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Review

A Systematic Review on the Solution Methodology of Singularly Perturbed Differential Difference Equations

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Abstract: This review paper contains computational methods or solution methodologies for singularly perturbed differential difference equations with negative and/or positive shifts in a spatial variable. This survey limits its coverage to singular perturbation equations arising in the modeling of neuronal activity and the methods developed by numerous researchers between 2012 and 2022. The review covered singularly perturbed ordinary delay differential equations with small or large negative shift(s), singularly perturbed ordinary differential–differential equations with mixed shift(s), singularly perturbed delay partial differential equations with small or large negative shift(s) and singularly perturbed partial differential–difference equations of the mixed type. The main aim of this review is to find out what numerical and asymptotic methods were developed in the last ten years to solve such problems. Further, it aims to stimulate researchers to develop new robust methods for solving families of the problems under consideration.

Keywords: singularly perturbed problems; differential–difference equations; systematic review

MSC: 34K10; 34K27; 65L03; 65L11; 65M06; 65M20; 65M50; 65M60

Citation: Duressa, G.F.; Daba, I.T.; Deressa, C.T. A Systematic Review on the Solution Methodology of Singularly Perturbed Differential Difference Equations. *Mathematics* **2023**, *11*, 1108. https://doi.org/10.3390/math11051108

Academic Editor: Yury Shestopalov

Received: 7 February 2023
Revised: 19 February 2023
Accepted: 20 February 2023
Published: 22 February 2023

1. Background of the Problem

Many scientific problems describe the relations between causes and their effects. The study of this relation in the subject of the perturbations theory has a long history [1]. Despite this long history, the topic is still in a state of irrepressible development and is termed as the theory of singular perturbation problems (SPPs). The SPPs containing a small parameter value (say ε), where $0 < \varepsilon \ll 1$ received remarkable attention from mathematicians and physicists. Scholars working on the solution methodologies of SPPs have carried out numerous studies and reviews. A survey on the asymptotic and numerical methods for solving SPPs was conducted by [2]. In 2002, ref. [3] reviewed the work of numerous researchers in SPPs from 1984 to 2000. Ref. [4] reviewed solution methodology for singularly perturbed partial differential equations. Ref. [5] carried out a survey on computational techniques for solving singularly perturbed boundary value problems. Ref. [6] briefly reviewed the computational methods developed to solve various classes of SPPs. Ref. [7] reviewed the development of computational methods for solving singularly perturbed (SP) boundary value problems. Ref. [8] discussed the numerical analysis of singularly perturbed convection–diffusion–reaction problems that appeared in 2008–2012, mainly focused on layer-adapted meshes. Ref. [9] reviewed singularly perturbed differential equations with turning point and interior layers. Ref. [10] discussed the review of singularly perturbed delay differential equations. This systematic review briefly assesses the solution methodologies on singularly perturbed differential–difference equations (SPDDEs).

The differential equations in which the highest order derivative is multiplied by a small positive parameter and contains a delay parameter (negative shift) and/or advance parameter(positive shift) is known as a singularly perturbed differential–difference equation

(SPDDE). Such problems frequently arise in modeling biosciences, vibrational models in control theory, physiological processes, diseases, economics, engineering, and so on. A few application fields are the mathematical modeling of population dynamics [11], immune response [12], variational problem in control theory [13,14], model of HIV infection [15–17], activation of neuronal variability [17,18], modeling of biological oscillators [19], mathematical ecology [20], models for physiological processes [21,22], evolutionary biology [23], neuronal variability [24], and others.

Due to the dual presence of singular perturbation (ε) and shift arguments in the SPDDEs, it is very difficult to obtain oscillation-free solutions on a uniform mesh unless using specially designed meshes. A thoughtful examination of the results from the conventional numerical methods, such as the finite difference method (FDM), the finite element method (FEM), the finite volume method(FVM), the spline method, and other methods, on uniform meshes as $\varepsilon \to 0$ fails for a satisfactory numerical solution, and the truncation error becomes unbounded unless a large number of mesh points or adaptive layer mesh is used in the approximation process [25]. This shows that the classical numerical method is computationally costly and inefficient. Sometimes, the increase in mesh points also causes the resulting systems of algebraic equations to be ill-conditioned. This drawback motivates researchers to develop robust numerical methods for SPDDEs. In this context, the fitted operator method (FOM) and fitted mesh method (FMM) are popular techniques to overcome the drawbacks of classical numerical methods. For more details about FOMs and FMMs, refer to the books and articles [18,25–27] and the references therein.

In this review, we investigate the solution methodology for the class of singularly perturbed delay ordinary differential equations (SPDODEs) with small or large negative shift(s), singularly perturbed ordinary differential–differential equations (SPODDEs) with mixed shift(s), singularly perturbed delay partial differential equations(SPDPDEs) with small or large negative shift(s), and singularly perturbed partial differential–difference equations (SPPPDDEs) of the mixed type that were solved from 2012 to 2022 using different numerical and asymptotic methods.

2. Models Depicting Singular Perturbation of Difference–Differential Problems

Several real-life problems are described by singularly perturbed differential–difference with mixed shifts of which the following are the major ones to consider for this particular study.

2.1. The Modeling of the Activation of a Neuron [28]

The authors in [28] generalized the Stein's model in terms of SPODDEs to consider the time evolution trajectories of the membrane potential:

$$\frac{\gamma^2}{2}u''(x) + (\lambda - x)u''(x) + \sigma_e u(x + a_e) + \sigma_i u(x - a_i - (\sigma_e + \sigma_i)u(x) = -1,$$

subject to the boundary conditions:

$$u(x) = 0, x \notin (x_1, x_2),$$

where the values $x = x_1$ and $x = x_2$ relate to the inhibitory reversal potential and to the threshold value of membrane potential for action potential generation, respectively, and to the non-derivative terms related to excitatory and inhibitory synaptic inputs.

2.2. Neuronal Variability [29]

The authors in [29] generalized the Stein's model and proposed the following mathematical model in terms of SPPPDDEs to consider the time evolution trajectories of the membrane potential:

$$-\frac{\partial z}{\partial t} = \frac{\gamma^2}{2}\frac{\partial^2 z}{\partial x^2} + (\sigma D - \frac{x}{\lambda})\frac{\partial z}{\partial x} + \tau_s z(x + a_s, t) + w_s z(x + i_s, t) - (\tau_s + w_s)z(x, t),$$

where the non-derivative terms are allied to the superposition of excitatory and inhibitory inputs.

3. Criteria for Including Studies and Selection Procedure

3.1. Literature Search

The relevant studies were identified by the use of electronic databases: Web of Science, SCOPUS, and PubMed. In addition, the relevant articles were collected from different Internet sources via Google Scholar, ResearchGate, and Sci-hub, library genius. Whenever possible, search organizers were used to align the initial survey results more thoroughly with the eligibility criteria. For example, studies written in English and published on SCOPUS/Web of Science (SCIE/SSCI) indexed journals during the years 2012 to 2022 were included. The search was completed in 30 January 2022.

3.2. Screening Process

Electronic and manual searches identified 496 potentially relevant studies and screened for retrieval via title, abstract, keywords, and references. Then, 78 studies included in this review were screened according to the screening criteria given in Table 1. Figure 1 shows various steps in the process of selecting studies and depicting how we ended up with the 78 original studies we further analyzed.

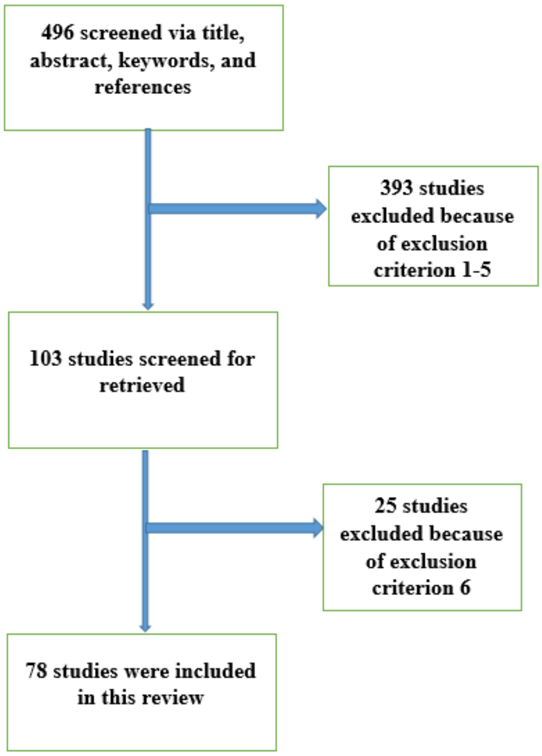

Figure 1. Review inclusion flowchart.

Table 1. Inclusion and exclusion criteria of studies included in the review.

Criterion	Include	Exclude
1. Studies focusing on	SPODDEs with small or large delay, small mixed shifts and small delays	SPODDEs without shift(s)
2. Studies focusing on	SPPPDDEs with small or large delay, small mixed shifts and small delays	SPPPDDEs without shift(s)
3. Boundary conditions	Dirichlet BC	Non-Dirichlet BC
4. Publication year	2012–2022	Before 2012
5. Language	English	Non-English
6. Indexed	on SCOPUS/Web of science /PubMed	not SCOPUS/Web of science /PubMed

4. Developments toward Solution Methodology for SPDDEs

In this paper, we discuss a survey in chronological order on the asymptotic and numerical treatment of SPDDEs of both ordinary and partial differential equations. For the sake of convenience, we divide this survey into twelve parts chronologically. We would like to apologize if there are any omissions, which are totally unintentional.

4.1. Developments toward Solution Methodology for SPODDEs

In this subsection, we give a brief description of the developed numerical methods for SPODDE of the form on the domain $D = (-1, 1)$:

$$-\varepsilon^2 \frac{d^2u(x)}{dx^2} + a(x)\frac{du(x)}{dx} + b(x)u(x-\delta) + c(x)u(x,) + d(x)u(x+\eta) = f(x), x \in D, \quad (1)$$

subject to the following interval boundary conditions (IBCs):

$$\begin{aligned} u(x) &= \varsigma_1(x), \ x \in [-\delta, 0], \\ u(x) &= \varsigma_2(x), \ x \in [1, 1+\eta], \end{aligned} \quad (2)$$

where $0 < \varepsilon \ll 1$ is a singular perturbation parameter, δ is delay, and η is an advance parameter satisfying either $\delta, \eta \leq \varepsilon$ or $\delta, \eta \geq \varepsilon$. For the existence and uniqueness of the solution, the functions $a(x), b(x), c(x), d(x), u(x), \varsigma_1(x)$, and $\varsigma_2(x)$ are assumed to be sufficiently smooth and bounded with $b(x) + c(x) + d(x) \geq \theta > 0$ for all $x \in \overline{D}$ and for some positive constant θ.

As can be seen from Tables 2 and 3, most of the solution methodologies presented to solve problems (1) and (2) were developed based on uniform mesh except for three works presented by [30–33]. Further, the majority of the methods are finite-difference-based. This implies that it is possible to think of other alternative techniques to solve the governing equation of the problem under consideration on either uniform or adaptive mesh discretization techniques.

Table 2. Various methods and mesh used to solve Equations (1) and (2).

Author(s)	Solution Methodology	Meshes
[34]	Exponentially fitted FDM based on Il'in-Allen-Southwell fitting	Specially designed mesh
[30]	Fitted modified upwind finite difference method	Uniform mesh
[35]	Collocation in combination with matrices of Fibonacci polynomials	Uniform mesh
[36]	Domain decomposition method	Uniform mesh
[37]	Asymptotic-numerical method	Uniform mesh
[38]	Fitted non-standard finite difference method	Uniform mesh
[39]	Galerkin method with exponential fitting	Uniform mesh
[40]	Fourth order finite difference method	Uniform mesh
[41]	Mixed FDM via domain decomposition	Uniform mesh
[42]	New exponentially fitted three term finite difference scheme	Uniform mesh
[43]	Fourth-order Runge–Kutta method	Uniform mesh
[44]	Numerical integration scheme using non polynomial interpolation function	Uniform mesh
[45]	Exponentially fitted non-standard FDM	Uniform mesh

Table 2. *Cont.*

Author(s)	Solution Methodology	Meshes
[46]	Exponentially fitted operator finite difference method with Richardson extrapolation	Uniform mesh
[47]	Hybrid of the midpoint upwind FDM and the central FDM	Piecewise uniform Shishkin mesh
[31]	Hybrid finite difference scheme with the cubic spline	Piecewise uniform Shishkin mesh

Table 3. Various methods and mesh used to solve Equations (1) and (2).

Author(s)	Solution Methodology	Meshes
[32]	FDM	Uniform mesh
[48]	Finite element method	Bakhvalov-S-mesh
[33]	Non-standard FDM	Uniform mesh
[49]	Finite difference approach with a parametric spline	Uniform mesh
[50]	Fitted non-polynomial spline approach	Uniform mesh
[51]	Successive complementary expansion method (SCEM)	Uniform mesh
[52]	Haar wavelet collocation method	Uniform mesh

4.2. Developments toward Solution Methodology for SP Convection Diffusion Problem with Large Shift in Space

In this subsection, we want to look at the static SPP given by

$$-\varepsilon\frac{d^2u(x)}{dx^2} - b(x)\frac{du(x)}{dx} + c(x)u(x) + d(x)u(x-1) = f(x), x \in (0,2), \tag{3}$$

subject to the following boundary conditions (BCs):

$$u(2) = 0,\ u(x) = \phi(x), x \in (-1,0], \tag{4}$$

where $0 < \varepsilon \ll 1, b(x) \geq \beta > 0, d \geq 0, c - \frac{b' L_\infty(1,2)}{2} \geq \gamma > 0$.

The results in Table 4 reveal that all the methods developed to solve the problem under consideration in Equations (3) and (4) followed a uniform mesh discretization approach. However, only scholars in [53] applied the nonuniform or adaptive mesh approach, particularly the Shshikin mesh technique. It is also observable from the results in the table that very few methods have been developed to solve the problem under consideration in Equations (3) and (4). Hence, the solution methodology development for the problem is at its infant stage.

Table 4. Various methods and mesh used to solve Equations (3) and (4).

Author(s)	Solution Methodology	Meshes
[53]	Asymptotic initial value technique (AIVT)	Piece-wise uniform Shishkin mesh
[54]	Exponentially fitted FDM	Uniform mesh
[55]	Fourth FDM	Uniform mesh
[56]	Cubic spline in compression method	Uniform mesh

4.3. Developments toward Solution Methodology for SP Reaction Diffusion Problem with Large Shift in Space

In this subsection, we want to look at the static SPP given by

$$-\varepsilon^2 \frac{d^2u(x)}{dx^2} + a(x)u(x) + b(x)u(x-1) = f(x), x \in (0,2), \tag{5}$$

with IBCs:

$$u(2) = L, u(x) = \phi(x), x \in (-1,0], \tag{6}$$

where $0 < \varepsilon \ll 1, a(x) \geq \alpha > 0, \beta_0 \leq b(x) \leq \beta < 0, \alpha + \beta_0 \geq \eta > 0,\ \forall x \in [0,2]$.

The results presented in Table 5 belong to the solution methodologies developed for Equations (5) and (6). It reveals that most numerical methods developed were based on nonuniform discretization techniques, namely Shihikin-, Bakhavlov-, and Shishkin–Bakhavlov-type discretization techniques. Furthermore, all the methods developed are based on finite difference approximation techniques except [57], a finite element method.

Table 5. Various methods and mesh used to solve Equations (5) and (6).

Author(s)	Solution Methodology	Meshes
[57]	Non-symmetric discontinuous Galerkin FEM	Shishkin polynomial Shishkin(pS) Bakhvalov–Shishkin (BS) modified Bakhvalov–Shishkin (mBS-) mesh
[58]	Hybrid finite difference scheme	Piece-wise uniform Shishkin mesh
[59]	Exponentially fitted numerical scheme via domain decomposition	Uniform mesh
[60]	Classical FDM	Piece-wise uniform Shishkin mesh
[61]	Numerov FDM	Uniform mesh
[62]	Iterative method	Shishkin mesh and Bakhvalov Shishkin mesh (BS mesh).
[63]	Numerov method	Uniform Mesh.
[64]	Central FDM	Uniform Mesh.

4.4. Developments toward Solution Methodology for SP Reaction Diffusion Problem with Small Shift

In this subsection, we want to look at the static SPP given by

$$-\varepsilon \frac{d^2u(x)}{dx^2} + a(x)u(x-\delta) + b(x)u(x) = f(x), x \in (0,1), \tag{7}$$

subject to the following IBCs:

$$u(1) = L, \quad u(x) = \phi(x), -\delta \leq x \leq 0. \tag{8}$$

Table 6 consists of the results of the solution methodology developed for the SP reaction–diffusion problem given in Equations (7) and (8). As can be seen from the review result, very few numerical methods have been developed to solve the problems described by the governing Equations (7) and (8), and they are all based on uniform discretization techniques. The methods are mainly finite difference methods and numerical integration techniques. Hence, one can look for finite elements and other quadrature techniques based on adaptive mesh approaches.

Table 6. Various methods and mesh used to solve Equations (7) and (8).

Author(s)	Solution Methodology	Meshes
[65]	Non-polynomial cubic spline method	Uniform mesh
[55]	Fourth FDM	Uniform mesh
[66]	Fourth order exponentially FDM	Uniform mesh
[67]	Trapezoidal rule	Uniform mesh
[68]	Simpson rule	Uniform mesh

4.5. Developments toward Solution Methodology for SP Convection Diffusion Problem with Negative Shift

In this subsection, we review numerical schemes developed for SPP given by

$$-\varepsilon\frac{d^2u(x)}{dx^2}+a(x)\frac{du(x-\delta)}{dx}+b(x)u(x)=f(x),x\in(0,1), \tag{9}$$

with IBCs:

$$u(1)=A,\;\; u(x)=\psi(x),\;\; -\delta\leq x\leq 0. \tag{10}$$

Table 7 reveals that only a few finite-difference-based solution methodologies on a uniform mesh discretization approach were generally developed to solve the problem of the family of SPDDEs described by the governing equation in (9) and (10). This implies that this area needs the attention of scholars working in this and related research areas.

Table 7. Various methods and mesh used to solve Equations (9) and (10).

Author(s)	Solution Methodology	Meshes
[69]	Exponential spline method	Uniform mesh
[70]	Non-polynomial spline method	Uniform mesh
[71]	Novel FDM	Uniform mesh

4.6. Developments toward Solution Methodology for SP Convection Diffusion Problem with Negative Shift

In this subsection, we want to look at the static SPP given by

$$-\varepsilon\frac{d^2u(x)}{dx^2}+a(x)\frac{du(x)}{dx}+b(x)u(x-\delta)=f(x),x\in(0,1), \tag{11}$$

with IBCs:

$$u(1)=\beta,\;\; u(x)=\phi(x),\;\; -\delta\leq x\leq 0. \tag{12}$$

As seen from Table 8, the family of SP problems described by the governing equation in (11) and (12) is solved by using various numerical methods, namely fitted operator finite difference methods, spline intention methods, and new Lioville–Green transform methods. Furthermore, all the techniques were developed based on uniform mesh discretization techniques. Like others, finite element approaches and adaptive mesh techniques can be considered an alternative to solve the problem.

Table 8. Various methods and mesh used to solve Equations (11) and (12).

Author(s)	Solution Methodology	Meshes
[72]	Tension splines method	Uniform mesh
[73]	New Liouville–Green Transform method	Uniform mesh
[74]	Exponentially fitted spline method	Uniform mesh
[75]	Exponentially fitted FDM	Equidistant mesh

4.7. Developments toward Solution Methodology for SPODDE with Negative Shifts

In this subsection, we give a brief description of the developed numerical methods for SPODDE of the form :

$$-\varepsilon\frac{d^2u(x)}{dx^2}+a(x)\frac{du(x-\delta)}{dx}+b(x)u(x-\delta)+c(x)u(x)=f(x),x\in(0,1), \tag{13}$$

subject to IBCs:

$$u(x)=\phi(x),\ x\in[-\delta,0],\ \ u(1)=\psi(1). \tag{14}$$

Table 9 summarizes the solution methodologies developed to solve SP problems involving small negative shifts both in convection and reaction terms given in (13) and (14). Almost all the developed methods are the families of fitted operator finite difference methods except [76], the B-spline collocation method. In this case, the peculiarity is that some of the fitted operator methods are developed on layer adaptive meshes, which are unique approaches.

Table 9. Various methods and mesh used to solve Equations (13) and (14).

Author(s)	Solution Methodology	Meshes
[77]	Non-standard mid-point upwind FDM, Standard mid-point upwind FDM, Non-standard mid-point upwind FDM	Uniform mesh, Shiskin mesh, Shiskin mesh
[78]	Exponentially fitted operator Mid-point upwind FDM	Uniform mesh
[79]	Exponentially fitted upwind FDM with Richardson extrapolation technique	Uniform mesh
[80]	Central FDM	Uniform mesh
[76]	B-spline collocation method	Piecewise uniform Shishkin mesh

4.8. Developments toward Solution Methodology for SPPPDDEs with Mixed Shifts

In this subsection, we give a brief description of the developed numerical methods for SPPPDDE of the form on the domain $\Im=\Im_x\times\Im_t=(0,1)\times(0,T]$ for some fixed number $T>0$:

$$\begin{aligned}&\frac{\partial w(x,t)}{\partial t}-\varepsilon^2\frac{\partial^2 w(x,t)}{\partial x^2}+\omega(x)\frac{\partial w(x,t)}{\partial x}+\varpi(x)\zeta(x-\delta,t)+\varrho(x)w(x,t)+\varphi(x)w(x+\eta,t)\\&=f(x,t),(x,t)\in\Im,\end{aligned} \tag{15}$$

subject to the following initial interval boundary conditions (I-IBCs):

$$\begin{aligned}&w(x,0)=w_0(x),x\in\overline{\Im}_x,\\&w(x,t)=w_1(x,t),\ -\delta\le x\le 0,t\in(0,T],\\&w(x,t)=w_2(x,t),\ 1\le x\le 1+\eta,t\in(0,T],\end{aligned} \tag{16}$$

where $0<\varepsilon\ll 1$, δ is the delay, and η is the advance parameter satisfying either $\delta,\eta\le\varepsilon$ or $\delta,\eta\ge\varepsilon$.

Tables 10 and 11 summarize the solution methodologies for singularly perturbed families of partial differential–difference equations given in (15) and (16). The majority of the methods developed are mainly from the families of finite difference methods except for a few methods, namely [81–86]. From the point of view of the discretization techniques, almost all have used either the implicit Euler method or the Crank–Nicholson method for temporal discretization, whereas both uniform and nonuniform are used for spatial mesh discretization.

Table 10. Various methods and mesh used to solve Equations (15) and (16).

Author(s)	Numerical Scheme: Temporal Direction	Spatial Direction	Meshes
[87]	Implicit Euler method	FDM	Uniform mesh
[88]	Implicit Euler method	Non-standard FDM	Special type of mesh
[87]	Implicit Euler method	Combined FDM made out of modified upwind and central difference schemes	Uniform mesh
[89]	Crank–Nicolson FDM	Midpoint upwind FDM	Piecewise-uniform Shishkin mesh
[90]	Implicit Euler FDM	Hybrid of midpoint upwind FDM and classical central FDM	Piecewise-uniform Shishkin mesh
[91]	Backward Euler formula	Exponentially fitted FDMs	Uniform mesh
[92]	Implicit Runge–Kutta method	Non-standard FDM	Uniform mesh
[81]	Implicit Euler method	Extended cubic B-spline basis functions	Uniform mesh
[93]	Implicit Euler method	Exponentially fitted operator FDM	Uniform mesh
[94]	Backward Euler method	New FDM	Uniform mesh
[95]	Implicit Euler method	Central FDM	Uniform mesh

Table 11. Various Methods and Mesh used to solve Equations (15) and (16).

Author(s)	Numerical Scheme: Temporal Direction	Spatial Direction	Meshes
[96]	Crank–Nicolson method	Quadratic B-spline collocation method	Exponentially graded
[82]	Implicit Euler method	Specially designed FDM	Uniform mesh
[97]	Implicit Euler method	Hybrid computational method consisting of midpoint upwind FDM and cubic spline in tension method	Piecewise-uniform Shishkin mesh
[83]	Crank–Nicolson method	Non-standard FDM	Uniform mesh
[84]	Crank–Nicolson method	Modified cubic B-spline basis functions	Shishkin mesh
[85]	Implicit Euler method	Cubic B-collocation method	Uniform mesh
[86]	Implicit Euler method	Cubic spline in tension method	Uniform mesh

4.9. Developments toward Solution Methodology for SPDPDEs with Large Delay in Space

In this subsection, we give a brief description of the developed numerical methods for SPDPDEs of the form on the domain $D = \Omega_x \times \Omega_t = (0,2) \times (0,T]$ for some fixed number $T > 0$:

$$\frac{\partial y(x,t)}{\partial t} - \varepsilon \frac{\partial^2 y(x,t)}{\partial x^2} + r(x)y(x,t) + s(x)y(x-1,t) = g(x,t), (x,t) \in D, \tag{17}$$

subject to the following I-IBCs:

$$\begin{aligned} &y(x,0) = y_0(x), x \in \overline{\Omega}_x, \\ &y(x,t) = \phi(x,t), \ -1 \le x \le 0, t \in (0,T], \\ &y(2,t) = \psi(2,t), \ t \in (0,T]. \end{aligned} \tag{18}$$

As can be seen from Table 12, only three types of solution methodologies have been developed for the SP reaction–diffusion partial differential equation with a large negative shift given by the governing equation in (17) and (18). All are designed on an adaptive mesh discretization approach which guarantees the parameter uniformity of the methods.

Table 12. Various methods and mesh used to solve Equations (17) and (18).

Author(s)	Numerical Scheme: Temporal Direction	Spatial Direction	Meshes
[98]	Implicit Euler method	Central FDM	Piecewise-uniform Shishkin mesh
[99]	Crank–Nicolson method	FDM	Piecewise-uniform Shishkin mesh
[100]	Discontinuous Galerkin method	β-weighted continuous Galerkin FEM	Duran- and S-type meshes

4.10. Developments toward Solution Methodology for SPDPDEs with Small Negative Shift in Space

In this subsection, we give a brief description of the developed numerical methods for SPDPDEs of the form on the domain $\Lambda = \Gamma_x \times \Gamma_t = (0,1) \times (0,T]$ for some fixed number $T > 0$:

$$\frac{\partial z(x,t)}{\partial t} - \varepsilon\frac{\partial^2 z(x,t)}{\partial x^2} + a(x)\frac{\partial z(x,t)}{\partial x} + r(x)z(x,t) + s(x)z(x-\delta,t) = h(x,t), (x,t) \in \Lambda, \tag{19}$$

subject to the following I-IBCs:

$$\begin{aligned} &z(x,0) = z_0(x), x \in \overline{\Gamma}_x, \\ &z(x,t) = \gamma(x,t), -\delta \leq x \leq 0, t \in \Gamma_t, \\ &z(1,t) = \zeta(1,t), t \in \Gamma_t. \end{aligned} \tag{20}$$

The solution methodologies developed to solve the SP convection–diffusion PDEs with a small negative shift given in (19) and (20) are summarized in Table 13. The methods applied the Crank–Nicholson, implicit Runge–Kutta, implicit Euler, and θ-methods for the mesh discretization of the temporal discretization.

Table 13. Various methods and mesh used to solve Equations (19) and (20).

	Numerical Scheme		
Author(s)	**Temporal Direction**	**Spatial Direction**	**Meshes**
[101]	Crank–Nicolson method	Hybrid method is designed using mid-point upwind with central FDM	Piecewise -uniform Shishkin mesh
[102]	Implicit Runge–Kutta method	Non-standard FDM	Uniform mesh
[103]	θ-method	Exponentially cubic spline method	Uniform mesh
[104]	Implicit Euler method	Hybrid numerical scheme consisting of the midpoint upwind method and the cubic spline method	Piecewise -uniform Shishkin mesh

4.11. Developments toward Solution Methodology for SP Convection-Diffusion Parabolic Equations Involving Small Shifts

In this subsection, we give a brief description of the developed numerical methods for SP convection–diffusion parabolic equations involving small shifts of the form on the domain $D = \Omega_x \times \Omega_t = (0,1) \times (0,T]$ for some fixed number $T > 0$:

$$\frac{\partial y(x,t)}{\partial t} - \varepsilon\frac{\partial^2 y(x,t)}{\partial x^2} + a(x)\frac{\partial y(x-\delta,t)}{\partial x} + r(x)y(x,t) + s(x)y(x-\delta,t) = g(x,t), (x,t) \in D, \tag{21}$$

subject to the following I-IBCs:

$$\begin{aligned} &y(x,0) = y_0(x), x \in \overline{\Omega}_x, \\ &y(x,t) = \phi(x,t), -\delta \leq x \leq 0, t \in (0,T], \\ &y(1,t) = \psi(1,t), t \in (0,T]. \end{aligned} \tag{22}$$

As can be observed from Table 14, only a single solution methodology, namely the non-standard finite difference method with the θ-method on a uniform mesh discretization approach, has been developed for solving the SP convection diffusion parabolic partial differential equation given in (21) and (22) and involving a small negative shift both in convection and reaction terms. This indicates the existence of a huge gap in developing a solution method for the families of the problem under consideration.

Table 14. Various methods and mesh used to solve Equations (21) and (22).

	Numerical Scheme		
Author(s)	Temporal Direction	Spatial Direction	Meshes
[105]	θ-method	Non-standard FDM with Richardson extrapolation	Uniform mesh

4.12. Developments toward Solution Methodology for Singularly Perturbed Parabolic Delay Differential Equation (SPPDDE) with Discontinuous Coefficients

In this subsection, we surveyed the numerical method developed to solve the following SPPDDE with discontinuous coefficients and source terms on the domain $\cup = \Im^- \cup \Im^+ = (0,1) \times (0,T] \cup (1,2) \times (0,T]$, where $\Im^- = (0,1) \times (0,T]$, $\Im^+ = (1,2) \times (0,T]$, $\partial\cup = \overline{\cup}\backslash\cup$, and T is some fixed positive time:

$$\varepsilon\frac{\partial^2 z(x,t)}{\partial x^2} + u(x)\frac{\partial z(x,t)}{\partial x} - s(x)z(x-1,t) - r(x)z(x,t) - \frac{\partial z(x,t)}{\partial t} = \gamma(x,t), \tag{23}$$

subject to the following I-IBCs:

$$\begin{aligned} z(x,t) &= \xi_0(x), x \in [0,2], \\ z(x,t) &= \xi_1(x,t), \text{ in}[-1,0] \times [0,T], \\ z(2,t) &= \xi_2(2,t), t \in [0,2], \end{aligned} \tag{24}$$

where $0 < \varepsilon \ll 1$, $s(x)$, and $r(x)$ are sufficiently smooth functions such that $0 < \lambda \leq u(x), s(x) < 0, r(x) > 0$, and $s(x) + r(x) \geq 0$, $\forall\, x \in [0,2]$. Further, we consider that

$$\begin{aligned} u(x) &= \begin{cases} u_1(x), \text{ if } 0 \leq x \leq 1, \\ u_2(x), \text{ if } 1 < x \leq 2, \end{cases} \\ v(x) &= \begin{cases} v_1(x), \text{ if } (x,t) \in \Im^-, \\ v_2(x), \text{ if } (x,t) \in \Im^+, \end{cases} \\ &-\lambda_1^* < u_1(x) < -\lambda_1 < 0, \ -\lambda_2^* > u_2(x) > \lambda_2 > 0, \ |[u]| \leq C, |[v]| \leq C, \end{aligned} \tag{25}$$

where $\lambda = \min\{\lambda_1, \lambda_2\}$, and $\lambda^* = \max\{\lambda_1^*, \lambda_2^*\}$.

Table 15 summarizes the solution methodologies developed for solving the SP parabolic PDEs containing a large negative shift and with discontinuous coefficients and source terms given in (23)–(25). There are only three methods developed so far for solving the problem under consideration, which indicates that it is a potential area for scholars to work on.

Table 15. Various methods and mesh used to solve Equations (23)–(25).

	Numerical Scheme		
Author(s)	Temporal Direction	Spatial Direction	Meshes
[106]	Backward Euler method	Upwind FDM	Piecewise-uniform Shishkin mesh
[107]	Implicit FDM	Hybrid scheme composition of a central difference scheme and a midpoint upwind scheme	Piecewise-uniform Shishkin mesh
[108]	Implicit Euler method	Cubic-spline in compression method	Uniform mesh

5. Conclusions and Further Directions

The class of SPDODEs with small or large negative shift(s), SPODDEs with mixed shift(s), SPDPDEs with small or large negative shift(s), and SPPPDDEs of the mixed type have been researched because of their numerous applications in many mathematical models. The future behaviors of these problems are assumed to be described by their solutions.

However, it is not easy to solve SPDDEs analytically due to the presence of a thin boundary layer in the solution. Therefore, it is desirable to develop numerical methods, more precisely $\varepsilon-$uniform convergent that solves SPDDEs effectively and efficiently. This survey indicates that a wide variety of studies in the last ten years were mainly based on the development of parameter uniform numerical methods than asymptotic method for SPDDEs. The primary contribution of this survey is the investigation of the numerical and asymptotic methods numerous researchers developed between 2012 and 2022 to solve SPDDEs.

Designing a low-cost uniformly convergent numerical method for such problems is always a desirable task [109] and an active topic of the current research area. Most of the numerical methods developed for SPDDEs were based on finite-difference schemes or spline schemes, except for one paper of the finite-element method (FEM). One can consider FEM to obtain better results in case of irregular boundaries. Spline techniques have become popular and the ultimate tool to achieve the goal. However, the survey reveals that the spline schemes considered for solving SPDDEs are up to the third order only.

Thus, we believe that more than third-order spline techniques for SPDDEs are one of the possible directions of future research work. One can try to extend the techniques used for solving SPDDEs to develop robust numerical schemes for multiple turning point problems, non-linear problems, higher-order problems, and so on. This review paper will serve as the building block for scholars working in this area to develop new robust computational methods for solving SPDDEs.

Author Contributions: Conceptualization, G.F.D.; validation, G.F.D., I.T.D. and C.T.D.; formal analysis, G.F.D., I.T.D. and C.T.D.; investigation,G.F.D., I.T.D. and C.T.D.; resources, I.T.D. and C.T.D.; data curation, G.F.D., I.T.D. and C.T.D.; writing—original draft preparation, I.T.D.; writing—review and editing, G.F.D. and C.T.D.; visualization, I.T.D. and C.T.D.; supervision, G.F.D.; project administration, G.F.D.; funding acquisition, G.F.D. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: There are no associated data, arising from this work.

Acknowledgments: The authors would like to thank the anonymous referees for their helpful comments that improved the quality of this paper.

Conflicts of Interest: The authors declare no conflict of interest.

Abbreviations

The following abbreviations are used in this paper:

SPODDEs	Singularly perturbed ordinary differential–difference equations
SPDODEs	Singularly perturbed delay ordinary differential equation
SPDPDEs	Singularly perturbed delay partial differential equations
SPPPDDEs	Singularly perturbed parabolic partial differential–difference equations
SPDDEs	Singularly perturbed differential–difference equations
SP	Singularly perturbed
SPPs	Singularly perturbed problems
FDM	Finite difference method
FEM	Finite element method
FVM	Finite volume method
FOM	Fitted operator method
FMM	Fitted mesh method
BC	Boundary condition
IBC	Initial boundary condition
I-IBC	Initial interval boundary condition

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