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Article

Analytical Solution of the Local Fractional KdV Equation

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Abstract: This research work is dedicated to solving the n-generalized Korteweg–de Vries (KdV) equation in a fractional sense. The method is a combination of the Sumudu transform and the Adomian decomposition method. This method has significant advantages for solving differential equations that are both linear and nonlinear. It is easy to find the solutions to fractional-order PDEs with less computing labor.

Keywords: KdV equation; local fractional calculus; Sumudu transform; Adomian decomposition method

MSC: 34A34; 65M06; 26A33

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1. Introduction

Many researchers have dedicated a significant amount of time to flourish structured approaches for handling nonlinear versions of partial differential equations which emerge in real-life problems related to science and engineering [1–4]. In mathematical writings, the most repeatedly occurring derivative-operators in the fractional form are the Caputo and Riemann–Liouville derivatives [5–7]. It has been proven that there is a good harmony between fractional-order derivatives and real-life problems rather than integer-order derivatives. Fractional order differential equations represent many tricky physical and natural prodigies.

In this research article, the nonlinear PDE Korteweg–de Vries (KdV) [8–11] has been investigated. We use the Korteweg–de-Vries (KdV) equation to formulate it mathematically in the study of waves on shallow water surfaces. This equation was established by Boussinesq [12] and reclaimed by Diederik Korteweg and Gustav de Vries (1895) [13]. We can express the local fractional Korteweg–De Vries (KdV) equation as follows:

$$D_t^{(1)}v + vD_x^{(1)}v + D_x^{(3)}v = 0. \tag{1}$$

The KdV equation is crucial to solving many real-world issues, such as in magma flow, surface waves, Rossby waves, internal waves in a fluid with a stratified density, and plasma waves [13]. The exact solution to the KdV equation was first provided by Kath and Smyth [9] using the inverse scattering transform. For further information, see the KdV equation and associated work [10,14,15].

Additionally, researchers have scrutinized various features of local fractional differential equations by combining nonidentical techniques and methods. Abdel-Rady [16] has incorporated the natural transform method with the Adomain decomposition method, namely the Adomain decomposition transform method, to resolve the nonlinear PDE [17–22]. From the motivation of the above-stated approach, we apply the local fractional Sumudu Adomian decomposition method to investigate the local fractional Korteweg–de Vries (KdV) equation [23–27].

The Sumudu transform and Adomian decomposition method were combined to create the local fractional Sumudu decomposition method. This method has significant advantages for solving differential equations. With less computing labour, this method makes it easy to find the solutions to fractional differential equations that are both linear and nonlinear. This method's ability to solve the nonlinear fractional differential equation without using He's polynomials for the nonlinear variables is one of its strongest features. There are no constraining assumptions or linearization in the proposed approach.

A number of local fractional differential equations, including the local fractional KdV equation, the local fractional Tricomi equation, the local fractional heat conduction equation, and others, have been applied to the Cantor fractal sets in recent papers by Yang et al. [28–31] to describe natural phenomena in fractal-like media. Numerous analytical techniques have been developed to address nonlinear problems, including the local fractional Fourier series approach, the Yang–Laplace transform method, the variational iteration method for local fractional derivatives, and the variational iteration transform method [4,11,32–39].

The primary goal of the work is to use a fractal Sumudu–Adomian decomposition method to solve the n-generalized KdV equation in a fractional sense. Consider the following n-generalized fractional KdV equation.

$$D_t^{(\alpha)} v + v^n D_x^{(\alpha)} + D_x^{(3\alpha)} = F(x,t), 0 < \alpha \leq 1, \tag{2}$$

with

$$v(x,0) = f(x). \tag{3}$$

The organization of the article is as follows:

Segment 1: Introduction
Segment 2: Ground Work
Segment 3: Existence and uniqueness of solution of fractional KdV equation
Segment 4: Analysis of the local fractional Sumudu decomposition method (LFSDM)
Segment 5: Application of the local fractional Sumudu decomposition method (LFSDM) and graphical representation of the solution by using MATLAB software
Segment 6: Conclusions

2. Ground Work

In this piece of work, elementary definitions and vague abstractions of the calculus of the fractional domain and Sumudu transform are given.

Definition 1. *(see [23–27]) For a real-valued function $\varphi(r\ )$ such that*

$$|\varphi(r\ ) - \varphi(r_0\ )| < \in^{\alpha},$$

the local fractal derivative at $r = r_0$ is defined as

$$\varphi^{(\alpha)}(r\ ) = \left.\frac{d^{\alpha}\varphi(r)}{dr^{\alpha}}\right|_{r=r_0} = \lim_{r \to r_0} \frac{\Delta^{\alpha}(\varphi(r\ ) - \varphi(r_0\ ))}{(r - r_0)^{\alpha}}, \tag{4}$$

where

$$\Delta^{\alpha}(\varphi(r\ ) - \varphi(r_0\ )) \cong \Gamma(1+\alpha)(\varphi(r\ ) - \varphi(r_0\ )).$$

Definition 2. *The local fractal integral of $\varphi(r\ )$ of order α in the interval (a,b) is defined as*

$$
\begin{aligned}
{}_aI_b{}^{(\alpha)}\varphi(r) &= \tfrac{1}{\Gamma(1+\alpha)} \int_a^b \varphi(\tau)(d\tau)^\alpha \\
&= \tfrac{1}{\Gamma(1+\alpha)} \lim_{\Delta\tau\to 0} \sum_{j=0}^{j=M-1} \zeta(\tau_j)(\Delta\tau_j)^\alpha,
\end{aligned} \tag{5}
$$

where $\Delta\tau_j = \tau_{j+1} - \tau_j$, $\Delta\tau = max\,\{\Delta\tau_1, \Delta\tau_2, \Delta\tau_3, \ldots\}$ *and* $[\tau_j, \tau_{j+1}], j = 0, \ldots, M-1, \tau_0 = a, \tau_M = b$ *is a segregation of* (a, b) *[30,31].*

Definition 3. *The Mittag–Leffler function, sine function, and cosine function are defined as [30,31]:*

$$E_\alpha(r^\alpha) = \sum_{k=0}^{+\infty} \frac{r^{k\alpha}}{\Gamma(k\alpha+1)}, 0 < \alpha \le 1, \tag{6}$$

$$\sin_\alpha(r^\alpha) = \sum_{k=0}^{+\infty} (-1)^k \frac{r^{(2k+1)\alpha}}{\Gamma((2k+1)\alpha+1)}, 0 < \alpha \le 1, \tag{7}$$

$$\cos_\alpha(r^\alpha) = \sum_{k=0}^{+\infty} (-1)^k \frac{r^{2k\alpha}}{\Gamma(2k\alpha+1)}, 0 < \alpha \le 1. \tag{8}$$

The following formulae related to the local fractional derivatives and the integral of non-differentiable functions are mentioned in [30,31].

$$\frac{d^\alpha}{dr^\alpha} \frac{r^{k\alpha}}{\Gamma(k\alpha+1)} = \frac{r^{(k-1)\alpha}}{\Gamma((k-1)\alpha+1)}, \tag{9}$$

$$\frac{d^\alpha}{dr^\alpha} E_\alpha(r^\alpha) = E_\alpha(r^\alpha), \tag{10}$$

$$\frac{d^\alpha}{dr^\alpha} \sin_\alpha(r^\alpha) = \cos_\alpha(r^\alpha), \tag{11}$$

$$\frac{d^\alpha}{dr^\alpha} \cos_\alpha(r^\alpha) = \sin_\alpha(r^\alpha), \tag{12}$$

$${}_0I_r^\alpha \frac{r^{k\alpha}}{\Gamma(k\alpha+1)} = \frac{r^{(k+1)\alpha}}{\Gamma((k+1)\alpha+1)}. \tag{13}$$

Here, we provide definitions and some properties of the local fractional Sumudu transform.

New transform operator ${}^{LF}S_\alpha : \varphi(r) \to F_\alpha(l)$ is defined as:

$${}^{LF}S_\alpha\left\{\sum_{k=0}^{\infty} a_k r^{k\alpha}\right\} = \sum_{k=0}^{\infty} \Gamma(1+k\alpha) a_k l^{k\alpha}. \tag{14}$$

Few cases are:

$${}^{LF}S_\alpha\{E_\alpha(i^\alpha r^\alpha)\} = \sum_{k=0}^{\infty} i^{k\alpha} l^{k\alpha}. \tag{15}$$

$${}^{LF}S_\alpha\left\{\frac{r^\alpha}{\Gamma(1+\alpha)}\right\} = l^\alpha. \tag{16}$$

Definition 4. *The local fractional Sumudu transform of function* $\varphi(r)$ *of order* α *is elucidated as*

$${}^{LF}S_\alpha\{\varphi(r)\} = F_\alpha(l) = \frac{1}{\Gamma(\alpha+1)} \int_0^\infty E_\alpha(-r^\alpha l^{-\alpha}) \frac{\varphi(r)}{l^\alpha} (dr)^\alpha, 0 < \alpha \le 1. \tag{17}$$

The local fractional inverse Sumudu transform is elucidated as

$$^{LF}S_{\alpha}^{-1}\{F_{\alpha}(l)\} = \varphi(r\,), 0 < \alpha \leq 1. \tag{18}$$

Definition 5. *(linearity) If $^{LF}S_{\alpha}\{\varphi(r\,)\} = F_{\alpha}(l)$ and $^{LF}S_{\alpha}\{\psi(r\,)\} = P_{\alpha}(l)$, then we have,*

$$^{LF}S_{\alpha}\{\varphi(r\,) + \psi(r\,)\} = F_{\alpha}(l) + P_{\alpha}(l). \tag{19}$$

Proof. From the Definition 4, we obtain the result. □

Definition 6. *(1) (The local fractional Sumudu transform of the local-fractional derivative)*

If $^{LF}S_{\alpha}\{\varphi(r)\} = F_{\alpha}(l)$ then we have,

$$^{LF}S_{\alpha}\left\{\frac{d^{n\alpha}\varphi(r)}{dr^{n\alpha}}\right\} = \frac{1}{l^{n\alpha}}\left[F_{\alpha}(l) - \sum_{k=0}^{n-1} l^{k\alpha}\varphi^{k\alpha}(0)\right]. \tag{20}$$

when $n = 1$ and $n = 2$ in (20), we obtain,

$$^{LF}S_{\alpha}\left\{\frac{d^{\alpha}\varphi(r)}{dr^{\alpha}}\right\} = \frac{1}{l^{\alpha}}[F_{\alpha}(l) - \varphi(0\,)]. \tag{21}$$

$$^{LF}S_{\alpha}\left\{\frac{d^{2\alpha}\varphi(r)}{dr^{2\alpha}}\right\} = \frac{1}{l^{2\alpha}}\left[F_{\alpha}(l) - \varphi(0\,) - l^{\alpha}\varphi^{(\alpha)}(0\,)\right]. \tag{22}$$

(2) (The local fractional Sumudu transform of the local-fractional integral) If $^{LF}S_{\alpha}\{\varphi(\alpha)\} = F_{\alpha}(l)$ then we have,

$$^{LF}S_{\alpha}\left\{{}_0I_r^{(\alpha)}\varphi(r)\right\} = l^{\alpha}F_{\alpha}(l). \tag{23}$$

Proof. see [23]. □

Definition 7. *(local fractional convolution) If $^{LF}S_{\alpha}\{\varphi(r\,)\} = F_{\alpha}(l)$ and $^{LF}S_{\alpha}\{\psi(r\,)\} = P_{\alpha}(l)$, then we have,*

$$^{LF}S_{\alpha}\{\varphi(r) * \psi(r\,)\} = l^{\alpha}F_{\alpha}(l)P_{\alpha}(l), \tag{24}$$

with

$$\varphi(r) * \psi(r\,) = \frac{1}{\Gamma(\alpha+1)}\int_0^{\infty} \varphi(w\,)\psi(r-w\,)(dr)^{\alpha}. \tag{25}$$

Proof. see [23]. □

3. Existence and Uniqueness of the Solution of the Local Fractional KdV Equation

Generalized Korteweg–de Vries (KdV) equation taken into account is as follows

$$D_t^{(\alpha)}v + v^n D_x^{(\alpha)}v + D_x^{(3\alpha)}v = F(x,t), \tag{26}$$

$$v(x,0) = v_0(x), \tag{27}$$

now we can write the system in operator form as:

$$L_{\alpha}[v((x,t)\,)] = \phi[v((x,t)\,)], \tag{28}$$

subject to the initial condition

$$v(x,0) = v_0(x), \tag{29}$$

where,

$$L_\alpha = \frac{\partial^\alpha}{\partial t^\alpha}$$

and

$$\phi[v(x,t)] = F(x,t) - v^n D_x^{(\alpha)} v + D_x^{(3\alpha)} v.$$

Theorem 1. *Let the function defined by*

$$\phi[v(x,t)] = F(x,t) - v^n D_x^{(\alpha)} v + D_x^{(3\alpha)} v,$$

be local, fractional, and continuous, which also satisfies the Lipschitz condition i.e.

$$|\phi[v_1(x,t)] - \phi[v_2(x,t)]| \le \eta^\alpha |v_1(x,t) - v_2(x,t)|, \tag{30}$$

$$0 \le \alpha \le 1, 0 < \eta < 1.$$

Then the system

$$\begin{aligned} L_\alpha[v((x,t)\,)] &= \phi[v((x,t)\,)], \\ v(x,0) &= v_0(x), \end{aligned} \tag{31}$$

has a unique solution in $C_\alpha[a,b]$, where C_α is a space of a continuous function with a fractal derivative of order α.

Proof. Let the map $P : C_\alpha[a,b] \to C_\alpha[a,b]$ be defined by

$$P[v(x,t)] = v_0(x) + \frac{1}{\Gamma(1+\alpha)} \int_\alpha^\tau \phi[v(x,s)](ds)^\alpha, \tag{32}$$

we first prove by induction that

$$\begin{aligned} &\|P^n\{v_1((x,t)\,)\} - P^n\{v_2((x,t)\,)\}\|_\alpha \\ &\le \frac{\eta^{n\alpha}|b^\alpha - a^\alpha|}{\Gamma^n(1+\alpha)}^n \|v_1((x,t)\,) - v_2((x,t)\,)\|_\alpha, n = 1,2,3\ldots \end{aligned} \tag{33}$$

for $n = 1$, we get

$$\begin{aligned} &\|P\{v_1((x,t)\,)\} - P\{v_2((x,t)\,)\}\|_\alpha \\ &\le \left| \frac{1}{\Gamma(1+\alpha)} \int_\alpha^\tau \phi[v_1(x,s)] - \phi[v_2(x,s)](ds)^\alpha \right|, \end{aligned}$$

$$\begin{aligned} &\|P\{v_1((x,t)\,)\} - P\{v_2((x,t)\,)\}\|_\alpha \\ &\le \left| \frac{1}{\Gamma(1+\alpha)} \int_\alpha^\tau \eta^\alpha |v_1((x,s)\,) - v_2((x,s)\,)|(ds)^\alpha \right|. \end{aligned}$$

This implies that

$$\begin{aligned} &\|P\{v_1((x,t)\,)\} - P\{v_2((x,t)\,)\}\|_\alpha \\ &\le \frac{\eta^\alpha|b^\alpha - a^\alpha|}{\Gamma(1+\alpha)} \|v_1((x,t)\,) - v_2((x,t)\,)\|_\alpha. \end{aligned} \tag{34}$$

Assume the equality holds for $n = k$

$$\begin{aligned} &\left\|P^K\{v_1((x,t)\,)\} - P^K\{v_2((x,t)\,)\}\right\|_\alpha \\ &\le \frac{\eta^{k\alpha}|b^\alpha - a^\alpha|}{\Gamma^k(1+\alpha)}^k \|v_1((x,t)\,) - v_2((x,t)\,)\|_\alpha, \end{aligned} \tag{35}$$

for $n = k + 1$, consider

$$\begin{aligned} &\left\|P^{k+1}\{v_1((x,t)\,)\} - P^{k+1}\{v_2((x,t)\,)\}\right\|_\alpha \\ &\le \left| \frac{1}{\Gamma(1+\alpha)} \int_\alpha^\tau \phi[P^k\{v_1(x,s)\}] - \phi[P^k\{v_2(x,s)\}](ds)^\alpha \right|, \end{aligned}$$

further it can be written as,

$$\begin{aligned}&\left\|P^{k+1}\{v_1((x,t)\,)\}-P^{k+1}\{v_2((x,t)\,)\}\right\|_\alpha\\&\leq\left|\frac{1}{|\Gamma(1+\alpha)}\int_\alpha^\tau\eta^\alpha\left|P^k\{v_1((x,s)\,)\}-P^k\{v_2((x,s)\,)\}\right|(ds)^\alpha\right|.\end{aligned}$$

This implies that,

$$\begin{aligned}&\left\|P^{k+1}\{v_1((x,t)\,)\}-P^{k+1}\{v_2((x,t)\,)\}\right\|_\alpha\\&\leq\frac{\eta^{(k+1)\alpha}|b^\alpha-a^\alpha|}{\Gamma^{k+1}(1+\alpha)}^{k+1}\|v_1((x,t)\,)-v_2((x,t)\,)\|_\alpha,\end{aligned}\tag{36}$$

hence, it proves our assumptions.

Now, we have

$$\frac{\eta^{(k+1)\alpha}|b^\alpha-a^\alpha|}{\Gamma^{k+1}(1+\alpha)}^{k+1}\|v_1((x,t)\,)-v_2((x,t)\,)\|_\alpha\to 0\tag{37}$$

as $n\to\infty$.

Thus P^K has contraction on $C_\alpha[a,b]$. Therefore, the system persists to a unique solution. □

4. Analysis of the Local Fractional Sumudu Decomposition Method (LFSDM)

The local fractional non-homogeneous nonlinear KdV equation with the initial condition is given as follows:

$$D_t^{(\alpha)}v+v^nD_x^{(\alpha)}v+D_x^{(3\alpha)}v=F(x,t),\tag{38}$$

$$v(x,0)=v_0(x),\tag{39}$$

applying the local fractional Sumudu transform to the above equation, we obtain the following result.

$${}^{LF}S_\alpha\left[D_t^{(\alpha)}v\right]+{}^{LF}S_\alpha\left[v^nD_x^{(\alpha)}v\right]_\alpha+{}^{LF}S_\alpha\left[D_x^{(3\alpha)}v\right]={}^{LF}S_\alpha[F(x,t)].\tag{40}$$

Using the properties of the local fractional Sumudu transform, we attain

$$\begin{aligned}&{}^{LF}S_\alpha[v]=v(x,0)+w^\alpha\left[{}^{LF}S_\alpha[F(x,t)]\right]\\&-w^\alpha\left[{}^{LF}S_\alpha\left[v^nD_x^{(\alpha)}v\right]_\alpha+{}^{LF}S_\alpha\left[D_x^{(3\alpha)}v\right]\right],\end{aligned}\tag{41}$$

the following is the result of applying the inverse local fractional Sumudu transform on both sides of (41)

$$\begin{aligned}&v(x,t)=v(x,0)+{}^{LF}S_\alpha^{-1}\left\{w^\alpha\left[{}^{LF}S_\alpha[F(x,t)]\right]\right\}\\&-{}^{LF}S_\alpha^{-1}\left\{w^\alpha\left[{}^{LF}S_\alpha\left[v^nD_x^{(\alpha)}v\right]+{}^{LF}S_\alpha\left[D_x^{(3\alpha)}v\right]\right]\right\},\end{aligned}\tag{42}$$

now according to the Adomian decompositon method [17–19], we decompose $v(x,t)$ in infinite series given by

$$\sum_{r=0}^{\infty}v_r(x,t).\tag{43}$$

Substituting (43) in (42)

$$\begin{aligned}&\sum_{r=0}^{\infty}v_r(x,t)=v(x,0)+{}^{LF}S_\alpha^{-1}\left\{w^\alpha\left[{}^{LF}S_\alpha[F(x,t)]\right]\right\}\\&-{}^{LF}S_\alpha^{-1}\left\{w^\alpha\left[{}^{LF}S_\alpha\left(\sum_{r=0}^{\infty}A_r\right)+{}^{LF}S_\alpha\left[D_x^{(3\alpha)}\sum_{r=0}^{\infty}v_r(x,t)\right]\right]\right\}.\end{aligned}\tag{44}$$

Now comparing the terms of both sides of Equation (44), we achieve

$$v_0(x,t) = v(x,0) + {}^{LF}S_\alpha^{-1}\left\{w^\alpha\left[{}^{LF}S_\alpha[F(x,t)]\right]\right\}, \tag{45}$$

$$v_1(x,t) = -{}^{LF}S_\alpha^{-1}\left\{w^\alpha\left[{}^{LF}S_\alpha(A_0) + {}^{LF}S_\alpha\left[D_x^{(3\alpha)}v_0(x,t)\right]\right]\right\}, \tag{46}$$

$$v_2(x,t) = -{}^{LF}S_\alpha^{-1}\left\{w^\alpha\left[{}^{LF}S_\alpha(A_1) + {}^{LF}S_\alpha\left[D_x^{(3\alpha)}v_1(x,t)\right]\right]\right\}, \tag{47}$$

and so on.

Therefore the solution of Equation (38) is given by:

$$v(x,t) = \sum_{r=0}^{\infty} v_r(x,t) = v_0(x,t) + v_1(x,t) + v_2(x,t) + \dots \tag{48}$$

5. Appliaction

Here, we'll use the suggested technique, the local fractional Sumudu decomposition method (LFSDM) to resolve a few cases.

Example 1. *In this illustration, we examine the following local fractional non-homogeneous KdV Equation:*

$$D_t^{(\alpha)}v + v^n D_x^{(\alpha)}v + D_x^{(3\alpha)}v = E_\alpha(-x^\alpha), \tag{49}$$

with the initial condition

$$v(x,0) = E_\alpha(-x^\alpha), \tag{50}$$

where

$$E_\alpha(x^\alpha) = \sum_{k=0}^{+\infty} \frac{x^{k\alpha}}{\Gamma(k\alpha+1)}, 0 < \alpha \le 1. \tag{51}$$

From Equation (44), we can write

$$\begin{aligned}\sum_{r=0}^{\infty} v_r(x,t) &= E_\alpha(-x^\alpha) + {}^{LF}S_\alpha^{-1}\{w^\alpha[{}^{LF}S_\alpha[E_\alpha(-x^\alpha)]]\} \\ &-{}^{LF}S_\alpha^{-1}\left\{w^\alpha\left[{}^{LF}S_\alpha\left(\sum_{r=0}^{\infty}A_r\right) + {}^{LF}S_\alpha\left[D_x^{(3\alpha)}\sum_{r=0}^{\infty}v_r(x,t)\right]\right]\right\}.\end{aligned} \tag{52}$$

Now, comparing terms on both sides of Equation (52), one has

$$v_0(x,t) = E_\alpha(-x^\alpha) + E_\alpha(-x^\alpha)\frac{t^\alpha}{\Gamma(1+\alpha)}, \tag{53}$$

and

$$v_1(x,t) = -{}^{LF}S_\alpha^{-1}\left\{w^\alpha\left[{}^{LF}S_\alpha(A_0) + {}^{LF}S_\alpha\left[D_x^{(3\alpha)}v_0(x,t)\right]\right]\right\}, \tag{54}$$

further, the above equation can be written as

$$v_1(x,t) = -{}^{LF}S_\alpha^{-1}\left[w^{\alpha\,LF}S_\alpha\left\{-E_\alpha^{n+1}(-x^\alpha) - E_\alpha(-x^\alpha)\right\}\right], \tag{55}$$

$$v_1(x,t) = \frac{t^\alpha}{\Gamma(1+\alpha)}\left\{E_\alpha^{n+1}(-x^\alpha) + E_\alpha(-x^\alpha)\right\}, \tag{56}$$

similarly, one can find

$$v_2(x,t) = -{}^{LF}S_\alpha^{-1}\left[w^{\alpha\,LF}S_\alpha\left\{nv_0^{n-1}v_1 D_x^{(\alpha)}v_0 + v_0^n D_x^{(\alpha)}v_1 + D_x^{(3\alpha)}v_1\right\}\right], \tag{57}$$

$$v_2(x,t) = -{}^{LF}S_\alpha^{-1}\left[w^{\alpha\,LF}S_\alpha \left\{ \begin{array}{l} nE_\alpha^{n-1}(-x^\alpha)\frac{t^\alpha}{\Gamma(1+\alpha)}\left(E_\alpha^{n+1}(-x^\alpha)+E_\alpha(-x^\alpha)\right) \\ (-E_\alpha(-x^\alpha))+E_\alpha^{n}(-x^\alpha)\frac{t^\alpha}{\Gamma(1+\alpha)} \\ \left(-(n+1)E_\alpha^{n+1}(-x^\alpha)-E_\alpha(-x^\alpha)\right)+ \\ \frac{t^\alpha}{\Gamma(1+\alpha)}\left(-(n+1)^3E_\alpha^{n+1}(-x^\alpha)-E_\alpha(-x^\alpha)\right) \end{array} \right\} \right] \quad (58)$$

$$v_2(x,t) = \frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\left[\begin{array}{l} nE_\alpha^{2n+1}(-x^\alpha)+nE_\alpha^{n+1}(-x^\alpha)+ \\ (n+1)E_\alpha^{2n+1}(-x^\alpha)+E_\alpha^{n+1}(-x^\alpha) \\ +(n+1)^3E_\alpha^{n+1}(-x^\alpha)+E_\alpha(-x^\alpha) \end{array} \right], \quad (59)$$

and so on.

Therefore, the solution of Equation (49) is given by:

$$v(x,t) = \sum_{r=0}^{\infty} v_r(x,t) = v_0(x,t)+v_1(x,t)+v_2(x,t)+\ldots \quad (60)$$

Particular Case*: when we substitute $n=1$ in Equation (49), we get*

$$D_t^{(\alpha)}v + vD_x^{(\alpha)}v + D_x^{(3\alpha)}v = E_\alpha(-x^\alpha), \quad (61)$$

with the initial condition

$$v(x,0) = E_\alpha(-x^\alpha). \quad (62)$$

Then by Equation (56)

$$v_1(x,t) = \frac{t^\alpha}{\Gamma(1+\alpha)}\left\{E_\alpha^2(-x^\alpha)+E_\alpha(-x^\alpha)\right\}, \quad (63)$$

and from Equation (59)

$$v_2(x,t) = \frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\left[\ 3E_\alpha^3(-x^\alpha)+10E_\alpha^2(-x^\alpha)+E_\alpha(-x^\alpha)\ \right], \quad (64)$$

and so on.

Substituting the above values into Equation (60), we attain the requisite outcome of (49).

The geometrical interpretation of the above result is (Figures 1–6):

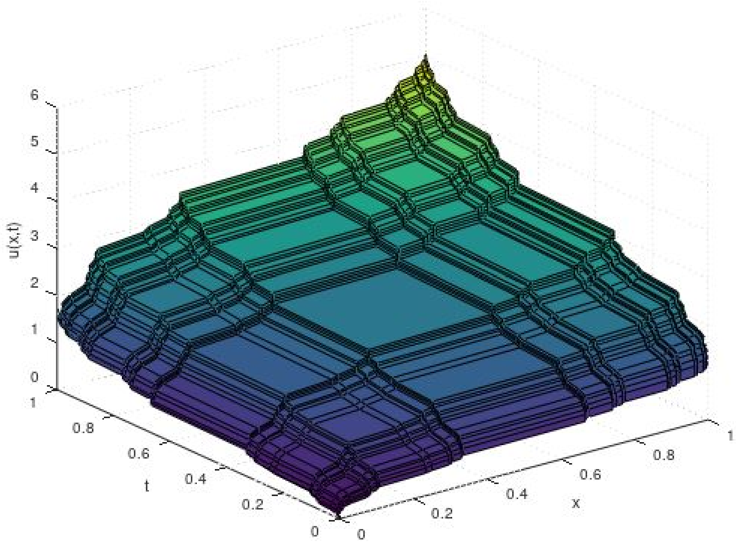

Figure 1. Outcome of Example 1 for $\alpha = 0.6309$ at $n = 2$.

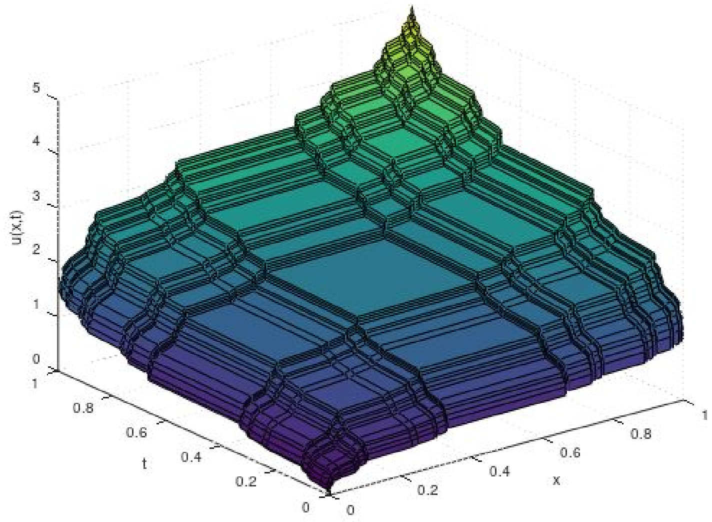

Figure 2. Outcome of Example 1 for $\alpha = 0.6309$ at n = 3.

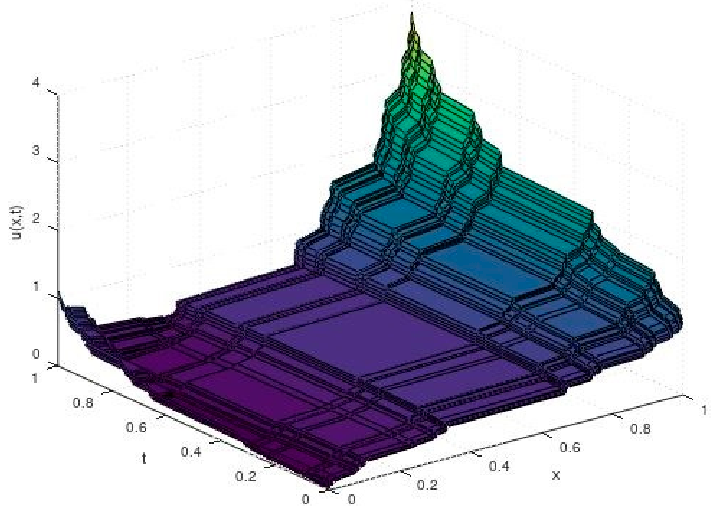

Figure 3. Outcome of Example 2 for $\alpha = 0.932$ at $n = 2$.

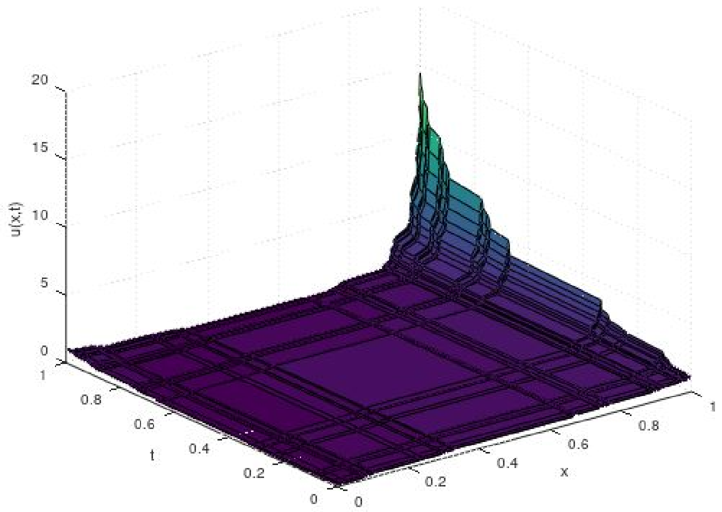

Figure 4. Outcome of Example2 for $\alpha = 0.932$ at $n = 3$.

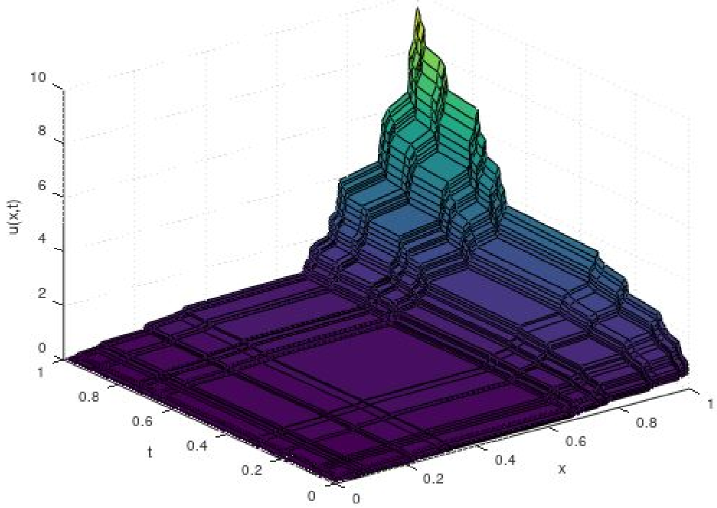

Figure 5. Outcome of Example 2 for $\alpha = 0.632$ at $n = 2$.

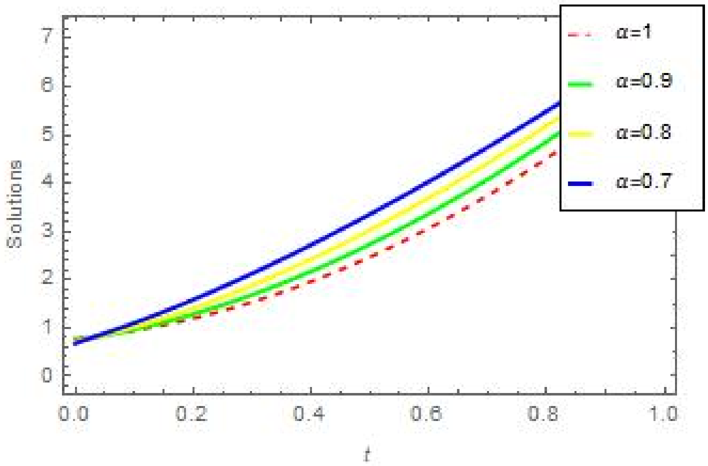

Figure 6. Outcome of Example 1 for different values of $\alpha = 1, 0.9, 0.8, 0.7$ at $n = 2$ and $x = 0.25$.

Example 2. *In this illustration, we consider the following local fractional homogeneous KdV equation:*

$$D_t^{(\alpha)} v + v^n D_x^{(\alpha)} v + D_x^{(3\alpha)} v = 0, \tag{65}$$

with the initial condition

$$v(x, 0) = \sin_\alpha(x^\alpha), \tag{66}$$

where the sine function is given by:

$$\sin_\alpha(x^\alpha) = \sum_{k=0}^{+\infty} (-1)^k \frac{x^{(2k+1)\alpha}}{\Gamma((2k+1)\alpha + 1)}, 0 < \alpha \leq 1. \tag{67}$$

From Equation (44), we get

$$\sum_{r=0}^{\infty} v_r(x, t) = \sin_\alpha(x^\alpha) - {}^{LF}S_\alpha^{-1} w^\alpha \left\{ \begin{array}{c} {}^{LF}S_\alpha\left(\sum_{r=0}^{\infty} A_r\right) + \\ {}^{LF}S_\alpha\left[D_x^{(3\alpha)} \sum_{r=0}^{\infty} v_r(x, t)\right] \end{array} \right\}. \tag{68}$$

Now, comparing terms on both sides of Equation (68), we get

$$v_0(x,t) = \sin_\alpha(x^\alpha), \tag{69}$$

$$v_1(x,t) = -{}^{LF}S_\alpha^{-1}\left\{w^\alpha\left[{}^{LF}S_\alpha(A_0) + {}^{LF}S_\alpha\left[D_x^{(3\alpha)}v_0(x,t)\right]\right]\right\}, \tag{70}$$

substituting the values of A_0 and $v_0(x,t)$ into the expression of $v_1(x,t)$, one gets the following.

$$v_1(x,t) = -{}^{LF}S_\alpha^{-1}\left[w^{\alpha\,LF}S_\alpha\{\sin_\alpha^n(x^\alpha)\cos_\alpha(x^\alpha) - \cos_\alpha(x^\alpha)\}\right],$$

$$v_1(x,t) = \frac{t^\alpha}{\Gamma(1+\alpha)}[-\sin_\alpha^n(x^\alpha)\cos_\alpha(x^\alpha) + \cos_\alpha(x^\alpha)], \tag{71}$$

similarly, we can find

$$v_2(x,t) = -{}^{LF}S_\alpha^{-1}\left[w^{\alpha\,LF}S_\alpha\left\{A_1 + D_x^{(3\alpha)}v_1\right\}\right], \tag{72}$$

now, substituting the values of A_1 and v_1 into $v_2(x,t)$, we get

$$v_2(x,t) = -{}^{LF}S_\alpha^{-1}\left[w^{\alpha\,LF}S_\alpha\left\{\frac{t^\alpha}{\Gamma(1+\alpha)}\left(\begin{array}{l} -2n\sin_\alpha^{2n-1}(x)^\alpha\cos_\alpha^2(x)^\alpha - \\ n(n-1)(n-2)\sin_\alpha^{n-3}(x)^\alpha \\ \cos_\alpha^4(x)^\alpha + \\ (6n^2-n)\sin_\alpha^{n-1}(x)^\alpha\cos_\alpha^2(x)^\alpha + \\ \sin_\alpha^{2n+1}(x)^\alpha - (3n+2)\sin_\alpha^{n+1}(x)^\alpha + \\ \sin_\alpha(x)^\alpha \end{array}\right)\right\}\right], \tag{73}$$

further,

$$v_2(x,t) = \frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\left[\begin{array}{l} 2n\sin_\alpha^{2n-1}(x)^\alpha\cos_\alpha^2(x)^\alpha + \\ n(n-1)(n-2)\sin_\alpha^{n-3}(x)^\alpha\cos_\alpha^4(x)^\alpha - \\ (6n^2-n)\sin_\alpha^{n-1}(x)^\alpha\cos_\alpha^2(x)^\alpha - \\ \sin_\alpha^{2n+1}(x)^\alpha + (3n+2)\sin_\alpha^{n+1}(x)^\alpha - \sin_\alpha(x)^\alpha \end{array}\right], \tag{74}$$

and so on.

Therefore the solution of Equation (65) is given by:

$$v(x,t) = \sum_{r=0}^{\infty} v_r(x,t) = v_0(x,t) + v_1(x,t) + v_2(x,t) + \ldots \tag{75}$$

Geometrical representation of the above solution is as follows (Figure 7):

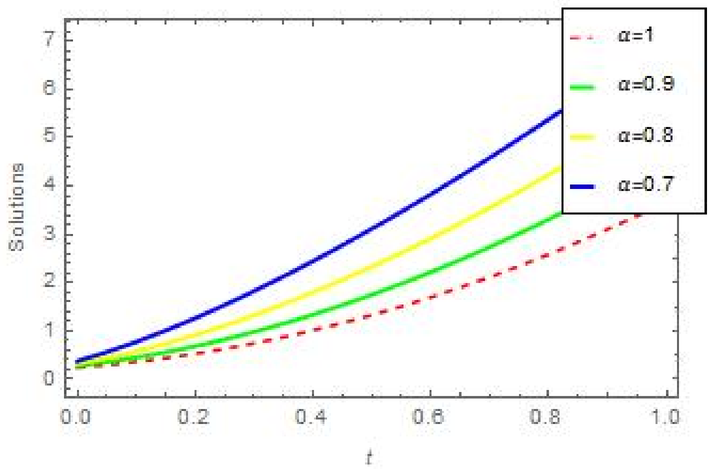

Figure 7. Outcome of Example 2 for $\alpha = 1, 0.9, 0.8, 0.7$ at $n = 2$ and $x = 0.25$.

6. Conclusions

This research work uses a method that merges the Sumudu transform and the Adomian decomposition method. Numerous fractional-order nonlinear PDEs can be solved effectively with this technique. This research is dedicated to solving the Korteweg–de Vries (KdV) equation. The applicability of the proposed method is represented in examples. Graphical representations of the solution demonstrate the relevance of the technique. In subsequent work, the method may be utilized to resolve coupled nonlinear PDEs and compare their results to demonstrate the competency of the technique.

Author Contributions: P.G. led the study, interpreted the results, and organized the required literature. J.G.P. wrote the manuscript, conducted all the numerical calculations, and made the graphs. K.S.A. and I.A. summarized the data for the tables, created the study site map, and formatted the final document. All authors have read and agreed to the published version of the manuscript.

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