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# Introduction to Completely Geometrically Integrable Maps in High Dimensions

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**Abstract:** We introduce here the concept of completely geometrically integrable self-maps of $n$-dimensional ($n \geq 2$) cells, cylinders and tori. This concept is the extension of the geometric integrability concept previously introduced for the self-maps of a rectangle in the plane. We formulate and prove here the criteria for the complete geometric integrability of maps on the cells, cylinders and tori of high dimensions. As a corollary of these results, we obtain, in particular, the generalization of the famous Sharkovsky's Theorem for the set of periods of periodic points of completely geometrically integrable self-maps of multidimensional cells.

**Keywords:** (completely) geometrically integrable map; quotient; local lamination; skew product; periodic point

**MSC:** 37E05; 37E10; 37D10; 37C25

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**Citation:** Efremova, L.S. Introduction to Completely Geometrically Integrable Maps in High Dimensions. *Mathematics* **2023**, *11*, 926. https://doi.org/10.3390/math11040926

Academic Editor: Marek Lampart

Received: 28 January 2023
Revised: 8 February 2023
Accepted: 9 February 2023
Published: 12 February 2023



## 1. Introduction

The theory of dynamical systems with continuous and discrete time gives powerful tools for the study of real phenomena [1–3]. The integrability problem is very much in demand in these investigations. There is a vast bibliography on integrable dynamical systems both with continuous (see, e.g., [4–10]) and discrete time (see, e.g., [11–17]). Originally, the concept of the integrability of dynamical systems with discrete time was introduced for systems obtained by the discretization of known differential equations [11,12,16,17]. However, there are discrete dynamical systems that do not belong to this class. We consider these systems.

This work is the direct continuation of the previous papers [18–22], where different aspects of the geometric integrability of self-maps of a compact rectangle in the plane or a two-dimensional cylinder are considered.

Formulate the definition of a geometrically integrable map on an invariant subset of two-dimensional cell, cylinder and torus (one can find the definition of a geometrically integrable map on an invariant subset of a compact plane rectangle and cylinder in the papers [18–22], respectively).

Let $M = M_1 \times M_2$ be a two-dimensional cell, cylinder or torus. Here, $M_1$, $M_2$ are closed intervals or circles.

**Definition 1.** *A map $F : M \to M$ is said to be geometrically integrable on a nonempty $F$-invariant set $A(F) \subseteq M$ if there exists a self-map $\psi$ of an arc $J \subseteq M_1$ and $\psi$-invariant set $B(\psi) \subseteq J$ such that the restriction $F_{|A(F)}$ is semiconjugate with the restriction $\psi_{|B(\psi)}$ by means of a continuous surjection $H : A(F) \to B(\psi)$, i.e., the following equality holds:*

$$H \circ F_{|A(F)} = \psi_{|B(\psi)} \circ H. \tag{1}$$

*The map $\psi_{|B(\psi)}$ is said to be the quotient of $F_{|A(F)}$.*

**Remark 1.** *Maps $F$ and $\psi$ in Definition 1 can be continuous or discontinuous. Moreover, in [23], Definition 1 is extended on the case of some multifunctions with noncompact domains in the plane $\mathbf{R}^2$.*

Formulate the geometric and analytic criteria (Theorem 1 and Theorem 2, respectively) of the geometric integrability for self-maps of two-dimensional cells, cylinders and tori (cf. [21,22]).

For this goal, we need the concept of a local lamination (which generalises the concepts of a lamination and a foliation) and its support (for definitions, see Section 3 and the paper [24] (Ch.1, § 1.2)). We also use natural projections $pr_1 : M \to M_1$ and $pr_2 : M \to M_2$.

**Theorem 1.** *Let $F : M \to M$, $A(F)$ be a nonempty closed $F$-invariant subset of $M$ ($A(F) \subseteq M$) satisfying*

$$pr_2(A(F)) = M_2. \tag{2}$$

*Let $J \subseteq M_1$ be an arc, $\psi$ be a self-map of $J$ and $B(\psi)$ be a closed $\psi$-invariant subset of $J$.*

*Then, $F_{|A(F)}$ is the geometrically integrable map with the quotient $\psi_{|B(\psi)}$ by means of a continuous surjection $H : A(F) \to B(\psi)$ such that for every $y \in M_2$, the map $H$ is an injection on $x$, if and only if $A(F)$ is the support of a continuous invariant lamination for $A(F) \neq M$ (of a continuous invariant foliation for $A(F) = M$) with fibres $\{\gamma_{x'}\}_{x' \in B(\psi)}$ that are pairwise disjoint graphs of continuous functions $x = x_{x'}(y)$ for every $y \in M_2$. Moreover, the inclusion*

$$F(\gamma_{x'}) \subseteq \gamma_{\psi(x')} \tag{3}$$

*holds (Formula (3) demonstrates the property of invariance of a local lamination).*

The dynamical system $\Psi : M \to M$ is said to be *a skew product* if

$$\Psi(x, y) = (\psi(x), g_x(y)), \text{ for all } (x; y) \in M. \tag{4}$$

Here, we set $g_x(y) = g(x, y)$.

The following Theorem 2 can be considered as the claim about the rectification of fibres of a local invariant lamination.

**Theorem 2.** *Let $F : M \to M$, $A(F)$ be a nonempty closed $F$-invariant subset of $M$ satisfying the equality (2). Let $J \subseteq M_1$ be an arc, $\psi$ be a self-map of $J$, and $B(\psi)$ be a closed $\psi$-invariant subset of $J$.*

*Then, $F_{|A(F)}$ is the geometrically integrable map with the quotient $\psi_{|B(\psi)}$ by means of a continuous surjection $H : A(F) \to B(\psi)$ such that for every $y \in I_2$, the map $H$ is an injection on $x$, if and only if there is a homeomorphism $\widetilde{H}$ that maps the set $A(F)$ on the set $B(\psi) \times M_2$ and reduces the restriction $F_{|A(F)}$ to the skew product $\Psi_{|B(\psi)\times M_2}$ satisfying the equality*

$$\Psi_{|B(\psi)\times M_2}(u, v) = (\psi_{|B(\psi)}(u), g_{x'}(v)), \;\; x' = pr_1 \circ \widetilde{H}^{-1}(u, v). \tag{5}$$

*Here, $\widetilde{H}^{-1} : B(\psi) \times M_2 \to A(F)$ is the inverse homeomorphism for $\widetilde{H}$,*

$$\textit{and} \;\; \widetilde{H}(x, y) = (H(x, y), y), \; \textit{for all} \; (x, y) \in A(F). \tag{6}$$

Detailed proofs of the above Theorems 1 and 2 for self-maps of a plane rectangle are given in the paper [21].

Note that a skew product with a two-dimensional phase space is the geometrically integrable map on the whole phase space under the natural projection $pr_1$. At the same time, there are examples of the geometrically integrable maps on the proper invariant subsets of two-dimensional phase spaces (see, e.g., [18,20,21]).

This paper is organised as follows. In Section 2, we describe skew products, introduce the concept of completely geometrically integrable maps in high dimensions and consider the main properties of these maps. In Section 3, we prove the geometric criterion for the complete geometric integrability of maps in high dimensions. In Section 4, we prove the analytic criterion for the complete geometric integrability of maps in high dimensions, and as a corollary of this criterion, we obtain the generalisation of the famous Sharkovsky's theorem for the set of periods of periodic points of the completely geometrically integrable maps on multidimensional cells.

## 2. Skew Products and Preliminary Properties of Completely Geometrically Integrable Maps in High Dimensions

We give here the description of skew products in high dimensions, following the paper [25].

Let $M^n = \prod_{i=1}^{n} M_i$, where $M_i$ is a closed interval or a circle for $i = 1, 2, \ldots, n$, and $n \geq 2$. Consider a map $\Psi : M^n \to M^n$ satisfying

$$\Psi(x_1, x_2, \ldots, x_n) = (\psi_1(x_1), \psi_{2,x_1}(x_2), \ldots, \psi_{n,x_1,\ldots,x_{n-1}}(x_n)), \tag{7}$$

where $(x_1, x_2, \ldots, x_n) \in M^n$, and

$$\psi_{j,x_1,\ldots,x_{j-1}}(x_j) = \psi_j(x_1, \ldots, x_{j-1}, x_j) \ \ (2 \leq j \leq n). \tag{8}$$

Map (7) is said to be *a skew product* with the phase space $M^n$.

We set

$$\widehat{M}^{j-1} = \prod_{i=1}^{j-1} M_i, \qquad \widehat{\psi}_{j-1} = (\psi_1, \psi_{2,x_1}, \ldots, \psi_{j-1,x_1,\ldots,x_{j-2}}).$$

$$\widehat{x}_{j-1} = (x_1, x_2, \ldots, x_{j-1}), \qquad (x_1, x_2, \ldots, x_j) = (\widehat{x}_{j-1}, x_j).$$

As follows from equalities (7) and (8), a map $\widehat{\psi}_j$, where $2 \leq j \leq n-1$, $n \geq 3$, is also a skew product with the phase space $\widehat{M}^j$.

We agree that the map $\widehat{\psi}_{n-1} : \widehat{M}^{n-1} \to \widehat{M}^{n-1}$ ($n \geq 2$) is the *quotient map (quotient)* of the skew product (7), and for every $\widehat{x}_{n-1} \in \widehat{M}^{n-1}$, the map $\psi_{n,\widehat{x}_{n-1}} : M_n \to M_n$ is the *fibre map* over a point $\widehat{x}_{n-1}$.

Let $pr_{\widehat{x}_{j-1}} : \widehat{M}^j \to \widehat{M}^{j-1}$ be the natural projection. Here, $2 \leq j \leq n$, $\widehat{M}^n = M^n$. Then, the following equality is correct:

$$pr_{\widehat{x}_{n-1}} \circ \Psi = \widehat{\psi}_{n-1} \circ pr_{\widehat{x}_{n-1}}. \tag{9}$$

The equality (9) means that the skew product $\Psi$ is semiconjugate with its quotient $\widehat{\psi}_{n-1}$.

By (7), we have for every $k \geq 2$:

$$\Psi^k(\widehat{x}_{n-1}, x_n) = (\widehat{\psi}^k_{n-1}(\widehat{x}_{n-1}), \ \psi_{n,\widehat{x}_{n-1},k}(x_n)), \tag{10}$$

where

$$\psi_{n,\widehat{x}_{n-1},k}(x_n) = \psi_{n,\widehat{\psi}^{k-1}_{n-1}(\widehat{x}_{n-1})} \circ \ldots \circ \psi_{n,\widehat{x}_{n-1}}(x_n).$$

Introduce the concept of the completely geometrically integrable map with the phase space of a high dimension.

**Definition 2.** *A map $F : M^n \to M^n$ ($n \geq 2$) satisfying*

$$F(x_1, x_2, \ldots, x_n) = (F_1(x_1, x_2 \ldots, x_n), \ldots, F_n(x_1, x_2 \ldots, x_n)) \tag{11}$$

*is said to be geometrically integrable on a nonempty $F$-invariant set $A^n(F) \subseteq M^n$ if there exists a map $\widehat{\psi}_{n-1} : \widehat{M}^{n-1} \to \widehat{M}^{n-1}$ and $\widehat{\psi}_{n-1}$-invariant set $A^{n-1}(\widehat{\psi}_{n-1}) \subseteq \widehat{M}^{n-1}$ such that $F_{|A^n(F)}$*

*is semiconjugate with* $\widehat{\psi}_{n-1|A^{n-1}(\widehat{\psi}_{n-1})}$ *by means of a continuous surjection* $H_n : A^n(F) \to A^{n-1}(\widehat{\psi}_{n-1})$*, i.e., the following equality holds:*

$$H_n \circ F_{|A^n(F)} = \widehat{\psi}_{n-1|A^{n-1}(\widehat{\psi}_{n-1})} \circ H_n. \tag{12}$$

*The map* $\widehat{\psi}_{n-1|A^{n-1}(\widehat{\psi}_{n-1})}$ *is said to be the (first) quotient of* $F_{|A^n(F)}$.

*Let, moreover, each j-th quotient* $\widehat{\psi}_{n-j|A^{n-j}(\widehat{\psi}_{n-j})}$ $(1 \leq j \leq n-2,\ n \geq 3)$ *of the map* $F_{|A^n(F)}$ *be geometrically integrable on the nonempty* $\psi_{n-j}$*-invariant set* $A^{n-j}(\widehat{\psi}_{n-j})$ *by means of a continuous surjection* $H_{n-j} : A^{n-j}(\widehat{\psi}_{n-j}) \to A^{n-j-1}(\widehat{\psi}_{n-j-1})$ *with its (first) quotient* $\widehat{\psi}_{n-j-1|A^{n-j-1}(\widehat{\psi}_{n-j-1})}$*, which is said to be the* $(j+1)$*-st quotient of* $F_{|A^n(F)}$*. Here,* $\widehat{\psi}_{n-j-1} : \widehat{M}^{n-j-1} \to \widehat{M}^{n-j-1}$*, and the set* $A^{n-j-1}(\widehat{\psi}_{n-j-1}) \subseteq \widehat{M}^{n-j-1}$ *is* $\widehat{\psi}_{n-j-1}$*-invariant and nonempty.*

*Then, the map* $F_{|A^n(F)}$ *is said to be completely geometrically integrable on the set* $A^n(F)$*.*

**Remark 2.** *The concept of the geometrically integrable map on a nonempty invariant subset of a two-dimensional cell, cylinder or torus coincides with the concept of its complete geometric integrability.*

**Remark 3.** *As it follows from equalities (7)–(9), a skew product* $\Psi : M^n \to M^n$ $(n \geq 2)$ *is a completely geometrically integrable map (on the phase space* $M^n$*) with j-th quotient* $\widehat{\psi}_{n-j} : \widehat{M}^{n-j} \to \widehat{M}^{n-j}$ *for* $1 \leq j \leq n-1$*.*

As it follows from Definitions 1 and 2, a topological conjugacy keeps the geometric integrability property of a map.

**Lemma 1.** *Let* $F : M^n \to M^n$ $(n \geq 2)$ *be a geometrically integrable map on a nonempty F-invariant set* $A^n(F) \subseteq M^n$ *with the quotient* $\widehat{\psi}_{n-1|A^{n-1}(\widehat{\psi}_{n-1})}$*. Here* $\psi_{n-1} : \widehat{M}^{n-1} \to \widehat{M}^{n-1}$*, and* $A^{n-1}(\widehat{\psi}_{n-1})$ *is a* $\widehat{\psi}_{n-1}$*-invariant nonempty set. Let a map* $F_* : M^n_* \to M^n_*$ *have a nonempty* $F_*$*-invariant set* $A^n_*(F_*) \subseteq M^n_*$ *so that* $F_{*|A^n_*(F_*)}$ *is topologically conjugate to* $F_{|A^n(F)}$*. Then,* $F_{*|A^n_*(F_*)}$ *is the geometrically integrable map with the same quotient* $\widehat{\psi}_{n-1|A^{n-1}(\widehat{\psi}_{n-1})}$*.*

In fact, let $H_* : A^n_*(F_*) \to A^n(F)$ be a homeomorphism that conjugates $F_{*|A^n_*(F_*)}$ and $F_{|A^n(F)}$. Then, the equality holds

$$F_{|A^n(F)} = H_* \circ F_{*|A^n_*(F_*)} \circ H_*^{-1},$$

where $H_*^{-1} : A^n(F) \to A^n_*(F_*)$ is the inverse homeomorphism for $H_*$.

Use the geometric integrability property of $F_{|A^n(F)}$ (as can be seen in equalities (1) and (12)). Then, we have

$$H_n \circ F_{|A^n(F)} = H_n \circ H_* \circ F_{*|A^n_*(F_*)} \circ H_*^{-1} = \widehat{\psi}_{n-1|A^{n-1}(\widehat{\psi}_{n-1})} \circ H_n.$$

This implies correctness of the equality

$$H_n \circ H_* \circ F_{*|A^n_*(F_*)} = \widehat{\psi}_{n-1|A^{n-1}(\widehat{\psi}_{n-1})} \circ H_n \circ H_*,$$

where $(H_n \circ H_*) : A_*(F^n_*) \to A^{n-1}(\widehat{\psi}_{n-1})$ is a continuous surjection. Therefore, $F_{*|A^n_*(F_*)}$ is the geometrically integrable map with the quotient $\widehat{\psi}_{n-1|A^{n-1}(\widehat{\psi}_{n-1})}$.

By Definitions 1 and 2, a topological conjugacy of maps in $\widehat{M}^{n-1}$ keeps the property of a map in dimension $(n-1)$ of being the quotient of a geometrically integrable map with $n$-dimensional phase space $M^n$.

**Lemma 2.** *Let* $F : M^n \to M^n$ $(n \geq 2)$ *be a geometrically integrable map on a nonempty F-invariant set* $A^n(F) \subseteq M^n$ *(under the continuous surjection* $H_n$*) with the quotient* $\widehat{\psi}_{n-1|A^{n-1}(\widehat{\psi}_{n-1})}$*.*

*Here, $\widehat{\psi}_{n-1}$ is a self-map of $\widehat{M}^{n-1}$ and $A^{n-1}(\widehat{\psi}_{n-1}) \subseteq \widehat{M}^{n-1}$ is a $\widehat{\psi}_{n-1}$-invariant nonempty set. Let $\widehat{\psi}_{*,(n-1)}$ be a self-map of $\widehat{M}^{n-1}$, and $A_*^{n-1}(\widehat{\psi}_{*,(n-1)}) \subseteq \widehat{M}^{n-1}$ be a $\widehat{\psi}_{*,(n-1)}$-invariant nonempty set. Let, moreover, $\widehat{\psi}_{*,(n-1)|A_*^{n-1}(\widehat{\psi}_{*,(n-1)})}$ be topologically conjugate with $\widehat{\psi}_{n-1|A^{n-1}(\widehat{\psi}_{n-1})}$ under a conjugating homeomorphism $h_* : A_*^{n-1}(\widehat{\psi}_{*,(n-1)}) \to A^{n-1}(\widehat{\psi}_{n-1})$. Then, $\widehat{\psi}_{*,(n-1)|A_*^{n-1}(\widehat{\psi}_{*,(n-1)})}$ is the quotient of $F_{|A^n(F)}$ with respect to the continuous surjection $(h_*^{-1} \circ H_n) : A^n(F) \to A_*^{n-1}(\widehat{\psi}_{*,(n-1)})$, where $h_*^{-1} : A^{n-1}(\widehat{\psi}_{n-1}) \to A_*^{n-1}(\widehat{\psi}_{*,(n-1)})$ is the inverse homeomorphism for $h_*$.*

In fact, the following equality is valid:

$$\widehat{\psi}_{n-1|A^{n-1}(\widehat{\psi}_{n-1})} = h_* \circ \widehat{\psi}_{*,(n-1)|A_*^{n-1}(\widehat{\psi}_{*,(n-1)})} \circ h_*^{-1}.$$

Hence,

$$h_*^{-1} \circ H_n \circ F_{|A^n(F)} = h_*^{-1} \circ \widehat{\psi}_{n-1|A^{n-1}(\widehat{\psi}_{n-1})} \circ H_n = \widehat{\psi}_{*,(n-1)|A_*^{n-1}(\widehat{\psi}_{*,(n-1)})} \circ h_*^{-1} \circ H_n.$$

Here, $(h_*^{-1} \circ H_n) : A^n(F) \to A_*^{n-1}(\widehat{\psi}_{*,(n-1)})$ is the continuous surjection. This means that $\widehat{\psi}_{*,(n-1)|A_*^{n-1}(\widehat{\psi}_{*,(n-1)})}$ is the quotient of $F_{|A^n(F)}$ with respect to the continuous surjection $h_*^{-1} \circ H_n$.

Using Lemmas 1 and 2, we obtain the following claim.

**Corollary 1.** *Let $F : M^n \to M^n$ ($n \geq 2$) be a completely geometrically integrable map on a nonempty $F$-invariant set $A^n(F) \subseteq M^n$. Let a map $F_* : M_*^n \to M_*^n$ on the $F_*$-invariant set $A_*^n(F_*) \subseteq \widehat{M}_*^n$ be topologically conjugate with $F_{|A^n(F)}$. Then, $F_{*|A_*^n(F_*)}$ is the completely geometrically integrable map.*

## 3. The Geometric Criterion for the Complete Geometric Integrability in High Dimensions

Give the definition of a local lamination (as can be seen in [24], Ch.1, § 1.2) for a manifold $M^n$, $n \geq 2$. Below, by a $C^0$-diffeomorphism, we mean a homeomorphism.

We say that the $C^r$-smooth (for $r \geq 1$) or continuous ($C^0$) $d$-dimensional ($1 \leq d \leq n-1$) manifold $L_\alpha$ is a *submanifold of the manifold* $M^n$ if $L_\alpha \subset M^n$ and this inclusion is $C^r$-regular embedding.

**Definition 3.** *Let $A$ be a subset of $M^n$ satisfying $A = \bigcup\limits_{\alpha} L_\alpha$, where $\alpha$ belongs to an index set; $C^r$-submanifolds $\{L_\alpha\}_\alpha$ of dimension $d$ are pairwise disjoint. The family of submanifolds $\{L_\alpha\}_\alpha$ is said to be $d$-dimensional $C^r$-local lamination without singularities if for every point $x \in A$ there exist a neighbourhood $U(x) \subset M^n$ and a $C^r$-diffeomorphism $\chi : U(x) \to \mathbf{R^n}$ ($\mathbf{R^n}$ is $n$-dimensional Euclidean space) such that every connected component of the intersection $U(x) \bigcap L_\alpha$ (if it is not empty) is mapping by means of the $C^r$-diffeomorphism $\chi$ into a $d$-dimensional hyperplane such that*

$$\chi_{|U(x) \bigcap L_\alpha} : U(x) \bigcap L_\alpha \to \chi(U(x) \bigcap L_\alpha)$$

*is a $C^r$-diffeomorphism on the image.*

*The set $A$ satisfying the above equality is said to be a support of the local lamination $L(A)$ and submanifolds $L_\alpha$ are said to be fibres. If $A$ is a closed set, $A \neq M^n$, then we refer to $d$-dimensional $C^r$-lamination; if $A = M^n$, then we refer to $d$-dimensional $C^r$-foliation.*

Prove the geometric criterion of the complete geometric integrability of a map. This result is based on the proof of the existence of one-dimensional continuous local laminations in invariant subsets of the spaces $M^n$, $\widehat{M}^{n-1}, \ldots, \widehat{M}^2$. We use further natural projections $pr_j : \widehat{M}^j \to M_j$, $2 \leq j \leq n$.

**Theorem 3.** *Let $F : M^n \to M^n$ $(n \geq 2)$, $A^n(F)$ be a nonempty closed $F$-invariant subset of $M^n$ satisfying*

$$pr_n(A^n(F)) = M_n. \tag{13}$$

*Let $\widehat{\psi}_{n-j}$ $(1 \leq j \leq n-1)$ be a self-map of $\widehat{M}^{n-j}$ and $A^{n-j}(\widehat{\psi}_{n-j})$ be a closed $\widehat{\psi}_{n-j}$-invariant subset of $\widehat{M}^{n-j}$ satisfying*

$$pr_{n-j}(A^{n-j}(\widehat{\psi}_{n-j})) = M_{n-j}. \tag{14}$$

*Then, $F_{|A^n(F)}$ is the completely geometrically integrable map with sequential quotients $\widehat{\psi}_{n-j|A^{n-j}(\widehat{\psi}_{n-j})}$ by means of continuous surjections $H_n : A^n(F) \to A^{n-1}(\widehat{\psi}_{n-1})$ and $H_{n-j} : A^{n-j}(\widehat{\psi}_{n-j}) \to A^{n-j-1}(\widehat{\psi}_{n-j-1})$ for $n \geq 3$, $1 \leq j \leq n-2$, satisfying:*

*$H_n$ is a one-to-one map on $\widehat{x}_{n-1}$ for every $x_n \in M_n$, and $H_{n-j}$ is a one-to-one map on $\widehat{x}_{n-j-1}$ for every $x_{n-j} \in M_{n-j}$, if and only if every set $A^n(F)$ and $A^{n-j}(\widehat{\psi}_{n-j})$ for the above $j$ is the support of a continuous invariant local lamination with fibres $\{\gamma_{\widehat{x}'_{n-1}} : \widehat{x}'_{n-1} \in A^{n-1}(\widehat{\psi}_{n-1})\}$ and $\{\gamma_{\widehat{x}'_{n-j-1}} : \widehat{x}'_{n-j-1} \in A^{n-j-1}(\widehat{\psi}_{n-j-1})\}$, respectively, which are pairwise disjoint graphs of continuous functions $\widehat{x}_{n-1} = \widehat{x}_{\widehat{x}'_{n-1}}(x_n)$ for every $x_n \in M_n$ and $\widehat{x}_{n-j-1} = \widehat{x}_{\widehat{x}'_{n-j-1}}(x_{n-j})$ for every $x_{n-j} \in M_{n-j}$, respectively.*

**Proof.** 1. Let $F_{|A^n(F)}$ be the completely geometrically integrable map with sequential quotients $\widehat{\psi}_{n-j|A^{n-j}(\widehat{\psi}_{n-j})}$ by means of continuous surjections $H_n : A^n(F) \to A^{n-1}(\widehat{\psi}_{n-1})$ and $H_{n-j} : A^{n-j}(\widehat{\psi}_{n-j}) \to A^{n-j-1}(\widehat{\psi}_{n-j-1})$ for $n \geq 3$, $1 \leq j \leq n-2$.

Then, for every $\widehat{x}'_{n-1} \in A^{n-1}(\widehat{\psi}_{n-1})$, there is a point $(\widehat{x}_{n-1}, x_n) \in A^n(F)$ satisfying

$$H_n(\widehat{x}_{n-1}, x_n) = \widehat{x}'_{n-1}; \tag{15}$$

and for every $\widehat{x}'_{n-j-1} \in A^{n-j-1}(\widehat{\psi}_{n-j-1})$, there is a point $(\widehat{x}_{n-j-1}, x_{n-j}) \in A^{n-j}(\widehat{\psi}_{n-j})$ satisfying

$$H_{n-j}(\widehat{x}_{n-j-1}, x_{n-j}) = \widehat{x}'_{n-j-1}. \tag{16}$$

Since $H_n$ is a one-to-one map on $\widehat{x}_{n-1}$ for every $x_n \in M_n$, and $H_{n-j}$ is a one-to-one map on $\widehat{x}_{n-j-1}$ for every $x_{n-j} \in M_{n-j}$, then, first, there are neighbourhoods

$$U^n(\widehat{x}_{n-1}, x_n) = \widehat{U}^{n-1}(\widehat{x}_{n-1}) \times U_n(x_n)$$

of a point $(\widehat{x}_{n-1}, x_n) \in A^n(F)$ and the unique continuous local implicit function

$$\widehat{x}^{loc}_{n-1} = \widehat{x}^{loc}_{\widehat{x}'_{n-1}}(x_n), \text{ where } \widehat{x}^{loc}_{\widehat{x}'_{n-1}} : U_n(x_n) \to \widehat{U}^{n-1}(\widehat{x}_{n-1}),$$

which is the solution of Equation (15); and, second, there are neighbourhoods

$$U^{n-j}(\widehat{x}_{n-j-1}, x_{n-j}) = \widehat{U}^{n-j-1,}(\widehat{x}_{n-j-1}) \times U_{n-j}(x_{n-j})$$

of points $(\widehat{x}_{n-j-1}, x_{n-j}) \in A^{n-j}(\widehat{\psi}_{n-j})$, and for every $1 \leq j \leq n-2$, $n \geq 3$, the unique continuous local implicit function

$$\widehat{x}^{loc}_{n-j-1} = \widehat{x}^{loc}_{\widehat{x}'_{n-j-1}}(x_{n-j}), \text{ where } \widehat{x}^{loc}_{\widehat{x}'_{n-j-1}} : U_{n-j}(x_{n-j}) \to \widehat{U}^{n-j-1}(\widehat{x}_{n-j-1}),$$

is the solution of Equation (16). Moreover, the following inclusions hold for graphs $\gamma^{loc}_{\widehat{x}'_{n-1}}$ and $\gamma^{loc}_{\widehat{x}'_{n-j-1}}$ of functions $\widehat{x}^{loc}_{n-1}$ and $\widehat{x}^{loc}_{n-j-1}$, respectively:

$$\gamma^{loc}_{\widehat{x}'_{n-1}} \subset A^n(F), \; \gamma^{loc}_{\widehat{x}'_{n-j-1}} \subset A^{n-j}(\widehat{\psi}_{n-j}).$$

Since every set $A^n(F)$ and $A^{n-j}(\widehat{\psi}_{n-j})$ is a compact, then in a finite number of steps we will construct continuous (global) implicit functions

$$\widehat{x}_{n-1} = \widehat{x}_{\widehat{x}'_{n-1}}(x_n) \text{ and } \widehat{x}_{n-j-1} = \widehat{x}_{\widehat{x}'_{n-j-1}}(x_{n-j}),$$

where by equalities (13) and (14), we have:

$$\widehat{x}_{\widehat{x}'_{n-1}} : M_n \to A^{n-1}(\widehat{\psi}_{n-1}), \text{ and } \widehat{x}_{\widehat{x}'_{n-j-1}} : M_{n-j} \to A^{n-j-1}(\widehat{\psi}_{n-j-1}).$$

Moreover, $\widehat{x}_{\widehat{x}'_{n-1}}$ is the solution of Equation (15) on $M_n$, and $\widehat{x}_{\widehat{x}'_{n-j-1}}$ is the solution of Equation (16) on $M_{n-j}$. Denote by $\gamma_{\widehat{x}'_{n-1}}$ the graph of the implicit function $\widehat{x}_{\widehat{x}'_{n-1}}$ and by $\gamma_{\widehat{x}'_{n-j-1}}$, the graph of the implicit function $\widehat{x}_{\widehat{x}'_{n-j-1}}$.

2. Since the map $H_n$ is one-to-one on $\widehat{x}_{n-1}$ for every $x_n \in M_n$, then by the above

$$\gamma_{\widehat{x}'_{n-1}} \bigcap \gamma_{\widehat{x}''_{n-1}} = \emptyset \text{ for } \widehat{x}'_{n-1} \neq \widehat{x}''_{n-1}. \tag{17}$$

Since maps $H_{n-j}$ are one-to-one on $\widehat{x}_{n-j-1}$ for every $x_{n-j} \in M_{n-j}$, then we have

$$\gamma_{\widehat{x}'_{n-j-1}} \bigcap \gamma_{\widehat{x}''_{n-j-1}} = \emptyset, \text{ for } \widehat{x}'_{n-j-1} \neq \widehat{x}''_{n-j-1}. \tag{18}$$

In the previous item 1, it was proven that

$$\bigcup_{\widehat{x}'_{n-1} \in A^{n-1}(\widehat{\psi}_{n-1})} \gamma_{\widehat{x}'_{n-1}} \subset A^n(F); \quad \bigcup_{\widehat{x}'_{n-j-1} \in A^{n-j-1}(\widehat{\psi}_{n-j-1})} \gamma_{\widehat{x}'_{n-j-1}} \subset A^{n-j}(\widehat{\psi}_{n-j}).$$

Since all maps $H_n : A^n(F) \to A^{n-1}(\widehat{\psi}_{n-1})$ and $H_{n-j} : A^{n-j}(\widehat{\psi}_{n-j}) \to A^{n-j-1}(\widehat{\psi}_{n-j-1})$ are continuous surjections then, first, for every point $(\widehat{x}_{n-1}, x_n) \in A^n(F)$, there is a unique point $\widehat{x}'_{n-1} \in A^{n-1}(\widehat{\psi}_{n-1})$ such that $(\widehat{x}_{n-1}, x_n) \in \gamma_{\widehat{x}'_{n-1}}$ (see Formula (17)), and, second, for every point $(\widehat{x}_{n-j-1}, x_{n-j}) \in A^{n-j}(\widehat{\psi}_{n-j})$, there exists a unique point $\widehat{x}'_{n-j-1} \in A^{n-j-1}(\widehat{\psi}_{n-j-1})$ satisfying $(\widehat{x}_{n-j-1}, x_{n-j}) \in \gamma_{\widehat{x}'_{n-j-1}}$ (see Formula (18)). These properties mean that the following inclusions hold:

$$A^n(F) \subset \bigcup_{\widehat{x}'_{n-1} \in A^{n-1}(\widehat{\psi}_{n-1})} \gamma_{\widehat{x}'_{n-1}}; \quad A^{n-j}(\widehat{\psi}_{n-j}) \subset \bigcup_{\widehat{x}'_{n-j-1} \in A^{n-j-1}(\widehat{\psi}_{n-j-1})} \gamma_{\widehat{x}'_{n-j-1}}.$$

Hence, the equalities are correct:

$$A^n(F) = \bigcup_{\widehat{x}'_{n-1} \in A^{n-1}(\widehat{\psi}_{n-1})} \gamma_{\widehat{x}'_{n-1}}; \tag{19}$$

$$A^{n-j}(\widehat{\psi}_{n-j}) = \bigcup_{\widehat{x}'_{n-j-1} \in A^{n-j-1}.(\widehat{\psi}_{n-j-1})} \gamma_{\widehat{x}'_{n-j-1}}. \tag{20}$$

Equalities (19) and (20) means that the sets $A^n(F)$ and $A^{n-j}(\widehat{\psi}_{n-j})$ are the supports of local laminations $L(F)$ and $L(\widehat{\psi}_{n-j})$ with fibres $\{\gamma_{\widehat{x}'_{n-1}} : \widehat{x}'_{n-1} \in A^{n-1}(\widehat{\psi}_{n-1})\}$ and $\{\gamma_{\widehat{x}'_{n-j-1}} : \widehat{x}'_{n-j-1} \in A^{n-j-1}(\widehat{\psi}_{n-j-1})\}$, respectively, (see Definition 3).

3. Prove the invariance of the constructed above local laminations. For certainty, we will carry out the reasoning for the local lamination with fibres

$$\{\gamma_{\widehat{x}'_{n-1}} : \widehat{x}'_{n-1} \in A^{n-1}(\widehat{\psi}_{n-1})\}.$$

In fact, the invariance of this local lamination means that the following property holds: for every $\widehat{x}'_{n-1} \in A^{n-1}(\widehat{\psi}_{n-1})$, there exists $\widehat{x}''_{n-1} \in A^{n-1}(\widehat{\psi}_{n-1})$ such that the inclusion

$F(\gamma_{\widehat{x}'_{n-1}}) \subset \gamma_{\widehat{x}''_{n-1}}$ holds. Suppose the contrary. Then, there is a fibre $\gamma_{\widehat{x}'_{n-1}}$ and points $(\widehat{x}^1_{n-1}, x^1_n)$, $(\widehat{x}^2_{n-1}, x^2_n) \in \gamma_{\widehat{x}'_{n-1}}$ satisfying

$$F(\widehat{x}^1_{n-1}, x^1_n) \in \gamma_{\widehat{x}''_{n-1}}, \quad F(\widehat{x}^2_{n-1}, x^2_n) \in \gamma_{\widehat{x}'''_{n-1}}$$

for some $\widehat{x}''_{n-1}, \widehat{x}'''_{n-1} \in A^{n-1}(\widehat{\psi}_{n-1})$, where $\widehat{x}''_{n-1} \neq \widehat{x}'''_{n-1}$. Using the equality (12), we obtain the inclusion

$$\{\widehat{x}''_{n-1}, \widehat{x}'''_{n-1}\} \subset H_n \circ F_{|A^n(F)}(\gamma_{\widehat{x}'_{n-1}}) =$$
$$= \widehat{\psi}_{n-1|A^{n-1}(\widehat{\psi}_{n-1})} \circ H_n(\gamma_{\widehat{x}'_{n-1}}) = \widehat{\psi}_{n-1|A^{n-1}(\widehat{\psi}_{n-1})}(\widehat{x}'_{n-1}).$$

This is impossible because $\widehat{\psi}_{n-1|A^{n-1}(\widehat{\psi}_{n-1})}$ is a single-valued map. Therefore, the real lamination $L(F)$ with fibres $\{\gamma_{\widehat{x}'_{n-1}} : \widehat{x}'_{n-1} \in A^{n-1}(\widehat{\psi}_{n-1})\}$ for $A^n(F) \neq M$ or the foliation for $A^n(F) = M$ is $F$-invariant. The proof of the invariance of local laminations $L(\widehat{\psi}_{n-j})$ for $1 \leq j \leq n-2$, $n \geq 3$, is analogous to the proof given in this item for the local lamination $L(F)$.

4. Prove the continuity of local laminations $L(F)$ and $L(\widehat{\psi}_{n-j})$. For certainty, give the proof for the local lamination $L(F)$. In fact, take a convergent sequence $\{(\widehat{x}^m_{n-1}, x^m_n)\}_{m\geq 1} \subset A^n(F)$. Let $(\widehat{x}^0_{n-1}, x^0_n)$ be its limit. Since $A^n(F)$ is a compact set, then $(\widehat{x}^0_{n-1}, x^0_n) \in A^n(F)$. Use the equality (19). Then, there are fibres $\{\gamma_{\widehat{x}^{1,m}_{n-1}}\}_{m\geq 1}$ and $\gamma_{\widehat{x}'_{n-1}}$ such that $(\widehat{x}^m_{n-1}, x^m_n) \in \gamma_{\widehat{x}^{1,m}_{n-1}}$ for every $m \geq 1$, and $(\widehat{x}^0_{n-1}, x^0_n) \in \gamma_{\widehat{x}'_{n-1}}$, i.e.,

$$\widehat{x}_{\widehat{x}^{1,m}_{n-1}}(x^m_n) = \widehat{x}^m_{n-1}, \quad \widehat{x}_{\widehat{x}'_{n-1}}(x^0_n) = \widehat{x}^0_{n-1}.$$

Thus, the following equality holds:

$$\lim_{m\to+\infty} \widehat{x}_{\widehat{x}^{1,m}_{n-1}}(x^m_n) = \widehat{x}_{\widehat{x}'_{n-1}}(x^0_n).$$

The last equality means that the sequence of continuous functions $\{x_{\widehat{x}^{1,m}_{n-1}}\}_{m\geq 1}$ continuously converges. In the set of continuous functions (defined on the compact interval or the circle $M_n$), the continuous convergence is equivalent to the uniform convergence (see [26], Ch.2, § 21, X). This means that $L(F)$ is a continuous local lamination, and the set of its fibres $\{\gamma_{\widehat{x}'_{n-1}} : \widehat{x}'_{n-1} \in A^{n-1}(\widehat{\psi}_{n-1})\}$ is a compact in $M^n$. Analogous considerations prove the continuity of invariant local laminations $L(\widehat{\psi}_{n-j})$.

5. Let each set $A^n(F)$ and $A^{n-j}(\widehat{\psi}_{n-j})$ be the support of the continuous invariant lamination $L(F)$ and $L(\widehat{\psi}_{n-j})$ for $A^n(F) \neq M^n$ and $A^{n-j}(\widehat{\psi}_{n-j}) \neq \widehat{M}^{n-j}$, respectively, (of the continuous invariant foliation $L(F)$ and $L(\widehat{\psi}_{n-j})$ for $A^n(F) = M^n$ and $A^{n-j}(\widehat{\psi}_{n-j}) = \widehat{M}^{n-j}$, respectively) with fibres

$$\{\gamma_{\widehat{x}'_{n-1}} : \widehat{x}'_{n-1} \in A^{n-1}(\widehat{\psi}_{n-1})\} \tag{21}$$

and

$$\{\gamma_{\widehat{x}'_{n-j-1}} : \widehat{x}'_{n-j-1} \in A^{n-j-1}(\widehat{\psi}_{n-j-1})\} \tag{22}$$

respectively, such that fibres (21) and (22) are pairwise disjoint graphs of continuous functions $\{\widehat{x}_{\widehat{x}'_{n-1}}(x_n)\}$ with the domain $M_n$ and $\{\widehat{x}_{\widehat{x}'_{n-j-1}}(x_{n-j})\}$ with the domain $M_{n-j}$, respectively.

Prove the geometric integrability of the map $F_{|A^n(F)}$. In fact, let $H_n$, where $H_n$: $A^n(F) \to A^{n-1}(\widehat{\psi}_{n-1})$, be the curvilinear projection satisfying the equality

$$H_n(\gamma_{\widehat{x}'_{n-1}}) = \widehat{x}'_{n-1}. \tag{23}$$

Then, by the above, $H_n$ is the injective map with respect to $\widehat{x}_{n-1}$ for every $x_n \in M_n$.

Note that continuity of the local lamination $L(F)$ implies the continuity of $H_n$. In fact, let $(\widehat{x}_{n-1}, x_n)$ be a point of the set $A^n(F)$. By the equality (19), $A^n(F)$ is the perfect set, i.e., $A^n(F)$ has no isolated points. Let $\{(\widehat{x}^m_{n-1}, x^m_n)\}_{m\geq 1} \subset A^n(F)$ be a sequence, convergent to a point $(\widehat{x}_{n-1}, x_n)$. Using (19) we find fibres $\{\gamma_{\widehat{x}^{1,m}_{n-1}}\}_{m\geq 1}$ and $\gamma_{\widehat{x}'_{n-1}}$ satisfying $(\widehat{x}^m_{n-1}, x^m_n) \in \gamma_{\widehat{x}^{1,m}_{n-1}}$ for every $m \geq 1$, and $(\widehat{x}_{n-1}, x_n) \in \gamma_{\widehat{x}'_{n-1}}$. Since the lamination (or the foliation) $L(F)$ is continuous, then using the equality (23), we obtain

$$\lim_{m\to+\infty} H_n(\widehat{x}^m_{n-1}, x^m_n) = \lim_{m\to+\infty} \widehat{x}^{1,m}_{n-1} = \widehat{x}'_{n-1} = H_n(\widehat{x}_{n-1}, x_n).$$

This means that the map $H_n : A^n(F) \to A^{n-1}(\widehat{\psi}_{n-1})$ is continuous.

Prove the equality (12). Let $(\widehat{x}_{n-1}, x_n) \in A^n(F)$. Then, $(\widehat{x}_{n-1}, x_n) \in \gamma_{\widehat{x}'_{n-1}}$ for some $\widehat{x}'_{n-1} \in A^{n-1}(\widehat{\psi}_{n-1})$. Denote by $C(\gamma_{\widehat{\psi}(\widehat{x}'_{n-1})})$ the subset of the fibre $\gamma_{\widehat{\psi}(\widehat{x}'_{n-1})}$ satisfying the equality

$$C(\gamma_{\widehat{\psi}(\widehat{x}'_{n-1})}) = F(\gamma_{\widehat{x}'_{n-1}}).$$

This equality holds by the invariance of the local lamination $L(F)$. Then, we have

$$H_n \circ F_{|A^n(F)}(\widehat{x}_{n-1}, x_n) = H_n \circ F_{|A^n(F)}(\gamma_{\widehat{x}'_{n-1}}) = H_n(C(\gamma_{\widehat{\psi}(\widehat{x}'_{n-1})})) =$$

$$= \widehat{\psi}(\widehat{x}'_{n-1}) = \widehat{\psi}_{|A^{n-1}(\widehat{\psi}_{n-1})} \circ H_n(\gamma_{\widehat{x}'_{n-1}}) = \widehat{\psi}_{|A^{n-1}(\widehat{\psi}_{n-1})} \circ H_n(\widehat{x}_{n-1}, x_n).$$

Thus, the equality (12) holds, and the map $F_{|A^n(F)}$ is integrable with the first quotient $\widehat{\psi}_{n-1|A^{n-1}(\widehat{\psi}_{n-1})}$. Analogously, the geometric integrability of maps $\widehat{\psi}_{n-j|A^{n-j}(\widehat{\psi}_{n-j})}$ for $n \geq 3$, $1 \leq j \leq n-2$ is proven. The proof of Theorem 3 is finished. □

**Remark 4.** *Nonlocal implicit functions are also used in the considerations of papers [20,21,27,28].*

**Remark 5.** *Theorem 3 generalises the geometric criteria of the integrability for maps in a plane rectangle from papers [18,19,21] and the sufficient conditions of the partial integrability of maps in the plane from [20] (compare with Theorem 1).*

## 4. The Analytic Criterion for the Complete Geometric Integrability in High Dimensions: Concluding Remarks

The main result of this part of the paper is the analytic criterion for the complete geometric integrability of the self-maps of multidimensional cells, cylinders and tori. This criterion is based on the possibility of reducing a map to a skew product.

**Theorem 4.** *Let $F : M^n \to M^n$ $(n \geq 2)$, $A^n(F)$ be a nonempty closed $F$-invariant subset of $M^n$ satisfying (13). Let $\widehat{\psi}_{n-j}$ $(1 \leq j \leq n-1)$ be a self-map of $\widehat{M}^{n-j}$, $A^{n-j}(\widehat{\psi}_{n-j})$ be a closed $\widehat{\psi}_{n-j}$-invariant subset of $\widehat{M}^{n-j}$ satisfying (14).*

*Then, $F_{|A^n(F)}$ is the completely geometrically integrable map with sequential quotients $\widehat{\psi}_{n-j|A^{n-j}(\widehat{\psi}_{n-j})}$ by means of continuous surjections $H_n : A^n(F) \to A^{n-1}(\widehat{\psi}_{n-1})$ and $H_{n-j} : A^{n-j}(\widehat{\psi}_{n-j}) \to A^{n-j-1}(\widehat{\psi}_{n-j-1})$ for $n \geq 3$, $1 \leq j \leq n-2$, satisfying:*

*$H_n$ is a one-to-one map on $\widehat{x}_{n-1}$ for every $x_n \in M_n$, and $H_{n-j}$ is a one-to-one map on $\widehat{x}_{n-j-1}$ for every $x_{n-j} \in M_{n-j}$, if and only if there are homeomorphisms $\widetilde{H}_n : A^n(F) \to A^{n-1}(\widehat{\psi}_{n-1}) \times M_n$ and $\widetilde{H}_{n-j} : A^{n-j}(\widehat{\psi}_{n-j}) \to A^{n-j-1}(\widehat{\psi}_{n-j-1}) \times M_{n-j}$ for $n \geq 3$, $1 \leq j \leq n-2$, which reduce the restrictions $F_{A^n(F)}$ and $\widehat{\psi}_{n-j|A^{n-j}(\widehat{\psi}_{n-j})}$, respectively, to skew products satisfying:*

$$\Psi_{n|A^{n-1}(\widehat{\psi}_{n-1})\times M_n}(\widehat{u}_{n-1}, u_n) = (\widehat{\psi}_{n-1|A^{n-1}(\widehat{\psi}_{n-1})}(\widehat{u}_{n-1}), \ \psi_{n,\widehat{x}'_{n-1}}(u_n)), \tag{24}$$

*where* $\widehat{x}'_{n-1} = pr_{\widehat{x}_{n-1}} \circ \widetilde{H}_n^{-1}(\widehat{u}_{n-1}, u_n)$, $\widetilde{H}_n^{-1} : A^{n-1}(\widehat{\psi}_{n-1}) \times M_n \to A^n(F)$ *is the inverse homeomorphism for* $\widetilde{H}_n$*;*

$$\begin{aligned} &\Psi_{n-j|A^{n-j-1}(\widehat{\psi}_{n-j-1})\times M_{n-j}}(\widehat{u}_{n-j-1}, u_{n-j}) = \\ &(\widehat{\psi}_{n-j-1|A^{n-j-1}(\widehat{\psi}_{n-j-1})}(\widehat{u}_{n-j-1}), \psi_{n-j,\widehat{x}'_{n-j-1}}(u_{n-j})), \end{aligned} \tag{25}$$

*where* $\widehat{x}'_{n-j-1} = pr_{\widehat{x}_{n-j-1}} \circ \widetilde{H}_{n-j}^{-1}(\widehat{u}_{n-j-1}, u_{n-j})$, $\widetilde{H}_{n-j}^{-1} : A^{n-j-1}(\widehat{\psi}_{n-j-1}) \times M_{n-j} \to A_{n-j}$ $(\widehat{\psi}_{n-j})$.

**Proof.** 1. Suppose that $F_{|A^n(F)}$ is the completely geometrically integrable map with sequential quotients $\widehat{\psi}_{n-j|A^{n-j}(\widehat{\psi}_{n-j})}$ by means of continuous surjections $H_n : A^n(F) \to A^{n-1}(\widehat{\psi}_{n-1})$ and $H_{n-j} : A^{n-j}(\widehat{\psi}_{n-j}) \to A^{n-j-1}(\widehat{\psi}_{n-j-1})$ for $n \geq 3$, $1 \leq j \leq n-2$.

Set

$$\begin{cases} \widehat{u}_{n-1} = H_n(\widehat{x}_{n-1}, x_n), \\ u_n = x_n. \end{cases} \tag{26}$$

Denote by $\widetilde{H}_n : A^n(F) \to A^{n-1}(\widehat{\psi}_{n-1}) \times M_n$ the map defined by Formula (26).

Set also that

$$\begin{cases} \widehat{u}_{n-j} = H_{n-j}(\widehat{x}_{n-j-1}, x_{n-j}), \\ u_{n-j} = x_{n-j}. \end{cases} \tag{27}$$

Let $\widetilde{H}_{n-j} : A^{n-j}(\widehat{\psi}_{n-j}) \to A^{n-j-1}(\widehat{\psi}_{n-j-1}) \times M_{n-j}$ be maps defined by Formula (27). Consider for certainty the map given by Formula (26) (the proof for maps defined by equalities (27) is analogous).

In fact, it is proven in Theorem 3 that $A^n(F)$ is the support of the continuous invariant lamination $L(F)$ for $A^n(F) \neq M^n$ (of the continuous invariant foliation for $A^n(F) = M^n$). Moreover, $H_n : A^n(F) \to A^{n-1}(\widehat{\psi}_{n-1})$ is the continuous curvilinear projection that maps every curvilinear fibre $\gamma_{\widehat{x}'_{n-1}} \in L(F)$ to the point $\widehat{x}'_{n-1} \in A^{n-1}(\widehat{\psi}_{n-1})$. Then, the map $\widetilde{H}_n : A^n(F) \to A^{n-1}(\widehat{\psi}_{n-1}) \times M_n$ defined by the equalities (26) is a continuous bijection. Since, moreover, $A^n(F)$ is the compact, and $M^n$ is the Hausdorff space, then $\widetilde{H}_n$ is a homeomorphism [29] (ch. 2, §6, item 2).

By equalities (26), we obtain $\widetilde{H}_n(\gamma_{\widehat{x}'_{n-1}}) = \{\widehat{x}'_{n-1}\} \times M_n$ for every fibre $\gamma_{\widehat{x}'_{n-1}} \in L(F)$, where $\widehat{x}'_{n-1} \in A^{n-1}(\widehat{\psi}_{n-1})$. Hence, the homeomorphism $\widetilde{H}_n$ rectifies the curvilinear fibres of the local lamination $L(F)$.

Let $\Psi_{n|A^{n-1}(\widehat{\psi}_{n-1})\times M_n}$ be the map in the space of variables $(\widehat{u}_{n-1}, u_n)$ that corresponds to the map $F_{|A^n(F)}$ in the space of variables $(\widehat{x}_{n-1}, x_n)$. Since $\widetilde{H}_n$ is a homeomorphism then $\Psi_{n|A^{n-1}(\widehat{\psi}_{n-1})\times M_n}$ is topologically conjugate to $F_{|A^n(F)}$ by means of $\widetilde{H}_n$, that is

$$\Psi_{n|A^{n-1}(\widehat{\psi}_{n-1})\times M_n} = \widetilde{H}_n \circ F_{|A^n(F)} \circ \widetilde{H}_n^{-1}. \tag{28}$$

Obtain the coordinate presentation for $\Psi_{n|A^{n-1}(\widehat{\psi}_{n-1})\times M_n}$ using (28). In fact, let $(\widehat{u}_{n-1}, u_n)$ be an arbitrary point of the set $A^{n-1}(\widehat{\psi}_{n-1}) \times M_n$. By Formula (26), the equality holds:

$$\widetilde{H}_n^{-1}(\widehat{u}_{n-1}, u_n) = (\widehat{x}'_{n-1}, x_n), \text{ where } x_n = u_n;$$

in addition, there exists $\widehat{u}_{n-1} \in A^{n-1}(\widehat{\psi}_{n-1})$, $\widehat{u}_{n-1} = \widehat{x}_{n-1}$, such that $(\widehat{x}'_{n-1}, x_n) \in \gamma_{\widehat{x}_{n-1}}$, and $\widehat{x}'_{n-1}$ is given by the formula

$$\widehat{x}'_{n-1} = pr_{\widehat{x}_{n-1}} \circ \widetilde{H}_n^{-1}(\widehat{u}_{n-1}, u_n). \tag{29}$$

Let

$$F_{|A^n(F)}(\widehat{x}_{n-1}, x_n) = (\widehat{f}_{n-1}(\widehat{x}_{n-1}, x_n), \psi_{n,\widehat{x}_{n-1}}(x_n)). \tag{30}$$

Here

$$\widehat{f}_{n-1}(\widehat{x}_{n-1}, x_n) = (f_1(\widehat{x}_{n-1}, x_n), f_2(\widehat{x}_{n-1}, x_n), \ldots, f_{n-1}(\widehat{x}_{n-1}, x_n)).$$

Use equalities (28)–(30). Then, we obtain:

$$F_{|A^n(F)}(\widehat{x}'_{n-1}, x_n) = (\widehat{f}_{n-1}(\widehat{x}'_{n-1}, x_n), \psi_{n,\widehat{x}'_{n-1}}(x_n)).$$

By the invariance of the local lamination $L(F)$, we have:

$$(\widehat{f}_{n-1}(\widehat{x}'_{n-1}, x_n), \psi_{n,\widehat{x}'_{n-1}}(x_n)) \in \gamma_{\widehat{\psi}_{n-1}(\widehat{x}_{n-1})}.$$

Therefore, using (26) we obtain

$$\begin{aligned}
&\widetilde{H}_n(\widehat{f}_{n-1}(\widehat{x}'_{n-1}, x_n), \psi_{n,\widehat{x}'_{n-1}}(x_n)) = \\
&= \Psi_{n|A^{n-1}(\widehat{\psi}_{n-1})\times M_n}(\widehat{f}_{n-1}(\widehat{x}'_{n-1}, x_n), \psi_{n,\widehat{x}'_{n-1}}(x_n)) = \\
&= (\widehat{\psi}_{n-1|A^{n-1}(\widehat{\psi}_{n-1})}(\widehat{x}_{n-1}), \psi_{n,\widehat{x}'_{n-1}}(x_n)).
\end{aligned}$$

Change $(\widehat{x}_{n-1}, x_n)$ on $(\widehat{u}_{n-1}, u_n)$. Then, we finally obtain

$$\Psi_{n|A^{n-1}(\widehat{\psi}_{n-1}\times M_n}(\widehat{u}_{n-1}, u_n) = (\widehat{\psi}_{n-1|A^{n-1}(\widehat{\psi}_{n-1})}(\widehat{u}_{n-1}), \psi_{n,\widehat{x}'_{n-1}}(u_n)),$$

where $\widehat{x}'_{n-1}$ is given by (29). Thus, Formula (24) is proven. Analogously, Formula (25) are proven.

2. Let homeomorphisms $\widetilde{H}_n : A^n(F) \to A^{n-1}(\widehat{\psi}_{n-1}) \times M_n$ and $\widetilde{H}_{n-j} : A^{n-j}(\widehat{\psi}_{n-j}) \to A^{n-j-1}(\widehat{\psi}_{n-j-1}) \times M_{n-j}$ for $n \geq 3$, $1 \leq j \leq n-2$, reduce restrictions $F_{|A^n(F)}$ and $\widehat{\psi}_{n-j|A^{n-j}(\widehat{\psi}_{n-j})}$ to skew products $\Psi_{n|A^{n-1}(\widehat{\psi}_{n-1})\times M_n}$ (see Formula (24)) and $\Psi_{n-j|A^{n-j-1}(\widehat{\psi}_{n-j-1})\times M_{n-j}}$ (see Formula (25)), respectively. This means that, first, $F_{|A^n(F)}$ and $\Psi_{n|A^{n-1}(\widehat{\psi}_{n-1})\times M_n}$ are topologically conjugate under the conjugating homeomorphism $\widetilde{H}_n$, and, second, each pair $\widehat{\psi}_{n-j|A^{n-j}(\widehat{\psi}_{n-j})}$ and $\Psi_{n-j|A^{n-j-1}(\widehat{\psi}_{n-j-1})\times M_{n-j}}$ consists of topologically conjugate maps under the conjugating homeomorp hism $\widetilde{H}_{n-j}$. Then, equalities hold:

$$F_{|A^n(F)} = \widetilde{H}_n^{-1} \circ \Psi_{n|A^{n-1}(\widehat{\psi}_{n-1})\times M_n} \circ \widetilde{H}_n; \tag{31}$$

$$\widehat{\psi}_{n-j|A^{n-j}(\widehat{\psi}_{n-j})} = \widetilde{H}_{n-j}^{-1} \circ \Psi_{n-j|A^{n-j-1}(\widehat{\psi}_{n-j-1})\times M_{n-j}} \circ \widetilde{H}_{n-j}. \tag{32}$$

Homeomorphisms $\widetilde{H}_n$ and $\widetilde{H}_{n-j}$ are bijections of $A^n(F)$ on $A^{n-1}(\widehat{\psi}_{n-1}) \times M_n$ and $A^{n-j}(\widehat{\psi}_{n-j})$ on $A^{n-j-1}(\widehat{\psi}_{n-j-1}) \times M_{n-j}$, respectively. Then, sets $A^{n-1}(\widehat{\psi}_{n-1}) \times M_n$ and $A^{n-j-1}(\widehat{\psi}_{n-j-1}) \times M_{n-j}$ are supports of the natural $\Psi_n$-invariant and $\Psi_{n-j}$-invariant local laminations, respectively, with fibres $\{\widehat{u}_{n-1}\} \times M_n$ for every $\widehat{u}_{n-1} \in A^{n-1}(\widehat{\psi}_{n-1})$ and $\{\widehat{u}_{n-j-1}\} \times M_{n-j}$ for every $\widehat{u}_{n-j-1} \in A^{n-j-1}(\widehat{\psi}_{n-j-1})$, respectively. Hence, $\widetilde{H}_n^{-1}(\{\widehat{u}_{n-1}\} \times M_n)$ for every $\widehat{u}_{n-1} \in A^{n-1}(\widehat{\psi}_{n-1})$ and $\widetilde{H}_{n-j}^{-1}(\{\widehat{u}_{n-j-1}\} \times M_{n-j})$ for every $\widehat{u}_{n-j-1} \in A^{n-j-1}(\widehat{\psi}_{n-j-1})$ are curvilinear fibres in $A^n(F)$ and $A^{n-j}(\widehat{\psi}_{n-j})$, respectively. Moreover, by equalities (26) and (31), every fibre $\widetilde{H}_n^{-1}(\{\widehat{u}_{n-1}\} \times M_n)$ is homeomorphic to $M_n$ and satisfies the equality

$$pr_n(\widetilde{H}_n^{-1}(\{\widehat{u}_{n-1}\} \times M_n)) = M_n.$$

In addition, by equalities (27) and (32), every fibre $\widetilde{H}_{n-j}^{-1}(\{\widehat{u}_{n-j-1}\} \times M_{n-j})$ is homeomorphic to $M_{n-j}$ and satisfies the equality

$$pr_{n-j}(\widetilde{H}_{n-j}^{-1}(\{\widehat{u}_{n-j-1}\} \times M_{n-j})) = M_{n-j}.$$

Therefore, $\widetilde{H}_n^{-1}(\{\widehat{u}_{n-1}\} \times M_n)$ and $\widetilde{H}_{n-j}^{-1}(\{\widehat{u}_{n-j-1}\} \times M_{n-j})$ are graphs of a continuous functions $\widehat{x}_{n-1} = \widehat{x}_{\widehat{x}'_{n-1}}(x_n)$ with the domain $M_n$ and $\widehat{x}_{n-j-1} = \widehat{x}_{\widehat{x}'_{n-j-1}}(x_{n-j})$ with the domain $M_{n-j}$, respectively. Denote these graphs by $\gamma_{\widehat{x}'_{n-1}}$ and $\gamma_{\widehat{x}'_{n-j-1}}$ respectively.

Since fibres $\{\{\widehat{u}_{n-1}\} \times M_n\}_{\widehat{u}_{n-1} \in A^{n-1}(\widehat{\psi}_{n-1})}$ are pairwise disjoint as well as fibres $\{\{\widehat{u}_{n-j-1}\} \times M_{n-j}\}_{\widehat{u}_{n-j-1} \in A^{n-j-1}(\widehat{\psi}_{n-j-1})}$ then the same property is valid for curvilinear fibres $\{\gamma_{\widehat{x}'_{n-1}}\}_{\widehat{x}'_{n-1} \in A^{n-1}(\widehat{\psi}_{n-1})}$ and $\{\gamma_{\widehat{x}'_{n-j-1}}\}_{\widehat{x}'_{n-j-1} \in A^{n-j-1}(\widehat{\psi}_{n-j-1})}$ respectively.

Maps $\widetilde{H}_n^{-1}$ and $\widetilde{H}_{n-j}^{-1}$ are bijections of $A^{n-1}(\widehat{\psi}_{n-1}) \times M_n$ on $A^n(F)$ and $A^{n-j-1}(\widehat{\psi}_{n-j-1}) \times M_{n-j}$ on $A^{n-j}(\widehat{\psi}_{n-j})$, respectively. Applying the topological conjugacy of maps $F_{|A^n(F)}$ and $\Psi_{n|A^{n-1}(\widehat{\psi}_{n-1}) \times M_n}$ (see equality (31)) as well as $\widehat{\psi}_{n-j|A^{n-j}(\widehat{\psi}_{n-j})}$ and $\Psi_{n-j|A^{n-j-1}(\widehat{\psi}_{n-j-1}) \times M_{n-j}}$ (see equality (32)), we obtain that $A^n(F)$ and $A^{n-j}(\widehat{\psi}_{n-j})$ are supports of invariant local laminations $L(F)$ with fibres $\{\gamma_{\widehat{x}'_{n-1}}\}_{\widehat{x}'_{n-1} \in A^{n-1}(\widehat{\psi}_{n-1})}$ and $L(\widehat{\psi}_{n-j})$ with fibres $\{\gamma_{\widehat{x}'_{n-j-1}}\}_{\widehat{x}'_{n-j-1} \in A^{n-j-1}(\widehat{\psi}_{n-j-1})}$, respectively.

Directly prove the correctness of the inclusions

$$F(\gamma_{\widehat{x}'_{n-1}}) \subseteq \gamma_{\widehat{\psi}_{n-1}(\widehat{x}'_{n-1})} \text{ and } \widehat{\psi}_{n-j}(\gamma_{\widehat{x}'_{n-j-1}}) \subseteq \gamma_{\widehat{\psi}_{n-j-1}(\widehat{x}'_{n-j-1})}. \tag{33}$$

Note that the correctness of first inclusion in (33) immediately follows from equality (31). In fact, let $\gamma_{\widehat{x}'_{n-1}}$, where $\widehat{x}'_{n-1} \in A^{n-1}(\widehat{\psi}_{n-1})$, be a curvilinear fibre of $L(F)$. Then, we have

$$\begin{aligned} &F(\gamma_{\widehat{x}'_{n-1}}) = \widetilde{H}_n^{-1} \circ \Psi_{n|A^{n-1}(\widehat{\psi}_{n-1}) \times M_n} \circ \widetilde{H}_n(\gamma_{\widehat{x}'_{n-1}}) = \\ &= \widetilde{H}_n^{-1} \circ \Psi_{n|A^{n-1}(\widehat{\psi}_{n-1}) \times M_n}(\{\widehat{u}_{n-1}\} \times M_n) \subseteq \\ &\subseteq \widetilde{H}_n^{-1}(\{\widehat{\psi}_{n-1}(\widehat{u}_{n-1})\} \times M_n) = \gamma_{\widehat{\psi}_{n-1}(\widehat{x}'_{n-1})}. \end{aligned}$$

Analogous considerations for sets $\widehat{\psi}_{n-j}(\gamma_{\widehat{x}'_{n-j-1}})$, where $\widehat{x}'_{n-j-1} \in A^{n-j-1}(\widehat{\psi}_{n-j-1})$, based on equality (32), prove a second inclusion in Formula (33).

Thus, the inclusions (33) hold, and local laminations $L(F)$ and $L(\widehat{\psi}_{n-j})$ are invariant. By Theorem 3, the map $F$ is completely geometrically integrable on the set $A^n(F)$ with sequential quotients $\widehat{\psi}_{n-j}$ on the sets $A^{n-j}(\widehat{\psi}_{n-j})$ for $1 \le j \le n-1$. Theorem 4 is proven. □

**Remark 6.** *Above Theorem 4 is the generalisation of the analytic criterion for the geometric integrability from papers [19,21] on the case of maps with the phase spaces of high dimensions (compare with Theorem 2).*

The obtained results can be applied to the study of dynamical properties of completely geometrically integrable maps. One of these applications deals with the description of the periodic point periods of these maps (for a two-dimensional case, see [20,21,30]).

Paying tribute to the memory of Professor Sharkovsky, we describe here the periods of periodic points of continuous completely geometrically integrable self-maps of $n$-dimensional cells.

**Theorem 5.** *Let $F : M^n \to M^n$ be a continuous completely geometrically integrable map on $n$-dimensional cell $M^n$ ($n \ge 2$) with sequential quotients $\widehat{\psi}_{n-j} : \widehat{M}^{n-j} \to \widehat{M}^{n-j}$ ($1 \le j \le n-1$) by means of continuous surjections $H_n : M^n \to \widehat{M}^{n-1}$ and $H_{n-j} : \widehat{M}^{n-j} \to \widehat{M}^{n-j-1}$ for $n \ge 3$, $1 \le j \le n-2$ satisfying:*

*$H_n$ is a one-to-one map on $\widehat{x}_{n-1}$ for every $x_n \in M_n$, and $H_{n-j}$ is a one-to-one map on $\widehat{x}_{n-j-1}$ for every $x_{n-j} \in M_{n-j}$.*

*Let $F$ contain a periodic orbit of a (least) period $m > 1$. Then, it contains also periodic orbits of every (least) period $n$, where $n$ precedes $m$ ($n \prec m$) in the Sharkovsky's order:*

$$1 \prec 2 \prec 2^2 \prec 2^3 \prec \ldots \prec \ldots \prec 2^2 \cdot 9 \prec 2^2 \cdot 7 \prec 2^2 \cdot 5 \prec 2^2 \cdot 3 \prec \ldots$$
$$\prec 2 \cdot 9 \prec 2 \cdot 7 \prec 2 \cdot 5 \prec 2 \cdot 3 \prec \ldots \prec 9 \prec 7 \prec 5 \prec 3.$$

**Proof.** In fact, by Theorem 4, $F$ satisfying the conditions of Theorem 5 is a completely geometrically integrable map. Moreover, $F$ is topologically conjugate to the skew product $\Psi$ given by Formula (7). Then, $\Psi$ has a periodic orbit of period $m > 1$. Use the generalisation of Sharkovsky Theorem for skew products on $n$-dimensional cells ($n \geq 2$) from [31]. Then, $\Psi$ has periodic orbits of every period $n$, where $n \prec m$ in the Sharkovsky's order. This means that $F$ possesses analogous properties.

Theorem 5 is proven. □

Finishing the paper, we formulate the following unsolved problem.

**Problem 1.** *Find sufficient conditions for the complete geometric integrability of a map*

$$F(\widehat{x}_{n-1}, x_n) = (\psi_1(x_1) + \mu_1(\widehat{x}_{n-1}, x_n);\ \psi_{2,x_1}(x_2) + \mu_2(\widehat{x}_{n-1}, x_n);$$
$$\ldots, \psi_{n-1,\widehat{x}_{n-2}}(x_{n-1}) + \mu_{n-1}(\widehat{x}_{n-1}, x_n);\ \psi_{n,\widehat{x}_{n-1}}(x_n))$$

*with the phase space $M^n$ for $n > 2$.*

This problem is solved for the maps of the above type with a compact plane rectangle and a cylinder (see [20,22,28]).

**Funding:** This research received no external funding.

**Informed Consent Statement:** Not applicable.

**Data Availability Statement:** Not applicable.

**Conflicts of Interest:** The author declares no conflict of interest.

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