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New Development of Variational Iteration Method Using Quasilinearization Method for Solving Nonlinear Problems

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Abstract: In this paper, we developed a new variational iteration method using the quasilinearization method and Adomian polynomial to solve nonlinear differential equations. The convergence analysis of our new method is also discussed under the Lipschitz continuity condition in Banach space. Some application problems are included to test the efficacy of our proposed method. The behavior of the method is investigated for different values of parameter t. This is a powerful technique for solving a large number of nonlinear problems. Comparisons of our technique were made with the available exact solution and existing methods to examine the applicability and efficiency of our approach. The outcome revealed that the proposed method is easy to apply and converges to the solution very fast.

Keywords: nonlinear ordinary differential equation; variational iteration method; quasilinearization method; Adomian decomposition method; Riccati equation; temperature distribution equation

MSC: 65L05

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1. Introduction

One of the challenging tasks in numerical analysis is solving nonlinear differential equations. These equation frequently appear in a wide variety of problems, including various applications in physics, engineering, biology, heat transfer within porous catalyst particles, and the modeling of chemical reactions. We have many iterative methods for solving these equations. One of the well-known methods is the variational iteration method, which was first introduced by He [1] and has proven to be a useful technique for solving both linear and nonlinear ODEs. Their applicability to different kinds of differential equations was given in [2–8]. Without linearization, discretization, or perturbation, this approach is directly applied to nonlinear ODEs. In [3], Abbasbandy proposed a new application of the VIM to solve a quadratic Riccati differential equation using the Adomian polynomial. Singh et al. [8] proposed an analytical technique for the solution of Lane–Emden equations using the Adomian polynomial [9], which is based on the VIM and control parameter h. Nilima et al. [6] proposed an algorithm based on the VIM for solving numerically the Bratu-type and the Lane–Emden equations, where the recursive schemes for approximate solutions are calculated by using the given boundary conditions. Recently, Hayani [10] developed a method using the combination of the variational iteration method and the homotopy analysis method to find the approximate solution to stiff systems of the initial-value problem for ODEs by using a sequence of subintervals and the step size.

The main motivation for this study is to introduce the quasilinearization method as a basis for the implementation of the VIM. The quasilinearization method was initially introduced by Bellman and Kalaba [11] as a generalization of Newton–Raphson's method. This is one of the most-powerful tools in which a sequence of linear differential equations is obtained from the nonlinear differential equations. Some applications of the quasilinearization method and the order of convergence of the method were discussed in [12–15].

In this work, the quasilinearization algorithm and Adomian polynomial were used to implement a new variational iteration method. To the best of our knowledge, no researchers have used the quasilinearization method and Adomian polynomial to develop the VIM for solving nonlinear differential equations. We propose a new variational iteration method using the QLM and ADM for solving ODEs. The convergence analysis of the method is discussed. We solved some application problems, such as the one-boundary-value problem and two-initial-value problems, to check the applicability and accuracy of the proposed technique. The behavior of the method was examined for different values of parameter t. We compared the obtained numerical results by our proposed iterative approach with the available exact solution, ADM [9], and modified VIM [3] to test the applicability and efficiency of our approach. The beauty of our method is that it easy to apply and converges to the solution very fast.

The outline of the paper is as follows: In Section 2, we discuss the development of a new variational iteration method by introducing a quasilinearization algorithm and the Adomian polynomial. In Section 3, the convergence analysis confirms the convergence to the solution discussed. In Section 4, some numerical experiments are performed and the proposed methods are compared with existing methods, and we observe that our method gives better results compared to the existing methods. Section 5 gives the concluding remarks.

2. Construction of Modified Iterative Method

In this section, we illustrate the basic concepts of He's variational iteration method to construct a new modified variational iteration method. Consider the general nonlinear ordinary differential equation:

$$Lu(t) + f(t, u(t)) = 0 \tag{1}$$

where L represent linear components and f represent nonlinear components.

From the variational iteration method, the correction functional can be constructed as

$$u_{m+1}(t) = u_m(t) + \int_0^t \lambda(x,t)[Lu_m(x) + f(t, \tilde{u}_m(x))]dx. \tag{2}$$

where λ is a general Lagrangian multiplier that can be easily identified by making the above correction functional stationery by considering the nonlinear function $f(t, \tilde{u}_m(x))$ as a restricted variation, i.e., $\delta f \tilde{u}_m = 0$; we obtain

$$\delta u_{m+1}(t) = \delta u_m(t) + \delta \int_0^t \lambda(x,t)[Lu_m(x) + f(t, \tilde{u}_m(x))]dx$$

and

$$\delta u_{m+1}(t) = \delta u_m(t) + \delta \int_0^t \lambda(x,t)[Lu_m(x)]dx.$$

In general, the Lagrangian multiplier λ, can be easily obtained by the stationary conditions derived from the above equation.

Using Lagrangian multiplier λ, the variational iteration formula can be obtained as

$$u_{m+1}(t) = u_m(t) + \int_0^t \lambda(x,t)[L(u_m(x)) + f(u_m(x))]dx. \tag{3}$$

Assume that the exact solution of (1) is $u_\alpha(t)$ and the initial approximation is u_0, which is sufficiently close to the exact solution $u_\alpha(t)$. Using the quasilinearization algorithm, (3) can be written as

$$u_{m+1}(t) = u_m(t) + \int_0^t \lambda(x,t)[L(u_m) + f(u_0) + (u_m - u_0)f_u(u_0) + g(u_m)]dx.$$

where

$$g(u_m) = f(u_m) - f(u_0) - (u_m - u_0)f_u(u_0) \tag{4}$$

and

$$u_{m+1}(t) = u_m(t) + \int_0^t \lambda(x,t)[L(u_m) + u_m f_u(u_0) + C + g(u_m)]dx. \tag{5}$$

where

$$C = f(u_0) - u_0 f_u(u_0). \tag{6}$$

Now, define the series solution

$$u_m = \sum_{i=0}^{m} y_i, \quad u_0 = y_0 \tag{7}$$

assuming that $g(u_m)$ is nonlinear. Using the Adomian decomposition method,

$$g(t, u_m) = g\left(x, \sum_{i=0}^{m} y_i\right) = \sum_{i=0}^{m} A_i \tag{8}$$

where A_i are Adomian polynomials [16] defined as

$$A_i = \frac{1}{i!}\frac{d^i}{dp^i}\left[g\left(t, \sum_{j=0}^{\infty} y_j p^j\right)\right]. \tag{9}$$

Using (7), (8), and (5), we obtain

$$\sum_{i=0}^{m+1} y_i(t) = \sum_{i=0}^{m} y_i(t) + \int_0^t \lambda(x,t)\left[L(\sum_{i=0}^{m} y_i(x)) + f_u(u_0)\sum_{i=0}^{m} y_i(x) + C + \sum_{i=0}^{m} A_i\right]dx$$

From this, we obtain the iterative scheme for solving (1):

$$y_{m+1}(t) = \int_0^t \lambda(x,t)\left[L(\sum_{i=0}^{m} y_i(x)) + f_u(u_0)\sum_{i=0}^{m} y_i(x) + C + \sum_{i=0}^{m} A_i\right]dx \quad, m = 0, 1, 2, \ldots \tag{10}$$

The iterations y_m, $m \geq 1$ are successively obtained by choosing an initial guess u_0. Hence, the mth-order approximation of the solution obtained from (10) is given by

$$u_m(t) = \sum_{i=0}^{m} y_i(t). \tag{11}$$

3. Convergence Analysis

In this section, we discuss the convergence analysis of our proposed method under the Lipschitz continuity condition in Banach spaces.

Theorem 1. *Suppose the nonlinear function $f(u)$ satisfies the Lipschitz condition $|f(u) - f(u^*)| \leq k|u - u^*|$ and there exist $R \in (0,1)$, then the series $\sum_{i=0}^{m} y_i(t)$ defined in (11) is convergent in Banach space $X = (C[0,1, ||u||])$ with the norm defined by*

$$||u|| = \max_{t\in[0,1]} |u(t)| \quad u \in X$$

Proof. From (10) and (11) for $m = 0, 1, 2\ldots$, we obtain

$$u_m = \sum_{i=0}^{m} y_i = u_{m-1} + \int_0^t \lambda(x,t)\left[L(\sum_{i=0}^{m-1} y_i(x)) + f_u(u_0)\sum_{i=0}^{m-1} y_i(x) + C + \sum_{i=0}^{m-1} A_i\right]dx.$$

For $r > s$ and for all $r, s \in N$,

$$\|u_r - u_s\| = \max_{t\in[0,1]} \mid (u_{r-1} - u_{s-1}) +$$
$$\int_0^t \lambda(x,t)\left[L\left(\sum_{i=0}^{r-1} y_i\right) - L\left(\sum_{i=0}^{s-1} y_i\right) + f_u(u_0)\left(\sum_{i=0}^{r-1} y_i - \sum_{i=0}^{s-1} y_i\right) + \left(\sum_{i=0}^{r-1} A_i - \sum_{i=0}^{s-1} A_i\right)\right]dx \mid$$

$$\begin{aligned}\|u_r - u_s\| \leq{}& \max_{t\in[0,1]}|u_{r-1} - u_{s-1}| + \\ & \max_{t\in[0,1]}\left|\int_0^t \lambda(x,t)\left[L(u_{r-1}) - L(u_{s-1}) + f_u(u_0)(u_{r-1} - u_{s-1}) + \left(\sum_{i=0}^{r-1} A_i - \sum_{i=0}^{s-1} A_i\right)\right]dx\right| \\ & \leq \|u_{r-1} - u_{s-1}\| + \max_{t\in[0,1]} \\ & \int_0^t \left|\lambda(x,t)\left[L(u_{r-1}) - L(u_{s-1}) + f_u(u_0)(u_{r-1} - u_{s-1}) + \left(\sum_{i=0}^{r-1} A_i - \sum_{i=0}^{s-1} A_i\right)\right]\right|dx.\end{aligned}$$

Using the relation $\sum_{i=0}^{m} A_i \leq f(u_m)$ and from [17], we obtain

$$\begin{aligned}\|u_r - u_s\| \leq{}& \|u_{r-1} - u_{s-1}\| + \\ & \max_{t\in[0,1]}\int_0^t |\lambda(x,t)[(L(u_{r-1}) - L(u_{s-1})) + f_u(u_0)(u_{r-1} - u_{s-1}) + (f(u_{r-1}) - f(u_{s-1}))]|dx.\end{aligned}$$

Since L is a continuous linear operator, then it is bounded in $(C[0,1], ||u||])$. Hence, there exists a real number γ such that

$$\|L(u_r) - L(u_s)\| \leq \gamma\|u_r - u_s\|. \tag{12}$$

and

$$M = \max_{t\in[0,1]}\int_0^t |\lambda(x,t)|dx. \tag{13}$$

From (12) and (13) and using the Lipschitz condition, (12) gives

$$\begin{aligned}\|u_r - u_s\| &\leq \|u_{r-1} - u_{s-1}\| + M\gamma\|u_{r-1} - u_{s-1}\| + M|f_u(u_0)|\|u_{r-1} - u_{s-1}\| \\ &\quad + Mk\|u_{r-1} - u_{s-1}\| \\ &= \{1 + M(\gamma + |f_u(u_0)| + k)\}\|u_{r-1} - u_{s-1}\| \\ &= R\|u_{r-1} - u_{s-1}\|.\end{aligned}$$

where $R = 1 + M(\gamma + |f_u(u_0)| + k)$. Setting $r = s+1$, we obtain

$$\|u_{s+1} - u_s\| \leq R\|u_s - u_{s-1}\| \leq R^2\|u_{s-1} - u_{s-2}\| \leq \ldots \leq R^s\|u_1 - u_0\|.$$

Using the triangular inequality for $r > s$, for all $r, s \in N$, we obtain

$$\begin{aligned}\|u_r - u_s\| =& \|(u_r - u_{r-1}) + (u_{r-1} - u_{r-2}) + (u_{r-2} - u_{r-3}) + \ldots + (u_{s+1} - u_s)\| \\ \leq& \|u_r - u_{r-1}\| + \|u_{r-1} - u_{r-2}\| + \|u_{r-2} - u_{r-3}\| + \ldots + \|u_{s+1} - u_s\| \\ \leq& R^s\left(1 + R + R^2 + \ldots + R^{r-s-1}\right)\|u_1 - u_0\| \\ =& R^s\left(\frac{1 - R^{r-s}}{1 - R}\right)\|y_1\|\end{aligned}$$

Since $0 < R < 1$, this gives

$$\|u_r - u_s\| \leq \left(\frac{R^s}{1 - R}\right)\|y_1\|$$

Letting $r, s \to \infty$, we obtain

$$\lim_{r,s\to\infty} \|u_r - u_s\| = 0$$

Therefore, $< u_m >$ is a Cauchy sequence in Banach space $(C[0,1], ||u||])$. Hence, $< u_m >$ converges to the solution. $\square$

4. Applications

In this section, we discuss the numerical results and the justification of our proposed method to check the occurrence, reliability, and applicability of our proposed method by performing some numerical examples of the nonlinear differential equation. We compared the obtained numerical results by our proposed iterative approach with the exact solution, ADM [9], and modified VIM [3].

Example 1. *Consider the following temperature distribution equation in a uniformly thick rectangular fin radiation to free space with higher order non-linearity [18,19].*

$$u''(t) - \eta u^4(t) = 0 \tag{14}$$

subject to the boundary conditions $u'(0) = 0,\ u(1) = 1.$

Solution: By using He's variational theory, the correction functional of this system can be constructed as

$$u_{m+1}(x) = u_m(x) + \int_0^t \lambda(x,t)\Big[u''_m(x) - \eta\tilde{u}^4_m(x)\Big]dx,\ m = 0,1,2,\ldots$$

where λ is the general Lagrangian multiplier and $\tilde{u}_m$ denotes restricted variations, i.e., $\delta\tilde{u}_m = 0$. By making the above correction functional stationary, we obtain the following stationary conditions:

$$i.e.,\quad 1 - \lambda'(t) = 0,\quad \lambda(x)|_{x=t} = 0,\quad \lambda''(x) = 0$$

Solving the above equations for λ, we obtain

$$\lambda = x - t$$

From (10), $the(m+1)$th iterative scheme gives

$$y_{m+1}(t) = \int_0^t \lambda(x,t)\left[\sum_{i=0}^{m} y''_i(x) + f_u(u_0)\sum_{i=0}^{m} y_i(x) + C + \sum_{i=0}^{m} A_i\right]dx,\ m = 0,1,2,\ldots \tag{15}$$

Using (4), (6), and (9), we obtain

$$g(u_m) = -\eta u_m^4 + 4\eta t^3 u_m - 3\eta t^4,\ \ C = 3\eta t^4$$
$$and\ A_i = \frac{1}{i!}\frac{d^i}{dp^i}\left[g\left(t, \sum_{j=0}^{\infty} y_j p^j\right)\right].$$

Using (15) with initial guess $u_0 = y_0 = t$ and $\eta = 0.1$, the fourth approximation of the solution of (14) is given by

$$u_4(t) = t + 0.00333333t^6 + 0.0000121212t^{11} + 4.797979797979796\ \times\ 10^{-8}t^{16}$$
$$+ 1.96408529741863\ \times\ 10^{-10}t^{21}.$$

We define the absolute residual error functions for all $t \in [0,1]$:

$$\begin{aligned} Ru_m(t) &= \mid u''_m(t) - \eta u^4_m(t) \mid, \\ rv_m(t) &= \mid v''_m(t) - \eta v^4_m(t) \mid, \\ rw_m(t) &= \mid, w''_m(t) - \eta w^4_m(t) \mid . \end{aligned}$$

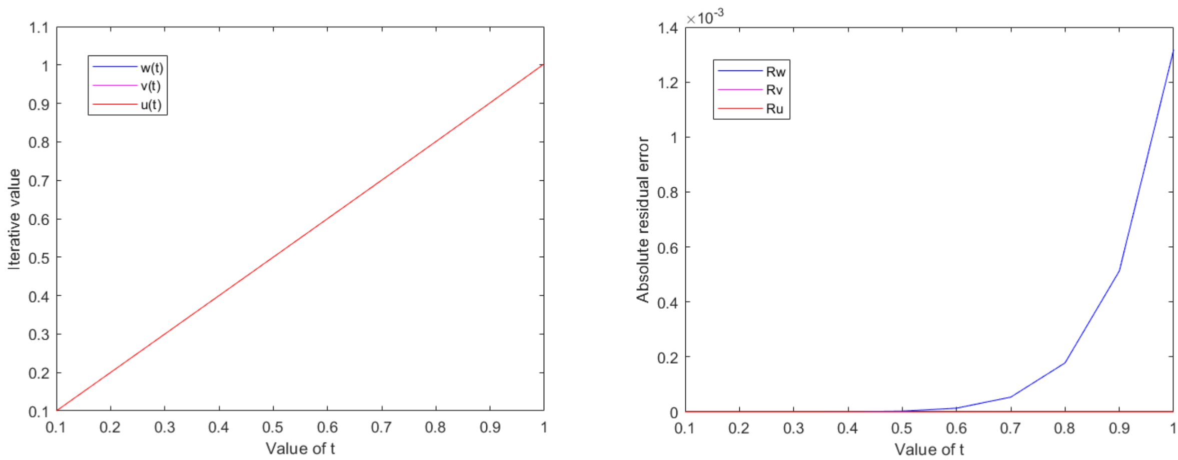
Here, u_m, v_m, and w_m represent the solution of the proposed method, existing ADM [9], and modified VIM [3], respectively. The absolute residual errors for the quantitative comparison of the numerical results of the proposed method with the results obtained by the existing ADM [9] and modified VIM [3] are shown in Table 1 for the different values of t.

The efficiency of our proposed method was tested over the existing ADM [9] and modified VIM [3] due to the absence of an exact solution. We included the absolute residual error to check the applicability of the proposed method. The comparison of the numerical results of (14) is shown in Table 1. We observed that the estimated fourth approximations by our proposed method have comparatively higher accuracy than the existing ADM [9] and modified VIM [3], as shown in Table 1 and Figure 1.

In Figure 1, Figure 1a represents the comparison of the numerical solutions obtained by the proposed method, ADM [9], and modified VIM [3] for $\eta = 0.1$ and $0 \leq t \leq 1$, and Figure 1b represents the comparison of absolute residual errors obtained by the proposed method, ADM [9], and modified VIM [3] for $\eta = 0.1$ and $0 \leq t \leq 1$.

Table 1. Comparison of numerical results for Example 1.

t	$u_4(t)$	$v_4(t)$	$w_4(t)$	Ru_4	Rv_4	Rw_4
0.1	0.10000000	0.10000000	0.10000000	3.39×10^{21}	1.69×10^{-21}	1.33×10^{-12}
0.2	0.20000021	0.20000021	0.20000021	2.71×10^{-20}	5.42×10^{-20}	6.83×10^{-10}
0.3	0.30000243	0.30000243	0.30000243	7.59×10^{-19}	4.34×10^{-19}	2.62×10^{-8}
0.4	0.40001365	0.40001365	0.40001365	4.34×10^{-19}	1.73×10^{-18}	3.49×10^{-7}
0.5	0.50005209	0.50005209	0.50005210	3.38×10^{-17}	3.12×10^{-17}	2.60×10^{-6}
0.6	0.60015556	0.60015556	0.60015561	2.52×10^{-15}	2.52×10^{-15}	0.00001342
0.7	0.70039240	0.70039240	0.70039264	1.02×10^{-13}	1.02×10^{-13}	0.00005370
0.8	0.80087486	0.80087486	0.80087590	2.52×10^{-12}	2.52×10^{-12}	0.00017824
0.9	0.90177528	0.90177528	0.90177908	4.26×10^{-11}	4.26×10^{-11}	0.00051280
1.0	1.00334550	1.00334550	1.00335758	5.36×10^{-10}	5.36×10^{-10}	0.00131687



(**a**) Proposed method vs. ADM vs. VIM (**b**) Absolute residual error

Figure 1. Comparison of numerical solutions and absolute residual errors obtained by the proposed method and existing methods for $0 \leq t \leq 1$ of Example 1.

Example 2. *Consider the following Riccati differential equation [3]:*

$$u'(t) = 2u(t) - u^2(t) + 1, \quad 0 \le t \le 1 \tag{16}$$

with the initial condition $u(0) = 0$.

The exact solution of (16) is

$$u(t) = 1 + \sqrt{2}Tanh\left(\sqrt{2}t + \frac{1}{2}\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)$$

Solution: From (10) and the $(m+1)$th iterative scheme, we obtain

$$y_{m+1}(t) = \int_0^t \lambda(x,t)\left[\sum_{i=0}^{m} y_i'(x) - 2\sum_{i=0}^{m} y_i(x) - 1 + f_u(u_0)\sum_{i=0}^{m} y_i(x) + C + \sum_{i=0}^{m} A_i\right]dx, \; m = 0,1,2,\ldots \tag{17}$$

The Lagrangian multiplier λ can be obtained by making the correction functional of (16) stationary:

$$i.e., \quad \lambda'(x) + 2\lambda(x) = 0, \;\; 1 + \lambda(x)|_{x=t} = 0$$

Solving the above equations for λ, we obtain

$$\lambda = -e^{2(t-x)}$$

Using (4), (6), and (9), we obtain

$$g(u_m) = u_m^2 - 2tu_m + t^2, \qquad C = -t^2$$
$$and, \quad A_i = \frac{1}{i!}\frac{d^i}{dp^i}\left[g\left(t, \sum_{j=0}^{\infty} y_j p^j\right)\right].$$

Using (17) with initial guess $u_0 = y_0 = t$, the fourth approximation of the solution of (16) is given by

$$u_4(t) = 2t + 2t^2 - 2t^4 - 1.6t^5 + 0.711111t^6 + 3.707937t^7 - 3.149206t^8 + 0.69982t^9.$$

The absolute residual errors for the quantitative comparison of the numerical results of the proposed method with the results obtained by the existing ADM [9] and modified VIM [3] as

$$Ru_m(t) =\mid u_m'(t) - 2u_m(t) + u_m^2(t) - 1 \mid,$$
$$rv_m(t) =\mid v_m'(t) - 2v_m(t) + v_m^2(t) - 1 \mid,$$
$$rw_m(t) =\mid w_m'(t) - 2w_m(t) + w_m^2(t) - 1 \mid.$$

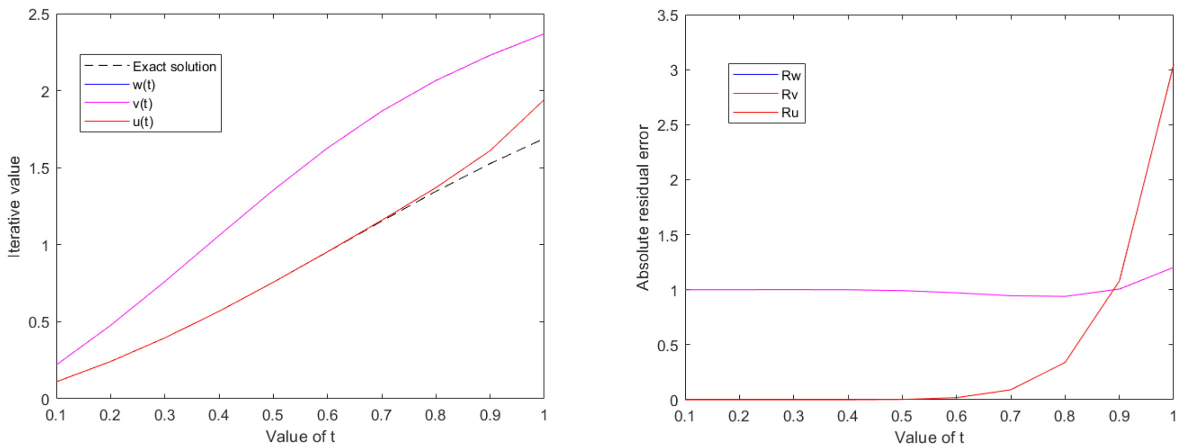
are shown in Table 2 for the different values of t.

The efficiency of our proposed method was tested with the existing ADM [9], modified VIM [3], and the available exact solution. Table 2 shows the applicability of our proposed method. We observed that the estimated fourth approximations by our proposed method have comparatively higher accuracy than the existing ADM [9] and modified VIM [3] and are close to the exact solutions, as shown in Table 2 and Figure 2. Furthermore, the absolute residual errors of our approximations, existing ADM [9], and modified VIM [3] are presented in Figure 2.

In Figure 2, Figure 2a represents the comparison of the numerical solutions obtained by the proposed method, ADM [9], and modified VIM [3] with the available exact solution for $0 \leq t \leq 1$, and Figure 2b represents the comparison of the absolute residual errors obtained by the proposed method, ADM [9], and modified VIM [3] for $0 \leq t \leq 1$.

Table 2. Comparison of numerical results for Example 2.

t	Exact Solution	$u_4(t)$	$v_4(t)$	$w_4(t)$	Ru_4	Rv_4	Rw_4
0.1	0.11029520	0.11029520	0.21978505	0.11029520	3.37×10^{-8}	1.00000153	3.37×10^{-8}
0.2	0.24197680	0.24197671	0.47637327	0.24197671	3.07×10^{-6}	1.00010509	3.07×10^{-6}
0.3	0.39510485	0.39510355	0.76104848	0.39510355	0.00002187	1.00049082	0.00002187
0.4	0.56781217	0.56781226	1.05952338	0.56781226	0.00015212	0.99959811	0.00015212
0.5	0.75601439	0.75611688	1.35414462	0.75611688	0.00273493	0.99208508	0.00273493
0.6	0.95356622	0.95455463	1.62751813	0.95455463	0.01912082	0.97226556	0.01912082
0.7	1.15294897	1.15860127	1.86660896	1.15860127	0.09083206	0.94524039	0.09083206
0.8	1.34636366	1.37046719	2.06611574	1.37046719	0.34008196	0.93920406	0.34008196
0.9	1.52691131	1.61103967	2.22992008	1.61103967	1.07927349	1.00638887	1.07927349
1.0	1.68949839	1.94266367	2.36966490	1.94266367	3.04564431	1.20296607	3.04564431



(**a**) Proposed method vs. ADM vs. VIM (**b**) Absolute residual error

Figure 2. Comparison of numerical solutions and absolute residual errors obtained by the proposed method and existing methods for $0 \leq t \leq 1$ of Example 2 with the available exact solution.

Example 3. *Consider the following nonlinear differential equation:*

$$u'(t) = u^2(t) + 1, \quad 0 \leq t \leq 1 \tag{18}$$

with the initial condition $u(0) = 0$.

The exact solution is $u(t) = Tan(t)$, and choose the initial guess $u_0(t) = t$.
Solution: From (10) and the $(m+1)$th iterative scheme, we obtain

$$y_{m+1}(t) = \int_0^t \lambda(x,t) \left[\sum_{i=0}^{m} y_i'(x) - 1 + f_u(u_0) \sum_{i=0}^{m} y_i(x) + C + \sum_{i=0}^{m} A_i \right] dx, \ m = 0, 1, 2, \ldots \tag{19}$$

The Lagrangian multiplier can be identified as $\lambda = -1$ by making the correction functional stationary. Using (4), (6), and (9),

$$g(u_m) = -u_m^2 + 2u_0 u_m - u_0^2, \quad C = t^2$$

$$and \quad A_i = \frac{1}{i!} \frac{d^i}{dp^i} \left[g\left(t, \sum_{j=0}^{\infty} y_j p^j \right) \right].$$

Using (19) with initial guess $u_0 = y_0 = t$, the fourth approximation of the solution of (18) is given by

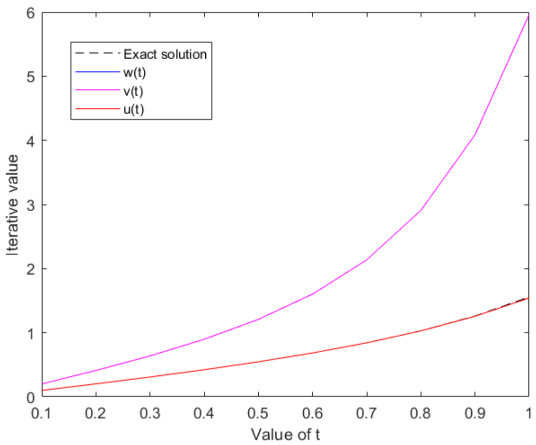
$$u_4(t) = t + 0.333333t^3 + 0.133333t^5 + 0.053968t^7 + 0.021869t^9.$$

The efficiency of our proposed method was tested with the existing ADM [9], modified VIM [3], and the available exact solution. Table 3 shows the applicability of our proposed method. We observed that the estimated fourth approximations by our proposed method have comparatively higher accuracy than the existing ADM [9] and modified VIM [3] and are close to the exact solutions, as shown in Table 3 and Figure 3. Furthermore, the absolute residual errors of our approximations, existing ADM [9], and modified VIM [3] are presented in Figure 3.

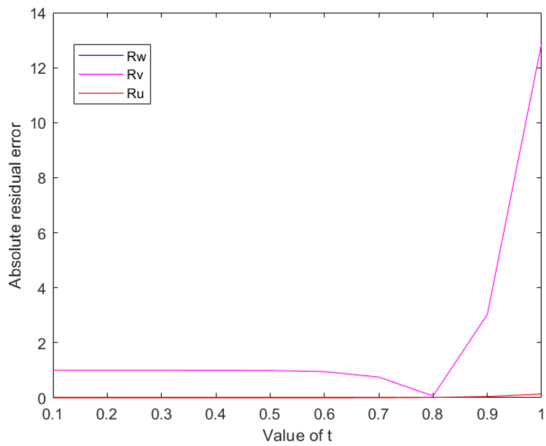
In Figure 3, Figure 3a represents the comparison of the numerical solutions obtained by the proposed method, ADM [9], and modified VIM [3] with the available exact solution $0 \leq t \leq 1$, and Figure 3b represents the comparison of the absolute residual errors obtained by the proposed method, ADM [9], and modified VIM [3] for $0 \leq t \leq 1$.

Table 3. Comparison of numerical results for Example 3.

t	Exact Solution	$u_4(t)$	$v_4(t)$	$w_4(t)$	Ru_4	Rv_4	Rw_4
0.1	0.10033467	0.10033467	0.20134409	0.10033467	9.78×10^{-12}	1.0000000	9.78×10^{-12}
0.2	0.20271004	0.20271004	0.41101941	0.20271004	1.01×10^{-8}	0.99999935	1.01×10^{-8}
0.3	0.30933625	0.30933623	0.63879462	0.30933623	5.92×10^{-7}	0.99996107	5.92×10^{-7}
0.4	0.42279322	0.42279282	0.89785420	0.42279282	0.00001073	0.99927694	0.00001073
0.5	0.54630249	0.54629767	1.20811287	0.54629767	0.00010286	0.99284423	0.00010286
0.6	0.68413681	0.68409916	1.60216886	0.68409916	0.00066015	0.95205369	0.00066015
0.7	0.84228838	0.84206970	2.13596069	0.84206970	0.00322294	0.75241906	0.00322294
0.8	1.02963856	1.02861057	2.90720815	1.02861057	0.01291904	0.06611274	0.01291904
0.9	1.26015822	1.25601754	4.08598759	1.25601754	0.04468657	3.03256885	0.04468657
1.0	1.55740772	1.54250441	5.96331570	1.54250441	0.13805001	12.88494362	0.13805001



(**a**) Proposed method vs. ADM vs. VIM



(**b**) Absolute residual error

Figure 3. Comparison of numerical solutions and absolute residual errors obtained by the proposed method and existing methods for $0 \leq t \leq 1$ of Example 3 with the available exact solution.

5. Conclusions

In this paper, we developed a new variational iteration method for finding the approximate series solution of nonlinear differential equations. The convergence analysis of the proposed method was also discussed. Some application problems were included to test the efficiency of our proposed method. This is a powerful technique for solving a large number of nonlinear problems. By computing the residual error, the applicability and accuracy of the proposed method were investigated. The outcome revealed that the proposed method is easy to apply and converges to the solution with fewer iterations.

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Abbreviations

The following abbreviations are used in this manuscript:

VIM	Variational iteration method
QLM	Quasilinearization method
ADM	Adomian decomposition method
ODE	Ordinary differential equation

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