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An Investigation on Existence, Uniqueness, and Approximate Solutions for Two-Dimensional Nonlinear Fractional Integro-Differential Equations

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Abstract: In this research, we provide sufficient conditions to prove the existence of local and global solutions for the general two-dimensional nonlinear fractional integro-differential equations. Furthermore, we prove that these solutions are unique. In addition, we use operational matrices of two-variable shifted Jacobi polynomials via the collocation method to reduce the equations into a system of equations. Error bounds of the presented method are obtained. Five test problems are solved. The obtained numerical results show the accuracy, efficiency, and applicability of the proposed approach.

Keywords: the mixed Riemann–Liouville integral; fixed-point theorems; shifted Jacobi polynomials; operational matrices; collocation method; error bound

MSC: 26A33, 33C45, 65N35

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1. Introduction

In the last decades, many problems, such as acoustic wave problems [1], groundwater pollution and groundwater flow problems [2–6], among others [7–10], have been shown by using fractional calculus. In addition, many engineering and physical problems, such as problems from control, electrochemistry, rheology, coupling and particle mechanics, viscoelasticity, electromagnetism fluid structure, and porous media (see e.g., [11–14]), have been mathematically formulated by fractional integro-differential equations (FIDEs). Recently, numerical methods for solving FIDEs have attracted the attention of many researchers. Taheri et al. [15] solved stochastic FIDEs by using the shifted Legendre spectral collocation method. Rahimkhani et al. [16] proposed the Bernoulli pseudo-spectral method for solving nonlinear Volterra FIDEs. Wang et al. [17] developed an approximate scheme based on fractional-order Euler functions to solve weakly singular FIDEs. Babaei et al. [18] considered a sixth-kind Chebyshev collocation method to solve a nonlinear quadratic FIDEs of variable order.

In the presented research, we focus on the following general two-dimensional nonlinear fractional integro-differential Equations (2D-NFIDEs):

$$\begin{aligned} &af_{yy}(x,y)+bf_{xx}(x,y)+cf_{yx}(x,y)+f(x,y)+\lambda I^{\varrho}f(x,y)=g(x,y)+\Theta(x,y)\\ &+\Lambda(x,y)+\rho(x,y)+\varphi(x,y), \end{aligned} \tag{1}$$

with the initial conditions of:

$$f(x,0)=d_1(x),\quad f(0,y)=d_2(y),\quad f_y(x,0)=d_3(x),\quad f_x(0,y)=d_4(y),\quad f_x(x,0)=d_5(x), \tag{2}$$

where $(x,y)\in\mathfrak{D}=[0,\ell_1)\times[0,\ell_2)$; $\varrho=(\varrho_1,\varrho_2)\in(0,\infty)\times(0,\infty)$; and a, b, c, λ are constants, and

$$\Theta(x,y) = \int_0^x \int_0^y k_1(x,t,y,s) f^{p_1}(t,s)\,\mathrm{d}s\,\mathrm{d}t,$$
$$\Lambda(x,y) = \frac{1}{\Gamma(\varrho_1)\Gamma(\varrho_2)} \int_0^x \int_0^y (x-t)^{\varrho_1-1}(y-s)^{\varrho_2-1} k_2(x,t,y,s) f^{p_2}(t,s)\,\mathrm{d}s\,\mathrm{d}t,$$
$$\rho(x,y) = \frac{1}{\Gamma(\varrho_1)\Gamma(\varrho_2)} \int_0^{\ell_1} \int_0^{\ell_2} (\ell_1-t)^{\varrho_1-1}(\ell_2-s)^{\varrho_2-1} k_3(x,t,y,s) f^{p_3}(t,s)\,\mathrm{d}s\,\mathrm{d}t,$$
$$\varphi(x,y) = \int_0^{\ell_1} \int_0^{\ell_2} k_4(x,t,y,s) f^{p_4}(t,s)\,\mathrm{d}s\,\mathrm{d}t.$$

Here, functions $d_i(.)$, $i = 1,2,3,4,5$, $k_j(x,t,y,s)$, $j = 1,2,3,4$, $g(x,y)$ are known, and $f(x,y)$ is unknown; $I^{\varrho} f(x,y)$ is the left-sided mixed Riemann–Liouville integral of order $\varrho = (\varrho_1, \varrho_2) \in (0,\infty) \times (0,\infty)$ of f denoted by [19]

$$I^{\varrho} f(x,y) = \frac{1}{\Gamma(\varrho_1)\Gamma(\varrho_2)} \int_0^x \int_0^y (x-t)^{\varrho_1-1}(y-s)^{\varrho_2-1} f(t,s)\,\mathrm{d}s\,\mathrm{d}t;$$

and $p_j \geq 1$, $j = 1,2,3,4$ are constants.

While several numerical techniques have been proposed for solving many different problems (see, for instance, [20–35] and references therein), there were few research studies that developed numerical methods for solving Equations (1) and (2). For example, Najafalizadeh and Ezzati [36] obtained approximate solutions of these equations by using operational matrices of two-dimensional block pulse functions (2D-BPFs) with the order of convergence $O(\frac{1}{N})$, $N \in \mathbb{N}$. Maleknejad et al. [37] applied operational matrices based on a hybrid of two-dimensional block-pulse functions and shifted Legendre polynomials (2D-HBPSLs) to solve the general 2D-NFIDEs. The order of convergence of this method was $O(\frac{1}{2^{2M-1} N^M M!})$.

According to the best of our knowledge, the existence and uniqueness of solutions for Equations (1) and (2) have not been discussed so far. In this research, we provide sufficient conditions to prove that there exist local and global solutions for the general 2D-NFIDEs. Then, we prove that the solutions of these equations are unique. Additionally, we prepare an efficient numerical approach to approximate solutions of the general 2D-NFIDEs with high accuracy.

The rest of this paper is organized as follows: in Section 2, some theorems for the existence and uniqueness of solutions of general 2D-NFIDEs are proved. In Section 3, an introduction of one- and two-variable shifted Jacobi polynomials (1D-SJPs and 2D-SJPs) is provided. Additionally, some operational matrices are introduced. In Section 4, by using the collocation method via these operational matrices, approximate solutions for Equations (1) and (2) are obtained. In Section 5, error bounds of approximations are obtained. In Section 6, five test problems are solved to show the accuracy of the proposed method. In Section 7, a conclusion is presented.

2. Existence and Uniqueness of Solutions

Now, by using Schauder's fixed-point theorem [38], a local existence of solutions of general 2D-NIDEFs is proved in a Banach space.

Theorem 1. *Suppose that*

(C1) $0 \leq t \leq x \leq \ell_1$, $0 \leq s \leq y \leq \ell_2$, $g, g_1, f, v \in C(\mathfrak{D}, \mathbb{R}^n)$, $k_1, k_2, k_3, k_4 \in C(\mathfrak{D} \times \mathfrak{D} \times \mathbb{R}^n, \mathbb{R}^n)$;

(C2) $\|f_{yy}(x,y) - v_{yy}(x,y)\| < \frac{\varepsilon}{24a}$, $\|f_{xx}(x,y) - v_{xx}(x,y)\| < \frac{\varepsilon}{24b}$, $\|f_{yx}(x,y) - v_{yx}(x,y)\| < \frac{\varepsilon}{24c}$, $\|I^{\varrho} f(x,y) - I^{\varrho} v(x,y)\| < \frac{\varepsilon}{24\lambda}$;

(C3) $\|g(x,y) - g_1(x,y)\| < \frac{\varepsilon}{6}$;

(C4) $\|k_i(x,t,y,s,f(t,s)) - k_i(x,t,y,s,v(t,s))\| < \frac{\varepsilon}{6\alpha\beta}$, $i = 1,4$, $0 < \alpha < \ell_1$, $0 < \beta < \ell_2$;

(C5) $\left\|k_j(x,t,y,s,f(t,s)) - k_j(x,t,y,s,v(t,s))\right\| < \frac{\varepsilon\Gamma(\varrho_1+1)\Gamma(\varrho_2+1)}{6\alpha^{\varrho_1}\beta^{\varrho_2}}$, $j = 2,3$, $0 < \alpha < \ell_1$, $0 < \beta < \ell_2$.

Then, there exists at least one solution for the 2D-NIDEF on $0 \le t \le \alpha$, $0 \le s \le \beta$.

Proof. Suppose that $\mathcal{D} = \{(x,t,y,s,f) : (x,t,y,s) \in \mathfrak{D}\times\mathfrak{D}, |f| \le b'\}$. Let $\left|f_{yy}(x,y)\right| \le \frac{b'}{16a}$, $|f_{xx}(x,y)| \le \frac{b'}{16b}$, $\left|f_{yx}(x,y)\right| \le \frac{b'}{16c}$, $|I^{\varrho} f(x,y)| \le \frac{b'}{16\lambda}$, $|g(x,y)| \le \frac{b'}{4}$,

$$\max\{|k_i(x_1,t,y_1,s,f(t,s))|, |k_i(x_2,t,y_2,s,f(t,s))|\} = \xi_i, \quad i = 1,2,3,4,$$

on $\mathcal{D}$. Choose $(\xi_1+\xi_4)\alpha\beta \le \frac{b'}{4}$, $\frac{(\xi_2+\xi_3)\alpha^{\varrho_1}\beta^{\varrho_2}}{\Gamma(\varrho_1+1)\Gamma(\varrho_2+1)} \le \frac{b'}{4}$. Consider $\Pi_0 = \{f : f \in C(\mathfrak{D}_0,\mathbb{R}^n)$, $|f| \le b'\}$ such that $\|f\| = \max\limits_{(x,y)\in\mathfrak{D}_0} |f(x,y)|$, $\mathfrak{D}_0 = [0,\alpha]\times[0,\beta]$. Clearly, Π_0 is bounded, closed, and convex. Now, for any $f \in \Pi_0$, define the operator

$$\begin{aligned}\mathcal{T}f(x,y) =& -af_{yy}(x,y) - bf_{xx}(x,y) - cf_{yx}(x,y) - \lambda I^{\varrho} f(x,y) + g(x,y) + \Theta(x,y)\\ &+ \Lambda(x,y) + \rho(x,y) + \varphi(x,y), \quad (x,y) \in \mathfrak{D}_0.\end{aligned} \tag{3}$$

It is clear that

$$\begin{aligned}|\Theta(x,y)| &\le \xi_1\alpha\beta,\\ |\Lambda(x,y)| &\le \frac{\xi_2\alpha^{\varrho_1}\beta^{\varrho_2}}{\Gamma(\varrho_1+1)\Gamma(\varrho_2+1)},\\ |\rho(x,y)| &\le \frac{\xi_3\alpha^{\varrho_1}\beta^{\varrho_2}}{\Gamma(\varrho_1+1)\Gamma(\varrho_2+1)},\\ |\varphi(x,y)| &\le \xi_4\alpha\beta.\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}|\mathcal{T}f(x,y)| \le& \left|af_{yy}(x,y)\right| + |bf_{xx}(x,y)| + \left|cf_{yx}(x,y)\right| + |\lambda I^{\varrho} f(x,y)| + |g(x,y)| + |\Theta(x,y)|\\ &+ |\Lambda(x,y)| + |\rho(x,y)| + |\varphi(x,y)|\\ \le& \frac{b'}{2} + (\xi_1+\xi_4)\alpha\beta + \frac{(\xi_2+\xi_3)\alpha^{\varrho_1}\beta^{\varrho_2}}{\Gamma(\varrho_1+1)\Gamma(\varrho_2+1)} \le b',\end{aligned}$$

which implies that $\mathcal{T}(\Pi_0) \subset \Pi_0$. Furthermore, for any (x_1,y_1), $(x_2,y_2) \in \mathfrak{D}_0$, such that $x_2 > x_1$ and $y_2 > y_1$, we obtain

$$\begin{aligned}|\mathcal{T}f(x_2,y_2) - \mathcal{T}f(x_1,y_1)| \le& a\left|f_{yy}(x_2,y_2) - f_{yy}(x_1,y_1)\right| + b|f_{xx}(x_2,y_2) - f_{xx}(x_1,y_1)|\\ &+ c\left|f_{yx}(x_2,y_2) - f_{yx}(x_1,y_1)\right| + \lambda|I^{\varrho} f(x_2,y_2) - I^{\varrho} f(x_1,y_1)|\\ &+ |g(x_2,y_2) - g(x_1,y_1)| + |\Theta(x_2,y_2) - \Theta(x_1,y_1)|\\ &+ |\Lambda(x_2,y_2) - \Lambda(x_1,y_1)| + |\rho(x_2,y_2) - \rho(x_1,y_1)|\\ &+ |\varphi(x_2,y_2) - \varphi(x_1,y_1)|.\end{aligned} \tag{4}$$

Additionally, we have

$$
\begin{aligned}
&|I^{\varrho}f(x_2,y_2)-I^{\varrho}f(x_1,y_1)| \\
&\le \frac{1}{\Gamma(\varrho_1)\Gamma(\varrho_2)}\bigg|\int_0^{x_2}\int_0^{y_2}(x_2-t)^{\varrho_1-1}(y_2-s)^{\varrho_2-1}f(t,s)\mathrm{d}s\mathrm{d}t \\
&-\int_0^{x_1}\int_0^{y_1}(x_1-t)^{\varrho_1-1}(y_1-s)^{\varrho_2-1}f(t,s)\mathrm{d}s\mathrm{d}t\bigg| \\
&\le \frac{1}{\Gamma(\varrho_1)\Gamma(\varrho_2)}\bigg|\int_0^{x_1}\int_0^{y_1}(x_2-t)^{\varrho_1-1}(y_2-s)^{\varrho_2-1}f(t,s)\mathrm{d}s\mathrm{d}t \\
&+\int_{x_1}^{x_2}\int_{y_1}^{y_2}(x_2-t)^{\varrho_1-1}(y_2-s)^{\varrho_2-1}f(t,s)\mathrm{d}s\mathrm{d}t \\
&-\int_0^{x_1}\int_0^{y_1}(x_1-t)^{\varrho_1-1}(y_1-s)^{\varrho_2-1}f(t,s)\,\mathrm{d}s\,\mathrm{d}t\bigg| \\
&\le \frac{1}{\Gamma(\varrho_1)\Gamma(\varrho_2)}\bigg|\int_0^{x_1}\int_0^{y_1}\Big((x_2-t)^{\varrho_1-1}(y_2-s)^{\varrho_2-1}-(x_1-t)^{\varrho_1-1}(y_1-s)^{\varrho_2-1}\Big)f(t,s)\,\mathrm{d}s\,\mathrm{d}t \\
&+\int_{x_1}^{x_2}\int_{y_1}^{y_2}(x_2-t)^{\varrho_1-1}(y_2-s)^{\varrho_2-1}f(t,s)\mathrm{d}s\mathrm{d}t\bigg| \\
&\le \frac{b'}{\Gamma(\varrho_1+1)\Gamma(\varrho_2+1)}\bigg(\int_0^{x_1}\int_0^{y_1}\Big((x_2-t)^{\varrho_1-1}(y_2-s)^{\varrho_2-1}-(x_1-t)^{\varrho_1-1}(y_1-s)^{\varrho_2-1}\Big)\mathrm{d}s\,\mathrm{d}t \\
&+\int_{x_1}^{x_2}\int_{y_1}^{y_2}(x_2-t)^{\varrho_1-1}(y_2-s)^{\varrho_2-1}\mathrm{d}s\mathrm{d}t\bigg) \\
&\le \frac{b'}{\Gamma(\varrho_1+1)\Gamma(\varrho_2+1)}\big((x_2-x_1)^{\varrho_1}(y_2-y_1)^{\varrho_2}-x_2^{\varrho_1}y_2^{\varrho_2}+x_1^{\varrho_1}y_1^{\varrho_2}-(x_2-x_1)^{\varrho_1}(y_2-y_1)^{\varrho_2}\big) \\
&=0.
\end{aligned}
$$

Therefore,

$$|\mathcal{T}f(x_2,y_2)-\mathcal{T}f(x_1,y_1)|=0. \tag{5}$$

Moreover, we can obtain

$$
\begin{aligned}
&|\Lambda(x_2,y_2)-\Lambda(x_1,y_1)| \le \frac{1}{\Gamma(\varrho_1)\Gamma(\varrho_2)}\bigg|\int_0^{x_2}\int_0^{y_2}(x_2-t)^{\varrho_1-1}(y_2-s)^{\varrho_2-1}k_2(x_2,t,y_2,s,f(t,s))\mathrm{d}s\mathrm{d}t \\
&-\int_0^{x_1}\int_0^{y_1}(x_1-t)^{\varrho_1-1}(y_1-s)^{\varrho_2-1}k_2(x_1,t,y_1,s,f(t,s))\mathrm{d}s\mathrm{d}t\bigg| \\
&=\frac{1}{\Gamma(\varrho_1)\Gamma(\varrho_2)}\bigg|\int_0^{x_1}\int_0^{y_1}(x_2-t)^{\varrho_1-1}(y_2-s)^{\varrho_2-1}k_2(x_2,t,y_2,s,f(t,s))\mathrm{d}s\mathrm{d}t \\
&+\int_{x_1}^{x_2}\int_{y_1}^{y_2}(x_2-t)^{\varrho_1-1}(y_2-s)^{\varrho_2-1}k_2(x_2,t,y_2,s,f(t,s))\mathrm{d}s\mathrm{d}t \\
&-\int_0^{x_1}\int_0^{y_1}(x_1-t)^{\varrho_1-1}(y_1-s)^{\varrho_2-1}k_2(x_1,t,y_1,s,f(t,s))\mathrm{d}s\mathrm{d}t\bigg| \\
&=\frac{1}{\Gamma(\varrho_1)\Gamma(\varrho_2)}\bigg|\int_0^{x_1}\int_0^{y_1}\Big((x_2-t)^{\varrho_1-1}(y_2-s)^{\varrho_2-1}k_2(x_2,t,y_2,s,f(t,s)) \\
&-(x_1-t)^{\varrho_1-1}(y_1-s)^{\varrho_2-1}k_2(x_1,t,y_1,s,f(t,s))\Big)\mathrm{d}s\mathrm{d}t \\
&+\int_{x_1}^{x_2}\int_{y_1}^{y_2}(x_2-t)^{\varrho_1-1}(y_2-s)^{\varrho_2-1}k_2(x_2,t,y_2,s,f(t,s))\mathrm{d}s\mathrm{d}t\bigg| \\
&\le \frac{1}{\Gamma(\varrho_1)\Gamma(\varrho_2)}\int_0^{x_1}\int_0^{y_1}\Big((x_2-t)^{\varrho_1-1}(y_2-s)^{\varrho_2-1}|k_2(x_2,t,y_2,s,f(t,s))| \\
&+(x_1-t)^{\varrho_1-1}(y_1-s)^{\varrho_2-1}|k_2(x_1,t,y_1,s,f(t,s))|\Big)\mathrm{d}s\mathrm{d}t \\
&+\frac{1}{\Gamma(\varrho_1)\Gamma(\varrho_2)}\int_{x_1}^{x_2}\int_{y_1}^{y_2}(x_2-t)^{\varrho_1-1}(y_2-s)^{\varrho_2-1}|k_2(x_2,t,y_2,s,f(t,s))|\mathrm{d}s\mathrm{d}t \\
&\le \frac{\xi_2}{\Gamma(\varrho_1)\Gamma(\varrho_2)}\bigg(\int_0^{x_1}\int_0^{y_1}\Big((x_2-t)^{\varrho_1-1}(y_2-s)^{\varrho_2-1}+(x_1-t)^{\varrho_1-1}(y_1-s)^{\varrho_2-1}\Big)\mathrm{d}s\mathrm{d}t \\
&+\int_{x_1}^{x_2}\int_{y_1}^{y_2}(x_2-t)^{\varrho_1-1}(y_2-s)^{\varrho_2-1}\mathrm{d}s\mathrm{d}t\bigg) \\
&\le \frac{\xi_2}{\Gamma(\varrho_1+1)\Gamma(\varrho_2+1)}\big((x_2-x_1)^{\varrho_1}(y_2-y_1)^{\varrho_2}-x_2^{\varrho_1}y_2^{\varrho_2}+x_1^{\varrho_1}y_1^{\varrho_2}+(x_2-x_1)^{\varrho_1}(y_2-y_1)^{\varrho_2}\big) \\
&\le \frac{2\xi_2}{\Gamma(\varrho_1+1)\Gamma(\varrho_2+1)}(x_2-x_1)^{\varrho_1}(y_2-y_1)^{\varrho_2}.
\end{aligned} \tag{6}
$$

Similarly,

$$|\Theta(x_2,y_2) - \Theta(x_1,y_1)| \le \xi_1(x_2y_2 - x_1y_1), \tag{7}$$

$$|\rho(x_2,y_2) - \rho(x_1,y_1)| \le \xi_3\left(\frac{\alpha^{\varrho_1}\beta^{\varrho_2}}{\Gamma(\varrho_1+1)\Gamma(\varrho_2+1)} - \frac{\alpha^{\varrho_1}\beta^{\varrho_2}}{\Gamma(\varrho_1+1)\Gamma(\varrho_2+1)}\right) = 0, \tag{8}$$

$$|\varphi(x_2,y_2) - \varphi(x_1,y_1)| \le \xi_4(\alpha\beta - \alpha\beta) = 0. \tag{9}$$

Applying inequalities (5)–(9) in (4) gives

$$\begin{aligned}|\mathcal{T}f(x_2,y_2) - \mathcal{T}f(x_1,y_1)| \le & |(\mathscr{F}f)(x_2,y_2) - (\mathscr{F}f)(x_1,y_1)| + \lambda|I^{\varrho}f(x_2,y_2) - I^{\varrho}f(x_1,y_1)| \\ & + |g(x_2,y_2) - g(x_1,y_1)| + \xi_1(x_2y_2 - x_1y_1) \\ & + \frac{2\xi_2}{\Gamma(\varrho_1+1)\Gamma(\varrho_2+1)}(x_2-x_1)^{\varrho_1}(y_2-y_1)^{\varrho_2},\end{aligned} \tag{10}$$

where

$$(\mathscr{F}f)(x,y) = -af_{yy}(x,y) - bf_{xx}(x,y) - cf_{yx}(x,y).$$

It is clear that the right-hand side of (10) tends to zero as $x_2 \to x_1$, $y_2 \to y_1$. Thus, $\mathcal{T} : \Pi_0 \to \Pi_0$ is equicontinuous. Therefore, by using the Arzela–Ascoli theorem [39], the compactness of the closure of $\mathcal{T}(\Pi_0)$ can be concluded.

Now, we need to show that $\mathcal{T}$ is continuous. For this propose, define

$$\mathcal{T}v(x,y) = (\mathscr{F}v)(x,y) - \lambda I^{\varrho}v(x,y) + g_1(x,y) + \Theta_v(x,y) + \Lambda_v(x,y) + \rho_v(x,y) + \varphi_v(x,y),$$

$$v(x,0) = d_1(x),\ v(0,y) = d_2(y),\ v_y(x,0) = d_3(x),\ v_x(0,y) = d_4(y),\ v_x(x,0) = d_5(x),$$

where $(x,y) \in \mathfrak{D}_0$, $v \in \Pi_0$, and

$$\begin{aligned}(\mathscr{F}v)(x,y) &= -av_{yy}(x,y) - bv_{xx}(x,y) - cv_{yx}(x,y), \\ \Theta_v(x,y) &= \int_0^x \int_0^y k_1(x,t,y,s,v(t,s))\mathrm{d}s\mathrm{d}t, \\ \Lambda_v(x,y) &= \frac{1}{\Gamma(\varrho_1)\Gamma(\varrho_2)}\int_0^x \int_0^y (x-t)^{\varrho_1-1}(y-s)^{\varrho_2-1}k_2(x,t,y,s,v(t,s))\mathrm{d}s\mathrm{d}t, \\ \rho_v(x,y) &= \frac{1}{\Gamma(\varrho_1)\Gamma(\varrho_2)}\int_0^\alpha \int_0^\beta (\alpha-t)^{\varrho_1-1}(\beta-s)^{\varrho_2-1}k_3(x,t,y,s,v(t,s))\mathrm{d}s\mathrm{d}t, \\ \varphi_v(x,y) &= \int_0^\alpha \int_0^\beta k_4(x,t,y,s,v(t,s))\mathrm{d}s\mathrm{d}t.\end{aligned}$$

Since k_i, $i = 1,2,3,4$, are uniformly continuous, we can write

$$\forall \varepsilon > 0,\ \exists \delta > 0 : \quad |f(x,y) - v(x,y)| < \delta.$$

Suppose that the assumptions (**C1**)–(**C5**) hold; therefore,

$$\begin{aligned}|\mathcal{T}f(x,y) - \mathcal{T}v(x,y)| \le & |(\mathscr{F}f)(x,y) - (\mathscr{F}v)(x,y)| + \lambda|I^{\varrho}f(x,y) - I^{\varrho}v(x,y)| \\ & + |g(x,y) - g_1(x,y)| + |\Theta(x,y) - \Theta_v(x,y)| \\ & + |\Lambda(x,y) - \Lambda_v(x,y)| + |\rho(x,y) - \rho_v(x,y)| \\ & + |\varphi(x,y) - \varphi_v(x,y)|.\end{aligned}$$

Furthermore, we can easily obtain the following inequalities:

$$|\Theta(x,y)-\Theta_v(x,y)|\le \frac{\varepsilon}{6\alpha\beta}\int_0^x\int_0^y \mathrm{d}s\mathrm{d}t \le \frac{\varepsilon}{6},$$
$$|\Lambda(x,y)-\Lambda_v(x,y)|\le \frac{1}{\Gamma(\varrho_1)\Gamma(\varrho_2)}\frac{\varepsilon\Gamma(\varrho_1+1)\Gamma(\varrho_2+1)}{6\alpha^{\varrho_1}\beta^{\varrho_2}}\int_0^x\int_0^y (x-t)^{\varrho_1-1}(y-s)^{\varrho_2-1}\mathrm{d}s\mathrm{d}t \le \frac{\varepsilon}{6},$$
$$|\rho(x,y)-\rho_v(x,y)|\le \frac{1}{\Gamma(\varrho_1)\Gamma(\varrho_2)}\frac{\varepsilon\Gamma(\varrho_1+1)\Gamma(\varrho_2+1)}{6\alpha^{\varrho_1}\beta^{\varrho_2}}\int_0^\alpha\int_0^\beta (\alpha-t)^{\varrho_1-1}(\beta-s)^{\varrho_2-1}\mathrm{d}s\mathrm{d}t = \frac{\varepsilon}{6},$$
$$|\varphi(x,y)-\varphi_v(x,y)|\le \frac{\varepsilon}{6\alpha\beta}\int_0^\alpha\int_0^\beta \mathrm{d}s\mathrm{d}t = \frac{\varepsilon}{6}.$$

Thus, we have

$$|\mathcal{T}f(x,y)-\mathcal{T}v(x,y)|\le\varepsilon,$$

and the proof is completed. □

In the following theorem, by using Tychonoff's fixed-point theorem [38], the global existence of solutions of the general 2D-NFIDEs will be discussed.

Theorem 2. *Suppose that*

(D1) $G_i\in C(\mathbb{R}_+^5,\mathbb{R}^n)$, $k_i\in C(\mathbb{R}_+^4\times\mathbb{R}^n,\mathbb{R}^n)$, $i=1,2,3,4$;

(D2) *For each* $(x,t,y,s)\in\mathbb{R}_+^4$, $G_i(x,t,y,s,u(t,s))$, $i=1,2,3,4$, *are monotonically non-decreasing in* u;

(D3) $|k_i(x,t,y,s,f(t,s))|\le G_i(x,t,y,s,|f(t,s)|)$, $(x,t,y,s,f(t,s))\in\mathbb{R}_+^4\times\mathbb{R}^n$, $i=1,2,3,4$;

(D4) $|(\mathscr{F}f)(x,y)|\le(\mathscr{F}u)(x,y)$.

Then, for every $x,y\ge 0$, *the generalized two-dimensional nonlinear fractional integro-differential equation*

$$u(x,y)=(\mathscr{F}u)(x,y)+\lambda' I^{\varrho}u(x,y)+q(x,y)+\Theta_u(x,y)+\Lambda_u(x,y)+\rho_u(x,y)+\varphi_u(x,y), \tag{11}$$

has a solution $u(x,y)$ *with initial conditions*

$$u(x,0)=d_1(x),\ u(0,y)=d_2(y),\ u_y(x,0)=d_3(x),\ u_x(0,y)=d_4(y),\ u_x(x,0)=d_5(x), \tag{12}$$

and

$$(\mathscr{F}u)(x,y)=-a\,u_{yy}(x,y)-b\,u_{xx}(x,y)-c\,u_{yx}(x,y),$$
$$I^{\varrho}u(x,y)=\frac{1}{\Gamma(\varrho_1)\Gamma(\varrho_2)}\int_0^x\int_0^y (x-t)^{\varrho_1-1}(y-s)^{\varrho_2-1}u(t,s)\mathrm{d}s\mathrm{d}t,$$
$$\Theta_u(x,y)=\int_0^x\int_0^y G_1(x,t,y,s,u(t,s))\mathrm{d}s\mathrm{d}t,$$
$$\Lambda_u(x,y)=\frac{1}{\Gamma(\varrho_1)\Gamma(\varrho_2)}\int_0^x\int_0^y (x-t)^{\varrho_1-1}(y-s)^{\varrho_2-1}G_2(x,t,y,s,u(t,s))\mathrm{d}s\mathrm{d}t,$$
$$\rho_u(x,y)=\frac{1}{\Gamma(\varrho_1)\Gamma(\varrho_2)}\int_0^{\ell_1}\int_0^{\ell_2} (\ell_1-t)^{\varrho_1-1}(\ell_2-s)^{\varrho_2-1}G_3(x,t,y,s,u(t,s))\mathrm{d}s\mathrm{d}t,$$
$$\varphi_u(x,y)=\int_0^{\ell_1}\int_0^{\ell_2} G_4(x,t,y,s,u(t,s))\mathrm{d}s\mathrm{d}t.$$

Additionally, for every $x,y\ge 0$ *and* $q(x,y)\in\mathbb{R}_+^2$, *such that* $|g(x,y)|\le q(x,y)$, *there exists a solution* $f(x,y)$ *for Equations* (1) *and* (2) *satisfying* $|f(x,y)|\le u(x,y)$ *and* $|\lambda|\le\lambda'$.

Proof. Let $\mathcal{Q}$ be a real space of all continuous functions from $(0,\infty)\times(0,\infty)$ into $\mathbb{R}^n$. The topology on $\mathcal{Q}$ is that induced by the family of pseudo-norms $\{\mathcal{Q}_{m',m}(f)\}_{m',m=1}^{\infty}$, where $\mathcal{Q}_{m',m}(f) = \sup_{0\leq x\leq m', 0\leq y\leq m} |f(x,y)|$ for $f \in \mathcal{Q}$. Consider $\{\mathcal{S}_{m',m}\}_{m',m=1}^{\infty}$ as a set of neighborhoods with $\mathcal{S}_{m',m} = \{f \in \mathcal{Q} : \mathcal{Q}_{m',m}(f) \leq 1\}$. Under this topology, $\mathcal{Q}$ is complete, locally convex, and a linear space.

Let

$$\mathcal{Q}_0 = \{f \in \mathcal{Q} : |f(x,y)| \leq u(x,y),\ x,y \geq 0\} \subseteq \mathcal{Q},$$

where $u(x,y)$ is a solution of Equations (11) and (12). Obviously, in the topology of $\mathcal{Q}$, $\mathcal{Q}_0$ is closed, convex, and bounded.

Note that a fixed point of Equations (11) and (12) corresponds to a solution of Equations (1) and (2). Since, in the topology of $\mathcal{Q}$, $\mathcal{T}$ is compact and $\mathcal{Q}_0$ is bounded, therefore, the closure of $T(\mathcal{Q}_0)$ is compact.

Considering assumptions (**D1**)–(**D4**) yields

$$\begin{aligned}|\Theta(x,y)| &\leq \int_0^x \int_0^y |k_1(x,t,y,s,f(t,s))| \mathrm{d}s\mathrm{d}t \leq \int_0^x \int_0^y G_1(x,t,y,s,|f(t,s)|)\mathrm{d}s\mathrm{d}t \\ &\leq \int_0^x \int_0^y G_1(x,t,y,s,u(t,s))\mathrm{d}s\mathrm{d}t = \Theta_u(x,y).\end{aligned}$$

Similarly,

$$|\Lambda(x,y)| \leq \Lambda_u(x,y), \quad |\rho(x,y)| \leq \rho_u(x,y), \quad |\varphi(x,y)| \leq \varphi_u(x,y), \quad |I^{\varrho} f(x,y)| \leq u(x,y).$$

Since $u(x,y)$ is a solution of Equations (11) and (12), the definition of $\mathcal{Q}_0$ yields $|\mathcal{T}f(x,y)| \leq u(x,y)$. Therefore, $\mathcal{T}(\mathcal{Q}_0) \subset \mathcal{Q}_0$. Now, by using Tychonoff's fixed-point theorem [38], we can deduce that $\mathcal{T}$ has a fixed point in $\mathcal{Q}_0$, and this completes the proof. □

In the following theorem, we prove that the general 2D-NFIDE has a unique solution.

Theorem 3. *Consider $k_i \in C(\mathfrak{D} \times \mathfrak{D} \times \mathbb{R}^n, \mathbb{R}^n)$ $(i = 1,2,3,4)$, $f \in C(\mathfrak{D}, \mathbb{R}^n)$. Assume that there exist $0 < L_j < 1$ $(j = 1,2,3)$ such that:*

$$\left|f_{yy}(x,y) - \overline{f}_{yy}(x,y)\right| \leq L_1\left|f(x,y) - \overline{f}(x,y)\right|, \tag{13}$$

$$\left|f_{xx}(x,y) - \overline{f}_{xx}(x,y)\right| \leq L_2\left|f(x,y) - \overline{f}(x,y)\right|, \tag{14}$$

$$\left|f_{yx}(x,y) - \overline{f}_{yx}(x,y)\right| \leq L_3\left|f(x,y) - \overline{f}(x,y)\right|, \tag{15}$$

$$\left|k_i(x,t,y,s,f(t,s)) - k_i(x,t,y,s,\overline{f}(t,s))\right| \leq \eta_i\left|f(t,s) - \overline{f}(t,s)\right|, \quad i = 1,2,3,4. \tag{16}$$

If

$$\left((aL_1 + bL_2 + cL_3) + \frac{\ell_1^{\varrho_1}\ell_2^{\varrho_2}}{\Gamma(\varrho_1+1)\Gamma(\varrho_2+1)}(\lambda + \eta_1 + \eta_2 + \eta_3 + \eta_4)\right) < 1, \tag{17}$$

then the general 2D-NIDEF has a unique solution.

Proof. Let

$$\mathcal{T}\overline{f}(x,y) = (\mathscr{F}\overline{f})(x,y) - \lambda I^{\varrho}\overline{f}(x,y) + g(x,y) + \bar{\Theta}(x,y) + \bar{\Lambda}(x,y) + \bar{\rho}(x,y) + \bar{\varphi}(x,y),$$

with

$$\overline{f}(x,0) = d_1(x), \quad \overline{f}(0,y) = d_2(y), \quad \overline{f}_y(x,0) = d_3(x), \quad \overline{f}_x(0,y) = d_4(y), \quad \overline{f}_x(x,0) = d_5(x),$$

and

$$
\begin{aligned}
(\mathscr{F}\overline{f})(x,y) &= -a\overline{f}_{yy}(x,y) - b\overline{f}_{xx}(x,y) - c\overline{f}_{yx}(x,y),\\
\overline{\Theta}(x,y) &= \int_0^x \int_0^y k_1(x,t,y,s)\overline{f}^{p_1}(t,s)\mathrm{d}s\mathrm{d}t,\\
\overline{\Lambda}(x,y) &= \frac{1}{\Gamma(\varrho_1)\Gamma(\varrho_2)} \int_0^x \int_0^y (x-t)^{\varrho_1-1}(y-s)^{\varrho_2-1}k_2(x,t,y,s)\overline{f}^{p_2}(t,s)\mathrm{d}s\mathrm{d}t,\\
\overline{\rho}(x,y) &= \frac{1}{\Gamma(\varrho_1)\Gamma(\varrho_2)} \int_0^{\ell_1} \int_0^{\ell_2} (\ell_1-t)^{\varrho_1-1}(\ell_2-s)^{\varrho_2-1}k_3(x,t,y,s)\overline{f}^{p_3}(t,s)\mathrm{d}s\mathrm{d}t,\\
\overline{\varphi}(x,y) &= \int_0^{\ell_1} \int_0^{\ell_2} k_4(x,t,y,s)\overline{f}^{p_4}(t,s)\mathrm{d}s\mathrm{d}t.
\end{aligned}
$$

for $(x,y) \in \mathfrak{D}$.

Using (13)–(16) yields

$$
\begin{aligned}
&\left|(\mathscr{F}f)(x,y) - (\mathscr{F}\overline{f})(x,y)\right| \le (aL_1 + bL_2 + cL_3)\left\|f - \overline{f}\right\|,\\
&\left|I^{\varrho}f(x,y) - \lambda I^{\varrho}\overline{f}(x,y)\right| \le \frac{\ell_1^{\varrho_1}\ell_2^{\varrho_2}}{\Gamma(\varrho_1+1)\Gamma(\varrho_2+1)}\left\|f - \overline{f}\right\|,\\
&|\Theta(x,y) - \bar{\Theta}(x,y)| \le \frac{\eta_1\ell_1^{\varrho_1}\ell_2^{\varrho_2}}{\Gamma(\varrho_1+1)\Gamma(\varrho_2+1)}\left\|f - \overline{f}\right\|,\\
&|\Lambda(x,y) - \bar{\Lambda}(x,y)| \le \frac{\eta_2\ell_1^{\varrho_1}\ell_2^{\varrho_2}}{\Gamma(\varrho_1+1)\Gamma(\varrho_2+1)}\left\|f - \overline{f}\right\|,\\
&|\rho(x,y) - \bar{\rho}(x,y)| \le \frac{\eta_3\ell_1^{\varrho_1}\ell_2^{\varrho_2}}{\Gamma(\varrho_1+1)\Gamma(\varrho_2+1)}\left\|f - \overline{f}\right\|,\\
&|\varphi(x,y) - \bar{\varphi}(x,y)| \le \frac{\eta_4\ell_1^{\varrho_1}\ell_2^{\varrho_2}}{\Gamma(\varrho_1+1)\Gamma(\varrho_2+1)}\left\|f - \overline{f}\right\|.
\end{aligned}
$$

Now, we can write

$$
\begin{aligned}
\left|\mathcal{T}f(x,y) - \mathcal{T}\overline{f}(x,y)\right| \le& \left|(\mathscr{F}f)(x,y) - (\mathscr{F}\overline{f})(x,y)\right| + \lambda\left|I^{\varrho}f(x,y) - \lambda I^{\varrho}\overline{f}(x,y)\right|\\
&+ |\Theta(x,y) - \bar{\Theta}(x,y)| + |\Lambda(x,y) - \bar{\Lambda}(x,y)|\\
&+ |\rho(x,y) - \bar{\rho}(x,y)| + |\varphi(x,y) - \bar{\varphi}(x,y)|\\
\le& \left(aL_1 + bL_2 + cL_3 + \frac{\ell_1^{\varrho_1}\ell_2^{\varrho_2}}{\Gamma(\varrho_1+1)\Gamma(\varrho_2+1)}(\lambda + \eta_1 + \eta_2 + \eta_3 + \eta_4)\right)\left\|f - \overline{f}\right\|,
\end{aligned}
$$

for any $(x,y) \in \mathfrak{D}$ and $f,\ \overline{f} \in C(\mathfrak{D}, \mathbb{R}^n)$. Therefore,

$$
\left\|\mathcal{T}f - \mathcal{T}\overline{f}\right\| \le \left(aL_1 + bL_2 + cL_3 + \frac{\ell_1^{\varrho_1}\ell_2^{\varrho_2}}{\Gamma(\varrho_1+1)\Gamma(\varrho_2+1)}(\lambda + \eta_1 + \eta_2 + \eta_3 + \eta_4)\right)\left\|f - \overline{f}\right\|.
$$

From (17), $\mathcal{T}$ is a contraction map in $C(\mathfrak{D}, \mathbb{R}^n)$, and thus, it has a unique fixed point. Therefore, $f \in C(\mathfrak{D}, \mathbb{R}^n)$ is a unique solution for the general 2D-NIDEF. □

3. The 1D-SJPs and 2D-SJPs and Their Operational Matrices

3.1. The 1D-SJPs

The 1D-SJPs are defined on the interval $[0,\ell)$ by

$$
\mathcal{J}_{\ell,l}^{(\tau,\varsigma)}(x) = \sum_{j=0}^{l} (-1)^{l-j} \frac{\Gamma(l+\varsigma+1)\Gamma(l+j+\tau+\varsigma+1)}{\Gamma(j+\varsigma+1)\Gamma(l+\tau+\varsigma+1)(l-j)!j!\ell^j} x^j.
$$

These polynomials are orthogonal on the interval $[0,\ell)$; therefore,

$$\int_0^{\ell_1} \mathcal{J}_{\ell,i}^{(\tau,\varsigma)}(x)\mathcal{J}_{\ell,i'}^{(\tau,\varsigma)}(x)w_{\ell}^{(\tau,\varsigma)}(x)\,\mathrm{d}x = \delta_{ii'}h_{\ell,i}^{(\tau,\varsigma)},$$

where $w_{\ell}^{(\tau,\varsigma)}(x) = x^{\varsigma}(\ell-x)^{\tau}$ is a weight function, $\delta_{ii'}$ is Kronecker delta, and

$$h_{\ell,l}^{(\tau,\varsigma)} = \frac{\ell^{\tau+\varsigma+1}\Gamma(l+\tau+1)\Gamma(l+\varsigma+1)}{(2l+\tau+\varsigma+1)l!\Gamma(l+\tau+\varsigma+1)}.$$

Additionally, these polynomials have the following property:

$$\frac{d^i}{dx^i}\mathcal{J}_{\ell,l}^{(\tau,\varsigma)}(x) = \frac{\Gamma(l+\tau+\varsigma+i+1)}{\Gamma(l+\tau+\varsigma+1)}\mathcal{J}_{\ell,l-i}^{(\tau+i,\varsigma+i)}(x). \tag{18}$$

The vector of 1D-SJPs is as follows:

$$\Psi(x) = \begin{pmatrix} \mathcal{J}_{\ell,0}^{(\tau,\varsigma)}(x) & \mathcal{J}_{\ell,1}^{(\tau,\varsigma)}(x) & \dots & \mathcal{J}_{\ell,N}^{(\tau,\varsigma)}(x) \end{pmatrix}^T. \tag{19}$$

3.2. 2D-SJPs and Function Approximation

The 2D-SJPs are defined on the domain $\mathfrak{D} = [0,\ell_1) \times [0,\ell_2)$ by

$$\mathcal{J}_{i,j}^{(\tau,\varsigma)}(x,y) = \mathcal{J}_{\ell_1,i}^{(\tau,\varsigma)}(x)\mathcal{J}_{\ell_2,j}^{(\tau,\varsigma)}(y), \quad i,j = 0,1,\dots,N.$$

These polynomials are orthogonal on $\mathfrak{D}$; therefore,

$$\int_0^{\ell_1}\int_0^{\ell_2} \mathcal{J}_{i,j}^{(\tau,\varsigma)}(x,y)\mathcal{J}_{i',j'}^{(\tau,\varsigma)}(x,y)\omega^{(\tau,\varsigma)}(x,y)\,\mathrm{d}y\,\mathrm{d}x = \delta_{ii'}\delta_{jj'}h_{\ell_1,i}^{(\tau,\varsigma)}h_{\ell_2,j}^{(\tau,\varsigma)},$$

where $\omega^{(\tau,\varsigma)}(x,y) = w_{\ell_1}^{(\tau,\varsigma)}(x)w_{\ell_2}^{(\tau,\varsigma)}(y)$ is a weight function.

By using 2D-SJPs, we can approximate a continuous function $f(x,y)$ on the domain $\mathfrak{D} = [0,\ell_1) \times [0,\ell_2)$ as follows:

$$f(x,y) \simeq f_N(x,y) = \sum_{i=0}^{N}\sum_{j=0}^{N} \hat{f}_{ij}\mathcal{J}_{i,j}^{(\tau,\varsigma)}(x,y) = \Psi^T(x,y)\hat{F} = \hat{F}^T\Psi(x,y), \tag{20}$$

where

$$\hat{F} = \begin{pmatrix} \hat{f}_{00} & \hat{f}_{01} & \dots\hat{f}_{0N} & \hat{f}_{10} & \hat{f}_{11} & \dots & \hat{f}_{1N} & \dots & \hat{f}_{N0} & \hat{f}_{N1} & \dots & \hat{f}_{NN} \end{pmatrix}^T,$$

with entries

$$\hat{f}_{ij} = \frac{1}{h_{\ell_1,i}^{(\tau,\varsigma)}h_{\ell_2,j}^{(\tau,\varsigma)}}\int_0^{\ell_1}\int_0^{\ell_2} f(x,y)\mathcal{J}_{i,j}^{(\tau,\varsigma)}(x,y)\omega^{(\tau,\varsigma)}(x,y)\,\mathrm{d}y\,\mathrm{d}x, \quad i,j = 0,1,\dots,N,$$

and

$$\begin{aligned}\Psi(x,y) = &\Big(\mathcal{J}_{0,0}^{(\tau,\varsigma)}(x,y),\dots\mathcal{J}_{0,N}^{(\tau,\varsigma)}(x,y),\mathcal{J}_{1,0}^{(\tau,\varsigma)}(x,y),\dots\mathcal{J}_{1,N}^{(\tau,\varsigma)}(x,y)\\ &\dots,\mathcal{J}_{N,0}^{(\tau,\varsigma)}(x,y),\dots\mathcal{J}_{N,N}^{(\tau,\varsigma)}(x,y)\Big)^T,\end{aligned} \tag{21}$$

are $(N+1)^2 \times 1$ vectors.

Additionally, we can expand a function $k(x,t,y,s)$ on the domain $\mathfrak{D}\times\mathfrak{D}$ with respect to 2D-SJPs as follows:

$$k(x,t,y,s) \simeq \Psi^T(x,y) K \Psi(t,s). \tag{22}$$

Here, K is a matrix with entries

$$K_{i,j} = \frac{\int_0^{\ell_1}\int_0^{\ell_2}\int_0^{\ell_1}\int_0^{\ell_2} \mathcal{J}^{(\tau,\varsigma)}_{q'[i],q''[i]}(x,y) k(x,t,y,s) \mathcal{J}^{(\tau,\varsigma)}_{q'[j],q''[j]}(t,s) \omega^{(\tau,\varsigma)}(x,y)\omega^{(\tau,\varsigma)}(t,s) \mathrm{d}s\mathrm{d}t\mathrm{d}y\mathrm{d}x}{h^{(\tau,\varsigma)}_{\ell_1,q'[i]} h^{(\tau,\varsigma)}_{\ell_2,q''[i]} h^{(\tau,\varsigma)}_{\ell_1,q'[j]} h^{(\tau,\varsigma)}_{\ell_2,q''[j]}},$$

where

$$q' = [0,\dots,0,1,\dots,1,\dots,N,\dots,N],$$
$$q'' = [0,\dots,N,0,\dots,N,\dots,0,\dots,N],$$

and $i,j = 1,\dots,(N+1)^2$.

3.3. Operational Matrices of Two-Dimensional Integration

In [40], the authors computed the one-dimensional integration of $\Psi(t)$ for $t\in[0,1)$. Similarly, we compute the one-dimensional integration of this vector for $t\in[0,\ell)$, as follows:

$$\int_0^x \Psi(t)\,\mathrm{d}t \simeq \mathbf{P}_x\,\Psi(x),$$

where $\mathbf{P}_x$ is a one-dimensional operational matrix of integration, defined in the following form:

$$\mathbf{P}_x = \begin{pmatrix} p_{00} & p_{01} & \cdots & p_{0N} \\ p_{10} & p_{11} & \cdots & p_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ p_{N0} & p_{N1} & \cdots & p_{NN} \end{pmatrix}, \tag{23}$$

with the following entries:

$$p_{kl} = \sum_{j=0}^{k}\left(\frac{(-1)^{k-j}\Gamma(l+\varsigma+1)\Gamma(k+\varsigma+1)\Gamma(k+j+\tau+\varsigma+1)\Gamma(\tau+1)}{h_l\Gamma(l+\tau+\varsigma+1)\Gamma(j+\varsigma+1)\Gamma(k+\tau+\varsigma+1)(j+1)!(k-j)!\ell^j}\right.$$
$$\left.\times\sum_{i=0}^{l}\frac{(-1)^{l-i}\Gamma(l+i+\tau+\varsigma+1)\Gamma(i+j+\varsigma+2)\ell}{\Gamma(i+\varsigma+1)\Gamma(i+j+\tau+\varsigma+3)i!(l-i)!}\right), \quad k,l=0,1,\dots,N.$$

Since $\Psi(x,y) = \Psi(x)\otimes\Psi(y)$, the two-dimensional integration of $\Psi(t,s)$ can be obtained as follows:

$$\int_0^x\int_0^y \Psi(t,s)\,\mathrm{d}s\,\mathrm{d}t \simeq (\mathbf{P}_x\otimes\mathbf{P}_y)\,\Psi(x,y), \qquad x\in[0,\ell_1),\ y\in[0,\ell_2), \tag{24}$$

where $\otimes$ denotes the Kronecker product; $\mathbf{P}_x\otimes\mathbf{P}_y$ is the $(N+1)^2\times(N+1)^2$ operational matrix of the two-dimensional integration; and $\mathbf{P}_x,\ \mathbf{P}_y$ are $(N+1)\times(N+1)$ one-dimensional operational matrices of integration, defined in Equation (23).

Additionally, it is easy to conclude the following result:

$$\int_0^{\ell_1}\int_0^{\ell_2} \Psi(t,s)\,\mathrm{d}s\,\mathrm{d}t = A_1\otimes A_2, \tag{25}$$

where

$$A_1 = \begin{pmatrix} a_0 & a_1 & \dots & a_N \end{pmatrix}^T, \qquad A_2 = \begin{pmatrix} a'_0 & a'_1 & \dots & a'_N \end{pmatrix}^T,$$

with the entries:

$$a_r = \sum_{j=0}^{r} (-1)^{r-j} \frac{\Gamma(r+\varsigma+1)\Gamma(r+j+\tau+\varsigma+1)\ell_1}{\Gamma(j+\varsigma+1)\Gamma(r+\tau+\varsigma+1)(r-j)!(j+1)!},$$
$$a'_r = \sum_{j=0}^{r} (-1)^{r-j} \frac{\Gamma(r+\varsigma+1)\Gamma(r+j+\tau+\varsigma+1)\ell_2}{\Gamma(j+\varsigma+1)\Gamma(r+\tau+\varsigma+1)(r-j)!(j+1)!},$$

for $r = 0, 1, \dots, N$.

3.4. Operational Matrices of Fractional-Order Integration

In [27], the authors defined an operational matrix of the Riemann–Liouville integral operator of order κ by

$$\mathbf{I}^{\ell,\kappa} = \begin{pmatrix} S_{00} & S_{01} & \dots & S_{0N} \\ S_{10} & S_{11} & \dots & S_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ S_{N0} & S_{N1} & \dots & S_{NN} \end{pmatrix},$$

with the entries

$$S_{lr} = \sum_{m=0}^{l} \Bigg(\frac{(-1)^{l-m}\Gamma(l+\varsigma+1)\Gamma(l+m+\tau+\varsigma+1)\Gamma(m+1)}{\Gamma(m+\varsigma+1)\Gamma(l+\tau+\varsigma+1)(l-m)!m!\Gamma(m+\kappa+1)\ell^m}$$
$$\times \sum_{m'=0}^{r} \frac{(-1)^{r-m'}(2r+\tau+\varsigma+1)\Gamma(r+1)\Gamma(r+m'+\tau+\varsigma+1)\Gamma(m+\kappa+m'+\varsigma+1)\Gamma(\kappa+1)\ell^{\kappa}}{\Gamma(r+\tau+1)\Gamma(m'+\varsigma+1)(r-m')!m'!\Gamma(m+\kappa+m'+\varsigma+\tau+2)} \Bigg),$$

for $l, r = 0, 1, \dots, N$.

Theorem 4 (see [34]). *Let $\varrho = (\varrho_1, \varrho_2) \in (0,\infty) \times (0,\infty)$ and $\Psi(x,y)$ be the vector of 2D-SJPs. Then*

$$I^{\varrho}\Psi(x,y) \simeq (\boldsymbol{I}^{\ell_1,\varrho_1} \otimes \boldsymbol{I}^{\ell_2,\varrho_2})\Psi(x,y), \qquad (x,y) \in [0,\ell_1) \times [0,\ell_2). \tag{26}$$

Here, $\boldsymbol{I}^{\ell_1,\varrho_1}$ and $\boldsymbol{I}^{\ell_2,\varrho_2}$ are operational matrices of a fractional Riemann–Liouville integration of orders ϱ_1 and ϱ_2, respectively.

Theorem 5 (see [34]). *Let $\kappa > 0$. Assume that $\Psi(s)$, defined in (19), is the vector of 1D-SJPs. Then,*

$$\frac{1}{\Gamma(\kappa)} \int_0^{\ell} (\ell - s)^{\kappa-1}\Psi(s)\,\mathrm{d}s = \Upsilon, \tag{27}$$

where $\Upsilon = \begin{pmatrix} \gamma_0 & \gamma_1 & \dots & \gamma_N \end{pmatrix}^T$ *and*

$$\gamma_r = \sum_{j=0}^{r} (-1)^{r-j} \frac{\Gamma(r+\varsigma+1)\Gamma(r+j+\tau+\varsigma+1)\ell^{\kappa-1}}{\Gamma(j+\varsigma+1)\Gamma(r+\tau+\varsigma+1)(r-j)!\Gamma(j+\kappa+1)}, \qquad r = 0, 1, \dots, N. \tag{28}$$

Theorem 6 (see [34]). *Let $\varrho_1,\ \varrho_2 > 0$. Assume that $\Psi(t,s)$, defined in (21), is the vector of 2D-SJPs. Then*

$$\frac{1}{\Gamma(\varrho_1)\Gamma(\varrho_2)}\int_0^{\ell_1}\int_0^{\ell_2}(\ell_1-t)^{\varrho_1-1}(\ell_2-s)^{\varrho_2-1}\Psi(t,s)\,\mathrm{d}s\,\mathrm{d}t = \Upsilon_1\otimes\Upsilon_2, \tag{29}$$

where

$$\Upsilon_1 = \begin{pmatrix} \gamma 1_0 & \gamma 1_1 & \dots & \gamma 1_N \end{pmatrix}^T, \qquad \Upsilon_2 = \begin{pmatrix} \gamma 2_0 & \gamma 2_1 & \dots & \gamma 2_N \end{pmatrix}^T,$$

and

$$\gamma 1_r = \sum_{j=0}^{r}(-1)^{r-j}\frac{\Gamma(r+\varsigma+1)\Gamma(r+j+\tau+\varsigma+1)\ell_1^{\varrho_1}}{\Gamma(j+\varsigma+1)\Gamma(r+\tau+\varsigma+1)(r-j)!\Gamma(j+\varrho_1+1)},$$
$$\gamma 2_r = \sum_{j=0}^{r}(-1)^{r-j}\frac{\Gamma(r+\varsigma+1)\Gamma(r+j+\tau+\varsigma+1)\ell_2^{\varrho_2}}{\Gamma(j+\varsigma+1)\Gamma(r+\tau+\varsigma+1)(r-j)!\Gamma(j+\varrho_2+1)},$$

for $r = 0,1,\dots,N$.

3.5. Operational Matrix of Product

Assume that $\Psi(x,y)$, defined in (21), is the vector of 2D-SJPs. In [34], Rashidinia et al. introduced the operational matrix of the product as follows:

$$\Psi(x,y)\Psi^T(x,y)\hat{F} \simeq \tilde{\hat{F}}\Psi(x,y), \tag{30}$$

for $(x,y)\in[0,\ell_1)\times[0,\ell_2)$. Here, $\tilde{\hat{F}}$ is the operational matrix of the product with the entries

$$\tilde{\hat{F}}_{m_1(N+1)+n_1+1,m_2(N+1)+n_2+1} = \frac{1}{h^{(\tau,\varsigma)}_{\ell_1,m_2}h^{(\tau,\varsigma)}_{\ell_2,n_2}}\sum_{j=0}^{N}\sum_{k=0}^{N}\hat{f}_{jk}v_{m_1jm_2}v_{n_1kn_2},$$

where

$$v_{m_1jm_2} = \int_0^{\ell_1}\mathcal{J}^{(\tau,\varsigma)}_{\ell_1,m_1}(x)\mathcal{J}^{(\tau,\varsigma)}_{\ell_1,j}(x)\mathcal{J}^{(\tau,\varsigma)}_{\ell_1,m_2}(x)w^{(\tau,\varsigma)}_{\ell_1}(x)\,\mathrm{d}x,$$
$$v_{n_1kn_2} = \int_0^{\ell_2}\mathcal{J}^{(\tau,\varsigma)}_{\ell_2,n_1}(y)\mathcal{J}^{(\tau,\varsigma)}_{\ell_2,k}(y)\mathcal{J}^{(\tau,\varsigma)}_{\ell_2,n_2}(y)w^{(\tau,\varsigma)}_{\ell_2}(y)\,\mathrm{d}y,$$

for $m_1,n_1,m_2,n_2 = 0,1,\dots,N$.

4. Method of Solution

Here, by using the method proposed in Section 3, we solve the general 2D-NFIDEs. First of all, we define

$$f_{yy}(x,y) \simeq f_{yy}^T\Psi(x,y), \tag{31}$$
$$f_{xx}(x,y) \simeq f_{xx}^T\Psi(x,y), \tag{32}$$
$$f_{yx}(x,y) \simeq f_{yx}^T\Psi(x,y), \tag{33}$$
$$g(x,y) \simeq \Psi^T(x,y)G, \tag{34}$$
$$k_i(x,t,y,s) \simeq \Psi^T(x,y)k_i\Psi(t,s), \quad i = 1,2,3,4, \tag{35}$$

$$d_1(x) = f(x,0) \simeq F_{x0}^T \Psi(x,y), \tag{36}$$
$$d_2(y) = f(0,y) \simeq F_{0y}^T \Psi(x,y), \tag{37}$$
$$d_3(x) = f_y(x,0) \simeq F_{yx0}^T \Psi(x,y), \tag{38}$$
$$d_4(y) = f_x(0,y) \simeq F_{x0y}^T \Psi(x,y), \tag{39}$$
$$d_5(x) = f_x(x,0) \simeq F_{xx0}^T \Psi(x,y). \tag{40}$$

Now, from the Appendix in [36], we can obtain:

$$f_{yy}^T = ((f^T - F_{x0}^T)(I \otimes \mathbf{P}_y)^{-1} - F_{yx0}^T)(I \otimes \mathbf{P}_y)^{-1}, \tag{41}$$
$$f_{xx}^T = ((f^T - F_{0y}^T)(\mathbf{P}_x \otimes I)^{-1} - F_{x0y}^T)(\mathbf{P}_x \otimes I)^{-1}, \tag{42}$$
$$f_{yx}^T = ((f^T - F_{0y}^T)(\mathbf{P}_x \otimes I)^{-1} - F_{xx0}^T)(I \otimes \mathbf{P}_y)^{-1}. \tag{43}$$

Using (26) for $I^{\varrho} f(x,y)$ yields

$$I^{\varrho} f(x,y) \simeq I^{\varrho} \hat{F}^T \Psi(x,y) = \hat{F}^T I^{\varrho} \Psi(x,y) = \hat{F}^T \left(\mathbf{I}^{\ell_1,\varrho_1} \otimes \mathbf{I}^{\ell_2,\varrho_2} \right) \Psi(x,y). \tag{44}$$

Additionally, by using (20) and (30), we have

$$f^2(x,y) \simeq \hat{F}^T \Psi(x,y) \Psi^T(x,y) \hat{F} = \underbrace{\hat{F}^T \tilde{\hat{F}}}_{\hat{F}_2} \Psi(x,y) = \hat{F}_2 \Psi(x,y),$$
$$f^3(x,y) \simeq \hat{F}^T \Psi(x,y) \hat{F}_2 \Psi(x,y) = \hat{F}^T \Psi(x,y) \Psi^T(x,y) \hat{F}_2^T = \underbrace{\hat{F}^T \widetilde{\hat{F}_2^T}}_{\hat{F}_3} \Psi(x,y) = \hat{F}_3 \Psi(x,y).$$

Similarly, we obtain

$$f^p(x,y) \simeq \hat{F}_p \Psi(x,y). \tag{45}$$

Now, using (24), (35), and (45) gives

$$\begin{aligned}
\Theta(x,y) &= \int_0^x \int_0^y k_1(x,t,y,s) f^{p_1}(t,s)\, \mathrm{d}s\, \mathrm{d}t \\
&\simeq \int_0^x \int_0^y \Psi^T(x,y) k_1 \Psi(t,s) \hat{F}_{p_1} \Psi(t,s)\, \mathrm{d}s\, \mathrm{d}t \\
&= \int_0^x \int_0^y \Psi^T(x,y) k_1 \widetilde{\hat{F}_{p_1}^T} \Psi(t,s)\, \mathrm{d}s\, \mathrm{d}t \\
&= \Psi^T(x,y) k_1 \widetilde{\hat{F}_{p_1}^T} \int_0^x \int_0^y \Psi(t,s)\, \mathrm{d}s\, \mathrm{d}t \\
&= \Psi^T(x,y) k_1 \widetilde{\hat{F}_{p_1}^T} (\mathbf{P}_x \otimes \mathbf{P}_y) \Psi(x,y).
\end{aligned} \tag{46}$$

Similarly, using (25), (35), and (45) for $\varphi(x,y)$, we can write

$$\begin{aligned}
\varphi(x,y) &= \int_0^{\ell_1} \int_0^{\ell_2} k_4(x,t,y,s) f^{p_4}(t,s)\, \mathrm{d}s\, \mathrm{d}t \\
&\simeq \int_0^{\ell_1} \int_0^{\ell_2} \Psi^T(x,y) k_4 \Psi(t,s) \hat{F}_{p_4} \Psi(t,s)\, \mathrm{d}s\, \mathrm{d}t \\
&= \int_0^{\ell_1} \int_0^{\ell_2} \Psi^T(x,y) k_4 \widetilde{\hat{F}_{p_4}^T} \Psi(t,s)\, \mathrm{d}s\, \mathrm{d}t \\
&= \Psi^T(x,y) k_4 \widetilde{\hat{F}_{p_4}^T} \int_0^{\ell_1} \int_0^{\ell_2} \Psi(t,s)\, \mathrm{d}s\, \mathrm{d}t \\
&= \Psi^T(x,y) k_4 \widetilde{\hat{F}_{p_4}^T} (A_1 \otimes A_2).
\end{aligned} \tag{47}$$

Additionally, using (26), (35), and (45), $\Lambda(x,y)$ can be determined as:

$$\begin{aligned}\Lambda(x,y) &= \frac{1}{\Gamma(\varrho_1)\Gamma(\varrho_2)}\int_0^x\int_0^y (x-t)^{\varrho_1-1}(y-s)^{\varrho_2-1}k_2(x,t,y,s)f^{p_2}(t,s)\,\mathrm{d}s\,\mathrm{d}t\\ &\simeq \frac{1}{\Gamma(\varrho_1)\Gamma(\varrho_2)}\int_0^x\int_0^y (x-t)^{\varrho_1-1}(y-s)^{\varrho_2-1}\Psi^T(x,y)k_2\Psi(t,s)\hat{F}_{p_2}\Psi(t,s)\,\mathrm{d}s\,\mathrm{d}t\\ &= \frac{1}{\Gamma(\varrho_1)\Gamma(\varrho_2)}\int_0^x\int_0^y (x-t)^{\varrho_1-1}(y-s)^{\varrho_2-1}\Psi^T(x,y)k_2\widetilde{\hat{F}^T_{p_2}}\Psi(t,s)\,\mathrm{d}s\,\mathrm{d}t\\ &= \Psi^T(x,y)k_2\widetilde{\hat{F}^T_{p_2}}\left(\frac{1}{\Gamma(\varrho_1)\Gamma(\varrho_2)}\int_0^x\int_0^y (x-t)^{\varrho_1-1}(y-s)^{\varrho_2-1}\Psi(t,s)\,\mathrm{d}s\,\mathrm{d}t\right)\\ &= \Psi^T(x,y)k_2\widetilde{\hat{F}^T_{p_2}}\left(\mathbf{I}^{\ell_1,\varrho_1}\otimes\mathbf{I}^{\ell_2,\varrho_2}\right)\Psi(x,y).\end{aligned} \tag{48}$$

Using (29), (35), and (45) for $\rho(x,y)$, we obtain

$$\begin{aligned}\rho(x,y) &= \frac{1}{\Gamma(\varrho_1)\Gamma(\varrho_2)}\int_0^{\ell_1}\int_0^{\ell_2} (\ell_1-t)^{\varrho_1-1}(\ell_2-s)^{\varrho_2-1}k_3(x,t,y,s)f^{p_3}(t,s)\,\mathrm{d}s\,\mathrm{d}t\\ &\simeq \frac{1}{\Gamma(\varrho_1)\Gamma(\varrho_2)}\int_0^{\ell_1}\int_0^{\ell_2} (\ell_1-t)^{\varrho_1-1}(\ell_2-s)^{\varrho_2-1}\Psi^T(x,y)k_3\Psi(t,s)\hat{F}_{p_3}\Psi(t,s)\,\mathrm{d}s\,\mathrm{d}t\\ &= \frac{1}{\Gamma(\varrho_1)\Gamma(\varrho_2)}\int_0^{\ell_1}\int_0^{\ell_2} (\ell_1-t)^{\varrho_1-1}(\ell_2-s)^{\varrho_2-1}\Psi^T(x,y)k_3\widetilde{\hat{F}^T_{p_3}}\Psi(t,s)\,\mathrm{d}s\,\mathrm{d}t\\ &= \Psi^T(x,y)k_3\widetilde{\hat{F}^T_{p_3}}\left(\frac{1}{\Gamma(\varrho_1)\Gamma(\varrho_2)}\int_0^{\ell_1}\int_0^{\ell_2} (\ell_1-t)^{\varrho_1-1}(\ell_2-s)^{\varrho_2-1}\Psi(t,s)\,\mathrm{d}s\,\mathrm{d}t\right)\\ &= \Psi^T(x,y)k_3\widetilde{\hat{F}^T_{p_3}}(\Upsilon_1\otimes\Upsilon_2).\end{aligned} \tag{49}$$

Now, by substituting (31)–(34), (36)–(44), and (46)–(49) into (1), a system of equations can be obtained as follows:

$$\begin{aligned}&af^T_{yy}\Psi(x,y)+bf^T_{xx}\Psi(x,y)+cf^T_{yx}\Psi(x,y)+\hat{F}^T\Psi(x,y)+\lambda\hat{F}^T\left(\mathbf{I}^{\ell_1,\varrho_1}\otimes\mathbf{I}^{\ell_2,\varrho_2}\right)\Psi(x,y)\\ &\simeq \Psi^T(x,y)G+\Psi^T(x,y)k_1\widetilde{\hat{F}^T_{p_1}}(\mathbf{P}_x\otimes\mathbf{P}_y)\Psi(x,y)+\Psi^T(x,y)k_2\widetilde{\hat{F}^T_{p_2}}\left(\mathbf{I}^{\ell_1,\varrho_1}\otimes\mathbf{I}^{\ell_2,\varrho_2}\right)\Psi(x,y)\\ &+\Psi^T(x,y)k_3\widetilde{\hat{F}^T_{p_3}}(\Upsilon_1\otimes\Upsilon_2)+\Psi^T(x,y)k_4\widetilde{\hat{F}^T_{p_4}}(A_1\otimes A_2).\end{aligned} \tag{50}$$

In the above system, the coefficients $\hat{f}_{mm'}$, $m,m'=0,1,\dots,N$ are unknown. Using the roots of $\mathcal{J}^{(\tau,\varsigma)}_{\ell_1,N+1}(x)$ and $\mathcal{J}^{(\tau,\varsigma)}_{\ell_2,N+1}(y)$ for an appropriate N determines these unknown coefficients. By collocating Equation (50) at points $\{(x_m,y'_m)\}_{m,m'=0}^N$, we obtain $(N+1)^2$ equations and solve this system using the Newton method. Therefore, we obtain the unknown coefficients and determine an approximate solution from (20).

5. Error Bounds

Let $\mathfrak{D}=[0,\ell_1)\times[0,\ell_2)$ and $L^2_{\omega^{(\tau,\varsigma)}}(\mathfrak{D})$ be a weighted space of square integrable functions on $\mathfrak{D}$. We recall the following inner product and norm on $L^2_{\omega^{(\tau,\varsigma)}}(\mathfrak{D})$ to discuss the convergence of the new method:

$$\begin{aligned}\langle f,g\rangle_{\omega^{(\tau,\varsigma)}} &= \int_0^{\ell_1}\int_0^{\ell_2} f(x,y)g(x,y)\omega^{(\tau,\varsigma)}(x,y)\,\mathrm{d}y\,\mathrm{d}x, \qquad \forall f,g\in L^2_{\omega^{(\tau,\varsigma)}}(\mathfrak{D}),\\ \|f\|_{\omega^{(\tau,\varsigma)}} &= \left(\int_0^{\ell_1}\int_0^{\ell_2}(f(x,y))^2\omega^{(\tau,\varsigma)}(x,y)\,\mathrm{d}y\,\mathrm{d}x\right)^{\frac{1}{2}}, \qquad \forall f\in L^2_{\omega^{(\tau,\varsigma)}}(\mathfrak{D}).\end{aligned}$$

Theorem 7. *Consider the following finite-dimensional polynomial space:*

$$\mathscr{P}_N = span\{\mathcal{J}^{(\tau,\varsigma)}_{m,m'}(x,y), \quad 0\le m,m'\le N\}.$$

Suppose that

$$\frac{\partial^i}{\partial x^{i_1}\partial y^{i_2}} f(x,y) \in C(\mathfrak{D}), \quad i_1 + i_2 = i,\ i = 0,1,\ldots,N.$$

If $f_N(x,y)$ is the best approximation from $\mathscr{P}_N$ to $f(x,y)$ and $\widetilde{f}_N(x,y)$ is the Taylor expansion of $f(x,y)$ of order N with respect to each variables x and y, then

$$\|f - f_N\|_{\omega^{(\tau,\varsigma)}} \leq \frac{\mu 2^{N+1}}{(N+1)!}\sqrt{(\ell_1\ell_2)^{\tau+\varsigma+1}B(\tau+1,\varsigma+1)}, \tag{51}$$

where

$$\mu = \max_{i=0,1,\ldots,N}\left\{\ell_1^{N+1-i}\ell_2^i \max_{(x,y)\in\mathfrak{D}}\left|\frac{\partial^{N+1}}{\partial x^{N+1-i}\partial y^i} f(x,y)\right|\right\}, \tag{52}$$

and $B(.,.)$ is a beta function.

Proof. Since $f_N(x,y)$ is the best approximation to $f(x,y)$, it is obvious that from the definition of best approximation, we have

$$\|f - f_N\|_{\omega^{(\tau,\varsigma)}} \leq \left\|f - \widetilde{f}_N\right\|_{\omega^{(\tau,\varsigma)}}. \tag{53}$$

The Taylor expansion of $f(x,y)$ about $(0^+,0^+)$ yields

$$\begin{aligned}
\left|f(x,y) - \widetilde{f}_N(x,y)\right| &= \left|f(x,y) - \sum_{r=0}^{N}\sum_{m=0}^{r}\frac{x^{r-m}y^m}{(r-m)!m!}\frac{\partial^r}{\partial x^{r-m}\partial y^m}f(0^+,0^+)\right| \\
&= \left|\sum_{r=0}^{N+1}\frac{x^{N+1-r}y^r}{(N+1-r)!r!}\frac{\partial^{N+1}}{\partial x^{N+1-r}\partial y^i}f(\eta_x,\eta_y)\right| \\
&\leq \sum_{r=0}^{N+1}\frac{\ell_1^{N+1-r}\ell_2^r}{(N+1-r)!r!}\left|\frac{\partial^{N+1}}{\partial x^{N+1-r}\partial y^r}f(\eta_x,\eta_y)\right| \\
&\leq \sum_{r=0}^{N+1}\frac{\ell_1^{N+1-r}\ell_2^r}{(N+1-r)!r!}\max_{(x,y)\in\mathfrak{D}}\left|\frac{\partial^{N+1}}{\partial x^{N+1-r}\partial y^r}f(x,y)\right| \\
&\leq \mu\sum_{r=0}^{N+1}\frac{1}{(N+1-r)!r!} \\
&= \frac{\mu}{(N+1)!}\sum_{r=0}^{N+1}\binom{N+1}{r} \\
&= \frac{\mu 2^{N+1}}{(N+1)!},
\end{aligned}$$

where $(\eta_x,\eta_y) \in [0,x]\times[0,y]$ and $(x,y)\in\mathfrak{D}$. Since $\widetilde{f}_N \in \mathscr{P}_N$, we can write

$$\begin{aligned}
\|f - f_N\|^2_{\omega^{(\tau,\varsigma)}} &\leq \int_0^{\ell_1}\int_0^{\ell_2}\left(\frac{\mu 2^{N+1}}{(N+1)!}\right)^2\omega^{(\tau,\varsigma)}(x,y)\,\mathrm{d}y\,\mathrm{d}x \\
&= \left(\frac{\mu 2^{N+1}}{(N+1)!}\right)^2(\ell_1\ell_2)^{\tau+\varsigma+1}(B(\tau+1,\varsigma+1))^2.
\end{aligned}$$

Taking the square roots of the above inequality gives the inequality (51). □

Definition 1. *A Jacobi-weighted Sobolev space of measurable functions is denoted by* $\mathscr{P}^{\varepsilon}_{\omega^{(\tau,\varsigma)}}(\mathfrak{D})$ *and is defined with the following norm and semi-norm:*

$$\|f\|_{\varepsilon,\omega^{(\tau,\varsigma)}} = \left(\sum_{l=0}^{\varepsilon} \left\| \partial_{\Delta}^{l} f \right\|^{2}_{\omega^{(\tau+l,\varsigma+l)}} \right)^{\frac{1}{2}} < \infty, \quad \Delta = (x,y), \quad \varepsilon \in \mathbb{N},$$
$$|f|_{\varepsilon,\omega^{(\tau,\varsigma)}} = \|\partial_{\Delta}^{\varepsilon} f\|_{\omega^{(\tau+\varepsilon,\varsigma+\varepsilon)}},$$

where

$$\partial_{\Delta}^{l} f = \frac{\partial^{l}}{\partial x^{l_1} \partial y^{l_2}} f, \quad l_1 + l_2 = l,$$

$$\omega^{(\tau+l,\varsigma+l)}(x,y) = \omega^{(\tau+l_1,\varsigma+l_1)}(x)\omega^{(\tau+l_2,\varsigma+l_2)}(y).$$

Theorem 8. *For any* $f \in \mathscr{P}^{\varepsilon}_{\omega^{(\tau,\varsigma)}}(\mathfrak{D})$, $\varepsilon \in \mathbb{N}$, *and* $0 \le \epsilon \le \varepsilon$, *we have*

$$\|f - f_N\|_{\epsilon,\omega^{(\tau,\varsigma)}} \le \eta (N(N+\tau+\varsigma)(1+\tau+\varsigma))^{\frac{\epsilon-\varepsilon}{2}} |f|_{\varepsilon,\omega^{(\tau,\varsigma)}}, \tag{54}$$

where η *is a positive constant.*

Proof. From (18), we can write

$$\begin{aligned}
\partial_{\Delta}^{l}(f(x,y) - f_N(x,y)) &= \sum_{j=0}^{\infty} \sum_{k=N+1}^{\infty} \hat{f}_{jk} \partial_{\Delta}^{l} \mathcal{J}^{(\tau,\varsigma)}_{\ell_1,j}(x) \mathcal{J}^{(\tau,\varsigma)}_{\ell_2,k}(y) \\
&\quad + \sum_{j=N+1}^{\infty} \sum_{k=0}^{\infty} \hat{f}_{jk} \partial_{\Delta}^{l} \mathcal{J}^{(\tau,\varsigma)}_{\ell_1,j}(x) \mathcal{J}^{(\tau,\varsigma)}_{\ell_2,k}(y) \\
&= \sum_{j=0}^{\infty} \sum_{k=N+1}^{\infty} \hat{f}_{jk} \nu_{j,l_1} \nu_{k,l_2} \mathcal{J}^{(\tau+l_1,\varsigma+l_1)}_{\ell_1,j-l_1}(x) \mathcal{J}^{(\tau+l_2,\varsigma+l_2)}_{\ell_2,k-l_2}(y) \\
&\quad + \sum_{j=N+1}^{\infty} \sum_{k=0}^{\infty} \hat{f}_{jk} \nu_{j,l_1} \nu_{k,l_2} \mathcal{J}^{(\tau+l_1,\varsigma+l_1)}_{\ell_1,j-l_1}(x) \mathcal{J}^{(\tau+l_2,\varsigma+l_2)}_{\ell_2,k-l_2}(y),
\end{aligned} \tag{55}$$

where

$$\nu_{j,l_1} = \frac{\Gamma(j+\tau+\varsigma+l_1+1)}{\Gamma(j+\tau+\varsigma+1)}, \qquad \nu_{k,l_2} = \frac{\Gamma(k+\tau+\varsigma+l_2+1)}{\Gamma(k+\tau+\varsigma+1)}. \tag{56}$$

Taking the $L^2_{\omega^{(\tau,\varsigma)}}$−norm of Equation (55) yields

$$\left\| \partial_{\Delta}^{l}(f - f_N) \right\|^{2}_{\omega^{(\tau+l,\varsigma+l)}} = \sum_{j=0}^{\infty} \sum_{k=N+1}^{\infty} \hat{f}^{2}_{jk} a_{j,k} + \sum_{j=N+1}^{\infty} \sum_{k=0}^{\infty} \hat{f}^{2}_{jk} a_{j,k} + 2 \sum_{j=N+1}^{\infty} \sum_{k=N+1}^{\infty} \hat{f}^{2}_{jk} a_{j,k}, \tag{57}$$

where

$$a_{j,k} = \nu^{2}_{j,l_1} \nu^{2}_{k,l_2} h^{(\tau+l_1,\varsigma+l_1)}_{\ell_1,j-l_1}(x) h^{(\tau+l_2,\varsigma+l_2)}_{\ell_2,k-l_2}(y).$$

Similarly,

$$\left\| \partial_{\Delta}^{l} f \right\|^{2}_{\omega^{(\tau+\varepsilon,\varsigma+\varepsilon)}} = \sum_{j=0}^{\infty} \sum_{k=N+1}^{\infty} \hat{f}^{2}_{jk} b_{j,k} + \sum_{j=N+1}^{\infty} \sum_{k=0}^{\infty} \hat{f}^{2}_{jk} b_{j,k} + 2 \sum_{j=N+1}^{\infty} \sum_{k=N+1}^{\infty} \hat{f}^{2}_{jk} b_{j,k}, \tag{58}$$

where

$$b_{j,k}=\nu_{j,\varepsilon_1}^2\nu_{k,\varepsilon_2}^2 h_{\ell_1,j-\varepsilon_1}^{(\tau+\varepsilon_1,\varsigma+\varepsilon_1)}(x)h_{\ell_2,k-\varepsilon_2}^{(\tau+\varepsilon_2,\varsigma+\varepsilon_2)}(y).$$

Using (18) and the Stirling formula

$$\Gamma(z+1)=\sqrt{2\pi z}z^z\mathrm{e}^{-z}\Big(1+O\Big(z^{-\frac{1}{5}}\Big)\Big), \tag{59}$$

and from

$$(m+\kappa)^{m+\kappa}=m^{m+\kappa}\mathrm{e}^{(m+\kappa)\sum\limits_{i=1}^{\infty}\frac{(-1)^i}{i}\left(\frac{\kappa}{m}\right)^i}, \tag{60}$$

we have

$$\frac{a_{j,k}}{b_{j,k}}\leq \eta j^{l_1-\varepsilon_1}k^{l_2-\varepsilon_2}(j+\tau+\varsigma)^{l_1-\varepsilon_1}(k+\tau+\varsigma)^{l_2-\varepsilon_2}. \tag{61}$$

From the relations (63)–(65), we obtain

$$\begin{aligned}
&|f(x,y)-f_N(x,y)|^2_{l,\omega^{(\tau,\varsigma)}}=\left\|\partial^l_\Delta(f-f_N)\right\|^2_{\omega^{(\tau+l,\varsigma+l)}}\\
&=\sum_{j=0}^{\infty}\sum_{k=N+1}^{\infty}\frac{a_{j,k}}{b_{j,k}}b_{j,k}\hat{f}_{jk}^2+\sum_{j=N+1}^{\infty}\sum_{k=0}^{\infty}\frac{a_{j,k}}{b_{j,k}}b_{j,k}\hat{f}_{jk}^2+2\sum_{j=N+1}^{\infty}\sum_{k=N+1}^{\infty}\frac{a_{j,k}}{b_{j,k}}b_{j,k}\hat{f}_{jk}^2\\
&\leq\sum_{j=0}^{\infty}\sum_{k=N+1}^{\infty}\eta j^{l_1-\varepsilon_1}k^{l_2-\varepsilon_2}(j+\tau+\varsigma)^{l_1-\varepsilon_1}(k+\tau+\varsigma)^{l_2-\varepsilon_2}b_{j,k}\hat{f}_{jk}^2\\
&+\sum_{j=N+1}^{\infty}\sum_{k=0}^{\infty}\eta j^{l_1-\varepsilon_1}k^{l_2-\varepsilon_2}(j+\tau+\varsigma)^{l_1-\varepsilon_1}(k+\tau+\varsigma)^{l_2-\varepsilon_2}b_{j,k}\hat{f}_{jk}^2\\
&+2\sum_{j=N+1}^{\infty}\sum_{k=N+1}^{\infty}\eta j^{l_1-\varepsilon_1}k^{l_2-\varepsilon_2}(j+\tau+\varsigma)^{l_1-\varepsilon_1}(k+\tau+\varsigma)^{l_2-\varepsilon_2}b_{j,k}\hat{f}_{jk}^2\\
&\leq\eta N^{l_2-\varepsilon_2}(1+\tau+\varsigma)^{l_1-\varepsilon_1}(N+\tau+\varsigma)^{l_2-\varepsilon_2}\sum_{j=0}^{\infty}\sum_{k=N+1}^{\infty}b_{j,k}\hat{f}_{jk}^2\\
&+\eta N^{l_1-\varepsilon_1}(N+\tau+\varsigma)^{l_1-\varepsilon_1}(1+\tau+\varsigma)^{l_2-\varepsilon_2}\sum_{j=N+1}^{\infty}\sum_{k=0}^{\infty}b_{j,k}\hat{f}_{jk}^2\\
&+2\eta N^{l_1-\varepsilon_1}N^{l_2-\varepsilon_2}(N+\tau+\varsigma)^{l_1-\varepsilon_1}(N+\tau+\varsigma)^{l_2-\varepsilon_2}\sum_{j=N+1}^{\infty}\sum_{k=N+1}^{\infty}b_{j,k}\hat{f}_{jk}^2\\
&\leq\eta N^{l_n-\varepsilon}(N+\tau+\varsigma)^{l_n-\varepsilon}(1+\tau+\varsigma)^{l_n-\varepsilon}\left(\sum_{j=0}^{\infty}\sum_{k=N+1}^{\infty}b_{j,k}\hat{f}_{jk}^2+\sum_{j=N+1}^{\infty}\sum_{k=0}^{\infty}b_{j,k}\hat{f}_{jk}^2+2\sum_{j=N+1}^{\infty}\sum_{k=N+1}^{\infty}b_{j,k}\hat{f}_{jk}^2\right)\\
&=\eta N^{l_n-\varepsilon}(N+\tau+\varsigma)^{l_n-\varepsilon}(1+\tau+\varsigma)^{l_n-\varepsilon}\left\|\partial^l_\Delta f\right\|^2_{\omega^{(\tau+\varepsilon,\varsigma+\varepsilon)}}\\
&=\eta N^{l_n-\varepsilon}(N+\tau+\varsigma)^{l_n-\varepsilon}(1+\tau+\varsigma)^{l_n-\varepsilon}|f|^2_{\varepsilon,\omega^{(\tau,\varsigma)}},
\end{aligned}$$

for any $l_n\leq\varepsilon_n$, where

$$l_n-\varepsilon_n=\min_{i=1,2}\{l_i-\varepsilon_i\},\quad 0\leq l_i\leq\varepsilon_i\leq\varepsilon,\ i=1,2.$$

Therefore, we obtain

$$\|f-f_N\|_{\epsilon,\omega^{(\tau,\varsigma)}}\leq\eta(N(N+\tau+\varsigma)(1+\tau+\varsigma))^{\frac{\epsilon-\varepsilon}{2}}|f|_{\varepsilon,\omega^{(\tau,\varsigma)}},\quad 0\leq\epsilon\leq\varepsilon.$$

□

Theorem 9. *For any $f \in \mathscr{P}^{\varepsilon}_{\omega^{(\tau,\varsigma)}}(\mathfrak{D})$, $\varepsilon \in \mathbb{N}$, and $0 \leq \epsilon \leq \varepsilon$, we have*

$$\left\| \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial y^2}\right)_N \right\|_{\epsilon,\omega^{(\tau,\varsigma)}} \leq \eta_1 (N(N+\tau+\varsigma)(1+\tau+\varsigma))^{\frac{\epsilon-\varepsilon}{2}} |f|_{\varepsilon,\omega^{(\tau,\varsigma)}}, \tag{62}$$

where η_1 is a positive constant.

Proof. From (18) and (56), we have

$$\begin{aligned}
\partial^l_\Delta\left(\frac{\partial^2 f(x,y)}{\partial y^2} - \left(\frac{\partial^2 f(x,y)}{\partial y^2}\right)_N\right) &= \sum_{j=0}^{\infty}\sum_{k=N+1}^{\infty} \hat{f}_{jk}\partial_\Delta^{l_1+l_2+2}\mathcal{J}^{(\tau,\varsigma)}_{\ell_1,j}(x)\mathcal{J}^{(\tau,\varsigma)}_{\ell_2,k}(y) \\
&+ \sum_{j=N+1}^{\infty}\sum_{k=0}^{\infty} \hat{f}_{jk}\partial_\Delta^{l_1+l_2+2}\mathcal{J}^{(\tau,\varsigma)}_{\ell_1,j}(x)\mathcal{J}^{(\tau,\varsigma)}_{\ell_2,k}(y) \\
&= \sum_{j=0}^{\infty}\sum_{k=N+1}^{\infty} \hat{f}_{jk}\nu_{j,l_1}\nu_{k,l_2+2}\mathcal{J}^{(\tau+l_1,\varsigma+l_1)}_{\ell_1,j-l_1}(x)\mathcal{J}^{(\tau+l_2+2,\varsigma+l_2+2)}_{\ell_2,k-l_2-2}(y) \\
&+ \sum_{j=N+1}^{\infty}\sum_{k=0}^{\infty} \hat{f}_{jk}\nu_{j,l_1}\nu_{k,l_2+2}\mathcal{J}^{(\tau+l_1,\varsigma+l_1)}_{\ell_1,j-l_1}(x)\mathcal{J}^{(\tau+l_2+2,\varsigma+l_2+2)}_{\ell_2,k-l_2-2}(y).
\end{aligned}$$

By taking the $L^2_{\omega^{(\tau,\varsigma)}}-$norm of the above equation, we obtain

$$\left\|\partial^l_\Delta\left(\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial y^2}\right)_N\right)\right\|^2_{\omega^{(\tau+l,\varsigma+l)}} = \sum_{j=0}^{\infty}\sum_{k=N+1}^{\infty}\hat{f}^2_{jk}c_{j,k} + \sum_{j=N+1}^{\infty}\sum_{k=0}^{\infty}\hat{f}^2_{jk}c_{j,k} + 2\sum_{j=N+1}^{\infty}\sum_{k=N+1}^{\infty}\hat{f}^2_{jk}c_{j,k}, \tag{63}$$

where

$$c_{j,k} = \nu^2_{j,l_1}\nu^2_{k,l_2+2}h^{(\tau+l_1,\varsigma+l_1)}_{\ell_1,j-l_1}(x)h^{(\tau+l_2+2,\varsigma+l_2+2)}_{\ell_2,k-l_2-2}(y).$$

Similarly,

$$\left\|\partial^l_\Delta\left(\frac{\partial^2 f}{\partial y^2}\right)\right\|^2_{\omega^{(\tau+\varepsilon,\varsigma+\varepsilon)}} = \sum_{j=0}^{\infty}\sum_{k=N+1}^{\infty}\hat{f}^2_{jk}d_{j,k} + \sum_{j=N+1}^{\infty}\sum_{k=0}^{\infty}\hat{f}^2_{jk}d_{j,k} + 2\sum_{j=N+1}^{\infty}\sum_{k=N+1}^{\infty}\hat{f}^2_{jk}d_{j,k}, \tag{64}$$

where

$$d_{j,k} = \nu^2_{j,\varepsilon_1}\nu^2_{k,\varepsilon_2+2}h^{(\tau+\varepsilon_1,\varsigma+\varepsilon_1)}_{\ell_1,j-\varepsilon_1}(x)h^{(\tau+\varepsilon_2+2,\varsigma+\varepsilon_2+2)}_{\ell_2,k-\varepsilon_2-2}(y).$$

Using (18), (59), and (60), we obtain

$$\frac{c_{j,k}}{d_{j,k}} \leq \eta_1 j^{l_1-\varepsilon_1}k^{l_2-\varepsilon_2-2}(j+\tau+\varsigma)^{l_1-\varepsilon_1}(k+\tau+\varsigma)^{l_2-\varepsilon_2-2}. \tag{65}$$

From the relations (63)–(65), we can write

$$
\begin{aligned}
&\left|\frac{\partial^2 f(x,y)}{\partial y^2}-\left(\frac{\partial^2 f(x,y)}{\partial y^2}\right)_N\right|^2_{l,\omega^{(\tau,\varsigma)}}=\left\|\partial^l_\Delta\left(\frac{\partial^2 f}{\partial y^2}-\left(\frac{\partial^2 f}{\partial y^2}\right)_N\right)\right\|^2_{\omega^{(\tau+l,\varsigma+l)}}\\
&=\sum_{j=0}^{\infty}\sum_{k=N+1}^{\infty}\frac{c_{j,k}}{d_{j,k}}d_{j,k}\hat{f}^2_{jk}+\sum_{j=N+1}^{\infty}\sum_{k=0}^{\infty}\frac{c_{j,k}}{d_{j,k}}d_{j,k}\hat{f}^2_{jk}+2\sum_{j=N+1}^{\infty}\sum_{k=N+1}^{\infty}\frac{c_{j,k}}{d_{j,k}}d_{j,k}\hat{f}^2_{jk}\\
&\le\sum_{j=0}^{\infty}\sum_{k=N+1}^{\infty}\eta_1 j^{l_1-\varepsilon_1}k^{l_2-\varepsilon_2-2}(j+\tau+\varsigma)^{l_1-\varepsilon_1}(k+\tau+\varsigma)^{l_2-\varepsilon_2-2}d_{j,k}\hat{f}^2_{jk}\\
&+\sum_{j=N+1}^{\infty}\sum_{k=0}^{\infty}\eta_1 j^{l_1-\varepsilon_1}k^{l_2-\varepsilon_2-2}(j+\tau+\varsigma)^{l_1-\varepsilon_1}(k+\tau+\varsigma)^{l_2-\varepsilon_2-2}d_{j,k}\hat{f}^2_{jk}\\
&+2\sum_{j=N+1}^{\infty}\sum_{k=N+1}^{\infty}\eta_1 j^{l_1-\varepsilon_1}k^{l_2-\varepsilon_2-2}(j+\tau+\varsigma)^{l_1-\varepsilon_1}(k+\tau+\varsigma)^{l_2-\varepsilon_2-2}d_{j,k}\hat{f}^2_{jk}\\
&\le\eta_1 N^{l_2-\varepsilon_2-2}(1+\tau+\varsigma)^{l_1-\varepsilon_1}(N+\tau+\varsigma)^{l_2-\varepsilon_2-2}\sum_{j=0}^{\infty}\sum_{k=N+1}^{\infty}d_{j,k}\hat{f}^2_{jk}\\
&+\eta_1 N^{l_1-\varepsilon_1}(N+\tau+\varsigma)^{l_1-\varepsilon_1}(1+\tau+\varsigma)^{l_2-\varepsilon_2-2}\sum_{j=N+1}^{\infty}\sum_{k=0}^{\infty}d_{j,k}\hat{f}^2_{jk}\\
&+2\eta_1 N^{l_1-\varepsilon_1}N^{l_2-\varepsilon_2-2}(N+\tau+\varsigma)^{l_1-\varepsilon_1}(N+\tau+\varsigma)^{l_2-\varepsilon_2-2}\sum_{j=N+1}^{\infty}\sum_{k=N+1}^{\infty}d_{j,k}\hat{f}^2_{jk}\\
&\le\eta_1 N^{l_n-\varepsilon}(N+\tau+\varsigma)^{l_n-\varepsilon}(1+\tau+\varsigma)^{l_n-\varepsilon}\left(\sum_{j=0}^{\infty}\sum_{k=N+1}^{\infty}d_{j,k}\hat{f}^2_{jk}+\sum_{j=N+1}^{\infty}\sum_{k=0}^{\infty}d_{j,k}\hat{f}^2_{jk}+2\sum_{j=N+1}^{\infty}\sum_{k=N+1}^{\infty}d_{j,k}\hat{f}^2_{jk}\right)\\
&=\eta_1 N^{l_n-\varepsilon}(N+\tau+\varsigma)^{l_n-\varepsilon}(1+\tau+\varsigma)^{l_n-\varepsilon}\left\|\partial^l_\Delta f\right\|^2_{\omega^{(\tau+\varepsilon,\varsigma+\varepsilon)}}\\
&=\eta_1 N^{l_n-\varepsilon}(N+\tau+\varsigma)^{l_n-\varepsilon}(1+\tau+\varsigma)^{l_n-\varepsilon}|f|^2_{\varepsilon,\omega^{(\tau,\varsigma)}},
\end{aligned}
$$

where

$$l_n-\varepsilon_n=\min_{i=1,2}\{l_i-\varepsilon_i\},\quad 0\le l_i\le\varepsilon_i\le\varepsilon,\ i=1,2,$$

for any $l_n\le\varepsilon_n$. Therefore,

$$\left\|\frac{\partial^2 f}{\partial y^2}-\left(\frac{\partial^2 f}{\partial y^2}\right)_N\right\|_{\epsilon,\omega^{(\tau,\varsigma)}}\le\eta\big(N(N+\tau+\varsigma)(1+\tau+\varsigma)\big)^{\frac{\epsilon-\varepsilon}{2}}|f|_{\varepsilon,\omega^{(\tau,\varsigma)}},\quad 0\le\epsilon\le\varepsilon.$$

□

Theorem 10. *For any $f\in\mathscr{P}^{\varepsilon}_{\omega^{(\tau,\varsigma)}}(\mathfrak{D})$, $\varepsilon\in\mathbb{N}$, and $0\le\epsilon\le\varepsilon$, we can conclude that*

$$\left\|\frac{\partial^2 f}{\partial x^2}-\left(\frac{\partial^2 f}{\partial x^2}\right)_N\right\|_{\epsilon,\omega^{(\tau,\varsigma)}}\le\eta_2\big(N(N+\tau+\varsigma)(1+\tau+\varsigma)\big)^{\frac{\epsilon-\varepsilon}{2}}|f|_{\varepsilon,\omega^{(\tau,\varsigma)}},\tag{66}$$

where η_2 is a positive constant.

Proof. The proof of this theorem is similar to the proof of Theorem 9. □

Theorem 11. *For any $f\in\mathscr{P}^{\varepsilon}_{\omega^{(\tau,\varsigma)}}(\mathfrak{D})$, $\varepsilon\in\mathbb{N}$, and $0\le\epsilon\le\varepsilon$, we have*

$$\left\|\frac{\partial^2 f}{\partial y\partial x}-\left(\frac{\partial^2 f}{\partial y\partial x}\right)_N\right\|_{\epsilon,\omega^{(\tau,\varsigma)}}\le\eta_3\big(N(N+\tau+\varsigma)(1+\tau+\varsigma)\big)^{\frac{\epsilon-\varepsilon}{2}}|f|_{\varepsilon,\omega^{(\tau,\varsigma)}},\tag{67}$$

where η_3 is a positive constant.

Proof. The proof of this theorem is similar to the proof of Theorem 9. □

Remark 1. *Inequality (54) implies that if N tends to infinity, then $f - f_N \to 0$.*

6. Numerical Results

Here, we solve five examples tested by Maple 2018. The number of bases are denoted by $\mathfrak{B}$. The absolute errors and maximum absolute errors are obtained by

$$
\begin{aligned}
&|f(x,y) - f_N(x,y)|, \qquad (x,y) \in [0,\ell_1) \times [0,\ell_2),\ N \in \mathbb{N},\\
&MAE := \max_{i,j=0,1,\ldots,N} \{ |f(x_i,y_j) - f_N(x_i,y_j)| \},
\end{aligned}
$$

respectively, where (x_i, y_j) are roots of 2D-SJPs in $\mathfrak{D} = [0,\ell_1) \times [0,\ell_2)$ for different values of τ and ς.

Moreover, using

$$
\max_{j=0,1,\ldots,N} \{ |f(x,y_j) - f_N(x,y_j)| \}, \qquad x \in [0,\ell_1),
$$

we plot maximum absolute errors where y_j are roots of 1D-SJPs in $[0,\ell_2)$ for $j = 0,1,\ldots,N$.

Example 1. *Consider the following 2D-NFIDE studied by [36]:*

$$
f_{yx}(x,y) + f(x,y) + I^{(\frac{7}{2},\frac{11}{2})} f(x,y) = g(x,y) + \Theta(x,y) + \Lambda(x,y) + \rho(x,y) + \varphi(x,y),
$$

with the initial conditions

$$
f(x,0) = f(0,y) = f_y(y,0) = 0,\ f_x(0,y) = y,\ f_y(x,0) = x,
$$

where $(x,y) \in [0,1) \times [0,1)$ and

$$
\begin{aligned}
\Theta(x,y) &= \int_0^x \int_0^y (yt - xs) f^2(t,s)\, \mathrm{d}s\, \mathrm{d}t,\\
\Lambda(x,y) &= \frac{1}{\Gamma(\frac{7}{2})\Gamma(\frac{11}{2})} \int_0^x \int_0^y (x-t)^{\frac{5}{2}} (y-s)^{\frac{9}{2}} \log(s-t) f(t,s)\, \mathrm{d}s\, \mathrm{d}t,\\
\rho(x,y) &= \frac{1}{\Gamma(\frac{7}{2})\Gamma(\frac{11}{2})} \int_0^1 \int_0^1 (1-t)^{\frac{5}{2}} (1-s)^{\frac{9}{2}} y(t-s) f^2(t,s)\, \mathrm{d}s\, \mathrm{d}t,\\
\varphi(x,y) &= \int_0^1 \int_0^1 (1+y)(t^2 - s^2) f^2(t,s)\, \mathrm{d}s\, \mathrm{d}t,\\
g(x,y) &= \frac{4096}{127702575\pi} x^{\frac{9}{2}} y^{\frac{13}{2}} + xy - \frac{524288}{1552224799125\pi} y + 1.
\end{aligned}
$$

The exact solution is $f(x,y) = yx$.

Tables 1 and 2 report the obtained numerical results and absolute errors, respectively, using the new approach and choosing $\tau = \varsigma = 0$ and $N = 2,3$. Additionally, Table 3 reports maximum absolute errors by selecting various values of τ, ς and $N = 2$. These tables show that by choosing $\mathfrak{B} = (N+1)^2 = 16$ numbers of 2D-SJPs, our obtained results are more accurate than the results reported in [36,37] and use $\mathfrak{B} = N^2M^2 = 16$ and $\mathfrak{B} = m^2 = 4096$ numbers of 2D-HBPSLs and 2D-BPFs, respectively, for solving this problem. From Figure 1, the accuracy and efficiency of proposed method is illustrated.

Table 1. Numerical results with $\tau = \varsigma = 0$ for Example 1.

$x = y$	Exact Solution	2D-SJPs		2D-HBPSLs [37]	2D-BPFs [36]
		$N = 2$ $\mathfrak{B} = 9$	$N = 3$ $\mathfrak{B} = 16$	$M = N = 2$ $\mathfrak{B} = 16$	$N = 64$ $\mathfrak{B} = 4096$
0	0	-1.45834×10^{-8}	-1.93165×10^{-9}	-5.39368×10^{-10}	6.06689×10^{-5}
0.2	0.04	0.04	0.04	0.04	0.0379181
0.4	0.16	0.16	0.16	0.16	0.1578
0.6	0.36	0.36	0.36	0.36	0.359706
0.8	0.64	0.64	0.64	0.640001	0.643637
0.99	0.9801	0.980099	0.9801	0.980101	0.978529
Max error	0	1.908184×10^{-5}	2.081128×10^{-7}	1.185071×10^{-5}	2.09569×10^{-3}

Table 2. Absolute errors with $\tau = \varsigma = 0$ for Example 1.

$x = y$	2D-SJPs		2D-HBPSLs [37]
	$N = 2$ $\mathfrak{B} = 9$	$N = 3$ $\mathfrak{B} = 16$	$M = N = 2$ $\mathfrak{B} = 16$
0	1.458338×10^{-8}	1.931649×10^{-9}	5.393684×10^{-10}
0.1	4.839152×10^{-9}	2.920296×10^{-11}	1.049377×10^{-8}
0.2	1.620866×10^{-8}	3.409101×10^{-10}	4.526134×10^{-8}
0.3	3.955156×10^{-8}	5.296828×10^{-11}	1.037633×10^{-7}
0.4	7.048566×10^{-8}	1.150183×10^{-10}	1.859998×10^{-7}
0.5	1.093869×10^{-7}	2.445786×10^{-10}	2.674698×10^{-7}
0.6	1.613894×10^{-7}	5.844627×10^{-10}	4.343939×10^{-7}
0.7	2.363855×10^{-7}	4.051722×10^{-9}	5.646382×10^{-7}
0.8	3.490255×10^{-7}	3.485633×10^{-8}	6.582027×10^{-7}
0.9	5.187180×10^{-7}	1.445669×10^{-7}	7.150874×10^{-7}

Table 3. Maximum absolute errors with $N = 2$ for Example 1.

(τ, ς)	MAE	(τ, ς)	MAE
$(0, 0)$	1.908184×10^{-5}	$(1, 1)$	5.525558×10^{-5}
$(1, 2)$	1.657651×10^{-4}	$(2, 1)$	1.682304×10^{-5}
$(2, 2)$	6.110782×10^{-5}	$(3, 2)$	2.426797×10^{-5}

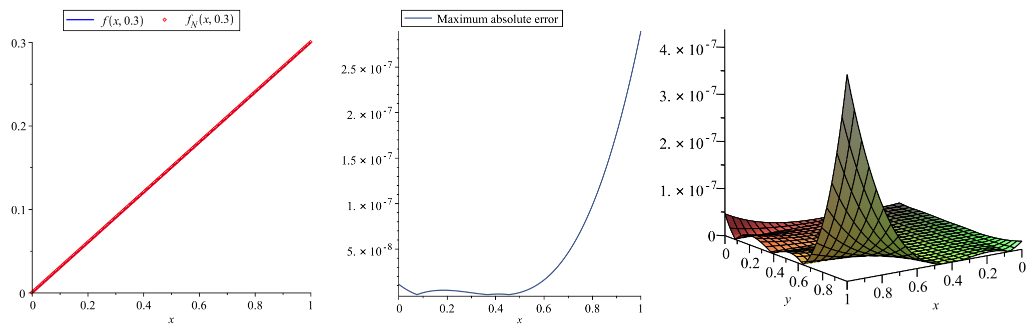

Figure 1. Plots of the exact and approximate solutions (**left**), maximum absolute error (**middle**) at $y = 0.3$, and absolute error (**right**) obtained by the 2D-SJPs with $N = 3$ and $\tau = \varsigma = 0$ for Example 1.

Example 2. *Consider the following 2D-NFIDE studied by [36]:*

$$f_{yy}(x,y) + f_{yx}(x,y) + f(x,y) + I^{(\frac{5}{2},1)} f(x,y) = g(x,y) + \rho(x,y) + \varphi(x,y),$$

with initial conditions

$$f(x,0) = f_x(x,0) = f_y(x,0) = f_x(0,y) = 0,\ f(0,y) = \frac{y^2}{4},$$

where $(x,y) \in [0,1) \times [0,1)$ *and*

$$\rho(x,y) = \frac{1}{\Gamma(\frac{5}{2})\Gamma(1)} \int_0^1 \int_0^1 (1-t)^{\frac{3}{2}} y^{\frac{8}{3}} x^{\frac{7}{2}} t^3 f(t,s)\, \mathrm{d}s\, \mathrm{d}t,$$
$$\varphi(x,y) = \int_0^1 \int_0^1 \frac{3360}{46} xy(x+y) f^2(t,s)\, \mathrm{d}s\, \mathrm{d}t,$$
$$g(x,y) = -\frac{608}{153153\sqrt{\pi}} x^{\frac{7}{2}} y^{\frac{8}{3}} + \frac{1}{4}(x^3+1)y^2 + \frac{1}{2}(x^3+1) - \frac{3}{2}xy^2 + \frac{2(16x^3+231)x^{\frac{5}{2}}y^3}{10395\sqrt{\pi}}.$$

The exact solution is $f(x,y) = \frac{y^2}{4}(x^3+1)$.

Tables 4 and 5 report the obtained numerical results and absolute errors, respectively, using the new approach and choosing $\tau = \varsigma = 0$ *and* $N = 2, 3$. *Additionally, Table 6 reports maximum absolute errors by selecting various values of* τ, ς *and* $N = 2$. *These tables show that by choosing* $\mathfrak{B} = (N+1)^2 = 16$ *numbers of 2D-SJPs, our obtained results are more accurate than the results reported in [36,37] and use* $\mathfrak{B} = N^2M^2 = 36$ *and* $\mathfrak{B} = m^2 = 1024$ *numbers of 2D-HBPSLs and 2D-BPFs, respectively, for solving this problem. In Figure 2, the accuracy and efficiency of proposed method is illustrated.*

Table 4. Numerical results with $\tau = \varsigma = 0$ for Example 2.

$x = y$	Exact Solution	2D-SJPs		2D-HBPSLs [37]	2D-BPFs [36]
		$N = 2$, $\mathfrak{B} = 9$	$N = 3$, $\mathfrak{B} = 16$	$N = 2, M = 3$, $\mathfrak{B} = 36$	$m = 32$, $\mathfrak{B} = 1024$
0	0	3.50087×10^{-8}	-1.75105×10^{-11}	-1.70722×10^{-8}	5.31008×10^{-5}
0.2	0.01008	0.00990005	0.01008	0.0100625	9.04921×10^{-3}
0.4	0.04256	0.0419991	0.04256	0.0426493	0.035166
0.6	0.10944	0.110691	0.10944	0.109233	0.099042
0.8	0.24192	0.244764	0.24192	0.242178	0.208004
0.99	0.482773	0.471861	0.482772	0.481524	0.411787
Max error	0	5.746197×10^{-5}	4.748580×10^{-8}	3.570347×10^{-4}	7.0986×10^{-2}

Table 5. Absolute errors with $\tau = \varsigma = 0$ for Example 2.

$x = y$	2D-SJPs		2D-HBPSLs [37]
	$N = 2$, $\mathfrak{B} = 9$	$N = 3$, $\mathfrak{B} = 16$	$N = 2, M = 3$, $\mathfrak{B} = 36$
0	3.500869×10^{-8}	1.751047×10^{-11}	1.707223×10^{-8}
0.1	9.992949×10^{-6}	1.282426×10^{-12}	5.623240×10^{-6}
0.2	1.799480×10^{-4}	1.250557×10^{-11}	1.752097×10^{-5}
0.3	4.950320×10^{-4}	7.129280×10^{-11}	3.919885×10^{-5}
0.4	5.608532×10^{-4}	3.100506×10^{-10}	8.934775×10^{-5}
0.5	3.355656×10^{-6}	1.029452×10^{-9}	3.884051×10^{-4}
0.6	1.251167×10^{-3}	2.924318×10^{-9}	2.071935×10^{-4}
0.7	2.676069×10^{-3}	7.479405×10^{-9}	2.249140×10^{-4}
0.8	2.844359×10^{-3}	1.753510×10^{-8}	2.583723×10^{-4}
0.9	8.713086×10^{-4}	7.185571×10^{-8}	4.152332×10^{-4}

Table 6. Maximum absolute errors with $N = 2$ for Example 2.

(τ, ς)	MAE	(τ, ς)	MAE
$(0,0)$	5.746197×10^{-5}	$(1,1)$	2.502457×10^{-3}
$(1,2)$	2.374060×10^{-2}	$(2,1)$	1.116405×10^{-2}
$(2,2)$	3.419813×10^{-3}	$(3,2)$	9.826693×10^{-3}

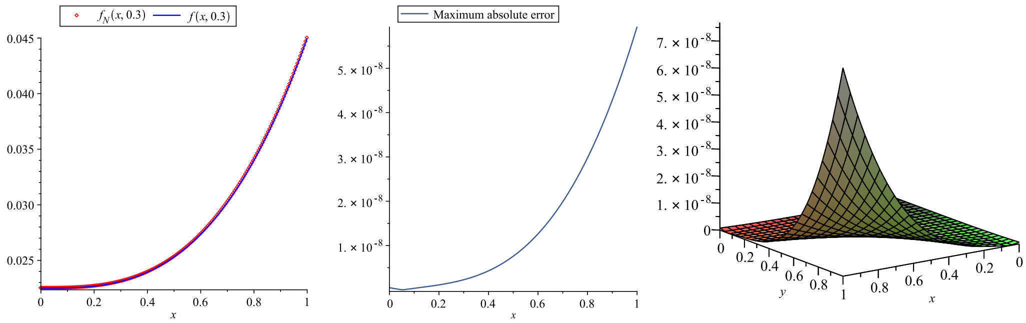

Figure 2. Plots of the exact and approximate solutions (**left**), maximum absolute error (**middle**) at $y = 0.3$, and absolute error (**right**) obtained by the 2D-SJPs with $N = 3$ and $\tau = \varsigma = 0$ for Example 2.

Example 3. *Consider the following 2D-NFIDE:*

$$f_{yy}(x,y) + f_{yx}(x,y) + f(x,y) = g(x,y) + \rho(x,y),$$

with initial conditions

$$f(x,0) = f_x(x,0) = f_y(x,0) = \mathrm{e}^x, \; f(0,y) = \mathrm{e}^y,$$

where $(x,y) \in [0,2) \times [0,2)$

$$\rho(x,y) = \frac{1}{\Gamma(\frac{3}{2})\Gamma(\frac{3}{2})} \int_0^2 \int_0^2 (2-t)^{\frac{1}{2}}(2-s)^{\frac{1}{2}}(x+y)(t^2+s^2)f(t,s)\,\mathrm{d}s\,\mathrm{d}t,$$
$$g(x,y) = 3\mathrm{e}^{x+y} - 4(x+y)\left(\frac{11}{\pi} - \frac{9\mathrm{e}^2\sqrt{2}\mathrm{erf}(\sqrt{2})}{\sqrt{\pi}} + \frac{7\mathrm{e}^4(\mathrm{erf}(\sqrt{2}))^2}{8}\right).$$

The exact solution is $f(x,y) = \mathrm{e}^{x+y}$. *Note that* $\mathrm{erf}(x) = \frac{2\int_0^x \mathrm{e}^{-x^2}\,\mathrm{d}x}{\sqrt{\pi}}$.

Tables 7 and 8 report the obtained numerical results and absolute errors, respectively, using the new approach and choosing $\tau = \varsigma = 0$ *and* $N = 4, 5$. *These tables show that by choosing* $\mathfrak{B} = (N+1)^2 = 36$ *numbers of 2D-SJPs, our obtained results are more accurate than the results reported in [37] and use* $\mathfrak{B} = N^2M^2 = 64$ *numbers of 2D-HBPSLs for solving this problem. From Figure 3, the accuracy and efficiency of the proposed method is illustrated.*

Table 7. Numerical results with $\tau = \varsigma = 0$ for Example 3.

$x = y$	Exact Solution	2D-SJPs $N = 4$ $\mathfrak{B} = 25$	2D-SJPs $N = 5$ $\mathfrak{B} = 36$	2D-HBPSLs [37] $N = 2, M = 4$ $\mathfrak{B} = 64$
0	1	0.999498	1.00004	0.99811
0.2	1.49182	1.49209	1.49181	1.49284
0.4	2.22554	2.22543	2.22556	2.22493
0.6	3.32012	3.31979	3.32012	3.31909
0.8	4.95303	4.95305	4.953	4.95461
1	7.38906	7.38946	7.38905	7.37414
1.2	11.0232	11.0233	11.0232	11.0301
1.4	16.4446	16.4439	16.4446	16.4394
1.6	24.5325	24.5318	24.5325	24.5242
1.8	36.5982	36.5992	36.5983	36.6095
Max error	0	3.352898×10^{-4}	2.293543×10^{-5}	1.118645×10^{-2}

Table 8. Absolute errors with $\tau = \varsigma = 0$ for Example 3.

$x = y$	2D-SJPs $N = 4$ $\mathfrak{B} = 25$	2D-SJPs $N = 5$ $\mathfrak{B} = 36$	2D-HBPSLs [37] $N = 2, M = 4$ $\mathfrak{B} = 64$
0	5.018269×10^{-4}	3.878596×10^{-5}	1.890271×10^{-3}
0.2	2.646095×10^{-4}	1.323019×10^{-5}	1.016898×10^{-3}
0.4	1.156163×10^{-4}	1.971321×10^{-5}	6.116798×10^{-4}
0.6	3.222169×10^{-4}	1.006197×10^{-6}	1.030940×10^{-3}
0.8	1.401881×10^{-5}	2.886721×10^{-5}	1.578274×10^{-3}
1	4.053898×10^{-4}	7.535908×10^{-6}	1.491810×10^{-2}
1.2	1.410474×10^{-4}	3.456475×10^{-5}	6.949542×10^{-3}
1.4	7.246987×10^{-4}	6.085008×10^{-7}	5.233810×10^{-3}
1.6	7.533610×10^{-4}	7.863965×10^{-5}	8.312774×10^{-3}
1.8	9.833575×10^{-4}	3.723796×10^{-5}	1.126113×10^{-2}

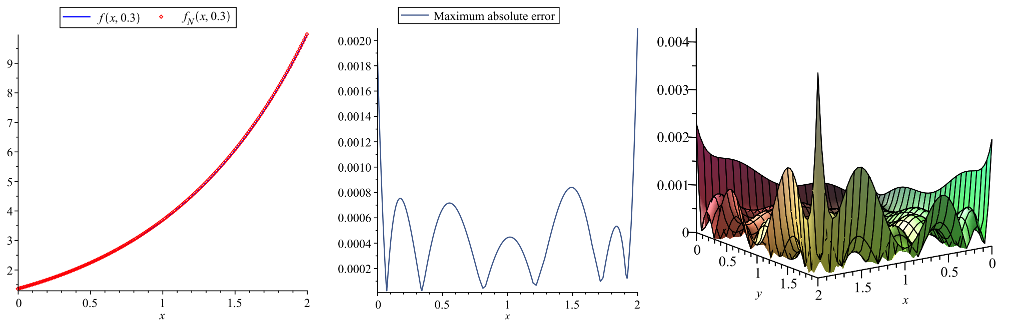

Figure 3. Plots of the exact and approximate solutions (**left**), maximum absolute error (**middle**) at $y = 0.3$, and absolute error (**right**) obtained by the 2D-SJPs with $N = 5$ and $\tau = \varsigma = 0$ for Example 3.

Example 4. *Consider the following 2D-NFIDE:*

$$f_{xx}(x,y) + f(x,y) + I^{(\frac{3}{2},1)} f(x,y) = g(x,y) + \Theta(x,y),$$

with initial conditions

$$f(x,0) = f(0,y) = f_y(x,0) = f_x(0,y) = f_x(x,0) = 0,$$

where $(x,y) \in [0,1) \times [0,1)$ *and*

$$\Theta(x,y) = \int_0^x \int_0^y (yt - xs) f^2(t,s)\, \mathrm{d}s\, \mathrm{d}t,$$

$$\begin{aligned} g(x,y) = &-\frac{1}{360\pi^{\frac{9}{2}}}\Bigg(y((-360x^2 - 720)\sin(\pi y) + x^6 y^3)\pi^{\frac{9}{2}} + 24y^2\left(\cos(\pi y)^2 - \frac{1}{2}\right)x^6\pi^{\frac{5}{2}} \\ &+ \frac{768\pi^2(\pi y\cos(\pi y) - \sin(\pi y))x^{\frac{7}{2}}}{7} \\ &- 27x^6\left(\frac{1}{9}\pi^{\frac{3}{2}} y\sin(\pi y)\cos(\pi y)\left(-2\pi^2 y^2 + 13\right) + \cos(\pi y)^2\sqrt{\pi} - \sqrt{\pi}\right)\Bigg). \end{aligned}$$

The exact solution is $f(x,y) = x^2 y \sin(\pi y)$.

Tables 9 and 10 report the obtained numerical results and absolute errors, respectively, using the new approach and choosing $\tau = \varsigma = 0$ *and* $N = 3,4$. *Additionally, Table 11 reports maximum absolute errors by selecting various values of* τ, ς *and* $N = 2$. *These tables show that by choosing* $\mathfrak{B} = (N+1)^2 = 25$ *numbers of 2D-SJPs, our obtained results are more accurate than the results obtained by the 2D-HBPSL method [37] and use* $\mathfrak{B} = N^2M^2 = 36$ *bases for solving this problem. In Figure 4, the accuracy and efficiency of the proposed method is illustrated.*

Table 9. Numerical results with $\tau = \varsigma = 0$ for Example 4.

$x = y$	Exact Solution	2D-SJPs		2D-HBPSLs [37]	
		$N = 3$, $\mathfrak{B} = 16$	$N = 4$, $\mathfrak{B} = 25$	$M = N = 2$, $\mathfrak{B} = 16$	$N = 2$, $M = 3$, $\mathfrak{B} = 36$
0	0	1.42563×10^{-6}	-1.53250×10^{-7}	0.00304418	-5.40176×10^{-8}
0.2	0.00470228	0.00503905	0.00461625	0.00860387	0.00502286
0.4	0.0608676	0.0606074	0.0615685	0.0582732	0.0592075
0.6	0.205428	0.201651	0.203813	0.2084	0.206665
0.8	0.300946	0.309229	0.302467	0.253177	0.299392
Max error	0	4.183049×10^{-3}	2.691559×10^{-4}	1.686288×10^{-2}	6.519558×10^{-3}

Table 10. Absolute errors with $\tau = \varsigma = 0$ for Example 4.

$x = y$	2D-SJPs		2D-HBPSLs [37]	
	$N = 3$, $\mathfrak{B} = 16$	$N = 4$, $\mathfrak{B} = 25$	$M = N = 2$, $\mathfrak{B} = 16$	$N = 2$, $M = 3$, $\mathfrak{B} = 36$
0	1.425634×10^{-6}	1.532497×10^{-7}	3.044182×10^{-3}	5.401763×10^{-8}
0.1	2.359796×10^{-6}	5.799964×10^{-5}	1.304790×10^{-6}	8.310324×10^{-5}
0.2	3.367631×10^{-4}	8.603064×10^{-5}	3.901591×10^{-3}	3.205808×10^{-4}
0.3	6.244970×10^{-4}	3.609740×10^{-4}	6.081376×10^{-3}	6.170073×10^{-4}
0.4	2.601698×10^{-4}	7.008446×10^{-4}	2.594409×10^{-3}	1.660119×10^{-3}
0.5	2.479625×10^{-3}	5.263376×10^{-5}	1.670057×10^{-2}	2.207590×10^{-3}
0.6	3.776819×10^{-3}	1.615412×10^{-3}	2.972021×10^{-3}	1.236545×10^{-3}
0.7	3.370614×10^{-4}	1.772158×10^{-3}	3.193363×10^{-2}	1.625204×10^{-5}
0.8	8.282794×10^{-3}	1.521422×10^{-3}	4.776857×10^{-2}	1.553977×10^{-3}
0.9	9.162584×10^{-3}	4.275932×10^{-3}	5.981687×10^{-3}	4.222491×10^{-4}

Table 11. Maximum absolute errors with $N = 2$ for Example 4.

(τ,ς)	*MAE*	(τ,ς)	*MAE*
$(0,0)$	4.183049×10^{-3}	$(1,1)$	7.254076×10^{-3}
$(1,2)$	6.743181×10^{-3}	$(2,1)$	4.492349×10^{-3}
$(2,2)$	4.903053×10^{-3}	$(3,2)$	3.081034×10^{-3}

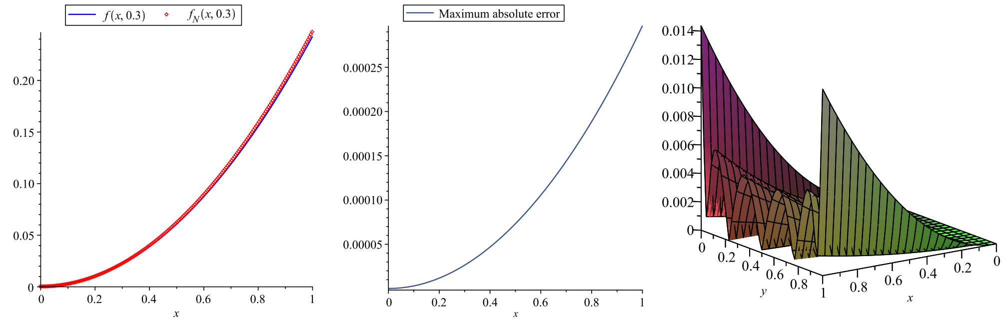

Figure 4. Plots of the exact and approximate solutions (**left**), maximum absolute error (**middle**) at $y = 0.3$, and absolute error (**right**) obtained by the 2D-SJPs with $N = 4$ and $\tau = \varsigma = 0$ for Example 4.

Example 5. *Consider the following 2D-NFIDE:*

$$f_{yy}(x,y) + f_{yx}(x,y) + f(x,y) + I^{(\frac{3}{2},1)} f(x,y) = g(x,y) + \Theta(x,y) + \varphi(x,y),$$

with initial conditions

$$f(x,0) = 0,\ f(0,y) = \sin(\pi y),\ f_y(x,0) = \pi \mathrm{e}^{x},\ f_x(0,y) = \sin(\pi y),\ f_x(x,0) = 0,$$

where $(x,y) \in [0,1) \times [0,1)$ *and*

$$\Theta(x,y) = \int_0^x \int_0^y xysf^2(t,s)\,\mathrm{d}s\,\mathrm{d}t,$$
$$\varphi(x,y) = \int_0^1 \int_0^1 x^2y^3t^2 f^2(t,s)\,\mathrm{d}s\,\mathrm{d}t,$$
$$g(x,y) = -\frac{1}{8\pi^{\frac{5}{2}}}\Bigg(\bigg(xy^2\Big(x-\frac{1}{2}\Big)\mathrm{e}^{2x} + \mathrm{e}^2x^2y^3 - x^2y^3 + \frac{1}{2}xy^2 - 8\mathrm{e}^x\sin(\pi y)\bigg)\pi^{\frac{5}{2}}$$
$$+\bigg(\Big(-2\sin(\pi y)yx\Big(x-\frac{1}{2}\Big)\mathrm{e}^{2x} - \sin(\pi y)yx + 8\mathrm{e}^x\mathrm{erf}(\sqrt{x})\Big)\cos(\pi y) - 8\mathrm{e}^x\mathrm{erf}(\sqrt{x})\bigg)\pi^{\frac{3}{2}}$$
$$-8\pi^{\frac{7}{2}}\mathrm{e}^x\cos(\pi y) + 8\pi^{\frac{9}{2}}\mathrm{e}^x\sin(\pi y) - (-1+\cos(\pi y))\bigg(x\Big(\frac{1}{2} + \Big(x-\frac{1}{2}\Big)\mathrm{e}^{2x}\Big)\sqrt{\pi}\cos(\pi y)$$
$$+x\sqrt{\pi}\Big(x-\frac{1}{2}\Big)\mathrm{e}^{2x} + \frac{1}{2}x\sqrt{\pi} + 16\pi\sqrt{x}\bigg)\Bigg).$$

The exact solution is $f(x,y) = \mathrm{e}^x\sin(\pi y)$.

Tables 12 and 13 report the obtained numerical results and absolute errors, respectively, using the new approach and choosing $\tau = \varsigma = 0$ *and* $N = 3,4$. *Additionally, Table 14 reports maximum absolute errors by selecting various values of* τ, ς *and* $N = 3$. *These tables show that by choosing* $\mathfrak{B} = (N+1)^2 = 25$ *numbers of 2D-SJPs, our obtained results are more accurate than the results obtained by the 2D-HBPSL method [37] and use* $\mathfrak{B} = N^2M^2 = 36$ *bases for solving this problem. In Figure 5, the accuracy and efficiency of proposed method is illustrated.*

Table 12. Numerical results with $\tau = \varsigma = 0$ for Example 5.

$x = y$	Exact Solution	2D-SJPs		2D-HBPSLs [37]	
		$N = 3$ $\mathfrak{B} = 16$	$N = 4$ $\mathfrak{B} = 25$	$M = N = 2$ $\mathfrak{B} = 16$	$N = 2,\ M = 3$ $\mathfrak{B} = 36$
0	0	−0.0498894	0.00126408	0.106466	−0.0269479
0.2	0.717923	0.743648	0.718851	0.644168	0.713975
0.4	1.41881	1.39875	1.41815	1.33223	1.3479
0.6	1.73294	1.70905	1.73227	1.43579	1.40505
0.8	1.30814	1.35737	1.31117	0.694829	0.506292
Max error	0	4.900771×10^{-3}	2.890626×10^{-3}	7.161353×10^{-1}	1.625743

Table 13. Absolute errors with $\tau = \varsigma = 0$ for Example 5.

$x = y$	2D-SJPs		2D-HBPSLs [37]	
	$N = 3$ $\mathfrak{B} = 16$	$N = 4$ $\mathfrak{B} = 25$	$M = N = 2$ $\mathfrak{B} = 16$	$N = 2,\ M = 3$ $\mathfrak{B} = 36$
0	4.988938×10^{-2}	1.264079×10^{-3}	1.064657×10^{-1}	2.694793×10^{-2}
0.1	1.314614×10^{-2}	2.544488×10^{-4}	1.500519×10^{-2}	7.810507×10^{-3}
0.2	2.572567×10^{-2}	9.282027×10^{-4}	7.375466×10^{-2}	3.947639×10^{-3}
0.3	7.178138×10^{-3}	6.978147×10^{-4}	1.226548×10^{-1}	3.448407×10^{-2}
0.4	2.005857×10^{-2}	6.560525×10^{-4}	8.657965×10^{-2}	7.090727×10^{-2}
0.5	3.495032×10^{-2}	1.518547×10^{-3}	1.553134×10^{-1}	3.482695×10^{-1}
0.6	2.389001×10^{-2}	6.632893×10^{-4}	2.971477×10^{-1}	3.278872×10^{-1}
0.7	1.206528×10^{-2}	1.561235×10^{-3}	4.595629×10^{-1}	5.951170×10^{-1}
0.8	4.922751×10^{-2}	3.025826×10^{-3}	6.133115×10^{-1}	8.018483×10^{-1}
0.9	3.323691×10^{-2}	2.322792×10^{-3}	7.485747×10^{-1}	1.408296

Table 14. Maximum absolute errors with $N = 3$ for Example 5.

(τ, ς)	MAE	(τ, ς)	MAE
$(0, 0)$	4.90077×10^{-3}	$(1, 1)$	3.494379×10^{-2}
$(1, 2)$	1.159618×10^{-1}	$(2, 1)$	1.718043×10^{-2}
$(2, 2)$	6.089228×10^{-2}	$(3, 2)$	3.330454×10^{-2}

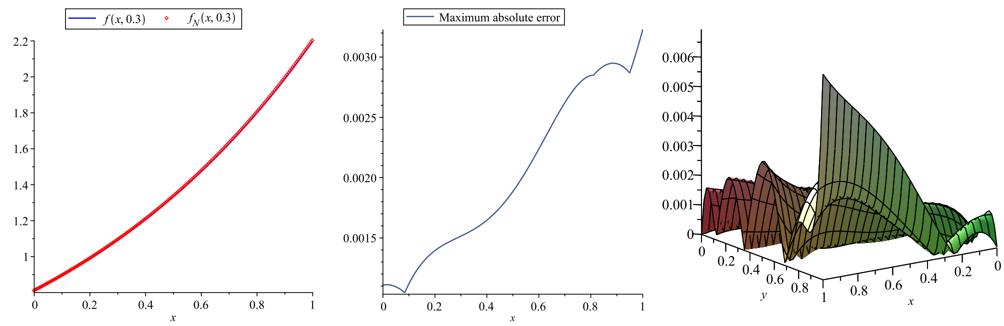

Figure 5. Plots of the exact and approximate solutions (**left**), maximum absolute error (**middle**) at $y = 0.3$, and absolute error (**right**) obtained by the 2D-SJPs with $N = 4$ and $\tau = \varsigma = 0$ for Example 5.

7. Conclusions

In this research, sufficient conditions for the existence and uniqueness of local and global solutions of general 2D-NFIDEs were provided. Additionally, the collocation method and operational matrices based on 2D-SJPs were used for solving these equations. Moreover, error bounds of the proposed method were obtained. We showed that the order of convergence of the method is $O\left(\frac{1}{(N(N+\tau+\varsigma))^{\frac{\varepsilon-\epsilon}{2}}}\right)$ in the Jacobi-weighted Sobolev space. Finally, we evaluated the presented method by solving five test problems. The obtained numerical results showed that a favorable approximate solution can be obtained by using lower numbers of basis functions.

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