2}{3}$ in (2), we obtain an almost-poly-Norden structure.

Example 1. *Conceive the* 4*-tuples real space* $\mathbb{R}^4$ *and identify a map,*

$$\tilde{J}_{\mathcal{M}} : \mathbb{R}^4 \to \mathbb{R}^4;$$

$$\tilde{J}_{\mathcal{M}}(x_1, x_2, y_1, y_2) = (C_{a,b}x_1, C_{a,b}x_2, (a - C_{a,b})y_1, (a - C_{a,b})y_2),$$

where $C_{a,b} = \frac{a+\sqrt{a^2-6b}}{2}$. *We can easily see that* $\tilde{J}_{\mathcal{M}}$ *satisfies* $\tilde{J}^2_{\mathcal{M}} - a\tilde{J}_{\mathcal{M}} + \frac{3}{2}bI = 0$. *That is,* $(\mathbb{R}^4, \tilde{J}_{\mathcal{M}})$ *is an example of ACM manifolds.*

If $(\mathcal{M}, \tilde{g})$ is a semi-Riemannian manifold given with an ACM structure, such that the metric $\tilde{g}$ is $\tilde{J}_{\mathcal{M}}-$compatible,

$$\tilde{g}(\tilde{J}_{\mathcal{M}}\tilde{X}, \tilde{J}_{\mathcal{M}}\tilde{Y}) = a\tilde{g}(\tilde{J}_{\mathcal{M}}\tilde{X}, \tilde{Y}) - \frac{3}{2}b\tilde{g}(\tilde{X}, \tilde{Y}) \tag{3}$$

equivalent to

$$\tilde{g}(\tilde{J}_{\mathcal{M}}\tilde{X}, \tilde{Y}) = \tilde{g}(\tilde{X}, \tilde{J}_{\mathcal{M}}\tilde{Y}), \tag{4}$$

for every $\tilde{X}, \tilde{Y} \in \Gamma(T\mathcal{M})$; therefore, $(\mathcal{M}, \tilde{J}_{\mathcal{M}}, \tilde{g})$ is called an almost-complex metallic semi-Riemannian (briefly ACMSR) manifold.

Proposition 1 ([21])**.** *If* $\tilde{J}_{\mathcal{M}}$ *is an ACM structure on* $\mathcal{M}$*, then*

$$J_{\pm} = \pm\left(\frac{2}{2C_{a,b} - a}\tilde{J}_{\mathcal{M}} - \frac{2a}{2C_{a,b} - a}I\right)$$

are two AC structures on $\mathcal{M}$*. Inversely, if* J *is an AC structure on* $\mathcal{M}$*, then*

$$\tilde{J}_{\mathcal{M}} = \frac{a}{2}I \pm \left(\frac{2C_{a,b} - a}{2}J\right)$$

are two ACM structures on $\mathcal{M}$*, where* $C_{a,b} = \frac{a+\sqrt{a^2-6b}}{2}$.

An ACM structure, $\tilde{J}_{\mathcal{M}}$, is called integrable if its Nijenhuis tensor field

$$N_{\tilde{J}_{\mathcal{M}}}(X, Y) = [\tilde{J}_{\mathcal{M}}X, \tilde{J}_{\mathcal{M}}Y] - \tilde{J}_{\mathcal{M}}[\tilde{J}_{\mathcal{M}}X, Y] - \tilde{J}_{\mathcal{M}}[X, \tilde{J}_{\mathcal{M}}Y] + \tilde{J}^2_{\mathcal{M}}[X, Y]$$

vanishes [21].

3. Submanifolds of Almost-Complex Metallic Manifolds

Let $\mathcal{N}$ be an n-dim submanifold of an $(n+s)$-dim ACMSR manifold $(\mathcal{M}, \tilde{J}_{\mathcal{M}}, \tilde{g})$. We express the induced Riemannian metric on $\mathcal{N}$ by g. For any $X \in \Gamma(T\mathcal{N})$ and $U \in \Gamma(T\mathcal{N}^{\perp})$, we put

$$\tilde{J}_{\mathcal{M}} X = hX + \psi X, \tag{5}$$

$$\tilde{J}_{\mathcal{M}} U = RU + SU, \tag{6}$$

where hX is the tangential part of $\tilde{J}_{\mathcal{M}} X$, ψX is the normal part of $\tilde{J}_{\mathcal{M}} X$, RU is the tangential part of $\tilde{J}_{\mathcal{M}} U$, and SU is the normal part of $\tilde{J}_{\mathcal{M}} U$.

From (4)–(6), it is easy to see that

$$g(\tilde{J}_{\mathcal{M}} X, Y) = g(X, \tilde{J}_{\mathcal{M}} Y), \ \forall X, Y \in \Gamma(T\mathcal{N}), \tag{7}$$

$$g(SU, V) = g(U, SV), \ \forall U, V \in \Gamma(TM^{\perp}).$$

In addition, ψ and R are related by

$$g(\psi X, U) = g(X, RU).$$

∇ and $\tilde{\nabla}$ are the Levi-Civita connections on $\mathcal{N}$ and $\mathcal{M}$, respectively. For any $X, Y \in \Gamma(T\mathcal{N})$ and an orthonormal basis $\{N_1, N_2, \ldots, N_s\}$ of $T\mathcal{N}^{\perp}$, where $i, j \in \{1, 2, \ldots, s\}$, the Gauss and Weingarten formulas are the following:

$$\tilde{\nabla}_X Y = \nabla_X Y + \sum_{i=1}^{s} t_i(X, Y) N_i,$$

$$\tilde{\nabla}_X N_i = -A_{N_i} X + \sum_{j=1}^{s} \tau_{ij}(X) N_j.$$

Here, A_{N_i} is the shape operator in the direction of N_i given by $g(A_{N_i} X, Y) = h_i(X, Y)$, and h_i are the second fundamental tensors.

Lemma 1. *Let $(\mathcal{M}, \tilde{J}_{\mathcal{M}}, \tilde{g})$ be an ACMSR manifold. For any $\tilde{X}, \tilde{Y}, \tilde{Z} \in \Gamma(T\mathcal{M})$, we have*

$$g((\tilde{\nabla}_{\tilde{X}} \tilde{J}_{\mathcal{M}})\tilde{Y}, \tilde{Z}) = g(\tilde{Y}, (\tilde{\nabla}_{\tilde{X}} \tilde{J}_{\mathcal{M}})\tilde{Z}).$$

Proposition 2. *Let $\mathcal{N}$ be an n-dim submanifold of an $(n+s)$-dim ACMSR manifold $(\mathcal{M}, \tilde{J}_{\mathcal{M}}, \tilde{g})$. For any $X, Y, Z \in \Gamma(T\mathcal{M})$, we have*

$$g((\nabla_X h)Y, Z) = g(Y, (\nabla_X h)Z).$$

Let the normal space $T\mathcal{N}^{\perp}$ of an n-dim submanifold $\mathcal{N}$ in an $(n+s)$-dim ACMSR $(\mathcal{M}, \tilde{J}_{\mathcal{M}}, \tilde{g})$ have an orthonormal basis such as $\{N_1, N_2, \ldots, N_s\}$. Then, for $\forall X \in \Gamma(T\mathcal{N})$, $\tilde{J}_{\mathcal{M}} X$ and $\tilde{J}_{\mathcal{M}} N_i$ $(1 \leq i \leq s)$ can be written, respectively, in the following forms:

$$\tilde{J}_{\mathcal{M}} X = hX + \sum_{i=1}^{s} \mu_i(X) N_i, \tag{8}$$

$$\tilde{J}_{\mathcal{M}} N_i = \xi_i + \sum_{j=1}^{s} \eta_{ij} N_j, \tag{9}$$

where h is a tensor field of type $(1,1)$ on $\mathcal{N}$, which modifies tangent vector field X on $\mathcal{N}$ to the tangential part of $\tilde{J}_{\mathcal{M}}X$, and μ_i are real 1-forms and ξ_i vector fields on $\mathcal{N}$; η_{ij} are differentiable valued functions on $\mathcal{N}$, equipped with an $s \times s$ matrix stated by $(\eta_{ij})_{1 \le i,j \le s}$.

As $g(\tilde{J}_{\mathcal{M}}X, N_i) = g(X, \tilde{J}_{\mathcal{M}}N_i)$, and $g(\tilde{J}_{\mathcal{M}}N_i, N_j) = g(N_i, \tilde{J}_{\mathcal{M}}N_j)$, then by using (3) and (7), we have

Lemma 2. *In a submanifold $\mathcal{N}$ of an ACMSR manifold $(\mathcal{M}, \tilde{J}_{\mathcal{M}}, \tilde{g})$, we have*

$$\mu_i(X) = g(\tilde{J}_{\mathcal{M}}X, N_i) = g(X, \xi_i), \tag{10}$$

$$g(hX, hY) = ag(X, hY) - \frac{3}{2}bg(X, Y) - \sum_{i=1}^{s} \mu_i(X)\mu_i(Y), \tag{11}$$

$$\eta_{ij} = \eta_{ji}, \tag{12}$$

$\forall X, Y \in \Gamma(T\mathcal{N})$, *where* $1 \le i, j \le s$.

Proposition 3. *Let $\mathcal{N}$ be an n-dim submanifold of an $(n+s)$-dim ACMSR manifold $(\mathcal{M}, \tilde{J}_{\mathcal{M}}, \tilde{g})$. Then, there exists a structure $(h, g, \mu_i, \xi_i, (\eta_{ij})_{s\times s})$ on $\mathcal{N}$, induced by the ACM structure of $\mathcal{M}$, which satisfies*

$$h^2X = ahX - \frac{3}{2}bX - \sum_{i=1}^{s} \mu_i(X)\xi_i, \tag{13}$$

$$\mu_i(hX) = a\mu_i(X) - \sum_{j=1}^{s} \eta_{ij}\mu_j(X), \tag{14}$$

$$\mu_j(\xi_i) = a\eta_{ij} - \frac{3}{2}b\omega_{ij} - \sum_{\lambda=1}^{s} \eta_{i\lambda}\eta_{\lambda j}, \tag{15}$$

$$h\xi_i = a\xi_i - \sum_{j=1}^{s} \eta_{ij}\xi_j,$$

$\forall X \in \Gamma(T\mathcal{N})$.

Proof. Fulfilling $\tilde{J}_{\mathcal{M}}$ to both sides of Equation (8), and applying (2), we have

$$a\tilde{J}_{\mathcal{M}}X - \frac{3}{2}bX = \tilde{J}_{\mathcal{M}}(hX) + \sum_{i=1}^{s} \mu_i(X)\tilde{J}_{\mathcal{M}}N_i,$$

which implies

$$ahX + a\sum_{i=1}^{s} \mu_i(X)N_i - \frac{3}{2}bX = h^2X + \sum_{i=1}^{s} \mu_i(hX)N_i + \sum_{i=1}^{s} \mu_i(X)\left(\xi_i + \sum_{j=1}^{s} \eta_{ij}N_j\right).$$

If we equalize the last equation's tangential and normal components, we get (13) and (14), respectively. By virtue of (9), we indite

$$g(\tilde{J}_{\mathcal{M}}N_i, \tilde{J}_{\mathcal{M}}N_j) = g(\xi_i, \xi_j) + \sum_{\lambda=1}^{s} \eta_{i\lambda}\eta_{\lambda j}. \tag{16}$$

In addition, from (3), we have

$$g(\tilde{J}_{\mathcal{M}}N_i, \tilde{J}_{\mathcal{M}}N_j) = a\eta_{ij} - \frac{3}{2}b\omega_{ij}. \tag{17}$$

Then, by applying (12), (16), and (17), we obtain (15). Lastly, taking $X = \xi_i$ in (14) gives (16). □

Proposition 4. *Let $\mathcal{N}$ be an n-dim submanifold of an $(n+s)$-dim ACMSR manifold $(\mathcal{M}, \tilde{J}_{\mathcal{M}}, \tilde{g})$. Consequently, the equations below hold:*

$$hA_{N_i}X + \nabla_X\xi_i - \sum_{j=1}^{s}\eta_{ij}A_{N_j}X - \sum_{j=1}^{s}\tau_{ij}(X)\xi_j = 0,$$

$$X(\eta_{i\lambda}) + h_\lambda(X,\xi_i) + h_j(X,\xi_\lambda) + \sum_{j=1}^{s}(\eta_{ij}\tau_{j\lambda}(X) - \eta_{j\lambda}\tau_{ij}(X)) = 0.$$

Theorem 1. *Let $\mathcal{N}$ be an n-dim submanifold of an $(n+s)$-dim ACMSR manifold $(\mathcal{M}, \tilde{J}_{\mathcal{M}}, \tilde{g})$. If h is parallel, with regard to the Levi-Civita connection on $\mathcal{N}$, and ξ_i $(1 \leq i \leq s)$ are linearly independent, then $\mathcal{N}$ is totally geodesic.*

Proof. From

$$(\nabla_X h)Y = \sum_{i=1}^{s}[g(\psi Y, N_i)A_{N_i}X + t_i(X,Y)RN_i],$$

for every $Z \in \Gamma(T\mathcal{N})$, we are able to write

$$\sum_{i=1}^{s}[g(\psi Y, N_i)g(A_{N_i}X, Z) + t_i(X,Y)g(RN_i, Z)] = 0,$$

which denotes

$$\sum_{i=1}^{s}\mu_i(Y)t_i(X,Z) = -\sum_{i=1}^{s}\mu_i(Z)t_i(X,Y),$$

by applying (10). When we replace the functions of X and Z in the above equation, we have

$$\sum_{i=1}^{s}\mu_i(Y)t_i(X,Z) = -\sum_{i=1}^{s}\mu_i(X)t_i(Y,Z).$$

If we consider that $\sum_{i=1}^{s}\mu_i(Y)t_i(X,Z)$ is both symmetric and skew-symmetric for X and Y, we have

$$\sum_{i=1}^{s}\mu_i(Y)t_i(X,Z) = 0;$$

that is:

$$\sum_{i=1}^{s}g(Y, t_i(X,Z)RN_i) = 0.$$

$RN_i = \xi_i$ $(1 \leq i \leq s)$ are linearly independent, and this completes the proof. □

3.1. Invariant Submanifolds of Almost-Complex Metallic Manifolds

Definition 1. *Let $\mathcal{N}$ be an n-dim submanifold of an $(n+s)$-dim ACMSR manifold $(\mathcal{M}, \tilde{J}_{\mathcal{M}}, \tilde{g})$. If $\tilde{J}_{\mathcal{M}}(T_p\mathcal{N}) \subset T_p\mathcal{N}$ for any point $p \in \mathcal{N}$, then $\mathcal{N}$ is called an invariant submanifold.*

Suppose $\mathcal{N}$ is an n-dim invariant submanifold of an $(n+s)$-dim ACMSR manifold. Then, from (8), we have $\mu_i = 0$ for $(1 \leq i \leq s)$. The opposite of the previous statement is also true.

Proposition 5. *Let $\mathcal{N}$ be an n-dim invariant submanifold of an $(n+s)$-dim ACMSR manifold $(\mathcal{M}, \tilde{J}_{\mathcal{M}}, \tilde{g})$. Then, the matrix $E = (\eta_{ij})_{s\times s}$ of the induced structure $(h, g, \mu_i, \xi_i, (\eta_{ij})_{s\times s})$ is an ACM matrix; that is to say:*

$$E^2 = aE - \frac{3}{2}bI_s,$$

where I_s states the unit matrix of order s.

Proof. As $\mathcal{N}$ is an n-dim invariant submanifold, then from (15), we obtain

$$\sum_{\lambda=1}^{s} \eta_{i\lambda}\eta_{\lambda j} = a\eta_{ij} - \frac{3}{2}b\omega_{ij},$$

for $1 \leq i, j \leq s$.

If we state the matrix $(\eta_{ij})_{s\times s}$ by E, we complete the proof. □

Proposition 6. *Let $(h, g, \mu_i, \xi_i, (\eta_{ij})_{s\times s})$ be the induced structure on an n-dim submanifold $\mathcal{N}$ of an $(n+s)$-dim ACMSR manifold $(\mathcal{M}, \tilde{J}_{\mathcal{M}}, \tilde{g})$. Then, $\mathcal{N}$ is an invariant submanifold if and only if the induced structure (h, g) on $\mathcal{N}$ is an ACM structure.*

Proof. Suppose that $\mathcal{N}$ is an invariant submanifold. As $\mu_i = 0$ for $(1 \leq i \leq s)$, then from (11) and (13), we can see that

$$h^2X = ahX - \frac{3}{2}bX,$$

$$g(hX, hY) = ag(X, hY) - \frac{3}{2}bg(X, Y),$$

for all $X, Y \in \Gamma(T\mathcal{N})$, which insinuates that (h, g) is an ACM structure on $\mathcal{N}$.

Inversely, if (h, g) is an ACM structure on $\mathcal{N}$, then, from (13), we write

$$\sum_{i=1}^{s} \mu_i(X)\xi_i = 0,$$

for all $X \in \Gamma(T\mathcal{N})$, which gives

$$\mu_i(X) = 0.$$

Therefore, $\mathcal{N}$ is an invariant submanifold. □

3.2. Anti-invariant Submanifolds of Almost-Complex Metallic Manifolds

Definition 2. *Let $\mathcal{N}$ be a submanifold of an ACMSR manifold $(\mathcal{M}, \tilde{J}_{\mathcal{M}}, \tilde{g})$. If $\tilde{J}_{\mathcal{M}}(T_p\mathcal{N}) \subset (T_p\mathcal{N})^{\perp}$ for any point $p \in \mathcal{N}$, then $\mathcal{N}$ is called an anti-invariant submanifold.*

Suppose that $\mathcal{N}$ is an n-dim anti-invariant submanifold of an $(n+s)$-dim ACMSR manifold $(\mathcal{M}, \tilde{J}_{\mathcal{M}}, \tilde{g})$. In this situation, from (8) and (9), for any $X \in \Gamma(T\mathcal{N})$, $\tilde{J}_{\mathcal{M}}X$ and $\tilde{J}_{\mathcal{M}}N_i$ can be expressed in the following order:

$$\tilde{J}_{\mathcal{M}}X = \sum_{i=1}^{s} \mu_i(X)N_i,$$

$$\tilde{J}_{\mathcal{M}}N_i = \xi_i + \sum_{j=1}^{s} \eta_{ij}N_j.$$

Proposition 7. *Let $\mathcal{N}$ be an n-dim anti-invariant submanifold of an $(n+s)$-dim ACMSR manifold $(\mathcal{M}, \tilde{J}_{\mathcal{M}}, \tilde{g})$. In this case, there exists a structure $(g, \mu_i, \xi_i, (\eta_{ij})_{s\times s})$ on $\mathcal{N}$, induced by the ACM structure of $\mathcal{M}$, which satisfies*

$$X = -\frac{2}{3b}\sum_{i=1}^{s}\mu_i(X)\xi_i,$$

$$\mu_i(X) = \frac{1}{a}\sum_{j=1}^{s}\eta_{ij}\mu_j(X),$$

$$\mu_j(\xi_i) = a\eta_{ij} - \frac{3}{2}b\omega_{ij} - \sum_{\lambda=1}^{s}\eta_{i\lambda}\eta_{\lambda j},$$

$$\xi_i = \frac{1}{a}\sum_{j=1}^{s}\eta_{ij}\xi_j,$$

$$g(X,Y) = -\frac{2}{3b}\sum_{i=1}^{s}\mu_i(X)\mu_i(Y),$$

for every $X, Y \in \Gamma(T\mathcal{N})$.

Proof. Proposition 7 can be proved similarly to Proposition 3. □

3.3. Slant Submanifolds of Almost-Complex Metallic Manifolds

Assume that $\mathcal{N}$ is an n-dim submanifold of an $(n+s)$-dim ACMSR manifold $(\mathcal{M}, \tilde{J}_{\mathcal{M}}, \tilde{g})$. For all $X \in \Gamma(T\mathcal{N})$, by using the Cauchy–Schwartz inequality, we can write

$$|g(\tilde{J}_{\mathcal{M}}X, hX)| \leq \|\tilde{J}_{\mathcal{M}}X\|\|hX\|.$$

Thus, there exists a function $\theta : \Gamma(T_x\mathcal{N}) \to [0, \pi]$, such that

$$g(\tilde{J}_{\mathcal{M}}X_x, hX_x) = cos\theta(X_x)\|\tilde{J}_{\mathcal{M}}X_x\|\|hX_x\|,$$

for any $x \in \mathcal{N}$ and any nonzero tangent vector $X_x \in \Gamma(T_x\mathcal{N})$. The angle $\theta(X_x)$, between $\tilde{J}_{\mathcal{M}}X_x$ and hX_x is called the Wirtinger angle of X, and it verifies

$$cos\theta(X_x) = \frac{g(\tilde{J}_{\mathcal{M}}X_x, hX_x)}{\|\tilde{J}_{\mathcal{M}}X_x\|\|hX_x\|} = \frac{\|hX_x\|}{\|\tilde{J}_{\mathcal{M}}X_x\|}.$$

Definition 3. *Let $\mathcal{N}$ be an n-dim submanifold of an $(n+s)$-dim ACMSR manifold $(\mathcal{M}, \tilde{J}_{\mathcal{M}}, \tilde{g})$. If the angle $\theta(X_x)$ between $\tilde{J}_{\mathcal{M}}X_x$ and hX_x is constant for any $x \in \mathcal{N}$ and $X_x \in \Gamma(T_x\mathcal{N})$, $\mathcal{N}$ is called the slant submanifold, and it verifies:*

$$cos\theta = \frac{g(\tilde{J}_{\mathcal{M}}X, hX)}{\|\tilde{J}_{\mathcal{M}}X\|\|hX\|} = \frac{\|hX\|}{\|\tilde{J}_{\mathcal{M}}X\|}. \tag{18}$$

The slant submanifolds in the ACMSR manifold $(\mathcal{M}, \tilde{J}_{\mathcal{M}}, \tilde{g})$, with the slant angle $\theta = 0$ and $\theta = \frac{\pi}{2}$, are the invariant and anti-invariant submanifolds, respectively. A slant submanifold $\mathcal{N}$ in $\mathcal{M}$ is called a proper slant submanifold, which is neither invariant nor anti-invariant.

Proposition 8. *Let $\mathcal{N}$ be an n-dim submanifold of an $(n+s)$-dim ACMSR manifold $(\mathcal{M}, \tilde{J}_{\mathcal{M}}, \tilde{g})$. If $\mathcal{N}$ is a slant submanifold with the slant angle θ, then for any $X, Y \in \Gamma(T_x\mathcal{N})$:*

$$g(hX, hY) = cos^2\theta\{ag(X, \tilde{J}_{\mathcal{M}}Y) - \frac{3}{2}bg(X,Y)\},$$

$$g(NX, NY) = sin^2\theta\{ag(X, \tilde{J}_{\mathcal{M}}Y) - \frac{3}{2}bg(X, Y)\}.$$

Proof. From (18),
$$cos\theta = \frac{\|hX\|}{\|\tilde{J}_{\mathcal{M}}X\|},$$
we can write
$$cos\theta^2\|\tilde{J}_{\mathcal{M}}X\|^2 = \|hX\|^2,$$
$$cos\theta^2 g(\tilde{J}_{\mathcal{M}}X, \tilde{J}_{\mathcal{M}}X) = g(hX, hX),$$
if $X + Y$ is written instead of X in the last equation, and considered together with Equation (3):
$$cos\theta^2 g(\tilde{J}_{\mathcal{M}}X + \tilde{J}_{\mathcal{M}}Y, \tilde{J}_{\mathcal{M}}X + \tilde{J}_{\mathcal{M}}Y) = g(hX + hY, hX + hY),$$
$$cos\theta^2(g(\tilde{J}_{\mathcal{M}}X, \tilde{J}_{\mathcal{M}}X) + 2g(\tilde{J}_{\mathcal{M}}X, \tilde{J}_{\mathcal{M}}Y) + g(\tilde{J}_{\mathcal{M}}Y, \tilde{J}_{\mathcal{M}}Y)) = g(hX, hX) + 2g(hX, hY) + g(hY, hY),$$
$$cos\theta^2 g(\tilde{J}_{\mathcal{M}}X, \tilde{J}_{\mathcal{M}}Y) = g(hX, hY),$$
$$cos\theta^2\{ag(X, \tilde{J}_{\mathcal{M}}Y) - \frac{3}{2}bg(X, Y)\} = g(hX, hY).$$

In addition,
$$g(\tilde{J}_{\mathcal{M}}X, \tilde{J}_{\mathcal{M}}Y) = g(hX + NX, hY + NY),$$
$$g(\tilde{J}_{\mathcal{M}}X, \tilde{J}_{\mathcal{M}}Y) = g(hX, hY) + g(NX, NY),$$
we can write
$$\begin{aligned} g(NX, NY) &= g(\tilde{J}_{\mathcal{M}}X, \tilde{J}_{\mathcal{M}}Y) - g(hX, hY) \\ &= g(\tilde{J}_{\mathcal{M}}X, \tilde{J}_{\mathcal{M}}Y) - cos\theta^2 g(\tilde{J}_{\mathcal{M}}X, \tilde{J}_{\mathcal{M}}Y) \\ &= sin\theta^2 g(\tilde{J}_{\mathcal{M}}X, \tilde{J}_{\mathcal{M}}Y) \\ &= sin\theta^2\{ag(X, \tilde{J}_{\mathcal{M}}Y) - \frac{3}{2}bg(X, Y)\}. \end{aligned}$$

□

Theorem 2. *Let $\mathcal{N}$ be an n-dim submanifold of an $(n+s)$-dim ACMSR manifold $(\mathcal{M}, \tilde{J}_{\mathcal{M}}, \tilde{g})$. $\mathcal{N}$ is slant if and only if*
$$h^2 = \epsilon(ah - \frac{3}{2}bI),$$
where $\epsilon = cos^2\theta$.

Proof. Let $\mathcal{N}$ be a slant submanifold, in this case, for all $X, Y \in \Gamma(T_x\mathcal{N})$:
$$\begin{aligned} g(h^2X, Y) &= g(h(hX), Y) \\ &= g(\tilde{J}_{\mathcal{M}}(hX), Y) \\ &= g(hX, \tilde{J}_{\mathcal{M}}Y) \\ &= g(hX, hY) \\ &= cos^2\theta(ag(X, \tilde{J}_{\mathcal{M}}Y) - \frac{3}{2}bg(X, Y)) \\ &= cos^2\theta g(a\tilde{J}_{\mathcal{M}}X - \frac{3}{2}bX, Y). \end{aligned}$$

We obtain
$$h^2X = cos^2\theta(a\tilde{J}_{\mathcal{M}}X - \frac{3}{2}bX, Y)$$
thus, $\epsilon = cos^2\theta$.

Conversely, let there exist a $\epsilon \in [0,1]$, such that $h^2 = \epsilon(ah - \frac{3}{2}bI)$. Then, for any $X \in \Gamma(T_x\mathcal{N})$:

$$\begin{aligned} cos\theta &= \frac{g(\tilde{J}_{\mathcal{M}}X, hX)}{\|\tilde{J}_{\mathcal{M}}X\|\|hX\|} \\ &= \frac{g(X, h^2X)}{\|\tilde{J}_{\mathcal{M}}X\|\|hX\|} \\ &= \frac{g(X, \epsilon(a\tilde{J}_{\mathcal{M}} - \frac{3}{2}bI)X)}{\|\tilde{J}_{\mathcal{M}}X\|\|hX\|} \\ &= \epsilon\frac{ag(X, \tilde{J}_{\mathcal{M}}X) - \frac{3}{2}bg(X,X)}{\|\tilde{J}_{\mathcal{M}}X\|\|hX\|} \\ &= \epsilon\frac{g(\tilde{J}_{\mathcal{M}}X, \tilde{J}_{\mathcal{M}}Y)}{\|\tilde{J}_{\mathcal{M}}X\|\|hX\|} \\ &= \epsilon\frac{\|\tilde{J}_{\mathcal{M}}X\|}{\|hX\|}. \end{aligned}$$

Therefore, we obtain $\epsilon = cos^2\theta = constant$. This completes the proof. □

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