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Article

Relative Controllability and Ulam–Hyers Stability of the Second-Order Linear Time-Delay Systems

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Abstract: We introduce the delayed sine/cosine-type matrix function and use the Laplace transform method to obtain a closed form solution to IVP for a second-order time-delayed linear system with noncommutative matrices A and Ω. We also introduce a delay Gramian matrix and examine a relative controllability linear/semi-linear time delay system. We have obtained the necessary and sufficient condition for the relative controllability of the linear time-delayed second-order system. In addition, we have obtained sufficient conditions for the relative controllability of the semi-linear second-order time-delay system. Finally, we investigate the Ulam–Hyers stability of a second-order semi-linear time-delayed system.

Keywords: stability; controllability; delay systems; fractional calculus

MSC: 93B05; 60H17; 93C25; 34K30; 34K35

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1. Introduction

Khusainov et al. [1] studied the IVP problem for a second-order linear pure delay differential equation of the form:

$$\begin{cases} z''(t) + \Omega^2 z(t-\tau) = h(t), & t \geq 0,\ \tau > 0, \\ z(t) = \varphi(t),\ z'(t) = \varphi'(t), & -\tau \leq t \leq 0, \end{cases} \tag{1}$$

where $h : [0,\infty) \to \mathbb{R}^n$, Ω is an $n \times n$ invertible matrix, τ is the time delay and φ is an arbitrary two times continuously differentiable function. The solution to (1) has a closed form explicit representation [1] (Theorem 2):

$$\begin{aligned} z(t) &= (\cos_\tau \Omega t)\varphi(-\tau) + \Omega^{-1}(\sin_\tau \Omega t)\varphi'(-\tau) \\ &\quad + \Omega^{-1}\int_{-\tau}^{0} \sin_\tau \Omega(t-\tau-s)\varphi''(s)ds \\ &\quad + \Omega^{-1}\int_{0}^{t} \sin_\tau \Omega(t-\tau-s)h(s)ds, \end{aligned}$$

where $\cos_\tau \Omega : \mathbb{R} \to \mathbb{R}^{n\times n}$ and $\sin_\tau \Omega : \mathbb{R} \to \mathbb{R}^{n\times n}$ denote the delayed matrix cosine and the delayed matrix sine, respectively.

It should be emphasized that the pioneer works [1–11] led to many new results in differential equations with a delay of integer and non-integer order and a discrete system with delay; see [1–23]. These models have applications in oscillatory systems [24,25], computational mathematics [26], spatially extended fractional reaction–diffusion models [27], and so on.

Controllability of dynamical systems is one of the most important concept in control theory. In recent years, the controllability of delayed dynamical systems has been investigated by many authors. There are several recent works on controllability/control theory [28–36] and Ulam–Hyers stability [37,38] for delay differential equations in the literature.

However, to the best of our knowledge, there is no study dealing with relative controllability and stability of the following second-order linear differential time-delay equation:

$$\begin{cases} z''(t) + Az(t) + \Omega z(t-h) = Bu(t), \quad t \in (0,T], \\ z(t) = \varphi(t), \quad -h \leq t \leq 0, \\ z(0) = \varphi(t), \quad z'(0) = \varphi'(0), \end{cases} \tag{2}$$

and semilinear second-order time-delay systems:

$$\begin{cases} z''(t) + Az(t) + \Omega z(t-h) = Bu(t) + f(t, z(t)), \quad t \in (0,T], \\ z(t) = \varphi(t), \quad -h \leq t \leq 0, \\ z(0) = \varphi(t), \quad z'(0) = \varphi'(0), \end{cases} \tag{3}$$

where $A, \Omega \in \mathbb{R}^{d\times d}$, $B \in \mathbb{R}^{d\times r}$, $f \in C\Big([0,T] \times \mathbb{R}^d, \mathbb{R}^d\Big)$, $\varphi \in C^1\Big([-h,0], \mathbb{R}^d\Big)$. $u : [0,T] \to \mathbb{R}^r$ is a control function.

The main contributions are presented as follows:

- We introduce the delayed sine/cosine-type matrix functions $S^h(A,\Omega;t)$, $C^h(A,\Omega;t)$ by means of the determining function $Q^{A,\Omega}_{k,m}$.
- We give a closed-form solution of problem (2) with nonpermutable matrices A, Ω by using $S^h(A,\Omega;t)$, $C^h(A,\Omega;t)$.
- We give the necessary and sufficient condition for the relative controllability of the system (2).
- We provide a sufficient condition for the relative controllability of the semilinear system (3).
- Finally, we study the Ulam–Hyers stability of problem (3).

The rest of this paper is structured as follows. In Section 2, we introduce delayed sine/cosine-type matrix functions and study some of their properties, Laplace transform delayed sine/cosine-type matrix functions. In Section 3, we obtain an exact analytical solution of the second-order problem (2) using delayed sine/cosine-type matrix functions $C^h(t)$ and $S^h(t)$. In Section 4, we give a necessary and sufficient condition for the relative controllability of the linear second-order delay systems. In Section 5, we give sufficient conditions for the relative controllability of the semilinear delay second-order systems. Finally, in Section 6 we investigate the Ulam–Hyers-type stability of the second-order semilinear time-delay system. In Section 7, to illustrate the theoretical findings, we provide some examples.

2. Auxiliary Lemmas

We study a concept of delayed sine/cosine-type matrix functions. In this consept, the determining equation and the determining function $Q^{A,\Omega}_{k,m}$ play an important role, see [18,21]. We define the determining function $Q^{A,\Omega}_{k,m}$ by means of the following recurrence (determining) equation:

$$\begin{cases} Q^{A,\Omega}_{2k,m} = -AQ^{A,\Omega}_{2k-2,m} - \Omega Q^{A,\Omega}_{2k-2,m-1}, \\ Q^{A,\Omega}_{-2,m} = Q^{A,\Omega}_{2k,-1} = \Theta, \;\; Q^{A,\Omega}_{0,0} = I, \; Q^{A,\Omega}_{2k,0} = (-1)^k A^k, \\ k = 0,1,2,\ldots, m = 0,1,2,\ldots. \end{cases} \tag{4}$$

Using the determining equation (4) we can easily obtain an explicit form for $Q^{A,\Omega}_{k,m}$ in terms of A and Ω.

Lemma 1. *The determining function* $Q^{A,\Omega}_{k,m}$ *has the following explicit form:*

$$\begin{cases} Q_{2k,m} = \sum_{j=0}^{k} (-1)^{j+1} A^j \Omega Q_{2k-2-2j,m-1}, \;\; k \geq 1, \;\; m \geq 1, \\ Q^{A,\Omega}_{0,0} = I, \;\; Q^{A,\Omega}_{-2,m} = Q^{A,\Omega}_{2k,-1} = \Theta. \end{cases}$$

Proof. Indeed,

$$\begin{aligned} Q_{2k,m} &= \sum_{j=0}^{k} (-1)^{j+1} A^j \Omega Q_{2k-2-2j,m-1} \\ &= \sum_{j=1}^{k} (-1)^{j+1} A^j \Omega Q_{2k-2-2j,m-1} - \Omega Q_{2k-2,m-1} \\ &= -A \sum_{j=0}^{k-1} (-1)^{j+1} A^j \Omega Q_{2k-4-2j,m-1} - \Omega Q_{2k-2,m-1} \\ &= -A Q_{2k-2,m} - \Omega Q_{2k-2,m-1}. \end{aligned}$$

□

Next, we introduce delayed cosine/sine-type matrix functions C^h, S^h by means of the determining function $Q_{k,m}$.

Definition 1. *Delayed cosine/sine-type matrix functions* $C^h, S^h : [0, \infty) \to \mathbb{R}^d$ *are defined as follows:*

$$C^h(A, \Omega; t) = C^h(t) := \sum_{m=0}^{\infty} \sum_{k=m}^{\infty} Q^{A,\Omega}_{2k,m} \frac{(t - mh)^{2k}_+}{(2k)!},$$

$$S^h(A, \Omega; t) = S^h(t) := \sum_{m=0}^{\infty} \sum_{k=m}^{\infty} Q^{A,\Omega}_{2k,m} \frac{(t - mh)^{2k+1}_+}{(2k+1)!},$$

where $(t)_+ = \max\{0, t\}$.

It is clear that C^h, S^h can be rewritten in the following way:

$$C^h(t) := \begin{cases} \Theta, & -\infty < t < 0, \\ \sum_{k=0}^{\infty} (-1)^k A^k \frac{t^{2k}}{(2k)!}, & 0 \leq t < h, \\ \sum_{k=0}^{\infty} (-1)^k A^k \frac{t^{2k}}{(2k)!} + \sum_{k=1}^{\infty} Q_{2k,1} \frac{(t-h)^{2k}_+}{(2k)!}, & h < t \leq 2h, \\ \vdots & \vdots \\ \sum_{k=0}^{\infty} (-1)^k A^k \frac{t^{2k}}{(2k)!} + \sum_{k=1}^{\infty} Q_{2k,1} \frac{(t-h)^{2k}_+}{(2k)!} & \\ + \ldots + \sum_{k=m}^{\infty} Q_{2k,m} \frac{(t-mh)^{2k}_+}{(2k)!}, & mh \leq t < (m+1)h, \end{cases}$$

and

$$S^h(t) := \begin{cases} \Theta, & -\infty < t < 0, \\ \sum_{k=0}^{\infty} (-1)^k A^k \frac{t^{2k+1}}{(2k+1)!}, & 0 \le t < h, \\ \sum_{k=0}^{\infty} (-1)^k A^k \frac{t^{2k+1}}{(2k+1)!} + \sum_{k=1}^{\infty} Q_{2k,1} \frac{(t-h)_+^{2k+1}}{(2k+1)!}, & h < t \le 2h, \\ \vdots & \vdots \\ \sum_{k=0}^{\infty} (-1)^k A^k \frac{t^{2k+1}}{(2k+1)!} + \sum_{k=1}^{\infty} Q_{2k,1} \frac{(t-h)_+^{2k+1}}{(2k+1)!} & \\ + \ldots + \sum_{k=m}^{\infty} Q_{2k,m} \frac{(t-mh)_+^{2k+1}}{(2k+1)!}, & mh \le t < (m+1)h, \end{cases}$$

where $m \in \mathbb{Z}_0^{\infty}$, Θ is the $d \times d$ null matrix.

If we introduce the following functions:

$$C_m^h(t-mh) := (-1)^m \sum_{k=m}^{\infty} Q_{2k,m}^{A,\Omega} \frac{(t-mh)_+^{2k}}{(2k)!}, \quad S_m^h(t-mh) := (-1)^m \sum_{k=m}^{\infty} Q_{2k,m}^{A,\Omega} \frac{(t-mh)_+^{2k+1}}{(2k+1)!},$$

then C^h, S^h can be written in the following compact form:

$$\begin{aligned} C^h(t) &:= C_0^h(t) + C_1^h(t-h) + \ldots + C_m^h(t-mh), \;\; mh \le t < (m+1)h, \\ S^h(t) &:= S_0^h(t) + S_1^h(t-h) + \ldots + S_m^h(t-mh), \;\; mh \le t < (m+1). \end{aligned}$$

Here are some special cases of C^h, S^h.

- If $\Omega = \Theta$, and $A = \mathcal{A}^2$, then:

$$\begin{aligned} C^h\left(\mathcal{A}^2, \Theta; t\right) &= \sum_{k=0}^{\infty} (-1)^k \mathcal{A}^{2k} \frac{t^{2k}}{(2k)!} = \cos(\mathcal{A}t), \\ \mathcal{A}S^h\left(\mathcal{A}^2, \Theta; t\right) &= \mathcal{A}\sum_{k=0}^{\infty} (-1)^k \mathcal{A}^{2k} \frac{t^{2k+1}}{(2k+1)!} = \sin(\mathcal{A}t), \end{aligned}$$

  where $\cos(\mathcal{A}t)$ and $\sin(\mathcal{A}t)$ are matrix cosine and sine functions, respectively.
- If $A = \Theta$ and $\Omega = \mathcal{B}^2$ then we have:

$$\begin{aligned} C^h\left(\Theta, \mathcal{B}^2; t\right) &= \sum_{m=0}^{\infty} (-1)^m \mathcal{B}^{2m} \frac{(t-mh)_+^{2m}}{(2m)!} = \cos^h(\mathcal{B}; t), \\ \mathcal{B}S^h\left(\Theta, \mathcal{B}^2; t\right) &= \mathcal{B}\sum_{m=0}^{\infty} (-1)^m \mathcal{B}^{2m} \frac{(t-mh)_+^{2m+1}}{(2m+1)!} = \mathcal{B}\sin^h(\mathcal{B}; t), \end{aligned}$$

  where $\cos^h(\mathcal{B}; t)$ and $\sin^h(\mathcal{B}; t)$ are pure delayed matrix cosine and sine functions, respectively.
- If A and Ω are permutable, then $Q_{2k,m} = (-1)^k \begin{pmatrix} k \\ m \end{pmatrix} A^{k-m}\Omega^m, \;\; ,k \ge m$ and:

$$\begin{aligned} C^h(t) &= \sum_{m=0}^{\infty}\sum_{k=m}^{\infty} (-1)^k \begin{pmatrix} k \\ m \end{pmatrix} A^{k-m}\Omega^m \frac{(t-mh)_+^{2k}}{(2k)!}, \\ S^h(t) &= \sum_{m=0}^{\infty}\sum_{k=m}^{\infty} (-1)^k \begin{pmatrix} k \\ m \end{pmatrix} A^{k-m}\Omega^m \frac{(t-mh)_+^{2k+1}}{(2k+1)!}. \end{aligned}$$

Using the definitions of $C^h(t)$ and $S^h(t)$ we can easily obtain the following estimations.

Lemma 2. *For any $mh \leq t < (m+1)h$, $m \in \mathbb{N}$, the following estimations hold:*

$$\left\|C^h(t)\right\| \leq \cosh\left(\sqrt{\|\Omega\|}t\right)\cosh\left(\sqrt{\|A\|}t\right) := L_C(t),$$
$$\left\|S^h(t)\right\| \leq t\cosh\left(\sqrt{\|\Omega\|}t\right)\cosh\left(\sqrt{\|A\|}t\right) := tL_C(t).$$

Proof. Indeed,

$$\begin{aligned}
\left\|C^h(t)\right\| &\leq \sum_{k=0}^{\infty}\|A\|^k\frac{t^{2k}}{(2k)!} + \sum_{k=1}^{\infty}\begin{pmatrix} k \\ 1 \end{pmatrix}\|A\|^{k-1}\|\Omega\|\frac{(t-h)_+^{2k}}{(2k)!} \\
&+ \ldots + \sum_{k=m}^{\infty}\begin{pmatrix} k \\ m \end{pmatrix}\|A\|^{k-m}\|\Omega\|^m\frac{(t-mh)_+^{2k}}{(2k)!} \\
&\leq \sum_{k=0}^{\infty}\|A\|^k\frac{t^{2k}}{(2k)!} + \sum_{k=0}^{\infty}\begin{pmatrix} k+1 \\ 1 \end{pmatrix}\|A\|^{k}\|\Omega\|\frac{(t-h)_+^{2k+2}}{(2k+2)!} \\
&+ \ldots + \sum_{k=0}^{\infty}\begin{pmatrix} k+m \\ m \end{pmatrix}\|A\|^{k}\|\Omega\|^m\frac{(t-mh)_+^{2k+2m}}{(2k+2m)!} \\
&< \sum_{k=0}^{\infty}\|A\|^k\frac{t^{2k}}{(2k)!} + \|\Omega\|\frac{(t-h)_+^{2}}{2!}\sum_{k=0}^{\infty}\|A\|^k\frac{(t-h)_+^{2k}}{(2k)!} \\
&+ \ldots + \|\Omega\|^m\frac{(t-mh)_+^{2m}}{(2m)!}\sum_{k=0}^{\infty}\|A\|^k\frac{(t-mh)_+^{2k}}{(2k)!} \\
&< \cosh\left(\sqrt{\|\Omega\|}t^2\right)\cosh\left(\sqrt{\|A\|}t^2\right).
\end{aligned}$$

For the delayed sine matrix function $S^h(t)$, we have:

$$\begin{aligned}
\left\|S^h(t)\right\| &\leq \sum_{k=0}^{\infty}\|A\|^k\frac{t^{2k+1}}{(2k+1)!} + \sum_{k=1}^{\infty}\begin{pmatrix} k \\ 1 \end{pmatrix}\|A\|^{k-1}\|\Omega\|\frac{(t-h)_+^{2k+1}}{(2k+1)!} \\
&+ \ldots + \sum_{k=m}^{\infty}\begin{pmatrix} k \\ m \end{pmatrix}\|A\|^{k-m}\|\Omega\|^m\frac{(t-mh)_+^{2k+1}}{(2k+1)!} \\
&\leq \sum_{k=0}^{\infty}\|A\|^k\frac{t^{2k+1}}{(2k+1)!} + \sum_{k=0}^{\infty}\begin{pmatrix} k+1 \\ 1 \end{pmatrix}\|A\|^{k}\|\Omega\|\frac{(t-h)_+^{2k+3}}{(2k+3)!} \\
&+ \ldots + \sum_{k=0}^{\infty}\begin{pmatrix} k+m \\ m \end{pmatrix}\|A\|^{k}\|\Omega\|^m\frac{(t-mh)_+^{2k+2m+1}}{(2k+2m+1)!} \\
&< \sum_{k=0}^{\infty}\|A\|^k\frac{t^{2k+1}}{(2k+1)!} + \|\Omega\|\frac{(t-h)_+^{2}}{2!}\sum_{k=0}^{\infty}\|A\|^k\frac{(t-h)_+^{2k+1}}{(2k+1)!} \\
&+ \ldots + \|\Omega\|^m\frac{(t-mh)_+^{2m}}{(2m)!}\sum_{k=0}^{\infty}\|A\|^k\frac{(t-mh)_+^{2k+1}}{(2k+1)!} \\
&< t\cosh\left(\sqrt{\|\Omega\|}t\right)\cosh\left(\sqrt{\|A\|}t\right).
\end{aligned}$$

□

Definition 2. *System (2) is stable on* $[0,T]$ *in the Ulam–Hyers sense, if there exists* $C>0$ *such that for any* $\varepsilon>0$ *and for any function* $z^*(t)$ *satisfying inequality*

$$\left\| \frac{d}{dt}z^*(t) + Az^*(t) + \Omega z^*(t-h) - f(t) \right\| \le \varepsilon \tag{5}$$

and the initial conditions in (2), there is a solution $z(t)$*of (2) such that:*

$$\|z^*(t) - z(t)\| \le C\varepsilon$$

for every $t \in [0,T]$.

Theorem 1. *The following formulae hold:*

(a) *The function* $C^h(\cdot)$ *and* $S^h(\cdot)$ *are continuous on* $(0,+\infty)$.
(b) $\frac{d}{dt}C^h(t) = -AS^h(t) - \Omega S^h(t-h)$, $\frac{d}{dt}S^h(t) = C^h(t)$ *for all* $t\in\mathbb{R}$.
(c) $\frac{d^2}{dt^2}C^h(t) = -AC^h(t) - \Omega C^h(t-h)$, $\frac{d^2}{dt^2}S^h(t) = -AS^h(t) - \Omega S^h(t-h)$.

Proof. The proofs of items (a) and (c) are obvious. In fact, the proof of item (c) is based on property (b):

$$\begin{aligned}
\frac{d}{dt}C^h(t) &= \frac{d}{dt}\sum_{m=0}^{\infty}\sum_{k=m}^{\infty} Q_{2k,m}^{A,\Omega}\frac{(t-mh)_+^{2k}}{(2k)!} \\
&= \sum_{m=0}^{\infty}\sum_{k=m}^{\infty} Q_{2k,m}^{A,\Omega}\frac{(t-mh)_+^{2k-1}}{(2k-1)!} \\
&= \sum_{m=0}^{\infty}\sum_{k=m-1}^{\infty} Q_{2k+2,m}^{A,\Omega}\frac{(t-mh)_+^{2k+1}}{(2k+1)!} \\
&= -\sum_{m=0}^{\infty}\sum_{k=m}^{\infty} AQ_{2k,m}^{A,\Omega}\frac{(t-mh)_+^{2k+1}}{(2k+1)!} - \sum_{m=1}^{\infty}\sum_{k=m}^{\infty} \Omega Q_{2k,m-1}^{A,\Omega}\frac{(t-mh)_+^{2k+1}}{(2k+1)!} \\
&= -A\sum_{m=0}^{\infty}\sum_{k=m}^{\infty} Q_{2k,m}^{A,\Omega}\frac{(t-mh)_+^{2k+1}}{(2k+1)!} - \Omega\sum_{m=0}^{\infty}\sum_{k=m}^{\infty} Q_{2k,m}^{A,\Omega}\frac{(t-h-mh)_+^{2k+1}}{(2k+1)!} \\
&= -AS^h(t) - \Omega S^h(t-h),
\end{aligned}$$

and

$$\begin{aligned}
\frac{d}{dt}S^h(t) &= \frac{d}{dt}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty} Q_{2k,m}^{A,\Omega}\frac{(t-mh)_+^{2k+1}}{(2k+1)!} \\
&= \sum_{m=0}^{\infty}\sum_{k=0}^{\infty} Q_{2k,m}^{A,\Omega}\frac{(t-mh)_+^{2k}}{(2k)!} = C^h(t).
\end{aligned}$$

□

The main tool that is used in this section is LT $F(s) := L\{f(t)\} = \int_0^\infty e^{-st}f(t)dt$, $\mathrm{Re}\, s > a$. It is known that LT is defined for a function f that is exponentially bounded.

Lemma 3. *We have:*

$$L^{-1}\left\{(-1)^m\left(e^{-hs}\left(s^2I+A\right)^{-1}\Omega\right)^m s\left(s^2I+A\right)^{-1}\right\} = \sum_{k=m}^{\infty} Q_{2k,m}\frac{(t-mh)_+^{2k}}{(2k)!} = C_m^h(t),$$

$$L^{-1}\left\{(-1)^m\left(e^{-hs}\left(s^2I+A\right)^{-1}\Omega\right)^m \left(s^2I+A\right)^{-1}\right\} = \sum_{k=m}^{\infty} Q_{2k,m}\frac{(t-mh)_+^{2k+1}}{(2k+1)!} = S_m^h(t),$$

where $Q_{2k,m}$ is defined in (4).

Proof. For $n = 0$ by the well-known formulas, we have:

$$L^{-1}\left\{s\left(s^2I+A\right)^{-1}\right\} = C^h\left(-At^2\right),$$
$$L^{-1}\left\{e^{-sh}\left(s^2I+A\right)^{-1}\right\} = S^h\left(-A(t-h)^2\right), \quad t \geq h.$$

Let $Q_{2k,0} = (-A)^k$. For $n = 1$, one can use the convolution property of the LT to obtain:

$$\begin{aligned}
&-L^{-1}\left\{e^{-hs}\left(s^2I+A\right)^{-1}\Omega s\left(s^2I+A\right)^{-1}\right\} \\
&= -\left\{e^{-hs}\left(s^2I+A\right)^{-1}\Omega\right\} * \left\{s\left(s^2I+A\right)^{-1}\right\} \\
&= -\int_0^t S^h\left(-A(s-h)^2\right)\Omega C^h\left(-A(t-s)^2\right)ds \\
&= -\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-A)^j\Omega(-A)^k}{(2j+1)!(2k)!}\int_h^t (s-h)^{2j+1}(t-s)^{2k}ds \\
&= -\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}(-A)^j\Omega(-A)^k\frac{(t-h)_+^{2k+2j+2}}{(2k+2j+2)!} \\
&= \sum_{k=0}^{\infty}\sum_{j=0}^{k}(-1)^{j+1}A^j\Omega(-A)^{k-j}\frac{(t-h)_+^{2k+2}}{(2k+2)!} \\
&= \sum_{k=1}^{\infty}\underbrace{\sum_{j=0}^{k-1}(-1)^{j+1}A^j\Omega(-A)^{k-1-j}}_{Q_{2k,1}}\frac{(t-h)_+^{2k}}{(2k)!},
\end{aligned}$$

$$\begin{aligned}
&L^{-1}\left\{e^{-hs}\left(s^2I+A\right)^{-1}\Omega e^{-hs}\left(s^2I+A\right)^{-1}\Omega s\left(s^2I+A\right)^{-1}\right\} \\
&= \left\{e^{-hs}\left(s^2I+A\right)^{-1}\Omega\right\} * \left\{e^{-hs}\left(s^2I+A\right)^{-1}\Omega s\left(s^2I+A\right)^{-1}\right\} \\
&= -\int_0^t S^h\left(-A(s-h)^2\right)\Omega C_1^h\left(-A(t-h-s)^2\right)ds \\
&= -\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-A)^k\Omega Q_{2j,1}}{(2k+1)!(2j)!}\int_h^{t-h}(s-h)^{2k+1}(t-h-s)^{2j}ds \\
&= -\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}(-A)^j\Omega^j Q_{2k,1}\frac{(t-2h)_+^{2k+2j+2}}{(2k+2j+2)!} \\
&= \sum_{k=0}^{\infty}\sum_{j=0}^{k}(-1)^{j+1}A^j\Omega Q_{2k-2j,1}\frac{(t-2h)_+^{2k+2}}{(2k+2)!} \\
&= \sum_{k=2}^{\infty}\underbrace{\sum_{j=0}^{k-1}(-1)^{j+1}A^j\Omega Q_{2k-2-2j,1}}_{Q_{2k,2}}\frac{(t-2h)_+^{2k}}{(2k)!}.
\end{aligned}$$

Now, we are ready to use the mathematical induction. To do so, suppose that it holds for $n = m$. Then the convolution property of LT yields:

$$\begin{aligned}
& L^{-1}\left\{(-1)^{m+1}\left(e^{-hs}\left(s^2I + A\right)^{-1}\Omega\right)^{m+1} s\left(s^2I + A\right)^{-1}\right\} \\
&= L^{-1}\left\{(-1)e^{-hs}\left(s^2I + A\right)^{-1}\Omega\right\} * L^{-1}\left\{(-1)^m\left(e^{-hs}\left(s^2I + A\right)^{-1}\Omega\right)^m s\left(s^2I + A\right)^{-1}\right\} \\
&= -\int_0^t S^h\left(-A(s-h)^2\right)\Omega C_m^h\left(-A(t-s)^2\right)ds \\
&= -\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^j A^j \Omega (-1)^k Q_{2k,m}}{(2j+1)!(2k)!}\int_h^{t-mh}(s-h)^{2j+1}(t-mh-s)^{2k}ds \\
&= -\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}(-1)^j A^j \Omega (-1)^k Q_{2k,m}\frac{(t-(m+1)h)_+^{2k+2j+2}}{(2k+2j+2)!} \\
&= -\sum_{k=0}^{\infty}\sum_{j=0}^{k}(-1)^j A^j \Omega Q_{2k-2j,m}\frac{(t-(m+1)h)_+^{2k}}{(2k)!} \\
&= \sum_{k=0}^{\infty}\sum_{j=m+1}^{k+1}(-1)^{j+1} A^j \Omega Q_{2k-2j-2,m}\frac{(t-(m+1)h)_+^{2k}}{(2k)!} \\
&= \sum_{k=0}^{\infty} Q_{2k,m+1}\frac{(t-(m+1)h)_+^{2k}}{(2k)!}.
\end{aligned}$$

□

Lemma 4. *We have:*

$$S^h(t) = L^{-1}\left\{\left(s^2I + A + \Omega e^{-hs}\right)^{-1}\right\} = \sum_{m=0}^{\infty}\sum_{k=0}^{\infty} Q_{2k,m}\frac{(t-mh)_+^{2k+1}}{(2k+1)!}.$$

$$C^h(t) = L^{-1}\left\{s\left(s^2I + A + \Omega e^{-hs}\right)^{-1}\right\} = \sum_{m=0}^{\infty}\sum_{k=0}^{\infty} Q_{2k,m}\frac{(t-mh)_+^{2k}}{(2k)!}.$$

Proof. It is easy to make the following calculations:

$$\begin{aligned}
& L^{-1}\left\{s\left(s^2I + A + \Omega e^{-hs}\right)^{-1}\right\} \\
&= L^{-1}\left\{s\left(\left(s^2I + A\right)I + \left(s^2I + A\right)\left(s^2I + A\right)^{-1}\Omega e^{-hs}\right)^{-1}\right\} \\
&= L^{-1}\left\{\left(I + \left(s^2I + A\right)^{-1}\Omega e^{-hs}\right)^{-1} s\left(s^2I + A\right)^{-1}\right\} \\
&= L^{-1}\left\{\sum_{m=0}^{\infty} e^{-mhs}(-1)^m\left(\left(s^2I + A\right)^{-1}\Omega\right)^m s\left(s^2I + A\right)^{-1}\right\} \\
&= \sum_{m=0}^{\infty}(-1)^m L^{-1}\left\{e^{-mhs}\left(\left(s^2I + A\right)^{-1}\Omega\right)^m s\left(s^2I + A\right)^{-1}\right\}.
\end{aligned}$$

Hence, by the Lemma 3 we obtain:

$$L^{-1}\left\{s\left(s^2I + A + \Omega e^{-hs}\right)^{-1}\right\} = \sum_{m=0}^{\infty}\sum_{k=m}^{\infty} Q_{2k,m}\frac{(t-mh)_+^{2k}}{(2k)!}.$$

□

3. Exact Analytical Solution

We present the closed form of the analytical solution of the second-order problem (2) using delayed sine/cosine-type matrix functions $C^h(t)$ and $S^h(t)$.

Theorem 2. *The exact analytical solution of the IVP (2) can be represented as follows:*

$$z(t) = C^h(t)\varphi(0) + S^h(t)\varphi'(0) - \int_{-h}^{0} S^h(t-s-h)\Omega\varphi(s)ds + \int_{0}^{t} S^h(t-s)f(s)ds.$$

Proof. Suppose that the function f and the solution of (2) are bounded exponentially. By applying the LT to (2), we obtain the following equation:

$$L\{z''(t)\} + AL\{z(t)\} + \Omega L\{z(t-h)\} = L\{f(t)\}.$$

It follows that:

$$\begin{aligned}\left(s^2 I + A + \Omega e^{-hs}\right)Z(s) &= s\varphi(0) + \varphi'(0)\\ &\quad - \Omega\int_0^{\infty} e^{-st}z(t-h)dt + F(s),\end{aligned}$$

where $Z(s) = L\{z(t)\},\ F(s) = L\{f(t)\}$. For sufficiently large s, such that

$$\left\|A + \Omega e^{-hs}\right\| < s^{\mu},$$

the matrix $s^2 I + A + \Omega e^{-hs}$ is invertible and

$$\begin{aligned}Z(s) &= s\left(s^2 I + A + \Omega e^{-hs}\right)^{-1}\varphi(0)\\ &\quad + \left(s^2 I + A + \Omega e^{-hs}\right)^{-1}\varphi'(0)\\ &\quad - \left(s^2 I + A + \Omega e^{-hs}\right)^{-1}\Omega\Psi(s)\\ &\quad + \left(s^2 I + A + \Omega e^{-hs}\right)^{-1}F(s).\end{aligned}$$

By Lemma 4:

$$\begin{aligned}z(t) &= C^h(t)\varphi(0) + S^h(t)\varphi'(0)\\ &\quad - \int_{-h}^{0} S^h(t-s-h)\Omega\varphi(s)ds + \int_0^t S^h(t-s)f(s)ds,\end{aligned} \tag{6}$$

since:

$$\begin{aligned}&L^{-1}\left\{\left(s^2 I + A + \Omega e^{-hs}\right)^{-1}\Omega\Psi(s)\right\} = L^{-1}\left\{\left(s^2 I + A + \Omega e^{-hs}\right)^{-1}\right\} * L^{-1}\{\Omega\Psi(s)\}\\ &= \int_0^t S^h(t-s)\Omega\psi(s-h)ds = \int_0^h S^h(t-s)\Omega\varphi(s-h)ds\\ &= \int_{-h}^0 S^h(t-s-h)\Omega\varphi(s)ds.\end{aligned}$$

Now the assumption on the exponential boundedness can easily be removed by checking that (6) is a solution of (2). □

4. Controllability of Linear Delay System

Definition 3. *System (2) is called relatively controllable if for an arbitrary initial function* $\varphi \in C^1\left([0,T],\mathbb{R}^d\right)$*, in the terminal state* $z_f \in \mathbb{R}^d$*, there exists a control* $u \in L^2([0,T],\mathbb{R}^r)$ *such that a solution of (2)* $z(t) := z(t;u)$ *satisfies the condition* $z(T) = z_f$*.*

According to Theorem 2, a solution of (2) has the form:

$$z(t) = \begin{cases} C^h(t)\varphi(0) + S^h(t)\varphi'(0) - \int_{-h}^{0} S^h(t-s-h)\Omega\varphi(s)ds \\ \qquad + \int_0^t S^h(t-s)Bu(s)ds, \quad t \geq 0, \\ \qquad \varphi(t), \quad -h \leq t \leq 0. \end{cases}$$

We establish some sufficient and necessary condition for the relative controllability of (2) by using a delay Gramian matrix defined by:

$$\Gamma_0^T := \int_0^T S^h(T-s)BB^\intercal\left(S^h\right)^\intercal(T-s)ds.$$

It follows from the symmetric form of the matrix Γ_0^T that it is always non-negative.

Theorem 3. *The linear system (2) is controllable if and only if* Γ_0^T *is positive.*

Proof. **Necessity.** Assume that the linear system (2) is relatively controllable. Contrarily, suppose that Γ_0^T is not positive and definite; there is at least a nonzero vector $z \in \mathbb{R}^d$ such that $z^\intercal\Gamma_0^T z = 0$, which implies that:

$$\begin{aligned} 0 &= z^\intercal\Gamma_0^T z \\ &= \int_0^T z^\intercal S^h(T-s)BB^\intercal\left(S^h\right)^\intercal(T-s)zds \\ &= \int_0^T \left\|B^\intercal\left(S^h\right)^\intercal(T-s)z\right\|^2 ds = \int_0^T \left\|z^\intercal S^h(T-s)B\right\|^2 ds. \end{aligned}$$

It follows that:

$$z^\intercal S^h(T-s)B = 0, \quad \text{for all } 0 \leq s \leq T.$$

Since (2) is relatively controllable, from Definition 3, there is a control function $u_1(t)$ that steers the response from 0 to z. Then,

$$\begin{aligned} z &= \int_0^T S^h(T-s)Bu_1(s)ds, \\ z^\intercal z &= \int_0^T z^\intercal S^h(T-s)Bu_1(s)ds = 0. \end{aligned}$$

This contradicts $z \neq 0$. Thus, Γ_0^T is positive.

Sufficiency. Assume that Γ_0^T is positive. Then it is an invertible matrix. Hence, we can choose:

$$\begin{aligned} u(t) &= B^\intercal\left(S^h\right)^\intercal(T-t)\left(\Gamma_0^T\right)^{-1}p, \\ p &:= z_f - C^h(T)\varphi(0) - S^h(T)\varphi'(0) + \int_{-h}^{0} S^h(T-s-h)\Omega\varphi(s)ds, \end{aligned}$$

$$\begin{aligned}
z(T) &= C^h(T)\varphi(0) + S^h(T)\varphi'(0) - \int_{-h}^{0} S^h(T-s-h)\Omega\varphi(s)ds \\
&+ \int_0^T S^h(T-s)BB^\intercal \left(S^h\right)^\intercal (T-s)\left(\Gamma_0^T\right)^{-1} pds \\
&= C^h(T)\varphi(0) + S^h(T)\varphi'(0) - \int_{-h}^{0} S^h(T-s-h)\Omega\varphi(s)ds + p \\
&= z_f.
\end{aligned}$$

Thus, (2) is controllable. □

5. Controllability of Semilinear Delay Differential System

In this section, we prove the sufficient condition for the relative controllability of (3) using Krasnoselskii's FT theorem.

We impose the following assumptions:

(A_1) The function $f : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ is continuous and $L_f(t) \in L^1([0,T],\mathbb{R}^+)$ such that:

$$\|f(t,y) - f(t,z)\| \le L_f(t)\|y - z\|, \;\; y,z \in \mathbb{R}^d,$$
$$f(t,0) \in C\left([0,T],\mathbb{R}^d\right).$$

(A_2) The linear system is relatively controllable.

Theorem 4. *Suppose that (A_1) and (A_2) are satisfied. Then System (3) is relatively controllable provided that*

$$\|B\|T^3 L_C^2(T) M \left\|L_f\right\|_{L^1} < 1.$$

Proof. To examine the conditions for the Krasnoselskii theorem, we divide our proof into several steps.

Step 1. The control function

$$\begin{aligned}
u(t;z) = B^\intercal \left(S^h(T-t)\right)^\intercal \left(\Gamma_0^T\right)^{-1} \Big(z_f - C^h(T)\varphi(0) - S^h(T)\varphi'(0) \\
+ \int_{-h}^{0} S^h(T-s-h)\Omega\varphi(s)ds - \int_0^T S^h(T-s)f(s,z(s))ds\Big)
\end{aligned}$$

satisfies the Lipschitz and linear growth conditions.

Indeed, in light of (A_1), we obtain that:

$$\begin{aligned}
\|u(t;z)\| \le \|B\| \left\| \left(\Gamma_0^T\right)^{-1} \right\| (T-t)L_C(T-t)\Big(\left\|z_f\right\| + L_C(T)\|\varphi(0)\| + TL_C(T)\left\|\varphi'(0)\right\| \\
+ TL_C(T)\|\Omega\| \int_{-h}^{0} \|\varphi(s)\|ds + TL_C(T)\int_0^T (\|f(s,z(s)) - f(s,0)\| + \|f(s,0)\|)ds\Big) \\
\le M_u + L_u\|z\|_\infty,
\end{aligned}$$

where:

$$\begin{aligned}
M_u &:= \|B\| \left\| \left(\Gamma_0^T\right)^{-1} \right\| (T-t)L_C(T-t)\Big(\left\|z_f\right\| + L_C(T)\|\varphi(0)\| + TL_C(T)\left\|\varphi'(0)\right\| \\
&+ TL_C(T)\|\Omega\| \int_{-h}^{0} \|\varphi(s)\|ds + T^2 L_C(T) \max_{0\le s\le T}\|f(s,0)\|\Big), \\
L_u &:= MTL_C(T)\left\|L_f\right\|_{L^1}.
\end{aligned}$$

Similarly,

$$\begin{aligned}\|u(t;z)-u(t;z)\| &\le \int_0^T \left\|S^h(T-s)\right\| \|f(s,z(s))-f(s,y(s))\| ds \\ &\le \int_0^T \left\|S^h(T-s)\right\| L_f(s) ds \|z-y\|_\infty \\ &\le L_u \|z-y\|_\infty.\end{aligned}$$

Step 2: Define:

$$\begin{aligned}P_1 z(t) :=& \; C^h(t)\varphi(0)+S^h(t)\varphi'(0) \\ &-\int_{-h}^0 S^h(t-s-h)\Omega\varphi(s)ds+\int_0^t S^h(t-s)Bu(s;z)ds,\end{aligned} \tag{7}$$

$$P_2 z(t) := \int_0^t S^h(t-s) f(s,z(s)) ds. \tag{8}$$

Show that $P_1 z + P_2 y \in B_r$ for all $z, y \in B_r$.

$$\begin{aligned}&\|P_1 z(t)+P_2 y(t)\| \\ &\le \left\|C^h(t)\right\| \|\varphi(0)\| + \left\|S^h(t)\right\| \|\varphi'(0)\| \\ &+ \int_{-h}^0 \left\|S^h(t-s-h)\right\| \|\Omega\| \|\varphi(s)\| ds \\ &+ \int_0^t \left\|S^h(t-s)\right\| \|B\| \|u(s;z)\| ds \\ &+ \int_0^t \left\|S^h(t-s)\right\| \|B\| \|f(s,y(s))\| ds \\ &\le L_C(t)\|\varphi(0)\| + tL_C(t)\|\varphi'(0)\| \\ &+ hTL_C(T)\|\Omega\|\|\varphi\|_\infty \\ &+ \|B\|T^2 L_C(T)(M_u+L_u r) \\ &+ TL_C(T)(r+\|f(\cdot,0)\|_{L^1}).\end{aligned}$$

Step 3: We prove that P_1 is a contraction.

$$\begin{aligned}&\|P_1 z(t)-P_1 y(t)\| \\ &\le \int_0^t \left\|S^h(t-s)\right\| \|B\| \|u(s;z)-u(s;y)\| ds \\ &\le \|B\|T^2 L_C(T) L_u \|z-y\|_\infty.\end{aligned}$$

Step 4: We prove that P_2 is a continuous compact operator.

First, we prove that P_2 is continuous. To prove this, let $\{z_n\}$ be a sequence in $C\left([-h,T],\mathbb{R}^d\right)$ such that $z_n \to z$ as $n \to \infty$. Then:

$$\begin{aligned}&\|P_2 z_n(t)-P_2 z(t)\| \\ &\le \int_0^t \left\|S^h(t-s)\right\| \|f(s,z_n(s))-f(s,z(s))\| ds \\ &\le TL_C(T)\left\|L_f\right\|_{L^1} \|z_n-z\|_\infty.\end{aligned}$$

Taking the supremum of the left-hand side and letting $n \to \infty$, we obtain the continuity of P_2.

Second, we show that P_2 is uniformly bounded on B_r. For any $z \in B_r$, we have:

$$\begin{aligned}\|P_2 z(t)\| &\leq \int_0^t \left\|S^h(t-s)\right\| \|f(s,z(s))\| ds \\ &\leq TL_C(T) \int_0^t L_f(s)(r + \|f(s,0)\|) ds \\ &\leq TL_C(T) \left\|L_f\right\|_{L^1} (r + \|f(\cdot,0)\|_\infty),\end{aligned}$$

in other words, P_2 is uniformly bounded on B_r.

Third, we show that P_2 is equicontinuous. Indeed,

$$\begin{aligned}&\|P_2 z(t_2) - P_2 z(t_1)\| \\ &\leq \left\| \int_0^{t_2} S^h(t_2 - s) f(s,z(s)) ds - \int_0^{t_1} S^h(t_1 - s) f(s,z(s)) ds \right\| \\ &\leq \left\| \int_{t_1}^{t_2} S^h(t_2 - s) f(s,z(s)) ds \right\| \\ &+ \left\| \int_0^{t_1} \left[S^h(t_2 - s) - S^h(t_1 - s) \right] f(s,z(s)) ds \right\| \\ &=: I_1 + I_2.\end{aligned}$$

For I_1 we have:

$$\begin{aligned}I_1 &\leq TL_C(T) \int_{t_1}^{t_2} \left(L_f(s) r + \|f(s,0)\| \right) ds \\ &\to 0\end{aligned}$$

as $t_2 \to t_1$.

For I_2 we have:

$$\begin{aligned}I_2 &\leq \int_0^{t_1} \left\| S^h(t_2 - s) - S^h(t_1 - s) \right\| \left(L_f(s) r + \|f(s,0)\| \right) ds \\ &\to 0,\end{aligned}$$

as $S^h(t)$ is uniformly convergent on $[0,T]$. Therefore:

$$\|P_2 z(t_2) - P_2 z(t_1)\| \to 0 \quad \text{as} \quad t_2 \to t_1.$$

The Arzela–Ascoli theorem says that P_2 is compact on B_r.

Steps 1-4 say that:

(i) $P_1 z + P_2 y \in B_r$ for any $z, y \in B_r$;

(ii) P_1 is a contraction;

(iii) P_2 is continuous and compact.

Thus, by the Krasnoselskii theorem there exists a fixed point $z \in B_r$ such that $P_1 z + P_2 z = z$. Moreover, $P_1 z(T) + P_2 z(T) = z(T)$, that is, the system (3) is exact and controllable on $[0,T]$. □

6. Hyers–Ulam Stability of Semilinear Delay Differential System

In this section, we discuss stability in the Hyers–Ulam sense of (3) on the interval $[0,T]$.

Definition 4. *System (3) is said to be stable in the Hyers-Ulam sense on $[0,T]$ if there exists, for a given $\varepsilon > 0$, a function $\psi \in C\left([0,T], \mathbb{R}^d\right)$ satisfying the inequality*

$$\left\| \psi''(t) + A\psi(t) + \Omega\psi(t-h) - f(t,\psi(t)) \right\| \leq \varepsilon, \quad t \in [0,T], \tag{9}$$

and there exists a solution $z \in C\left([0,T],\mathbb{R}^d\right)$ *of (3) and a constant* $M > 0$ *such that*

$$\|\psi(t) - z(t)\| \leq M\varepsilon, \quad t \in [0,T].$$

Remark 1. *A function* $\psi \in C\left([0,T],\mathbb{R}^d\right)$ *is a solution of Inequality (9) if and only if there exists a function* $g \in C\left([0,T],\mathbb{R}^d\right)$ *such that:*

1. $\|g(t)\| \leq \varepsilon, \ \ t \in [0,T]$.
2. $\psi''(t) = -A\psi(t) - \Omega\psi(t-h) + f(t,\psi(t)) + g(t), \ \ t \in [0,T]$.

Lemma 5. *Let* $\psi \in C\left([0,T],\mathbb{R}^d\right)$ *be a solution of Inequality (9). Then,* ψ *is a solution of the inequality:*

$$\|\psi(t) - \psi^*(t)\| \leq \frac{\varepsilon t^2}{2} L_C(t), \tag{10}$$

where:

$$\begin{aligned}\psi^*(t) &= C^h(t)\varphi(0) + S^h(t)\varphi'(0) \\ &\quad - \int_{-h}^{0} S^h(t-s-h)\Omega\varphi(s)ds + \int_0^t S^h(t-s)f(s,\psi(s))ds.\end{aligned}$$

Proof. From Remark 1, the solution of the equation

$$\psi''(t) = -A\psi(t) - \Omega\psi(t-h) + f(t,\psi(t)) + g(t), \ \ t \in [0,T],$$

can be written as:

$$\begin{aligned}\psi(t) &= C^h(t)\varphi(0) + S^h(t)\varphi'(0) \\ &\quad - \int_{-h}^{0} S^h(t-s-h)\Omega\varphi(s)ds + \int_0^t S^h(t-s)f(s,\psi(s))ds \\ &\quad + \int_0^t S^h(t-s)g(s)ds.\end{aligned}$$

From Lemma 5, we obtain for all $t \in [0,T]$:

$$\|\psi(t) - \psi^*(t)\| \leq \int_0^t \left\|S^h(t-s)\right\| \|g(s)\| ds \leq \frac{\varepsilon t^2}{2} L_C(t).$$

□

Theorem 5. *Let* (A_1) *be satisfied. Then, System (3) is Hyers–Ulam stable. In other words:*

$$\|\psi - z\|_\infty \leq M\varepsilon,$$

where

$$M := \frac{T^2}{2} \frac{L_C(T)}{1 - TL_C(T)\left\|L_f\right\|_{L^1}}.$$

Proof. Assume that ψ is a solution of Inequality (9) and z is the unique solution of (3), that is,

$$\begin{aligned}z(t) &= C^h(t)\varphi(0) + S^h(t)\varphi'(0) \\ &\quad - \int_{-h}^{0} S^h(t-s-h)\Omega\varphi(s)ds + \int_0^t S^h(t-s)f(s,z(s))ds.\end{aligned}$$

From Lemma 5, similar to the proof of Theorem 4, and by virtue of (10), we obtain:

$$\begin{aligned}\|\psi(t)-z(t)\| &\le \|\psi(t)-\psi^*(t)\|+\|\psi^*(t)-z(t)\| \\ &\le \frac{\varepsilon t^2}{2}L_C(t) \\ &+\int_0^t \left\|S^h(t-s)\right\| \|f(s,\psi(s))-f(s,z(s))\|ds \\ &\le \frac{\varepsilon t^2}{2}L_C(t) \\ &+tL_C(t)\left\|L_f\right\|_{L^1}\|\psi-z\|_\infty.\end{aligned}$$

Therefore,

$$\|\psi-z\|_\infty \le \frac{\varepsilon T^2}{2}\frac{L_C(T)}{1-TL_C(T)\left\|L_f\right\|_{L^1}}.$$

Thus:

$$\|\psi-z\|_\infty \le M\varepsilon.$$

□

7. Examples

Example 1. *We study a linear delayed dynamical second-order control system of the form*

$$\begin{cases}\zeta''(t)=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\zeta(t)+\begin{bmatrix}-1&1&0\\1&0&1\\0&1&1\end{bmatrix}\zeta(t-h)\\ \qquad +\begin{bmatrix}0&0\\1&0\\0&1\end{bmatrix}u(t)+f(t,\zeta(t)),\\ \zeta(t)=\varphi(t), \quad -h\le t\le 0,\end{cases} \tag{11}$$

defined in a time interval $[0,T]$, $T>1$, *with one constant time delay* $h=1$. *Hence,* $d=3$, $r=2$ *and*

$$Q(T):=\{Q_{0,t},Q_{1,t},Q_{2,t}:t\in[0,T)\}.$$

Moreover, using the determining function given in [39], we have:

$$\begin{aligned}Q(T)&=[Q_1(0)\ Q_2(0)\ Q_2(h)\ Q_3(0)\ Q_3(h)\ Q_3(2h)]\\ &=\left[B\ AB\ \Omega B\ A^2B\ (A\Omega+\Omega A)B\ \Omega^2B\right].\end{aligned}$$

Substituting matrices A, Ω, *and* B *in to* $Q(T)$, *we easily obtain:*

$$rank Q(T)=rank\begin{bmatrix}0&0&0&0&1&0&0&0&2&0&-1&1\\1&0&1&0&0&1&1&0&0&2&0&1\\0&1&0&1&1&1&0&1&2&4&2&1\end{bmatrix}=3.$$

Hence, by [40] the linear dynamical system associated with (11) is relatively controllable in each $[0,T]$ *for* $T>1$. *Consequently, by Theorem 4 System (11) is relatively controllable under the condition that the function* $f(t,\zeta)$ *satisfies the Lipschitz condition and is uniformly bounded.*

Example 2. *Now, we study a deterministic model of population dynamics with delayed birth rates and delayed logistic terms. The following system was used in [41] to model the growth in population:*

$$\begin{aligned}\zeta''(t) &= \begin{bmatrix} -a_1 & 0 \\ 0 & -a_2 \end{bmatrix}\begin{bmatrix} \zeta_1(t) \\ \zeta_2(t) \end{bmatrix} + \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}\begin{bmatrix} \zeta_1(t-h) \\ \zeta_2(t-h) \end{bmatrix} \\ &+ \begin{bmatrix} P_1(t,\zeta_1(t),\zeta_2(t-h)) \\ P_2(t,\zeta_2(t),\zeta_1(t-h)) \end{bmatrix}, \quad 0 \leq t \leq T. \end{aligned} \tag{12}$$

Then, (12) can be turned into:

$$\zeta''(t) = A\zeta(t) + B\zeta(t-h) + P(t, \underset{\sim}{\zeta}(t), \zeta(t-h)).$$

The linear system

$$\zeta''(t) = A\zeta(t) + B\zeta(t-h)$$

associated with (12) is controllable; see [41]. Thus, for any Lipschitz continuous and uniformly bounded P, by Theorem 4 System (12) is relatively controllable on $[0, T]$.

Example 3. *Assume that $d = 2$, $h = 0.2$, and $T = 0.8$. Consider the problem of the relative controllability of the time-delay linear dynamical control system:*

$$\begin{cases} y''(t) = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} y(t-0.2) + \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} u(t), \quad 0 \leq t \leq 0.8 \\ y(t) = \begin{pmatrix} t \\ 2t \end{pmatrix}, \quad -0.2 \leq t \leq 0. \end{cases} \tag{13}$$

In this case $S_h(t) = e_h^{B(t-h)}$, which is defined in Definition 3. The delayed Grammian matrix of System (13) has the following explicit form:

$$\begin{aligned} W_0^{0.8} &= \int_0^{0.8} S_h(0.8-s)BB^\intercal S_h^\intercal(0.8-s)ds = \int_0^{0.8} e_h^{\Omega(0.6-s)} BB^\intercal e_h^{\Omega^\intercal(0.6-s)} ds \\ &= \int_0^{0.8} e_h^{\Omega(0.6-s)} B^2 e_h^{\Omega^\intercal(0.6-s)} ds \\ &= W_1 + W_2 + W_3 + W_4. \end{aligned}$$

Here,

$$\begin{aligned} W_1 = \int_0^{0.2} &\left(I + \Omega\frac{(0.6-s)}{1!} + \Omega^2\frac{(0.4-s)^2}{2!} + \Omega^3\frac{(0.2-s)^3}{3!} \right) \\ &\times B^2 \left(I + \Omega^\intercal\frac{(0.6-s)}{1!} + (\Omega^\intercal)^2\frac{(0.4-s)^2}{2!} + (\Omega^\intercal)^3\frac{(0.2-s)^3}{3!} \right) ds, \end{aligned}$$

$$\begin{aligned} W_2 = \int_{0.2}^{0.4} &\left(I + \Omega\frac{(0.6-s)}{1!} + \Omega^2\frac{(0.4-s)^2}{2!} \right) \\ &\times B^2 \left(I + \Omega^\intercal\frac{(0.6-s)}{1!} + (\Omega^\intercal)^2\frac{(0.4-s)^2}{2!} \right) ds, \end{aligned}$$

$$W_3 = \int_{0.4}^{0.6} \left(I + \Omega\frac{(0.6-s)}{1!} \right) B^2 \left(I + \Omega^\intercal\frac{(0.6-s)}{1!} \right) ds,$$

$$W_4 = \int_{0.6}^{0.8} B^2 ds.$$

By elementary computation, one can obtain:

$$W_1 = \begin{pmatrix} 0.192\,3 & 9.264\,8\times 10^{-2} \\ 9.264\,8\times 10^{-2} & 6.994\,5\times 10^{-2} \end{pmatrix}, \quad W_2 = \begin{pmatrix} 0.106\,17 & 4.138\,3\times 10^{-2} \\ 4.138\,3\times 10^{-2} & 8.557\,1\times 10^{-2} \end{pmatrix}$$
$$W_3 = \begin{pmatrix} 6.333\,3\times 10^{-2} & 1.133\,3\times 10^{-2} \\ 1.133\,3\times 10^{-2} & 6.066\,7\times 10^{-2} \end{pmatrix}, \quad W_4 = \begin{pmatrix} 0.05 & 0 \\ 0 & 0.05 \end{pmatrix}.$$

Therefore, we obtain:

$$W = \begin{pmatrix} 0.411\,8 & 0.145\,36 \\ 0.145\,36 & 0.266\,18 \end{pmatrix}, \qquad W^{-1} = \begin{pmatrix} 3.0082 & -1.6428 \\ -1.6428 & 4.6539 \end{pmatrix}.$$

By Theorem 3 this implies that the linear time delay system (13) is relatively controllable on $[0, 0.8]$.

Example 4. *Let* $h = 0.5$, $T = 1$, *and* $d = 2$. *Consider the relative controllability of the following linear time-delay differential controlled system:*

$$\begin{cases} y''(t) = \begin{pmatrix} 0.2 & 0 \\ 0 & 0.2 \end{pmatrix} y(t) + \begin{pmatrix} 0.2 & 0.1 \\ 0 & 0.3 \end{pmatrix} y(t-0.2) + \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} u(t), \ 0 \le t \le 1 \\ y(t) = \begin{pmatrix} t \\ 2t \end{pmatrix}, \ -0.2 \le t \le 0. \end{cases} \tag{14}$$

In this case, $S_h(t) = e^{At} e_h^{\widehat{\Omega}(t-h)}$, *which is defined in Definition 3. The delayed Grammian matrix of System (13) has the following explicit form:*

$$\widehat{\Omega} = e^{-Ah}\Omega = \begin{pmatrix} 1.8221 & 0.3644 \\ 0 & 1.4577 \end{pmatrix}$$

$$\begin{aligned} W_0^1 &= \int_0^1 S_h(t) B B^\intercal S_h^\intercal(t) ds = \int_0^1 e^{A(1-s)} e_h^{\widehat{\Omega}(1-s-h)} B B^\intercal e_h^{\widehat{\Omega}^\intercal(1-s-h)} e^{A^\intercal(1-s)} ds \\ &= W_1 + W_2. \end{aligned}$$

Here,

$$\begin{aligned} W_1 &= \frac{1}{4}\int_0^{0.5} e^{A(1-s)} \Big(I + \widehat{\Omega}(0.5-s)\Big)\Big(I + \widehat{\Omega}^\intercal(0.5-s)\Big) e^{A^\intercal(1-s)} ds, \\ W_2 &= \frac{1}{4}\int_{0.5}^{1} e^{A(1-s)} e^{A^\intercal(1-s)} ds \end{aligned}$$

Simple calculations show that:

$$W_1 = \begin{pmatrix} 0.4555 & 0.0307 \\ 0.0307 & 0.3656 \end{pmatrix}, \quad W_2 = \begin{pmatrix} 0.1384 & 0 \\ 0 & 0.1384 \end{pmatrix},$$
$$W_0^1 = \begin{pmatrix} 0.5939 & 0.0307 \\ 0.0307 & 0.35040 \end{pmatrix}.$$

Therefore, we obtain that W_0^1 *is invertible. This means that the linear determinisitic delay system corresponding to (13) is relatively exact and controllable on* $[0, 1]$. *By Theorem 4, System (13) is relatively controllable* $[0, 0.8]$.

8. Conclusions

In this article we

- Studied a problem of finding the exact analytical solution of continuous linear time-delay systems using the delayed sine/cosine-type matrix functions;

- Established sufficient and necessary conditions for the relative controllability of linear time-delay differential systems in terms of a delay Grammian matrix;
- Obtained sufficient conditions of the relative controllability and Hyers–Ulam stability of semilinear time-delay differential systems.

A possible direction in which to extend the results of this paper is toward fractional linear/semilinear impulsive systems of order $1 < \alpha < 2$. On the other hand, in the future, the same approach can be used for various types of random noises or disturbances in second-order dynamical stochastic systems.

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