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Integrable Systems: In the Footprints of the Greats

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Abstract: In his 1842 lectures on dynamics C.G. Jacobi summarized difficulties with differential equations by saying that the main problem in the integration of differential equations appears in the choice of right variables. Since there is no general rule for finding the right choice, it is better to introduce special variables first, and then investigate the problems that naturally lend themselves to these variables. This paper follows Jacobi's prophetic observations by introducing certain "meta" variational problems on semi-simple reductive groups G having a compact subgroup K. We then use the Maximum Principle of optimal control to generate the Hamiltonians whose solutions project onto the extremal curves of these problems. We show that there is a particular sub-class of these Hamiltonians that admit a spectral representation on the Lie algebra of G. As a consequence, the spectral invariants associated with the spectral curve produce a large number of integrals of motion, all in involution with each other, that often meet the Liouville complete integrability criteria. We then show that the classical integrals of motion associated, with the Kowalewski top, the two-body problem of Kepler, and Jacobi's geodesic problem on the ellipsoid can be all derived from the aforementioned Hamiltonian systems. We also introduce a rolling geodesic problem that admits a spectral representation on symmetric Riemannian spaces and we then show the relevance of the corresponding integrals on the nature of the curves whose elastic energy is minimal.

Keywords: symplectic; manifolds; Lie-Poisson bracket; Lie algebras; co-adjoint orbits; extremal curves; integrable systems

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1. Introduction

The theory of integrable systems begins with W.R. Hamilton who in 1835 pronounced that the equations of motion of an n body system conform to the principle of least action, and consequently can be represented as

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, i = 1, \ldots, n, \tag{1}$$

under the transformation

$$p_i = \frac{\partial T}{\partial \dot{q}_i}(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n), \dot{q}_i = \frac{dq_i}{dt}, i = 1, \ldots, n, \tag{2}$$

where $H = T + V$ is the total energy, with T the kinetic and V the potential energy of the system. He then observed that H is conserved along the solutions of the system. Hamilton's discovery gave rise to a new class of differential equations of the form

$$\frac{dx}{dt} = \frac{\partial H}{\partial y}(x(t), y(t)) \frac{dy}{dt} = -\frac{\partial H}{\partial x}(x(t), y(t)). \tag{3}$$

associated with any function H of $2n$ variables $x = x_1, \ldots, x_n$ and $y = y_1, \ldots, y_n$. Such equations became known as the *canonical equations*. Then the transformations $(x, y) \rightarrow$

(x', y') that preserved the canonical form of these equations were also called canonical, and the functions whose values were conserved by canonical systems became known as *integrals*.

Hamilton's discovery had an immediate impact on the scientific community of the nineteenth century. Canonical equations became the central object of study in the mathematics of that period with the contributions of J. Liouville, S.D. Poisson, C.G. Jacobi and H. Poincaré leading the way towards a new branch in mathematics known today as the theory of integrable systems. This theory was principally driven by a lasting interest in the existence of extra integrals of motion and the symmetries that are accountable for the existence of these integrals. Its defining moment may be attributed to S.D. Poisson who in 1809 [1] introduced his bracket (known since as the Poisson bracket)

$$\{f, g\} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial y_i} - \frac{\partial f}{\partial y_i}\frac{\partial g}{\partial x_i} \tag{4}$$

for functions f and g in the canonical variables $x_1, \dots, x_n, y_1, \dots, y_n$.

The introduction of the Poisson bracket greatly facilitated the emerging theory of that period. It provided an alternative definition of canonical systems as differential systems that satisfy

$$\frac{dx_i}{dt} = \{x_i, H\}, \text{ and } \frac{dy_i}{dt} = \{y_i, H\}, i = 1, \dots, n \tag{5}$$

and it also redefined integrals of motion associated with H as functions F that satisfy $\{F, H\} = 0$. It was Jacobi, however, who noticed the fundamental property of the Poisson bracket

$$\{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\} = 0, \tag{6}$$

that has been known ever since as the Jacobi's identity. It is then an easy consequence of Jacobi's identity that $F_3 = \{F_1, F_2\}$ is a third integral of motion for H for any two integrals F_1 and F_2 (known as Poisson's theorem [2]). Alternatively integrals of motion were detected through a suitable change of canonical coordinates. Jacobi characterized such changes of coordinates through a generating function $S(x, y')$. According to Jacobi $(x, y) \to (x', y')$ is canonical if and only if $y_i = \frac{\partial S}{\partial x_i}, x'_i = \frac{\partial S}{\partial y'}$. Poincaré characterized canonical change of coordinates in terms of differential forms: $(x, y) \to (x', y')$ is canonical if and only if $\sum_{i=1}^{n} x_i dy_i - x'_i dy'_i = dS$ for some function S.

From contemporary perspectives the theory of integrable systems begins with C.G. Jacobi and his seminal book *Lectures in Dynamics* [3]. Jacobi demostrated that the canonical Equation (3) can be integrated with the aid of a partial differential equation

$$H(x_1, \dots, x_n, \frac{dS}{dx_1}, \dots, \frac{dS}{dx_n}) = c, \tag{7}$$

in terms of an unknown function S. He showed that if a particular solution of (7) can be found in terms of n arbitrary constants of motion $h_1, \dots, h_n$ then $c = \phi(h_1, \dots, h_n)$ for some function ϕ, and the transformation

$$y_i = \frac{\partial S}{\partial x_i}, h' = \frac{\partial S}{\partial h_i} \tag{8}$$

transforms the canonical coordinates (x, y) into new canonical coordinates (h', h) relative to which the canonical Equation (3) are transformed into the equations

$$\frac{dh'}{dt} = \frac{d\phi}{dh}, \frac{dh}{dt} = -\frac{d\phi}{dh'} = 0,$$

whose solutions are given by

$$h'(t) = c_2 t + c_3, h(t) = c_1, c_2 = \frac{d\phi}{dh}. \tag{9}$$

Canonical coordinates whose solutions are given by (9) are called *action-angle coordinates* [4].

Equation $H(x_1, \ldots, x_n, \frac{\partial S}{\partial x_1}, \ldots, \frac{\partial S}{\partial x_n}) = c$ is known as *Jacobi's equation*. Poincaré referred to the above result as the first theorem of Jacobi in his treatise of celestial mechanics [2]. Jacobi's solution of the above partial differential equation in terms of the elliptic coordinates stands out as the most original and, perhaps, the most enigmatic contribution to the theory of canonical systems. Jacobi's use of elliptic coordinates suggested the existence of a special class of variational problems whose solutions can be described by Abelian integrals in some privileged system of coordinates, exemplified by the geodesic problem on the ellipsoid. In the absence of any apparent symmetries on the ellipsoid that account for the integrability of the geodesic problem, this result of Jacobi seemed particularly mysterious.

In Jacoby summary, the main problem in the integration of differential equations appears in the choice of right variables. Given no general rule for finding the right choice, it is better to introduce special variables first, and then investigate the problems that naturally lend themselves to these variables [3]. Jacobi, however, does not comment on another exceptional aspect of his discovery, namely the mysterious presence of partial differential equations for the problems of variational calculus, an issue that remained open for a long time.

Almost a hundred years later, C. Carathéodory in the introduction to his famous book on the calculus of variations [5] remarks that "neither Jacobi, nor his students, nor the many other prominent men who so brilliantly represented and advanced this discipline during the nineteenth century, thought in any way of the relationship between the calculus of variations and partial differential equation". H. Poincaré also sidestepped this issue by treating canonical systems as the solutions of a dynamical system

$$\frac{d}{dt}\sum_{k=1}^{2n} x_i \frac{dy_i}{d\alpha_k} - \frac{d}{d\alpha_k}\sum_{k=1}^{2n} x_i \frac{dy_i}{dt} = \frac{dF}{d\alpha_k}, k = 1, \ldots, 2n, \tag{10}$$

where $\alpha_1, \ldots, \alpha_{2n}$ denote the constants $x_i(t_0) = \alpha_i, y_i(t_0) = \alpha_{i+n}, i = 1, \ldots, n$. Since $\frac{dF}{d\alpha_k} = \sum_{i=1}^n \frac{\partial F}{\partial x_i}\frac{\partial x_i}{\partial \alpha_k} + \frac{\partial F}{\partial y_i}\frac{\partial y_i}{\partial \alpha_k}$ the above differential equation can be reformulated as

$$\sum_{i=1}^n (\frac{dx_i}{dt} - \frac{\partial F}{\partial y_i})\frac{dy_i}{d\alpha_k} - (\frac{dy_i}{dt} + \frac{\partial F}{\partial x_i})\frac{dx_i}{d\alpha_k} = 0,$$

which shows that Equations (3) and (10) have the same solutions. Poincaré equation used Equation (10) to show that a transformation $(x, y) \to (x', y')$ is canonical if and only if the differential form $\sum_{i=1}^n y_i dx_i$ satisfies $\sum_{i=1}^n y_i dx_i - y_i' dx_i' = dS$ for some function $S(x, x')$.

Among many other stellar advancements of that epoch, the following result of J. Liouville, reported in 1855 [6], seemed particularly influential for the present mathematics [4]. Liouville considered a differential system

$$\frac{dx}{dt} = \frac{\partial}{\partial y}F(t, x(t), y(t)), \frac{dy}{dt} = -\frac{\partial}{\partial x}F(t, x(t), y(t)) \tag{11}$$

associated with a function $F(t, x_1, \ldots, x_n, y_1, \ldots, y_n)$. He then assumed the existence of n integrals of motion $h_1(t, x, y), \ldots, h_n(t, x, y)$ such that the system of equations

$$h_1 = h_1(t, x, y), h_2 = h_2(t, x, y), \ldots, h_n = h_n(t, x, y)$$

can be solved for $y_1, \ldots, y_n$ in the variables $t, x_1, \ldots, x_n, h_1, \ldots h_n$. He also imposed the condition that $h_1, \ldots, h_n$ are in involution, that is,

$$\{h_i, h_j\} = \sum_{k=1}^{n} \frac{\partial h_i}{\partial x_k}\frac{\partial h_j}{\partial y_h} - \frac{\partial h_i}{\partial y_k}\frac{\partial h_j}{\partial x_k} = 0, 1 \geq i, j \leq n. \tag{12}$$

Liouville interpreted $\frac{dx_i}{dt} = \frac{\partial}{\partial y_i}F(t, x(t), y(t))$ as the exactness condition for the differential form $\sum_{i=1}^{n} y_i(t, x, h)dx_i - F(t, x, p(t, x, h))dt$ and concluded that there is a function $S(t, x, h)$ such that

$$\sum_{i=1}^{n} y_i dx_i - F(t, x, p(t, x, h))dt = \frac{\partial S}{\partial x_i}dx_i + \frac{dS}{dt}dt,$$

that is, $y_i = \frac{\partial S}{\partial x}, \frac{dS}{dt} + F(t, x, \frac{\partial S}{\partial x}) = 0$. But then S can be used as the generating function for the canonical transformation $(x, y) \to (h, h')$ where $h' = -\frac{\partial S}{\partial h}$. Liouville refers to

$$y_i = \frac{\partial S}{\partial x_i}, h'_i = -\frac{\partial S}{\partial h_i}, i = 1, \ldots, n \tag{13}$$

as a complete system. Indeed, in the new coordinates $h_1, \ldots, h_n$ remain constants of motion and therefore $\frac{dh_i}{dt} = 0, i = 1, \ldots, n$. Since $\frac{dh_i}{dt} = \frac{\partial F}{\partial h'}$, F is independent of h', that is, F is a function of t and h. But then $-\frac{\partial \tilde{F}}{\partial h}$ is a given function of time, and $h'(t)$ is given by its integral. When F is a function of x and y, and not explicitly dependent on time, then $\tilde{F}$ is only a function of h. Therefore, the general solution is given by

$$h(t) = h(0), h'(t) = \omega t + h'(0), \omega = -\frac{\partial \tilde{F}}{\partial h}. \tag{14}$$

This heritage from 19-th century mathematics forms a core of knowledge indispensable for problems of mathematical physics, symplectic geometry, calculus of variations and optimal control theory, and its unanswered questions still motivate much of the current research in integrable systems.

This paper will address the "hidden" symmetries that account for the existence of extra integrals of motion. We will show that the canonical integrable systems, such as Jacobi's geodesic problem on the ellipsoid, Neumann's mechanical problem on the sphere, Euler's top, and the associated heavy tops, all derive their constants of motion from certain "meta" systems on Lie groups that admit isospectral representations of the form

$$\frac{dL_\lambda}{dt}(t) = [M_\lambda(t), L_\lambda(t)] \tag{15}$$

on the Lie algebra $\mathfrak{g}$ of G.

We will confine our attention to semi-simple Lie groups G having a compact subgroup K, for then the Lie algebra $\mathfrak{g}$ admits a decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, where $\mathfrak{k}$ the Lie algebra of K and $\mathfrak{p}$ is the orthogonal complement of $\mathfrak{k}$ relative to the Killing form $Kl(A, B) = Tr(adA \circ ad(B)$. But then $[\mathfrak{p}, \mathfrak{k}] \subseteq \mathfrak{p}]$ and therefore $\mathfrak{g}$ as a vector space also carries the semi-direct Lie algebra $\mathfrak{g}_s$ associated with the semi-direct product $G_s = \mathfrak{p} \rtimes K$. We will then single out a class of left-invariant variational problems on G that admit an isospectral representation with

$$L_{\lambda,s} = L_{\mathfrak{p}} - \lambda L_{\mathfrak{k}} + (\lambda^2 - s)A, \tag{16}$$

where $s = 0$ in the semi-direct case and $s = 1$ in the semi-simple case, $L = L_{\mathfrak{k}} + Ł_{\mathfrak{p}}$, $L_{\mathfrak{k}} \in \mathfrak{k}, L_{\mathfrak{p}} \in \mathfrak{p}$, and where A is a fixed element in $\mathfrak{p}$. It is then known that the spectral invariants $\phi^k_{\lambda,s}(L) = Tr(L^k_{\lambda,s})$ are in involution relative to the canonical Poisson bracket on $\mathfrak{g}$, respectively on $\mathfrak{g}_s$. We will show that these invariants shed light on the hidden symmetries that surround many of the aforementioned integrable systems. In the process we will be able to demonstrate that the quest for the geometric origins behind the "mysterious" integrals of motions also leads to new and unexpected encounters with problems of Riemannian and sub-Riemannian geometry in which geometric control theory plays a major role.

2. Symplectic Background, Hamiltonian Systems

The theoretic framework upon which above claims are made is rooted in symplectic geometry. Below is a brief summary of the theoretical ingredients required for our main results.

Recall that a manifold M together with a non-degenerate and closed 2-form ω is called *symplectic*. The symplectic form yields a correspondence between functions and vector fields: to every function f there is a vector field $\vec{f}$ defined by $\omega(\vec{f}, X) = df(X)$ for all vector fields X on M. Then $\vec{f}$ is called the Hamiltonian vector field generated by f. Every symplectic manifold is even dimensional, and at each point of M there is a neighbourhood with coordinates $(x_1, \dots, x_n, p_1, \dots, p_n)$ on which Hamiltonian vector fields are given by

$$\vec{f} = \sum_{i=1}^{n} \frac{\partial f}{\partial p_i}\frac{\partial}{\partial x_i} - \frac{\partial f}{\partial x_i}\frac{\partial}{\partial p_i}. \tag{17}$$

This choice of coordinates in which $\vec{f}$ is given by (17) is called *symplectic*, or canonical in the terminology of the 19-th century.

Every cotangent bundle T^*M is a symplectic manifold with its canonical symplectic form, $\omega = dp \wedge dx$ in terms of the symplectic coordinates $(x_1, \dots, x_n, p_1, \dots, p_n)$. As a symplectic manifold the cotangent bundle is special, in the sense that it is also a vector bundle. Hence every vector field X on M can be lifted to a unique Hamiltonian vector field $\vec{f}_X$ in T^*M via the function $f_X(\xi) = \xi(X(x))$, $\xi \in T_x^*M$. Vector field $\vec{f}_X$ is called the *Hamiltonian lift of* X. The same procedure is applicable to any time varying vector field, and by extension to any differential system on M. Thus any differential system in M can be lifted to a Hamiltonian system in T^*M. This fact is also important for problems of optimal control where the Maximum Principle singles out the appropriate Hamiltonian lifts that govern the optimal solutions [7].

When the base manifold is a Lie group G, and when the underlying differential system is either left or right invariant, then there is a special system of coordinates based on the representation of T^*G as $G \times \mathfrak{g}^*$, with $\mathfrak{g}^*$ the dual of $\mathfrak{g}$. This coordinate system preserves the left invariant symmetries and elucidates the conserved quantities of the associated Hamiltonian systems. The passage to these coordinates and the associated formalism was amply documented in my earlier publications [7–9]. Below we will highlight the main points in this theory required for our results.

2.1. Left-Invariant Trivializations and the Symplectic Form

Having in mind applications that involve left-invariant variational systems the cotangent bundle T^*G and the tangent bundle TG will be viewed as the products $G \times \mathfrak{g}^*$ and $G \times \mathfrak{g}$ via the left-translations. More explicitly, tangent vectors $v \in T_gG$ will be identified with pairs $(g, X) \in G \times \mathfrak{g}$ via the relation $v = L_{g_*}X$, where L_{g_*} denotes the tangent map associated with the left translation $L_g(h) = gh$. Similarly, points $\xi \in T_g^*G$ will be identified with pairs $(g, \ell) \in G \times \mathfrak{g}^*$ via $\xi = \ell \cdot L_{g_*}^{-1}$. Then $T(T^*G)$, the tangent bundle of the cotangent bundle T^*G, will be identified with $(G \times \mathfrak{g}^*) \times (\mathfrak{g} \times \mathfrak{g}^*)$, with the understanding that an element $((g, \ell), (A, a)) \in (G \times \mathfrak{g}^*) \times (\mathfrak{g} \times \mathfrak{g}^*)$ denotes the tangent vector (A, a) at the base point (g, ℓ).

We will make use of the fact that $G \times \mathfrak{g}^*$ is a Lie group in its own right since $\mathfrak{g}^*$, as a vector space, is an abelian Lie group. Then left-invariant vector fields V in $G \times \mathfrak{g}^*$ will be denoted by $V(g, \ell) = (gA, a)$, (g, ℓ) in $G \times \mathfrak{g}^*$. In this setting the canonical symplectic form on T^*G is given by

$$\omega_{(g,\ell)}(V_1, V_2) = a_2(A_1) - a_1(A_2) - \ell([A_1, A_2]) \tag{18}$$

for any left-invariant vector fields $V_1 = (gA_1, a_1)$ and $V_2 = (gA_2, a_2)$ [7]. The above form is invariant under the left-translations in $G \times \mathfrak{g}^*$, and is especially revealing for the Hamiltonian vector fields generated by left-invariant functions on $G \times \mathfrak{g}^*$.

A function H on $G \times \mathfrak{g}^*$ is left-invariant if $H(hg, \ell) = H(g, \ell)$ for all $g, h \in G$ and all $\ell \in \mathfrak{g}^*$. That is, left-invariant functions coincide with functions of $\mathfrak{g}^*$. Each left-invariant vector field $X(g) = gA$ on G lifts to a linear function $\ell \to \ell(A)$ on $\mathfrak{g}^*$ because

$$h_X(\xi) = \xi(X(g)) = \ell \circ L_{g_*}^{-1} \circ (L_g)_*(A) = \ell(A), \xi \in T_g^* G.$$

Functions H on $\mathfrak{g}^*$ generate Hamiltonian vector fields $\vec{H}$ on $G \times \mathfrak{g}^*$ whose integral curves are the solutions of

$$\frac{dg}{dt}(t) = g(t) dH_{\ell(t)}, \quad \frac{d\ell}{dt}(t) = -\mathrm{ad}^* dH_{\ell(t)}(\ell(t)). \tag{19}$$

In a more general case, where H depends on both $g \in G$ and $\ell \in \mathfrak{g}^*$, the integral curves of $\vec{H}$ are the solutions of

$$\frac{dg}{dt}(t) = g(t) dH_{\ell(t)}, \quad \frac{d\ell}{dt}(t) = -\mathrm{ad}^* dH_{\ell(t)}(\ell(t)) - dH_g \circ L_{g_*}, \tag{20}$$

that can be easily shown through the relations

$$b(dH_\ell) + dH_g \circ L_{g_*} B = b(A) - a(B) - \ell[A, B].$$

This situation occurs in problems of mechanics in the presence of potential functions. For example, the movements of a three-dimensional rigid body with a potential function $V : SO(3) \to R$ are described by the Hamiltonian

$$H(R, \ell) = H_0(\ell) + V(\alpha_1, \alpha_2, \alpha_3)$$

on the cotangent bundle of $SO(3)$, where $\alpha_1, \alpha_2, \alpha_3$ denote the columns of the matrix transpose of the rotation R in $SO(3)$. For then the directional derivative of V in the direction RX is given by

$$dV(RX) = \sum_{i=1}^{3} \langle \frac{\partial V}{\partial \alpha_i} \wedge \alpha_i, X \rangle$$

where $\langle\, , \,\rangle$ denotes the standard inner product $-\frac{1}{2} Tr(XY)$ in $\mathfrak{so}(3)$. Thus $dH_g \circ dL_g = \sum_{i=1}^{3} \frac{\partial V}{\partial \alpha_i} \wedge \alpha_i$ and the equations of motion for H are given by

$$\frac{dg}{dt}(t) = g(t) dH_0(\ell(t)), \quad \frac{d\ell}{dt}(t) = -\mathrm{ad}^* dH_0(\ell(t))(\ell(t)) + \sum_{i=1}^{3} \alpha_i \wedge \frac{\partial V}{\partial \alpha_i}. \tag{21}$$

These equations extend to an "n-dimensional rigid body" with the Hamiltonian $H(R, \ell) = H_0(\ell) + V(\alpha_1, \ldots, \alpha_n)$ where

$$\begin{gathered} \tfrac{dR}{dt} = R(t)\Omega(t), \tfrac{dM}{dt} = [\Omega(t), M(t)] + \textstyle\sum_{i=1}^{n} \alpha_i \wedge \tfrac{\partial V}{\partial \alpha_i} \\ \mathcal{P}(\Omega(t)) = M(t), \alpha_i(t) = R^T(t) e_i, i = 1, \ldots, n. \end{gathered} \tag{22}$$

In this context, $M(t)$ is the generalization of the angular momentum, $\Omega(t)$ is the generalization of the angular velocity, $\mathcal{P}$ is the generalized inertia tensor, and $\sum_{i=1}^{n} \alpha_i \wedge \frac{\partial V}{\partial \alpha_i}$ is the external torque.

2.2. Poisson Manifolds, Coadjoint Orbits

Equation (19) lend themselves to an insightful description in terms of the Poisson structure on $\mathfrak{g}^*$ inherited from the symplectic form ω. Recall that a manifold M together with a bilinear, skew-symmetric form

$$\{\ ,\ \} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$$

that satisfies

$$\begin{aligned} &\{fg,h\} = f\{g,h\} + g\{f,h\}, \text{(Leibniz's rule), and} \\ &\{f,\{g,h\}\} + \{h,\{f,g\}\} + \{g,\{h,f\}\} = 0, \text{ (Jacobi's identity),} \end{aligned}$$

for all functions f, g, h on M, is called a Poisson manifold.

Every symplectic manifold is also a Poisson manifold with the Poisson bracket given by $\{f,g\}(p) = \omega_p(\vec{f}(p), \vec{g}(p)), p \in M$. However, the converse may not be true due to the fact that the Poisson bracket may be degenerate at some points of M. Nevertheless, each function f on M induces a Poisson vector field $\vec{f}$ through the formula $\vec{f}(g) = \{f,g\}$ as in the symplectic case. Poisson vector fields clarify the relation with symplectic manifolds through the following fundamental fact: every Poisson manifold is foliated by the orbits of its family of Poisson vector fields and each orbit is a symplectic submanifold of M with its symplectic form $\omega_p(\vec{f}, \vec{h}) = \{f,h\}(p)$ [7].

The dual $\mathfrak{g}^*$ of a Lie algebra $\mathfrak{g}$ is a Poisson manifold with the Poisson bracket

$$\{f,h\}(\ell) = \ell([dh, df]) \tag{23}$$

for any functions f and h on $\mathfrak{g}^*$. In the literature on integrable systems the bracket $\{f,h\}(\ell) = \ell([df, dh])$ is known as the Lie-Poisson bracket [10]. We have taken its negative to be compatible with the projections of left-invariant Hamiltonian vector fields on $\mathfrak{g}^*$ (and also to agree with the sign conventions in [7]).

It follows that each function H on $\mathfrak{g}^*$ defines a Poisson vector field $\vec{H}$ on $\mathfrak{g}^*$ via the formula $\vec{H}(f)(\ell) = \{H,f\}(\ell) = -\ell([dH, df])$ in which case the integral curves of $\vec{H}$ are the solutions of

$$\frac{d\ell}{dt}(t) = -\mathrm{ad}^* dH_{\ell(t)}(\ell(t)). \tag{24}$$

Thus, as we already mentioned above, each function H on $\mathfrak{g}^*$ may be simultaneously viewed as a Hamiltonian on T^*G, and a function on the Poisson space $\mathfrak{g}^*$. Of course, Poisson equations coincide with the projections of the Hamiltonian equations on $\mathfrak{g}^*$.

Solutions of Equation (24) are intimately linked with the coadjoint orbits of G through the following proposition. due to of A.A. Kirillov [11] (the proof is also given in [7]).

Proposition 1. *Let $\mathcal{F}$ denote the family of Poisson vector fields on $\mathfrak{g}^*$ and let $M = \mathcal{O}_{\mathcal{F}}(\ell_0)$ denote the orbit of $\mathcal{F}$ through a point $\ell_0 \in \mathfrak{g}^*$. Then M is equal to the connected component of the coadjoint orbit of G that contains ℓ_0. Consequently each coadjoint orbit is a symplectic submanifold of $\mathfrak{g}^*$.*

Recall that the coadjoint orbit of G through a point $\ell \in \mathfrak{g}^*$ is given by $\mathrm{Ad}^*_g(\ell) = \{\ell \circ \mathrm{Ad}_{g^{-1}}, \quad g \in G\}$.

The fact that the Poisson equations can be naturally restricted to coadjoint orbits implies useful reductions in the theory of Hamiltonian systems.

2.3. Representation of Coadjoint Orbits on Lie Algebras

On semi-simple Lie groups Poisson Equation (24) can be expressed on $\mathfrak{g}$ as

$$\frac{dL}{dt} = [dH, L], \tag{25}$$

because the Killing form, or any scalar multiple of it $\langle\,,\,\rangle$ is non-degenerate, and invariant, in the sense that, $\langle X, [Y, Z]\rangle = \langle [X, Y], Z\rangle, X, Y, Z \in \mathfrak{g}$, and can be used to identify $\mathfrak{g}$ with $\mathfrak{g}^*$ via the formula

$$\langle L, X\rangle = \ell(X), \ell \in \mathfrak{g}^*, X \in \mathfrak{g}.$$

Then coadjoint orbits are identified with the adjoint orbits and the Poisson vector fields $\vec{f}_X(\ell) = -\mathrm{ad}^* X(\ell)$ are identified with vector fields $\vec{X}(L) = [X, L]$. Each vector field $[X, L]$ is tangent to an orbit at L, and $\omega_L([X, L], [Y, L]) = \langle L, [Y, X]\rangle$, X, Y in $\mathfrak{g}$ is the symplectic form on each orbit $\mathcal{O}(L_0)$.

In a reductive semi-simple Lie group G there is also the semi-direct product $G_0 = \mathfrak{p} \rtimes K$ described earlier which generates its own coadjoint orbits on the dual of the Lie algebra $\mathfrak{g}_0$ of G_0. Recall that the Lie algebra $\mathfrak{g}_0$ of G_0 consists of pairs $(A, B), A \in \mathfrak{p}, B \in \mathfrak{k}$ together with the Lie bracket

$$[(A_1, B_1), (A_2, B_2)] = ([A_1, B_2] - [A_2, B_1], [B_1, B_2]).$$

When the elements $(A, B) \in \mathfrak{g}_0$ are identified with the sums the sums $A + B$ in $\mathfrak{g}$, $\mathfrak{g}$ as a vector space, carries a double Lie algebra; the semi-direct product Lie algebras $\mathfrak{g}_0$, and the semi-simple Lie algebra $\mathfrak{g}_1 = \mathfrak{g}$. We then have

$$[A + B, C + D] = [A, B]s + [A, D + [B, C] + [A, D], s = 0, 1,$$

for any A, C in $\mathfrak{p}$ and any B, D in $\mathfrak{k}$, with $s = 0$ in the semi-direct case, and $s = 1$ in the semi-simple case.

Since both $\mathfrak{g}$ and $\mathfrak{g}_0$ Lie algebras over the same vector space, the Poisson equations on $\mathfrak{g}_0^*$ can be also represented on $\mathfrak{g}_0$ via the quadratic form $\langle\,,\,\rangle$, but the resulting expression takes a slightly different form. To see the difference, let $dH = dH_{\mathfrak{p}} + dH_{\mathfrak{k}}$ and $L = L_{\mathfrak{p}} + L_{\mathfrak{k}}$ denote the decompositions of dH and L onto the factors $\mathfrak{p}$ and $\mathfrak{k}$. On the semi-direct product Poisson equations reduce to

$$\frac{dL_{\mathfrak{k}}}{dt} = [dH_{\mathfrak{k}}, L_{\mathfrak{k}}] + [dH_{\mathfrak{p}}, L_{\mathfrak{p}}], \frac{dL_{\mathfrak{p}}}{dt} = [dH_{\mathfrak{k}}, L_{\mathfrak{p}}]. \tag{26}$$

This equation can be combined with the equations for the semi-simple case in terms of the parameter s as

$$\frac{dL_{\mathfrak{k}}}{dt} = [dH_{\mathfrak{k}}, L_{\mathfrak{k}}] + [dH_{\mathfrak{p}}, L_{\mathfrak{p}}], \frac{dL_{\mathfrak{p}}}{dt} = [dH_{\mathfrak{k}}, L_{\mathfrak{p}}] + s[dH_{\mathfrak{p}}, L_{\mathfrak{k}}], s = 0, 1. \tag{27}$$

One can show that

$$P = Ad_h(P_0), Q = [Ad_h(P_0), X] + Ad_h(Q_0), (X, h) \in G_0 \tag{28}$$

is the coadjoint orbit through $P_0 \in \mathfrak{p}, Q_0 \in \mathfrak{k}$ under the action of $G_0 = \mathfrak{p} \rtimes K$ when $\ell_0 \in \mathfrak{g}_s^*$ is identified with $L_0 = P_0 + Q_0$ in $\mathfrak{g}_0$, and when $\ell = \mathrm{Ad}^*_{(X,h)}(\ell_0)$ is identified with $L = P + Q$ [7].

The adjoint orbits of a non-compact semi-simple Lie groups G can be realized as the cotangent bundles of flag manifolds [12], and the same has been shown recently for the coadjoint orbits under the action of the semi-direct products [13,14]. We will make use of that fact later on in the paper.

3. Affine-Quadratic Problems

As stated earlier, we will restrict our attention to semi-simple Lie groups G and compact subgroups K with zero centre. We refer to (G, K) as a reductive pair. Then $\mathfrak{g}$ and $\mathfrak{k}$ will denote their Lie algebras, and $\mathfrak{p}$ will denote the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$

relative to the Killing $Kl(A, B) = Tr(adA \circ ad(B))$ in $\mathfrak{g}$. Recall that Kl is non-degenerate and satisfies

$$Kl(A, [B, C]) = Kl([A, B], C), A, B, C \text{ in } \mathfrak{g}.$$

Hence $\mathfrak{p}$ is well defined and satisfies $[\mathfrak{p}, \mathfrak{k}] \subseteq \mathfrak{p}$ (in fact, $[\mathfrak{p}, \mathfrak{k}] = \mathfrak{p}$ because $\mathfrak{g}$ is semi-simple). We will also assume that $[\mathfrak{p}, \mathfrak{p}] = \mathfrak{k}$. Note that the Killing form is negative-definite on $\mathfrak{k}$ because K has zero centre [15], hence any negative scalar multiple $\langle\, , \rangle$ of it is positive definite on $\mathfrak{k}$. We shall assume that such a scalar product is fixed.

An affine quadratic problem is defined through a positive definite quadratic form Q on $\mathfrak{k}$, and a regular element A in the Cartan space $\mathfrak{p}$. An element A in $\mathfrak{p}$ is called regular if $\{X \in \mathfrak{p} : [A, X] = 0\}$ is an abelian subalgebra in $\mathfrak{p}$. The corresponding affine-quadratic problem consists of finding the solutions $g(t)$ in G of the affine control system

$$\frac{dg}{dt} = g(t)(A + U(t)), \tag{29}$$

generated by a square-integrable control $U(t)$ in $\mathfrak{k}$ that transfers a given state g_0 in G to a given terminal state g_1 in time T with a minimal energy $\frac{1}{2}\int_0^T Q(U(s))ds$. Any positive definite quadratic form Q is of the form $Q(U) = \frac{1}{2}\langle \mathcal{P}(U), U\rangle$ for some self-adjoint and positive linear operator $\mathcal{P}$ on $\mathfrak{k}$. Then there exists an orthonormal basis $U_1, \ldots, U_k$ in $\mathfrak{k}$ such that $\mathcal{P}$ is diagonal relative to it. That is, if $U(t) = \sum_{i=1}^k u_i(t)U_i$ then $\mathcal{P}(U(t)) = \sum_{i=1}^k c_i u_i(t)U_i$ for some constants $c_1, \ldots, c_k$. Then (29) can be rewritten as as

$$\frac{dg}{dt} = X_0(g) + \sum_{i=1}^k u_i(t)X_i(g), \tag{30}$$

where $X_0, \ldots, X_k$ are the left-invariant vector fields with $X_0(g) = gA$ and $X_i(g) = gU_i, i = 1, \ldots, k$, with $\frac{1}{2}\int_0^T \sum_{i=1}^k c_i u_i^2(t)\, dt$. the energy associated with each solution. The most natural case occurs when $\mathcal{P} = I$, that is, when $c_i = 1, i = 1, \ldots, k$. We will refer to this case as the canonical affine-quadratic problem.

When A is regular, then (29) is controllable, a consequence of our assumption $[\mathfrak{p}, \mathfrak{p}] = \mathfrak{k}$, that is, any terminal state g_1 can be reached in some finite time $T > 0$ from any initial state g_0. But then there is an optimal solution $(\bar{g}(t), \bar{U}(t))$ on the interval $[0, T]$ for which the energy of transfer $\int_0^T Q(\bar{U}(s))ds$ is minimal (see [7] for the proof). Therefore the above optimal control problem is well-posed.

To each affine-quadratic problem there is an analogous "shadow problem" defined on the semi-direct product $G_o = \mathfrak{p} \ltimes K$ defined by the same data as in the original problem. It follows that every affine space $\Gamma = \{A + U : U \in \mathfrak{k}\}$ that defines an affine left-invariant system on G also defines a corresponding left-invariant affine system on the semi-direct product G_0. Thus behind every affine quadratic optimal problem on G there is a corresponding affine-quadratic "shadow" problem on the semi-direct product G_s. The shadow problem is also well defined in the sense that optimal solutions exist on some interval $[0, T]$ for each pair of boundary points $g(0) = g_0$ and $g(T) = g_1$.

According to Pontriyagin's Maximum Principle every optimal trajectory generated by a bounded and measurable control is the projection of an extremal curve, and each extremal curve is an integral curve of a suitable Hamiltonian system on the cotangent bundle of the ambient space. The Maximum Principle is also valid for optimal problems with L^2 controls over affine systems with quadratic costs ([16]).

Let now $\mathbf{g}(\mathbf{t})$ be an optimal trajectory generated by a control $\mathbf{u}(\mathbf{t})$. According to the Maximum Principle, $\mathbf{g}(\mathbf{t})$ is the projection of an extremal curve $\xi(t)$ in T^*G along which the cost extended Hamiltonian

$$-\frac{\lambda}{2}\sum_{i=1}^k c_i u_i^2(t) + H_0(\xi) + \sum_{i=1}^k u_i(t)H_i(\xi(t)), \lambda = 0, 1$$

is maximal at $\mathbf{u}(\mathbf{t})$ relative to all competing controls $u(t)$. In this notation, each H_i is the Hamiltonian lift of X_i, i.e., $H_i(\xi(t)) = \xi(t)(X_i(g(t)))$. In the abnormal case, which we will not treat here, $\lambda = 0$, and the Maximum principle results in the constraints $H_i(\xi(t)) = 0, i = 1, \dots, k$. In the normal case, $\lambda = 1$, the maximality condition implies that the optimal controls are of the form $\mathbf{u}_i(t) = \frac{1}{c_i} H_i(\xi(t)), i = 1, \dots, k$. Consequently, optimal solutions are the projections of solution curves of a single Hamiltonian vector field $\vec{H}$ generated by the Hamiltonian

$$H(\xi) = \frac{1}{2} \sum_{i=1}^{k} \frac{1}{c_i} H_i^2(\xi) + H_0(\xi) = \frac{1}{2} \sum_{i=1}^{k} \frac{1}{c_i} (\ell(U_i))^2 + \ell(A). \tag{31}$$

Recall that each lift $H_i(\xi)$ is a linear function on $\mathfrak{g}^*$ given by $H_i(\xi) = \ell(U_i)$ with $H_0(\xi) = \ell(A)$. Thus H is left-invariant, hence its Hamiltonian equations are given by

$$\frac{dg}{dt} = X_0(g) + \sum_{i=1}^{n} \frac{1}{c_i} H_i(\ell(t)) X_i(g(t)), \frac{d\ell}{dt} = -ad^* dH(\ell(t))(\ell(t)).$$

The associated Poisson equations can be now written in $\mathfrak{g}$ as

$$\frac{dL_\mathfrak{k}}{dt} = [\mathcal{P}^{-1}(L_\mathfrak{k}), L_\mathfrak{k}] + [A, L_\mathfrak{p}], \frac{dL_\mathfrak{p}}{dt} = [\mathcal{P}^{-1}(L_\mathfrak{k}), L_\mathfrak{p}] + s[A, L_\mathfrak{k}], s = 0, 1, \tag{32}$$

after the identification of $\ell \in \mathfrak{g}^*$ with $L \in \mathfrak{g}$ via the scalar product $\langle\, , \rangle$, and the decomposition $L = L_\mathfrak{k} + L_\mathfrak{p}, L_\mathfrak{k} \in \mathfrak{k}, L_\mathfrak{p} \in \mathfrak{p}$ (Equation (27)). In the canonical case ($\mathcal{P} = I$) the preceding equations reduce to

$$\frac{dL_\mathfrak{k}}{dt} = [A, L_\mathfrak{p}], \frac{dL_\mathfrak{p}}{dt} = [L_\mathfrak{k}, L_\mathfrak{p}] + s[A, L_\mathfrak{k}], s = 0, 1. \tag{33}$$

Note that $s\langle L_\mathfrak{k}, L_\mathfrak{k}\rangle + \langle L_\mathfrak{p}, L_\mathfrak{p}\rangle$ is an integral for (32). This integral is a universal integral of motion in the sense that it remains constant for any left-invariant Hamiltonian on $\mathfrak{g}_s$.

3.1. Isospectral Representations

We now single out a remarkable class of affine-quadratic Hamiltonians that plays a prominent role in the theory of integrable systems. It consists of Hamiltonians $H = \frac{1}{2}\langle \mathcal{P}^{-1} L_\mathfrak{k}, L_\mathfrak{k}\rangle + \langle L_\mathfrak{p}, A\rangle$ that admit a spectral representation of the form

$$\begin{gathered} \tfrac{dL_\lambda}{dt} = [M_\lambda, L_\lambda], \\ \text{with } M_\lambda = \mathcal{P}^{-1}(L_\mathfrak{k}) - \lambda A, \text{ and } L_{\lambda,s}(L) = L_\mathfrak{p} - \lambda L_\mathfrak{k} + (\lambda^2 - s)B, \end{gathered} \tag{34}$$

for some element $B \in \mathfrak{p}$ that comutes with A, where $L_\mathfrak{p}$ and $L_\mathfrak{k}$ are the solutions of the Poisson Equation (32). Such a class is called *isospectral* and $L_\lambda(s)$ is called the associated spectral curve. This terminology has origins in J. Zimmerman's PhD thesis in 2002, in which he showed that the rolling sphere problem is isospectral [17]. We will return to Zimmerman's problem and relate its results to the canonical affine-quadratic problem [18].

For Hamiltonian systems that admit an isospectral representation, the discrete spectral invariants of L are replaced by the functional invariants $\phi^{(k)}_{\lambda,s}(L) = Trace(L^k_{\lambda,s}(L))$. Remarkably, the functional invariants $\phi^k_{\lambda,s}$ are in involution with each other, both with respect to the semi-simple and the semi-direct product Lie bracket, and in some instances generate a sufficient number of integrals of motion to ensure complete integrability ([7], 9.2). For instance, the family of functions

$$\mathcal{F}_0 = \{\phi^k_{\lambda,0}, k \geq 1, \lambda \in \mathbb{R}\} \cup \{h_X : [X, B] = 0, X \in \mathfrak{k}\}$$

is completely integrable on each coadjoint orbit in $\mathfrak{p} \rtimes K$ [19]. This means that H is completely integrable on each coadjoint orbit in $\mathfrak{p} \rtimes \mathfrak{k}$ whenever H is in involution with the

Hamiltonian lifts $h_X(L) = \langle L, X\rangle, X \in \mathfrak{k}, [X, B] = 0$. This implies that the canonical affine Hamiltonian is completely integrable on coadjoint orbits since each left-invariant vector field with values in the isotropy group of A is a symmetry for the canonical system. It is reasonable to expect that the analogous family of functions is also completely integrable on coadjoint orbits of G, but, to the best of my knowledge, the proofs have not yet appeared in the literature.

The focus on the affine-quadratic problem and the associated Hamiltonians allows for the following characterization of isospectral Hamiltonians (proved in [7]).

Theorem 1. *An affine Hamiltonian* $H = \frac{1}{2}\langle \mathcal{P}^{-1}L_\mathfrak{k}, L_\mathfrak{k}\rangle + \langle L_\mathfrak{p}, A\rangle$ *is isospectral if and only* $[\mathcal{P}^{-1}(L_\mathfrak{k}), B] = [L_\mathfrak{k}, A]$ *for some element* $B \in \mathfrak{p}$ *that commutes with* A*. In the isospectral case,* $L_\mathfrak{p} = sB$ *is an invariant set for Equation (32). On this set (32) are given by*

$$\frac{dL_\mathfrak{k}}{dt} = [\mathcal{P}^{-1}(L_\mathfrak{k}), L_\mathfrak{k}], \tag{35}$$

and admit the reduced spectral representation

$$\frac{d}{dt}(L_\mathfrak{k} - \lambda B) = [\mathcal{P}^{-1}(L_\mathfrak{k}) - \lambda A, L_\mathfrak{k} - \lambda B]. \tag{36}$$

This theorem shows that the fundamental results A.T. Fomenko, A. S. Mischenko, and V.V. Trofimov on integrable left-invariant Riemannian metrics on compact Lie groups [20,21] based on Manakov's seminal work on the n-dimensional Euler's top [22] are subordinate to the isospectral properties of the affine Hamiltonian system, in the sense that the spectral invariants of $L_\mathfrak{k} - \lambda B$ on $\mathfrak{k}$ are always in involution with a larger family of functions generated by the spectral invariants of $L_\lambda = -L_\mathfrak{p} + \lambda L_\mathfrak{k} + (\lambda^2 - s)B$ on $\mathfrak{g}_s$ associated with an affine Hamiltonian H.

3.2. Affine Hamiltonians and Mechanical Tops

Let us now draw comparisons between the semi-direct Poisson equations

$$\frac{dL_\mathfrak{k}}{dt} = [\mathcal{P}^{-1}(L_\mathfrak{k}(t)), L_\mathfrak{k}(t)] + [A, L_\mathfrak{p}(t)], \frac{dL_\mathfrak{p}}{dt} = [\mathcal{P}^{-1}(L_\mathfrak{k}(t)), L_\mathfrak{p}(t)] \tag{37}$$

and the "top-like" equations:

$$\frac{dR}{dt} = R(t)(\mathcal{P}^{-1}(M(t))), \frac{dM}{dt} = [\mathcal{P}^{-1}(M(t)), M(t)] + \sum_{i=1}^{n} \alpha_i(t) \wedge \frac{\partial V}{\partial \alpha_i}, \tag{38}$$

associated with the Hamiltonian $H = \frac{1}{2}\langle \mathcal{P}^{-1}(M), M\rangle + V(\alpha_1, \dots, \alpha_n)$. We will consider two cases- tops with linear potentials and tops with quadratic potentials.

Linear potentials. Equation (38) will be referred to *heavy top-like equations* when the potential energy V is generated by a linear Newtonian field, that is, when $V = -\sum_{i=1}^{n} c_i(\alpha_i, a)$, where a is a vector in $\mathbb{R}^n$, and $c_1, \dots, c_n$ are constants. When $a = 0$, the external torque $\sum_{i=1}^{n} \alpha_i(t) \wedge \frac{\partial V}{\partial \alpha_i}$ is equal to zero, and Equation (38) reduces to the Hamiltonian equation associated with a left-invariant Riemannian metric induced by the operator $\mathcal{P}$ (called the n-dimensional Euler's top in some Russian literature [20]).

Heavy top-like equations can be written more compactly as

$$\frac{dR}{dt} = R(t)\Omega(t), \frac{dM}{dt} = [\Omega(t), M(t)] + a \wedge p(t), \tag{39}$$

where $\Omega(t) = \mathcal{P}^{-1}M(t)$, and $p(t) = \sum_{i=1}^{n} c_i\alpha_i(t)$. Since $\alpha_i(t) = R(t)^T e_i$, $p(t)$ is a solution of $\frac{dp}{dt} = -\Omega(t)p(t)$. Hence each solution resides on the sphere $\{p \in R^n : ||p(t)|| = ||p(0)||\}$.

Our theorems below relate Equation (39) to the Poisson Equation (37) on the reductive Lie algebras $\mathfrak{so}(n+1)$ and $\mathfrak{so}(1,n)$ associated with reductive pairs (SO_ϵ, K) where SO_ϵ is $SO(n+1)$ when $\epsilon = 1$ and $SO(1,n)$ when $\epsilon = -1$, and $K = \{1\} \times SO(n)$.

We will tackle both cases simultaneously but first we will need to introduce additional notation and terminology. We will use $\mathfrak{so}_\epsilon$ to denote the Lie algebra of SO_ϵ endowed with the trace form $\langle A, B\rangle = -\frac{1}{2}Tr(AB)$. Relative to SO_ϵ we define its invariant bilinear form $(x,y)_\epsilon = x_0y_0 + \epsilon\sum_{i=1}^n x_iy_i$ in the ambient space R^{n+1}.

Then $a \otimes_\epsilon b, a \in R^{n+1}, b \in R^{n+1}$, will denote the matrix defined by

$$(a \otimes_\epsilon b)x = (a,x)_\epsilon b, x \in \mathbb{R}^{n+1}.$$

and $a \wedge_\epsilon b$ denotes the matrix $a \otimes_\epsilon b - b \otimes_\epsilon a$. Since

$$((a \wedge_\epsilon b)x, y)_\epsilon + (x, (a \wedge_\epsilon b)y)_\epsilon = 0,$$

$a \wedge_\epsilon b$ belongs to $\mathfrak{so}_\epsilon(n+1)$ for any a, b in $\mathbb{R}^{n+1}$. We then have

Theorem 2. *Heavy top-like Equation (39) are isomorphic to the Poisson Equation (37) on the coadjoint orbit through $P_0 = p(0) \wedge_\epsilon e_0, Q_0 = \begin{pmatrix} 0 & 0 \\ 0 & M(0) \end{pmatrix}$ under the coadjoint action of $\mathfrak{p}_\epsilon \ltimes SO(n)$. The passage to the affine Hamiltonian is via the following correspondences*

$$A = \epsilon a \wedge_\epsilon e_0, L_\mathfrak{p} = p \wedge_\epsilon e_0, L_\mathfrak{k} = \begin{pmatrix} 0 & 0 \\ 0 & M \end{pmatrix}, \mathcal{P}^{-1}(L_\mathfrak{k}) = \begin{pmatrix} 0 & 0 \\ 0 & \mathcal{P}^{-1}(M) \end{pmatrix}. \tag{40}$$

For a proof see [14]. The preceding theorem clarifies the presence of heavy tops in the Hamiltonian equations on Lie algebras [10]. It also clarifies the relation between the tops and elastic rods initiated by G. Kirchhoff known as the "kinetic analogues" [23,24]. It also proves that the classification of completely integrable elastic rods in [7,8] carries over to the heavy tops.

Quadratic potentials. We will now show that the tops with quadratic potential V are also present in the equations of affine Hamiltonians, but this time on the tangent bundle of $SL(n)$, or more precisely on the tangent bundle of the semi-direct product $sym^0(n) \rtimes SO(n)$ where $sym^0(n)$ denotes the space of symmetric $n \times n$ matrices with zero trace. For that purpose let

$$H(R,M) = \frac{1}{2}(\mathcal{P}^{-1}(M), M) + \frac{1}{2}\sum_{i=1}^n a_i\langle S\alpha_i, \alpha_i\rangle,$$

with $R \in SO(n), M \in so(n), R^Te_i = \alpha_i$, and S a symmetric $n \times n$. In accordance with (38) the Hamiltonian equations of $\vec{H}$ are given by

$$\frac{dR}{dt} = R(t)\Omega(t), \frac{dM}{dt} = [\Omega(t), M(t)] + \sum_{i=1}^n a_i\alpha_i(t) \wedge S\alpha_i(t), \tag{41}$$

where $\Omega(t) = \mathcal{P}^{-1}(M(t))$.

Theorem 3. *Top-like Equation (41) are isomorphic with the Poisson equations generated by the affine Hamiltonian $H = \frac{1}{2}\langle\mathcal{P}^{-1}(L_\mathfrak{k}), L_\mathfrak{k}\rangle + \langle L_\mathfrak{p}, S\rangle$ on the coadjoint orbit through $P_0 = \sum_{i=1}^n a_i(e_i \otimes e_i) - (\frac{1}{n}\sum_{i=1}^n a_i)I$ and $Q_0 = M(0)$ under the action of the semi-direct product $sym^0(n) \rtimes SO(n)$.*

Proof. Every solution $(M(t), R(t))$ of (41) generates symmetric matrices $L_{\mathfrak{p}}(t)$ and $X(t)$ given by

$$L_{\mathfrak{p}}(t) = Ad_{h(t)}P_0 = \sum_{i=1}^n a_i(\alpha_i(t) \otimes \alpha_i(t)) - \frac{1}{n}\sum_{i=1}^n a_i I,$$
$$X(t) = \mathrm{Ad}_{h(t)}Y(t), Y(t) = -\int_0^t \mathrm{Ad}_{h^{-1}(s)} S\, ds,$$

with $h(t) = R^T(t)$. Then,

$$\frac{dL_{\mathfrak{p}}}{dt} = -\sum_{i=1}^n a_i(\Omega\alpha_i \otimes \alpha_i + \alpha_i \otimes \Omega\alpha_i) = [\Omega, L_{\mathfrak{p}}],$$
$$\frac{dX}{dt} = [\Omega(t), X(t)] + \mathrm{Ad}_{h(t)}\dot{Y} = [\Omega(t), X(t)] - S.$$

Additionally,

$$[S, L_{\mathfrak{p}}(t)] = \sum_{i=1}^n (a_i(\alpha_i \otimes \alpha_i)S - Sa_i(\alpha_i \otimes \alpha_i)) =$$
$$\sum_{i=1}^n a_i\alpha_i \otimes S\alpha_i - a_i S\alpha_i \otimes \alpha_i = \sum_{i=1}^n a_i(\alpha_i \wedge S\alpha_i),$$

which in turn implies that (41) can be written as

$$\frac{dR}{dt} = R(t)\Omega(t), \frac{dM}{dt} = [\Omega(t), M(t)] + [S, L_{\mathfrak{p}}(t)].$$

Let now $Q(t) = [\mathrm{Ad}_{h(t)}(P_0), X(t)] + \mathrm{Ad}_{h(t)}Q_0 = [L_{\mathfrak{p}}(t), X(t)] + \mathrm{Ad}_{h(t)}Q_0$. Note first that

$$[[\Omega, L_{\mathfrak{p}}], X] = -[[X, \Omega], L_{\mathfrak{p}}] - [[L_{\mathfrak{p}}, X], \Omega]$$
$$= -[[X, \Omega], L_{\mathfrak{p}}] + [\Omega, Q] - [\Omega, \mathrm{Ad}_h Q_0].$$

Then,

$$\frac{dQ}{dt} = [[\Omega(t), L_{\mathfrak{p}}(t)], X(t)] + [L_{\mathfrak{p}}(t), \frac{dX}{dt}(t)] + [\Omega(t), \mathrm{Ad}_{h(t)}(Q_0] =$$
$$[\Omega(t), Q(t)] + [L_{\mathfrak{p}}, [X, \Omega(t)]] + [L_{\mathfrak{p}}, \frac{dX}{dt}] =$$
$$[\Omega(t), Q(t)] + [S, L_{\mathfrak{p}}].$$

Therefore $Q(t)$ and $M(t)$ satisfy the same differential equation. Hence $Q(t) = M(t)$ whenever $Q_0 = M(0)$. If we now rename $Q(t)$ as $L_{\mathfrak{k}}(t)$ we get the Poisson equations for the shadow Hamiltonian $H = \frac{1}{2}\langle \mathcal{P}^{-1}(L_{\mathfrak{k}}), L_{\mathfrak{k}}\rangle + \langle S, L_{\mathfrak{p}}\rangle$. □

The preceding theorem links isospectral Hamiltonians to the equations of the top under quadratic potentials and paves a way to the n-dimensional generalization of O. Bogoyavlensky's famous result on integrability of three-dimensional mechanical tops in the presence of a quadratic potential [25]. The path to isospectral Hamiltonians is provided by Manakov's observation that the inertia tensor $\langle \mathcal{P}(U), U\rangle$ for a rigid body is confined to the transformations $\mathcal{P}(U) = SU + US$, for some positive definite matrix S. For then, $[\mathcal{P}^{-1}(M), S^2] = [M, S]$. Indeed, in this situation $\mathcal{P}(U) = SU + US = M$, and

$$[\mathcal{P}^{-1}M, S^2] = [U, S^2] = [SU + US, S] = [M, S].$$

Hence the corresponding affine Hamiltonian $\hat{H} = \frac{1}{2}\langle \mathcal{P}^{-1}L_{\mathfrak{k}}, L_{\mathfrak{k}}\rangle + \langle S, L_{\mathfrak{p}}\rangle$ is isospectral on $\mathfrak{sl}(n)$ (Theorem 1). Since the equations of the Hamiltonian

$$H = \frac{1}{2}\langle \mathcal{P}^{-1}M, M\rangle + \sum_{i=1}^n a_i(\alpha_i, S\alpha_i)$$

corresponding to the top with quadratic potential $V = \frac{1}{2}\sum_{i=1}^{n} a_i(S\alpha_i, \alpha_i)$ can be identified with the Poisson equations of $\hat{H}$ on the coadjoint orbit through $L_{\mathfrak{p}} = P_0, L_{\mathfrak{k}} = M(0)$, the isospectral invariants of

$$L_\lambda = \sum_{i=1}^{n} a_i\alpha_i - \lambda M + \lambda^2 S \tag{42}$$

are integrals of motion for the top. (Theorem 3). Since $\{X \in \mathfrak{so}(n) : [X, S] = 0\} = 0$ for each non-singular symmetric matrix S, the spectral invariants of

$$L_\lambda(s) = \sum_{i=1}^{n} a_i(\alpha_i \otimes \alpha_i - \lambda M + (\lambda^2 - s)S \tag{43}$$

form a completely integrable family of functions on each coadjoint orbit in $\mathfrak{sl}(n)$ (semi-simple and semi-direct). the top with a quadratic potential is completely integrable in all dimensions.

3.3. Three-Dimensional Tops- Kirchhoff-Kowalewski Type

We will now turn our attention to the class of affine-quadratic systems of Kirchhoff-Kowalewski type on complex Lie algebras with a particular interest on the symmetries that account for the existence of Kowalewski's integral reported in her seminal paper on the motions of a rigid body around a fixed point under the influence of gravity [26]. We will follow our recent paper [27] and show that there is a natural Hamiltonian on $\mathfrak{sp}(4,\mathbb{C})$ that answers the fundamental questions raised by Kowalewki's paper, namely, what is the geometric rational behind her approach in which all the variables were treated as complex quantities, and secondly. what are the symmetries that account for the existence of not only her integral of motion, but also of similar integrals, known as Kowalewski type integrals, that subsequently appeared in the literature on integrable systems [8,28–31].

Theorem 2 suggests that the search for the answers to the above questions should begin with the Poisson equations associated with an affine-quadratic Hamiltonian on $\mathfrak{so}(4,\mathbb{C})$ since both $\mathfrak{so}(1,3)$ and $\mathfrak{so}(4)$ are real forms for $\mathfrak{so}(4,\mathbb{C})$ (see also [24]). We will show that Kowalewski's "mysterious" change of variables appear naturally in the passage from $\mathfrak{so}(4,\mathbb{C})$ to $\mathfrak{sl}(2,\mathbb{C}) \times \mathfrak{sl}(2,\mathbb{C})$ an important intermediate step towards the right Hamiltonian on $\mathfrak{sp}(4,\mathbb{C})$. The journey from $\mathfrak{so}(4,\mathbb{C})$ to $\mathfrak{sp}(4,\mathbb{C})$ to this remarkable Hamiltonian begins with

$$H = \frac{1}{2}(\frac{m_1}{\lambda_1} + \frac{m_2}{\lambda_2} + \frac{m_3}{\lambda_3}) + b_1p_1 + b_2p_2 + p_3b_3, \tag{44}$$

where $L = m_1A_1 + m_2A_2 + m_3A_3 + p_1B_1 + p_2B_2 + p_3B_3$ is the coordinate representation of a point L in $\mathfrak{so}(4,\mathbb{C})$ relative to an orthonormal basis $A_1, A_2, A_3, B_1, B_2, B_3$ that conforms to the following Lie bracket Table 1:

Table 1. Lie brackets for $s = 0, 1$.

$[\,,]$	A_1	A_2	A_3	B_1	B_2	B_3
A_1	0	$-A_3$	A_2	0	$-B_3$	B_2
A_2	A_3	0	$-A_1$	B_3	0	$-B_1$
A_3	$-A_2$	A_1	0	$-B_2$	B_1	0
B_1	0	$-B_3$	B_2	0	$-sA_3$	sA_2
B_2	B_3	0	$-B_1$	sA_3	0	$-sA_1$
B_3	$-B_2$	B_1	0	$-sA_2$	sA_1	0

Then

$$\frac{dL_{\mathfrak{k}}}{dt} = [dH_{\mathfrak{k}}, L_{\mathfrak{k}}] + [B, L_{\mathfrak{p}}], \frac{dL_{\mathfrak{p}}}{dt} = [dH_{\mathfrak{k}}, L_{\mathfrak{p}}] + s[B, L_{\mathfrak{k}}], s = 0, 1, \tag{45}$$

are the Poisson equations generated by H, where $B = b_1B_1 + b_2B_2 + b_3B_3$ denote the drift element in $\mathfrak{p}$, $dH_{\mathfrak{k}} = \sum_{i=1}^{3} \frac{m_i}{\lambda_i} A_i$, $L_{\mathfrak{k}} = \sum_{i=1}^{3} m_i A_i$, and $L_{\mathfrak{p}} = \sum_{i=1}^{3} p_i B_i$. The same equations can be also expressed as

$$\begin{aligned}
\frac{dm_1}{dt} &= m_2m_3\left(\frac{1}{\lambda_3} - \frac{1}{\lambda_2}\right) + p_2b_3 - p_3b_2, \\
\frac{dm_2}{dt} &= m_1m_3\left(\frac{1}{\lambda_1} - \frac{1}{\lambda_3}\right) + p_3b_1 - p_1b_3, \\
\frac{dm_3}{dt} &= m_1m_2\left(\frac{1}{\lambda_2} - \frac{1}{\lambda_1}\right) + p_1b_2 - p_2b_1, \\
\frac{dp_1}{dt} &= \frac{1}{\lambda_3} p_2m_3 - \frac{1}{\lambda_2} p_3m_2 + s(m_2b_3 - m_3b_2), \\
\frac{dp_2}{dt} &= \frac{1}{\lambda_1} p_3m_1 - \frac{1}{\lambda_3} p_1m_3 + s(m_3b_1 - m_1b_3), \\
\frac{dp_3}{dt} &= \frac{1}{\lambda_2} p_1m_2 - \frac{1}{\lambda_1} p_2m_1 + s(m_1b_2 - m_2b_1).
\end{aligned} \tag{46}$$

When $s = 0$ the above equations formally coincide with the equations of the top: $\frac{dM}{dt} = [\Omega(t), M(t)] + b \wedge p$, $\frac{dp}{dt} = -\Omega(t)p(t)$ (Equation (21)).

On $\mathfrak{g}_s$ there are two Casimirs:

$$I_1 = \langle L_{\mathfrak{p}}, L_{\mathfrak{p}} \rangle + s\langle L_{\mathfrak{k}}, L_{\mathfrak{k}} \rangle = (p,p) + s(m,m), I_2 = \langle L_{\mathfrak{k}}, L_{\mathfrak{p}} \rangle = (m,p),$$

Hence generic coadjoint orbits in $\mathfrak{g}_s$ are four-dimensional. Since each coadjoint orbit is symplectic, integrable cases occur whenever there is an extra integral of motion functionally independent of H, I_1, and I_2. Since the motion of the top is subordinate to the Poisson system of H on $\mathfrak{se}(3,\mathbb{C})$, the search for integrable tops reduces to the search for an additional integral of motion functionally independent from I_1,I_2 and H.

Let us now come to the conditions of Kowalewski

$$\lambda = \lambda_1 = \lambda_2 = 2\lambda_3, b_3 = 0. \tag{47}$$

and her "mysterious" variables

$$\begin{aligned}
&z_1 = m_1 + im_2, z_2 = m_1 - im_2, w_1 = p_1 + ip_2, \\
&w_2 = p_1 - ip_2, z_3 = im_3, w_3 = ip_3, \\
&b = b_1 + ib_2, \bar{b} = b_1 - ib_2.
\end{aligned} \tag{48}$$

After the substitutions, Equation (46) become

$$\begin{gathered}
\frac{dz_1}{dt} = -\frac{1}{\lambda} z_1z_3 + bw_3, \frac{dz_2}{dt} = \frac{1}{\lambda} z_2z_3 - \bar{b}w_3 \\
\frac{dz_3}{dt} = \frac{1}{2}(bw_2 - \bar{b}w_1) \\
\frac{dw_1}{dt} = \frac{1}{\lambda} z_1w_3 - \frac{2}{\lambda} z_3w_1 + sbz_3, \frac{dw_2}{dt} = -\frac{1}{\lambda} z_2w_3 + \frac{2}{\lambda} z_3w_2 - s\bar{b}z_3 \\
\frac{dw_3}{dt} = \frac{1}{2\lambda}(z_1w_2 - z_2w_1) + \frac{s}{2}(bz_2 - \bar{b}z_1),
\end{gathered} \tag{49}$$

from which it can be easily extracted that

$$I = \left(\frac{z_1^2}{2\lambda} - bw_1 + \frac{1}{2}s\lambda b^2\right)\left(\frac{z_2^2}{2\lambda} - \bar{b}w_2 + \frac{1}{2}s\lambda\bar{b}^2\right), \tag{50}$$

is an integral of motion. Following the terminology in [7] we will refer refer to this integral as the Kirchhoff-Kowalewski integral. It is only in the special case $s = 0$ and $\lambda = 2$ that this integral coincides with the integral of motion found by Kowalewski. The real versions of the Kirchhoff-Kowalewski integral were originally discovered by V. Kuznetsov and I.V. Komarov in their studies of the hydrogen atom [28,29].

Let us reveal the geometric rational behind Kowalewski's change of variables. The explanations are most naturally articulated through the root system in $\mathfrak{so}(4,\mathbb{C})$. Recall that any maximal commutative sub-algebra $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is called a Cartan subalgebra. All Cartan subalgebras in a semi-simple Lie algebra are conjugate, and hence all have the same dimension. The dimension of any Cartan algebra is the rank of $\mathfrak{g}$. The rank of $\mathfrak{so}_4(\mathbb{C})$ is two. Evidently each pair $(A_i, B_i), i = 1,2,3$, in Table 1 generates a Cartan algebra in

$\mathfrak{so}(4,\mathbb{C})$. Since these algebras are conjugate, there is no preferential choice. However, in regard to the equations of the top, there is a preferential choice when two moments of inertia are equal. In the case that $\lambda_1 = \lambda_2$ the natural choice is the Cartan algebra generated by the pair $\{A_3, B_3\}$.

An element α in the dual $\mathfrak{h}^*$ of a Cartan algebra $\mathfrak{h}$ is called a root if for some $v \in \mathfrak{g}$, $[h,v] = \alpha(h)v$ for all $h \in \mathfrak{h}$. An easy calculation shows that there are four roots $\pm\alpha_1$, $\pm\alpha_2$ given by

$$\alpha_1(xA_3 + yB_3) = -i(x+y), \alpha_2(xA_3 + yB_3) = -i(x-y), x, y \in \mathbb{C}. \tag{51}$$

The corresponding root spaces are one dimensional, and are generated by

$$\begin{aligned} C_1 &= \tfrac{1}{2}(A_1 - iA_2) + \tfrac{1}{2}(B_1 - iB_2), \alpha = \alpha_1, \\ C_2 &= \tfrac{1}{2}(A_1 + iA_2) + \tfrac{1}{2}(B_1 + iB_2), \alpha = -\alpha_1, \\ D_1 &= \tfrac{1}{2}(A_1 - iA_2) - \tfrac{1}{2}(B_1 - iB_2), \alpha = \alpha_2, \\ D_2 &= \tfrac{1}{2}(A_1 + iA_2) - \tfrac{1}{2}(B_1 + iB_2), \alpha = -\alpha_2. \end{aligned} \tag{52}$$

Together with $C_3 = \frac{i}{2}(A_3 + B_3)$ and $D_3 = \frac{i}{2}(A_3 - B_3)$ these matrices form a basis for $\mathfrak{so}(4,\mathbb{C})$. A simple calculation shows that

$$\begin{gathered} \alpha_1(C_3) = 1, \alpha_2(D_3) = 1, \alpha_1(D_3) = \alpha_2(C_3) = 0, \text{ hence,} \\ [C_3, C_1] = C_1, [C_3, C_2] = -C_2, [D_3, D_1] = D_1, [D_3, D_2] = -D_2. \end{gathered}$$

Furthermore, $[C_1, C_2] = -2C_3$, $[D_1, D_2] = -2D_3$, and $[C_i, D_j] = 0$, for all i and j.

The Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ spanned by C_1, C_2, C_3, and D_1, D_2, D_3 satisfy $[\mathfrak{g}_1, \mathfrak{g}_2] = 0$. and each is isomorphic to $\mathfrak{sl}(2,\mathbb{C})$ under the identification

$$C_1, D_1 \to \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, C_2, D_2 \to \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, C_3, D_3 \to \frac{1}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{53}$$

An easy calculation shows that the coordinates $a_1, a_2, a_3, b_1, b_2, b_3$ of an arbitrary point $X \in \mathfrak{so}(4,\mathbb{C})$ relative to the basis $A_1, A_2, A_3, B_1, B_2, B_3$ are transformed to the coordinates $c_1, c_2, c_3, d_1, d_2, d_3$ relative to the basis $C_1, C_2, C_3, D_1, D_2, D_3$ according to the following formulas:

$$\begin{aligned} c_1 &= \frac{1}{2}(a_1 + ia_2) + \frac{1}{2}(b_1 + ib_2), d_1 = \frac{1}{2}(a_1 + ia_2) - \frac{1}{2}(b_1 + ib_2), \\ c_2 &= \frac{1}{2}(a_1 - ia_2) + \frac{1}{2}(b_1 - ib_2), d_2 = \frac{1}{2}(a_1 - ia_2) - \frac{1}{2}(b_1 - ib_2). \\ c_3 &= -i(a_3 + b_3), d_3 = -i(a_3 - b_3). \end{aligned}$$

Let us now $\Phi : \mathfrak{so}(4,\mathbb{C}) \to \mathfrak{sp}(4,\mathbb{C})$ be given by

$$\begin{aligned} &\Phi(\textstyle\sum_{i=1}^3 (c_iC_i + d_iD_i)) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} -\frac{c_3}{2} & c_1 \\ -c_2 & \frac{c_3}{2} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} -\frac{d_3}{2} & d_1 \\ -d_2 & \frac{d_3}{2} \end{pmatrix} = \\ &I \otimes \tfrac{1}{2}\begin{pmatrix} -\frac{1}{2}(c_3 + d_3) & c_1 + d_1 \\ -(c_2 + d_2) & \frac{1}{2}(c_3 + d_3) \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \tfrac{1}{2}\begin{pmatrix} -\frac{1}{2}(c_3 - d_3) & c_1 - d_1 \\ -(c_2 - d_2) & \frac{1}{2}(c_3 - d_3) \end{pmatrix} = \\ &I \otimes \tfrac{1}{2}\begin{pmatrix} ia_3 & a_1 + ia_2 \\ -a_1 + ia_2 & -ia_3 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \tfrac{1}{2}\begin{pmatrix} ib_3 & b_1 + ib_2 \\ -b_1 + ib_2 & -ib_3 \end{pmatrix}. \end{aligned}$$

where $A \otimes B$ denotes the Kronecker product of matrices A and B.

To see that $\Phi(\mathfrak{so}(4,\mathbb{C})) \subset \mathfrak{sp}(4,\mathbb{C})$ recall first that $\mathfrak{sp}(4,\mathbb{C})$ consists of matrices M that satisfy $JMJ^{-1} = -M^T$, where J is the matrix that defines the symplectic form (z, Jw) on $\mathbb{C}^4$, i.e., $J^2 = -I$. It is easy to check that both $I \otimes \frac{1}{2}\begin{pmatrix} ia_3 & a_1 + ia_2 \\ -a_1 + ia_2 & -ia_3 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes$

$\frac{1}{2}\begin{pmatrix} ib_3 & b_1+ib_2 \\ -b_1+ib_2 & -ib_3 \end{pmatrix}$ satisfy $JMJ^{-1} = -M^T$ with $J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Since $J^2 = -I_2 \otimes I_2 = -I$, our claim follows.

We will identify $\mathfrak{sl}(2,\mathbb{C})$ with pure complex quaternions $\mathbb{Q}$ via the correspondence

$$\mathbf{q} = q_1\vec{i} + q_2\vec{j} + q_3\vec{k} \Leftrightarrow \mathbf{Q} = q_1\mathbf{E_1} + q_2\mathbf{E_2} + q_3\mathbf{E_3} = \begin{pmatrix} iq_3 & q_1+iq_2 \\ -q_1+iq_2 & -iq_3. \end{pmatrix}$$

Then the standard basis $\vec{i},\vec{j},\vec{k}$ in $\mathbb{Q}$ is identified with the matrices

$$\mathbf{E_1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \mathbf{E_2} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \mathbf{E_3} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},$$

and any element $X = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$ in $\mathfrak{sl}(2,\mathbb{C})$ is represented by the quaternion $\mathbf{X} = \frac{1}{2}\begin{pmatrix} ia_3 & a_1+ia_2 \\ -a_1+ia_2 & -ia_3 \end{pmatrix}$, $a_1 = b-c$, $a_2 = -(b+c)$, $a_3 = -2ia$.

Let now $\mathbf{A_i} = \frac{1}{2}\mathbf{E}_i$, $\mathcal{A}_i = I \otimes \mathbf{A}_i$, $\mathcal{B}_i = E_3 \otimes \mathbf{A_i}$, so that $\Phi(A_i) = \mathcal{A}_i$ and $\Phi(B_i) = \mathcal{B}_i$ for each i, $i = 1,2,3$, and let $\mathbf{so_4} = \Phi(\mathfrak{so}(4,\mathbb{C}))$. Matrices $\mathcal{A}_i, \mathcal{B}_i, i = 1,2,3$. form an orthonormal basis in $\mathbf{so}_4$ relative to the inner product $\langle X, Y\rangle = -Tr(XY)$ on $\mathfrak{sp}(4,\mathbb{C})$.

It is easy to verify that $\Phi : \mathfrak{so}(4,\mathbb{C}) \to \mathfrak{so}_4$ is a Poisson map. Therefore

$$\tilde{H}(\tilde{\ell}) = H(\Phi^*(\tilde{\ell})) = H(\ell), \Phi^*(\tilde{\ell}) = \ell, \tilde{\ell} \in \mathbf{so}_4, \tag{54}$$

for any function H on $\mathfrak{so}^*(4,\mathbb{C})$, where Φ^* denotes the dual map of Φ. After the identification of $\mathbf{so}_4^*$ with $\mathbf{so}_4$ via the trace form, the Poisson equations of $\tilde{H}$ associated with H in (44) become

$$\begin{aligned} \tfrac{d}{dt}(I \otimes \mathbf{Z}) &= [I \otimes \mathbf{\Omega}, I \otimes \mathbf{Z}] + [E_3 \otimes \mathbf{B}, E_3 \otimes \mathbf{W}] = I \otimes ([\mathbf{\Omega}, \mathbf{Z}] + [\mathbf{B}, \mathbf{W}]), \\ \tfrac{d}{dt}(E_3 \otimes \mathbf{W}) &= [I \otimes \mathbf{Z}, E_3 \otimes \mathbf{W}] + s[E_3 \otimes \mathbf{B}, I \otimes \mathbf{Z}] = E_3 \otimes ([\mathbf{\Omega}, \mathbf{W}] + s[\mathbf{B}, \mathbf{Z}]), \end{aligned}$$

or, in simpler form,

$$\frac{d\mathbf{Z}}{dt} = [\mathbf{\Omega}, \mathbf{Z}] + [\mathbf{B}, \mathbf{W}], \frac{d\mathbf{W}}{dt} = [\mathbf{\Omega}, \mathbf{W}] + s[\mathbf{B}, \mathbf{Z}], \tag{55}$$

where $Z = \frac{1}{2}\begin{pmatrix} im_3 & m_1+im_2 \\ -m_1+im_2 & -im_3 \end{pmatrix}$, $W = \frac{1}{2}\begin{pmatrix} ip_3 & p_1+ip_2 \\ -p_1+ip_2 & -ip_3 \end{pmatrix}$, $\mathbf{\Omega} = \frac{1}{2}\begin{pmatrix} \frac{1}{\lambda_3}z_3 & \frac{1}{\lambda_1}m_1 + \frac{i}{\lambda_2}m_2 \\ -\frac{1}{\lambda_1}m_1 + \frac{i}{\lambda_2}m_2 & -\frac{1}{\lambda_3}z_3 \end{pmatrix}$, and $\mathbf{B} = \frac{1}{2}\begin{pmatrix} ib_3 & b_1+ib_2 \\ -b_1+ib_2 & -ib_3 \end{pmatrix}$.

Now we see Kowalewski variables $z_1 = m_1 + im_2, z_2 = m_1 - im_2, w_1 = p_1 + ip_2, w_2 = p_1 - ip_2$ as the natural coordinates in this Poisson representation. Under Kowalewski's conditions $\lambda = \lambda_1 = \lambda_2 = 2\lambda_3$, $b_3 = 0$, Equation (55) reduce to Equation (49). The passage from H to $\tilde{H}$ reveals the geometric rational behind the ad-hoc change of variables in (48) and serves as a natural segue to our ultimate Hamiltonian on $\mathfrak{sp}(4,\mathbb{C})$.

3.4. Kowalewski's Conditions and Isospectral Representations

We now address the origins of the "enigmatic" conditions (47) through an extended affine-quadratic Hamiltonian

$$\mathcal{H} = \sum_{i=1}^{3} \frac{m_i^2}{2\lambda_i} + b_i p_i + c_i q_i \tag{56}$$

on $\mathfrak{sp}^*(4,\mathbb{C})$ defined by complex numbers $\lambda_1, \lambda_2, \lambda_3, b_1, b_2, b_3$, and c_1, c_2, c_3 and an extended basis $\mathcal{A}_i, \mathcal{B}_i, i = 1,2,3$, $\mathcal{A}_4 = \frac{1}{2}E_2 \otimes I$, and $\mathcal{C}_1 = E_1 \otimes \mathbf{A}_1, \mathcal{C}_2 = E_1 \otimes \mathbf{A}_2, \mathcal{C}_3 = E_1 \otimes \mathbf{A}_3$, where $E_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, E_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and $E_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

The reader can easily verify that $\mathfrak{g} = \mathfrak{sp}(4,\mathbb{C})$ has the following decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2, \mathfrak{k} = \mathfrak{k}_0 \oplus \mathbb{C}\mathcal{A}_4 \tag{57}$$

where $\mathfrak{k}_0$ is the Lie algebra spanned by $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$ and where $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are respectively the linear spans of $\mathcal{B}_i, i = 1,2,3$ and $\mathcal{C}_i, i = 1,2,3$. These spaces conform to the following Lie algebraic relations:

$$[\mathcal{A}_4, \mathfrak{k}_0] = 0, [\mathcal{A}_4, \mathfrak{p}_1] = \mathfrak{p}_2, [\mathcal{A}_4, \mathfrak{p}_2] = \mathfrak{p}_1, [\mathfrak{k}_o, \mathfrak{p}_1] = \mathfrak{p}_1, [\mathfrak{k}_0, \mathfrak{p}_2] = \mathfrak{p}_2,$$
$$[\mathfrak{p}_1, \mathfrak{p}_1] = \mathfrak{k}_0, [\mathfrak{p}_2, \mathfrak{p}_2] = \mathfrak{k}_0, [\mathfrak{p}_1, \mathfrak{p}_2] = \mathbb{C}\mathcal{A}_4.$$

After $\mathfrak{g}^*$ is identified with $\mathfrak{g}$ via the scalar product $\langle X, Y\rangle = -\frac{1}{2}Tr(XY)$ the above Hamiltonian can be written as

$$\mathcal{H} = \frac{1}{2}\langle \mathcal{P}(L^0_{\mathfrak{k}}), L^0_{\mathfrak{k}}\rangle + \langle L_{\mathfrak{p}}, \mathcal{A}\rangle, \tag{58}$$

where $L^0_{\mathfrak{k}} = m_1\mathcal{A}_1 + m_2\mathcal{A}_2 + m_3\mathcal{A}_3, L_{\mathfrak{p}} = \sum_{i=1}^3 p_i\mathcal{B}_i - q_i\mathcal{C}_i$ and $\mathcal{A} = \mathcal{B} + \mathcal{C}$, $\mathcal{B} = \sum_{i=1}^3 b_i\mathcal{B}_i$, $\mathcal{C} = \sum_{i=1}^3 c_i\mathcal{C}_i$ (note that $\langle\, ,\, \rangle$ is negative on $\mathfrak{p}_2$ which accounts for the negative signs in the expression for $L_{\mathfrak{p}}$).

Since $\mathfrak{g} = \mathfrak{sp}(4,\mathbb{C})$ is semi-simple the Poisson equations for $\mathcal{H}$ are given by

$$\frac{dL_{\mathfrak{k}}}{dt} = [d\mathcal{H}_{\mathfrak{k}}, L_{\mathfrak{k}}] + [\mathcal{A}, L_{\mathfrak{p}}], \frac{dL_{\mathfrak{p}}}{dt} = [d\mathcal{H}_{\mathfrak{k}}, L_{\mathfrak{p}}] + s[\mathcal{A}, L_{\mathfrak{k}}], s = 0, 1.$$

These equations can be written in a more succinct form as

$$\begin{aligned} \frac{d\mathbf{Z}}{dt} &= [\boldsymbol{\Omega}, \mathbf{Z}] + [\mathbf{B}, \mathbf{W}] + [\mathbf{C}, \mathbf{S}], \\ \frac{dm_4}{dt} &= -(\mathbf{C}\cdot\mathbf{W} + \mathbf{B}\cdot\mathbf{S}), \\ \frac{d\mathbf{W}}{dt} &= [\boldsymbol{\Omega}, \mathbf{W}] + s([\mathbf{B}, \mathbf{Z}] - m_4\mathbf{C}), \\ \frac{d\mathbf{S}}{dt} &= [\boldsymbol{\Omega}, \mathbf{S}] - s([\mathbf{C}, \mathbf{Z}] - \mathbf{m_4}\mathbf{B}). \end{aligned} \tag{59}$$

in terms of the following notations:

$$L_{\mathfrak{k}} = I \otimes \mathbf{Z} + m_4\mathcal{A}_4, L_{\mathfrak{p}} = E_3 \otimes \mathbf{W} - E_1 \otimes \mathbf{S},$$
$$d\mathcal{H} = I \otimes \boldsymbol{\Omega} + E_3 \otimes \mathbf{B} + E_1 \otimes \mathbf{C}, \boldsymbol{\Omega} = \sum_{\mathbf{i=1}}^{\mathbf{3}} \frac{\mathbf{m_i}}{\mathbf{\check{i}}}\mathbf{A_i},$$
$$\mathbf{B} = \sum_{i=1}^3 b_i\mathbf{A}_i, \mathbf{C} = \sum_{\mathbf{i=1}}^{\mathbf{3}} \mathbf{c_i}\mathbf{A_i}.$$

$\mathbf{Z} = \frac{1}{2}\begin{pmatrix} z_3 & z_1 \\ -z_2 & -z_3 \end{pmatrix}, \mathbf{W} = \frac{1}{2}\begin{pmatrix} w_3 & w_1 \\ -w_2 & -w_3 \end{pmatrix}, z_{1,2} = m_1 \pm im_2, z_3 = im_3,$

$w_{1,2} = p_1 \pm ip_2, , w_3 = ip_3$, as in the previous section, and $\mathbf{S} = \frac{1}{2}\begin{pmatrix} s_3 & s_1 \\ -s_2 & -s_3 \end{pmatrix}$, with $s_{1,2} = q_1 \pm iq_2, , s_3 = iq_3$.

We now come to the crux of the matter, the existence of integrals of motion for the above system. The intermediate question is the existence of an integral I of the form $I = \alpha_1 m_1 + \alpha_2 m_2 + \alpha_3 m_3 + \beta m_4$ for some constants $\alpha_1, \alpha_2, \alpha_3$, and β.

Proposition 2. *$I = \alpha_1 m_1 + \alpha_2 m_2 + \alpha_3 m_3 + \beta m_4$ is an integral of motion for $\mathcal{H}$ in exactly two cases: when $\lambda_1 = \lambda_2$ and $b_1 = b_2 = c_1 = c_2 = 0$, then $I = m_3$, and in the second case, when $\lambda_1 = \lambda_2, b_3 = c_3 = 0, b_1 = \pm ic_2, b_2 = \mp ic_1$, then $I = im_3 + m_4$. (for the proof see [27]).*

This first condition singles out the top of Lagrange, while the second condition is a precursor to Kowalewski's top as will be demonstrated below. Note that the second condition $b_1 = ic_2$ and $b_2 = -ic_1$ can be also written as $b = c$, and $\bar{b} = -\bar{c}$ where $b = b_1 + ib_2, \bar{b} = b_1 - ib_2, c = c_1 + ic_2$ and $\bar{c} = c_1 - ic_2$. Then $\mathbf{C} = \frac{1}{2}\begin{pmatrix} 0 & b \\ \bar{b} & 0 \end{pmatrix}$, and since it is orthogonal to $\mathbf{B} = \frac{1}{2}\begin{pmatrix} 0 & b \\ -\bar{b} & 0 \end{pmatrix}$, it will be denoted by $\mathbf{B}^\perp$.

We will say that (59) satisfies the preliminary condition of Kowalewski whenever $\lambda_1 = \lambda_2$ and $\mathbf{C} = \mathbf{B}^\perp$. It follows that the preliminary condition of Kowalewski is synonymous with the integral of motion $I = im_3 + m_4 = z_3 + m_4$. (This integral of motion was also discovered earlier by A.M Savu in [32]).

We will now assume that the preliminary condition holds and we will pursue conditions on the ratio $\delta = \frac{\lambda}{\lambda_3}$, where $\lambda = \lambda_1 = \lambda_2$, that guarantee extra integrals of motion for system (59). Note that in this situation $\mathcal{A} = \mathcal{B} + \mathcal{C} = E_3 \otimes \mathbf{B} + E_1 \otimes \mathbf{B}^\perp$. Systems that satisfy the preliminary condition of Kowalewski and also satisfy $\delta = 2$ will be said to satisfy the Kowalewsky conditions. The following proposition provides an important characterization of Kowalewski's conditions for both $s = 0$ and $s = 1$.

Proposition 3. *Assume that (59) satisfies the preliminary condition of Kowalewski and is restricted to the manifold $z_3 + m_4 = 0$. Then $\mathcal{H}$ satisfies the isospectrality condition $[d\mathcal{H}_\mathfrak{k}, B] = [L_\mathfrak{k}, \mathcal{A}]$(as in Theorem 1) for some matrix $B \in \mathfrak{p}$, with $[\mathcal{A}, B] = 0$ if only if (59) satisfies the Kowalewski conditions. In fact, $B = \lambda(E_1 \otimes \mathbf{B}^\perp + E_3 \otimes \mathbf{B}) = \lambda\mathcal{A}$ [27].*

Indeed, under Kowalewski's conditions

$$\Omega = \frac{1}{\lambda}Z + \frac{1}{2\lambda}z_3 E_3 \text{ and } L_\mathfrak{k} = I \otimes Z - z_3\mathcal{A}_4.$$

Therefore,

$$[d\mathcal{H}_\mathfrak{k}, B] = [I \times \Omega, B] = [I \times Z, \mathcal{A}] + \frac{z_3}{2}[I \times E_3, \mathcal{A}], \text{ and } [L_\mathfrak{k}, A] = [I \otimes Z, \mathcal{A}] - z_3[\mathcal{A}_4, \mathcal{A}].$$

Since $[\frac{z_3}{2} I \otimes E_3, \mathcal{A}] = -z_3[\mathcal{A}_4, \mathcal{A}]$, $[I \otimes \Omega, B] = [L_\mathfrak{k}, \mathcal{A}]$.

It follows from Theorem 1 that Kowalewski's condition is necessary and sufficient for the existence of isospectral representation

$$\frac{dL_\mu}{dt} = [M_\mu, L_\mu], L_\mu = L_\mathfrak{p} - \mu(L_\mathfrak{k}^0 - z_3\mathcal{A}_4) + (\mu^2 - s)\lambda\mathcal{A}$$

on the invariant manifold $z_3 + m_4 = 0$. Consequently, $\phi_k = Tr(L_\mu^{2k}) = \langle L_\mu^k, L_\mu^k \rangle$ are integrals of motion for (59), in involution with each other for each $s = 0$, or $s = 1$. Remarkably, the prototype of Kowalewski's integrals of motion is found among the above spectral invariants. (see also [33,34] for other spectral representations).

We will show the existence of Kowalewki's integral of motion directly from the equations

$$\begin{aligned} \frac{d\mathbf{Z}}{dt} &= [\mathbf{\Omega}, \mathbf{Z}] + [\mathbf{B}, \mathbf{U}] + [\mathbf{B}^\perp, \mathbf{V}], \\ \frac{d\mathbf{U}}{dt} &= [\mathbf{\Omega}, \mathbf{U}], \frac{d\mathbf{V}}{dt} = [\mathbf{\Omega}, \mathbf{V}], \end{aligned} \tag{60}$$

obtained from (59) under the change of variables $\mathbf{U} = \mathbf{W} - s\lambda\mathbf{B}, \mathbf{V} = \mathbf{S} + s\lambda\mathbf{B}^\perp$. Equation (60) may be seen as a semisimple extension of the Kowalewski-type gyrostat in two constant fields introduced in [35].

Equations (60) may be also expressed in terms of the coordinates as

$$\begin{array}{c} \frac{dz_1}{dt} = -\frac{1}{\lambda}z_1z_3 + b(u_3+v_3), \frac{dz_2}{dt} = \frac{1}{\lambda}z_2z_3 - \bar{b}(u_3-v_3), \\ \frac{dz_3}{dt} = \frac{1}{2}(bu_2 - \bar{b}u_1 + bv_2 + \bar{b}v_1), \\ \frac{du_1}{dt} = \frac{u_3z_1}{\lambda} - \frac{2u_1z_3}{\lambda}, \frac{du_2}{dt} = \frac{2u_2z_3}{\lambda} - \frac{z_2u_3}{\lambda}, \\ \frac{du_3}{dt} = \frac{1}{2\lambda}(z_1u_2 - u_1z_2) \\ \frac{dv_1}{dt} = \frac{v_3z_1}{\lambda} - \frac{2v_1z_3}{\lambda}, \frac{dv_2}{dt} = \frac{2v_2z_3}{\lambda} - \frac{z_2v_3}{\lambda}, \\ \frac{dv_3}{dt} = \frac{1}{2\lambda}(z_1v_2 - v_1z_2). \end{array} \tag{61}$$

One readily obtains the following fundamental equalities

$$\begin{array}{c} \frac{d}{dt}(u_1+v_1) = \frac{z_1}{\lambda}(u_3+v_3) - \frac{2z_3}{\lambda}(u_1+v_1), \\ \frac{d}{dt}(u_2-v_2) = \frac{2z_3}{\lambda}(u_2-v_2) - \frac{z_2}{\lambda}(u_3-v_3) \end{array} \tag{62}$$

Let now

$$e_1 = \frac{z_1^2}{2\lambda} - b(u_1+v_1), \text{ and } e_2 = \frac{z_2^2}{2\lambda} - \bar{b}(u_2-v_2).$$

Then

$$\begin{array}{c} \frac{de_1}{dt} = \frac{d}{dt}(\frac{z_1^2}{2\lambda} - b(u_1+v_1)) = \\ \frac{z_1}{\lambda}(-\frac{1}{\lambda}z_1z_3 + b(u_3+v_3)) - b(\frac{z_1}{\lambda}(u_3+v_3) - \frac{2z_3}{\lambda}(u_1+v_1)) = \\ -\frac{2z_3}{\lambda}(\frac{z_1^2}{2\lambda} - b(u_1+v_1)) = -\frac{2z_3}{\lambda}e_1, \end{array}$$

and

$$\begin{array}{c} \frac{de_2}{dt} = \frac{d}{dt}(\frac{z_2^2}{2\lambda} - \bar{b}(u_2-v_2)) = \\ \frac{z_2}{\lambda}(\frac{1}{\lambda}z_2z_3 - \bar{b}(u_3-v_3)) - \bar{b}(\frac{2z_3}{\lambda}(u_2-v_2) - \frac{z_2}{\lambda}(u_3-v_3) = \\ \frac{2z_3}{\lambda}(\frac{z_2^2}{2\lambda} - \bar{b}(u_2-v_2)) = \frac{2z_3}{\lambda}e_2. \end{array}$$

Hence, $c = e_1e_2$ is an integral of motion for (61) since

$$\frac{d}{dt}c = \frac{d}{dt}e_1e_2 = \frac{de_1}{dt}e_2 + e_1\frac{de_2}{dt} = 0.$$

An interested reader may want to show that the following are also integrals of motion

$$\begin{aligned} c_0 = &\quad ||\mathbf{V}^2||, c_1 = ||\mathbf{U}||^2, c_2 = \langle \mathbf{U}, \mathbf{V} \rangle, \\ c_3 = &\quad [\mathbf{U}, \mathbf{V}] \cdot (\lambda([\mathbf{B}, \mathbf{V}] + [\mathbf{B}^\perp, \mathbf{U}]) - z_3\mathbf{Z}) + \frac{1}{2}((\mathbf{V} \cdot \mathbf{Z})^2 - (\mathbf{U} \cdot \mathbf{Z})^2). \end{aligned}$$

The preceding calculation also draws attention to the following general fact:

Proposition 4. $c = (\frac{z_1^2}{2\lambda} - b(u_1+v_1))(\frac{z_2^2}{2\lambda} - \bar{b}(u_2-v_2))$ *is a constant of motion for any differential system in the variables* $z_i, u_i, v_i, i = 1,2,3$ *that satisfy*

$$\begin{array}{c} \frac{dz_1}{dt} = -\frac{1}{\lambda}z_1z_3 + b(u_3+v_3), \\ \frac{dz_2}{dt} = \frac{1}{\lambda}z_2z_3 - \bar{b}(u_3-v_3), \\ \frac{d}{dt}(u_1+v_1) = \frac{z_1}{\lambda}(u_3+v_3) - \frac{2z_3}{\lambda}(u_1+v_1), \\ \frac{d}{dt}(u_2-v_2) = \frac{2z_3}{\lambda}(u_2-v_2) - \frac{z_2}{\lambda}(u_3-v_3), \end{array} \tag{63}$$

independently of the equations that govern the evolution of u_3 *and* v_3*.*

To come back to the top of Kowalewski, note that $\mathbf{V} = 0$ is an invariant subsystem for (60). On this set, $\mathbf{S} = -\lambda s \mathbf{B}^{\perp}$, and c reduces to

$$c = (\frac{z_1^2}{2\lambda} - bu_1)(\frac{z_2^2}{2\lambda} - \bar{b}u_2) = (\frac{z_1^2}{2\lambda} - bw_1 + s\lambda b^2)(\frac{z_2^2}{2\lambda} - \bar{a}bw_2 + s\lambda \bar{b}^2), \tag{64}$$

and remains an integral of motion for the reduced system

$$\frac{d\mathbf{Z}}{dt} = [\mathbf{\Omega}, \mathbf{Z}] + [\mathbf{B}, \mathbf{U}], \frac{d\mathbf{U}}{dt} = [\mathbf{\Omega}, \mathbf{U}]), \tag{65}$$

with its fundamental relations (63)

$$\begin{aligned} \tfrac{dz_1}{dt} &= -\tfrac{1}{\lambda} z_1 z_3 + b u_3, \tfrac{dz_2}{dt} = \tfrac{1}{\lambda} z_2 z_3 - \bar{b} u_3, \\ \tfrac{d}{dt} u_1 &= \tfrac{z_1}{\lambda} u_3 - \tfrac{2z_3}{\lambda} u_1, \tfrac{d}{dt} u_2 = \tfrac{2z_3}{\lambda} u_2 - \tfrac{z_2}{\lambda} u_3. \end{aligned} \tag{66}$$

This reduced system coincides the Kirchhoff-Kowalewski system on $\mathfrak{se}(3, \mathbb{C})$ (Equation (49), $s = 0$, after u is replaced by w). Then

$$c = (\frac{z_1^2}{2\lambda} - bu_1)(\frac{z_2^2}{2\lambda} - \bar{b}u_2)$$

coincides with the integral of motion discovered by Kowalewski. The remaining isospectral integrals of motion $c_1 = ||\mathbf{U}||^2$ and $c_3 = (\mathbf{U} \cdot \mathbf{Z})$ coincide with the Casimirs on $\mathfrak{se}(3, \mathbb{C})$.

To recover the semi-simple form of the Kirchhoff-Kowalewski integral, let $\mathbf{Y} = \mathbf{U} + \frac{s}{2}\lambda \mathbf{B}$. In terms of $\mathbf{Z}$ and $\mathbf{Y}$ the preceding system becomes

$$\frac{d\mathbf{Z}}{dt} = [\mathbf{\Omega}, \mathbf{Z}] + [\mathbf{B}, \mathbf{Y}], \frac{d\mathbf{Y}}{dt} = [\mathbf{\Omega}, \mathbf{Y}] - \frac{s\lambda}{2}[\mathbf{\Omega}, \mathbf{B}]. \tag{67}$$

This system satisfies the same equations as the Kirchhoff-Kowalewski system except for $\frac{dy_3}{dt}$. Indeed,

$$\begin{gathered} -\tfrac{s\lambda}{2}[\mathbf{\Omega}, \mathbf{B}] = -\tfrac{s}{2}([\mathbf{Z}, \mathbf{B}] + z_3[\mathbf{A}_3, \mathbf{B}]) = \\ -s[\mathbf{Z}, \mathbf{B}] + \tfrac{s}{2}([\mathbf{Z}, \mathbf{B}] + z_3 \mathbf{B}^{\perp}) = s[\mathbf{B}, \mathbf{Z}] + \tfrac{s}{2}(\bar{b}z_1 - bz_2)\mathbf{A}_3. \end{gathered}$$

The remaining equations given by

$$\begin{aligned} \tfrac{dz_1}{dt} &= -\tfrac{1}{\lambda} z_1 z_3 + b y_3, \tfrac{dz_2}{dt} = \tfrac{1}{\lambda} z_2 z_3 - \bar{b} y_3, \\ \tfrac{dy_1}{dt} &= \tfrac{z_1}{\lambda} y_3 - \tfrac{2z_3}{\lambda} y_1 + sbz_3, \\ \tfrac{dy_2}{dt} &= \tfrac{2z_3}{\lambda} y_2 - \tfrac{z_2}{\lambda} y_3 - s\bar{b}z_3, \end{aligned} \tag{68}$$

are the same as (66), and consequently yield

$$c = (\frac{z_1^2}{2\lambda} - by_1 + \frac{s}{2}\lambda b^2)(\frac{z_2^2}{2\lambda} - \bar{b}y_2 + \frac{s}{2}\lambda \bar{b}^2)$$

as an integral of motion for the preceding system, as well as for the Kirchhoff-Kowalewski system (Equation (49)) when $\mathbf{Y}$ is replaced by $\mathbf{W}$.

The papers of V. Dragović and K. Kukić [30] and V. V. Sokolov [31] produce differential systems which admit Kowalewski type integrals different from the ones in this paper and yet follow the same integration procedure used by S. Kowalewski in her original paper. Remarkably, all these systems satisfy the fundamental relations (63) from which the existence of their extra integrals of motion could be easily ascertained.

4. Kepler, Jacobi, Neumann and Moser

Let us now return to $G = SL(n+1)$ and its Lie algebra $\mathfrak{sl}(n+1)$ endowed with the trace form $\langle X, Y \rangle = \frac{1}{2} Tr(XY)$. As a vector space V, the set of $(n+1) \times (n+1)$ matrices with zero trace admits several kinds of Lie algebras and each of these Lie algebras induces its own Poisson structure on V. The most common Lie algebra is $\mathfrak{sl}(n+1)$ itself. Then $K = SO(n+1)$ induces the orthogonal decomposition $\mathfrak{sl}(n+1) = sym_0 \oplus \mathfrak{so}(n+1)$ where sym_0 denotes the vector space of symmetric matrices in V. But then V also carries the semi-direct product structure $sym_0 \rtimes \mathfrak{so}(n+1)$.

However, $K = SO(p,q)$, $p + q = n + 1$, is also a closed subgroup of G and hence the pair $(SL(n+1), SO(p,q))$ induces its own Cartan decomposition $V = \mathfrak{p} \otimes \mathfrak{k}$, where $\mathfrak{p}$ is the orthogonal complement to $\mathfrak{k} = \mathfrak{so}(p,q)$. In fact K is the set of points in G fixed by the automorphism $\sigma(g) = D{g^T}^{-1} D^{-1}$ where D denotes diagonal matrix with its first p diagonal entries equal to 1 and the remaining q diagonal entries equal to -1. The set of points $g \in G$ such that $\sigma(g) = g$ satisfies $D = gDg^T$, that is, $g \in SO(p,q)$. It follows that its tangent map σ_* induces the above decomposition with

$$\mathfrak{k} = \{X \in V : DXD^T = -X\}, \mathfrak{p} = \{X \in V : DX^T D = X\}. \tag{69}$$

Consequently, matrices in $\mathfrak{p}$ are symmetric relative to the scalar product $(x,y)_{p,q} = (x, Dy)$, x, y in R^{n+1}.

We will now return to the canonical Hamiltonians

$$H(L) = \frac{1}{2} \langle L_{\mathfrak{k}}, L_{\mathfrak{k}} \rangle + \langle A, L_{\mathfrak{p}} \rangle$$

and their Poisson Equation (33) restricted to the coadjoint orbits through rank one matrices X_0 in $\mathfrak{sl}(n+1)$. We will consider two cases: the coadjoint orbit through a symmetric rank-one matrix X_0 of unit length under the action of $G_1 = sym_0 \rtimes SO(n+1)$, and the second case, the coadjoint orbit through rank-one matrix X_0 of unit length, symmetric relative to the Lorentzian inner product in R^{n+1} under the action of $\mathfrak{p} \rtimes SO(1,n)$. The above matrices can be naturally expressed in terms of the notations introduced earlier, the scalar product $(x,y)_\epsilon = x_0 y_0 + \epsilon \sum_{i=1}^n x_i y_i$, $\epsilon = \pm 1$, in the ambient space $\mathbb{R}^{n+1}$, and matrices $a \otimes_\epsilon b$ and $a \wedge_\epsilon b = a \otimes_\epsilon b - b \otimes_\epsilon a$. For then

$$X_0 = x_0 \otimes_\epsilon x_0 - \frac{(x_0, x_0)_\epsilon}{n+1} I.$$

If $(x_0, x_0)_\epsilon > 0$ then let $S^n_\epsilon = \{x \in R^{n+1} : (x,x)_\epsilon = (x_0, x_0)_\epsilon, x_0.0\}$. It follows that S^n_ϵ is the Euclidean sphere of radius $||x_0||$ when $\epsilon = 1$ and a hyperboloid of two sheets when $\epsilon = -1$. We have chosen S^n_{-1} to be the sheet defined by $x_0 > 0$.

Proposition 5. *The coadjoint orbit through* $X_0 = x_0 \otimes_\epsilon x_0 - \frac{(x_0, x_0)_\epsilon}{n+1} I$ *is symplectomorphic to the cotangent bundle of the real projective space* $\mathbb{P}^{n+1}$ *in the semi-simple case, and it is symplectomorphic to the cotangent bundle of* S^n_ϵ *in the semi-direct case.*

For the proof see [36]. Here it is implicitly understood that the cotangent bundles are identified with the tangent bundles via the ambient inner product $(\ ,\ )_\epsilon$. Then each tangent vector (x,y), $x \in S^n_\epsilon$, $(x,y)_\epsilon = 0$ is identified with $L_{\mathfrak{p}} = x \otimes_\epsilon x - \frac{(x_0, x_0)_\epsilon}{n+1} I$ in $\mathfrak{p}_\epsilon$ and $L_{\mathfrak{k}} = x \wedge_\epsilon y$ in $\mathfrak{k}_\epsilon$.

On the orbit through X_0, $H = \frac{1}{2}(x,x)_\epsilon (y,y)_\epsilon - \frac{1}{2}(Ax, x)_\epsilon$, and the associated Poisson equations are of the form

$$\frac{d}{dt}(x \wedge_\epsilon y) = [A, x \otimes_\epsilon x], \frac{d}{dt}(x \otimes_\epsilon x) = [x \wedge_\epsilon y, x \otimes_\epsilon x] \tag{70}$$

A simple calculation show that

$$\dot{x} = (x,x)_\epsilon y, \dot{y} = Ax - (\frac{(Ax,x)_\epsilon}{(x,x)_\epsilon} + (y,y)_\epsilon)x. \tag{71}$$

On the unit sphere, Equation (71) after A is replaced by $-A$ coincide with the equations for the mechanical problem of C. Neumann for a particle on the sphere moving under a quadratic potential [37]. The preceding equations for $\epsilon = -1$ could be analogously interpreted as the equations on the hyperboloid for a particle moving under quadratic potential [7].

The canonical affine-quadratic problem illuminates deep and beautiful connections between Kepler's gravitational problem, Jacobi's geodesic problem on the ellipsoid, and Neumann's mechanical problems.

Let us first examine the isospectral integrals associated with the spectral curve $L_\lambda = L_\mathfrak{p} - \lambda L_\mathfrak{k} + \lambda^2 A$ on the coadjoint orbit through rank-one matrices. The zero trace requirement is inessential for the calculations below and will be disregarded. Additionally, A will be replaced by $-A$ and L_λ will be rescaled by dividing by $-\lambda^2$ to read

$$L_\lambda = -\frac{1}{\lambda^2}L_\mathfrak{p} + \frac{1}{\lambda}L_\mathfrak{k} + A. = -\frac{1}{\lambda^2}x \otimes_\epsilon x + \frac{1}{\lambda}x \wedge_\epsilon y + A.$$

The spectrum of L_λ is then given by

$$0 = Det(zI - L_\lambda) = Det(zI - A)Det(I - (zI - A)^{-1}(-\frac{1}{\lambda^2}L_\mathfrak{p} + \frac{1}{\lambda}L_\mathfrak{k})),$$

Matrix $M = I - (zI - A)^{-1}(-\frac{1}{\lambda^2}L_\mathfrak{p} + \frac{1}{\lambda}L_\mathfrak{k})$ is of the form .

$$M = I + \frac{1}{\lambda^2}R_z x \otimes_\epsilon x - \frac{1}{\lambda}(R_z x \otimes_\epsilon y - R_z y \otimes_\epsilon x),$$

where $R_z = (zI - A)^{-1}$. We then have the following proposition

Lemma 1. $Det(M) = \frac{1}{\lambda^2}((R_z x, x)_\epsilon + (R_z x, x)_\epsilon (R_z y, y)_\epsilon - (R_z x, y)_\epsilon^2) + 1.$

For the proof see [7] (p. 200).

Corollary 1. *Function* $F(z) = (R_z x, x)_\epsilon + (R_z x, x)_\epsilon (R_z y, y)_\epsilon - (R_z x, y)_\epsilon^2, z \in \mathbb{R}$ *is an integral of motion for* H.

Function F is a rational function with poles at the eigenvalues of the matrix A. Hence, $F(z)$ is an integral of motion for H if and only if the residues of F are constants of motion for H.

In the Euclidean case the eigenvalues of A are real and distinct since A is symmetric and regular. Hence there is no loss in generality in assuming that A is diagonal. Let $a_0, \ldots, a_n$ denote its diagonal entries Then

$$F(z) = \sum_{k=0}^{n} \frac{F_k}{z - a_k},$$

where $F_0, \ldots, F_n$ denote the residues of F. It follows that

$$F(z) = \textstyle\sum_{k=0}^{n} \frac{x_k^2}{z-a_k} + \sum_{k=0}^{n}\sum_{j=0}^{n} \frac{x_k^2 y_j^2}{(z-a_k)(z-a_j)} - (\sum_{k=0}^{n} \frac{x_k y_k}{z-a_k})^2 =$$

$$\textstyle\sum_{k=0}^{n} \frac{x_k^2}{z-a_k} + \sum_{k=0}^{n}\sum_{j=0, j\neq k}^{n} \frac{x_k^2 y_j^2}{(z-a_k)(z-a_j)} - 2\sum_{k=0}^{n}\sum_{j=0, j\neq k}^{n} \frac{x_k y_k x_j y_j}{(z-a_k)(z-a_j)}.$$

Hence,

$$F_k = \lim_{z\to a_k}(z-a_k)F(z) =$$
$$x_k^2 + \sum_{j=0,j\neq k}^n \frac{x_j^2 y_k + x_k^2 y_j^2}{(a_k-a_j)} - 2\sum_{j=0,j\neq k}^n \frac{x_k y_k x_j y_j}{(a_k-a_j)} =$$
$$x_k^2 + \sum_{j=0,j\neq k}^n \frac{(x_j y_k - x_k y_j)^2}{(a_k-a_j)}, k = 0,\dots,n.$$

The preceding calculation yields the following proposition.

Proposition 6. *Each residue* $F_k = x_k^2 + \sum_{j=0,j\neq k}^n \frac{(x_j y_k - x_k y_j)^2}{(a_k-a_j)}, k = 0,\dots,n$ *is an integral of motion for Newmann's spherical system, and functions* $F_0,\dots,F_n$ *are in involution.*

These results coincide with the ones reported in [38–40], but the connection with the affine-quadratic problem shows that similar integrals of motion exist for the hyperbolic Neumann problem [7] (p. 191).

In the literature on integrable systems the integrals of motion for Neumann's problem are related to the integrals of motion for Jacobi's problem on the ellipsoid through the transformation of H. Knörrer that transforms the Neumann's equations on energy level $H = 0$ onto the equations of Jacobi on the ellipsoid [39,41]. Our exposition takes another route: we will instead show that an "elliptic" problem on the sphere is completely integrable with its integrals of motion as in Neumann's problem, and then we will show that the Hamiltonian equations for the elliptic problem on the sphere and Jacobi's problem on the ellipsoid are symplectomorphic. We will then use this symplectomorphism to show the existence of Jacobi's integrals of motion on the ellipsoid.

Let now $\mathcal{H} = \frac{1}{2}\langle D^{-1}(L_\mathfrak{k})D^{-1}, L_\mathfrak{k}\rangle + \langle D^{-1}, L_\mathfrak{p}\rangle$ denote an affine-quadratic Hamiltonian on $G = SL(n+1)$ defined by a diagonal matrix D with positive diagonal entries. As before, we will dispense with zero-trace requirements since they are inessential. The above Hamiltonian is generated by a positive definite operator $\mathcal{P}(X) = DXD, X \in \mathfrak{so}(n+1)$ and the drift $A = D^{-1}$. We will call this Hamiltonian elliptic for reasons that will be made clear later on. Let $B = -D$. Then

$$[\mathcal{P}^{-1}(L_\mathfrak{k}), B] = [\mathcal{P}^{-1}(L_\mathfrak{k}), -D] = -L_\mathfrak{k}D^{-1} + D^{-1}L_\mathfrak{k} = [L_\mathfrak{k}, D^{-1}] = [L_\mathfrak{k}, A].$$

Therefore $\mathcal{H}$ is isospectral (Proposition 1) and its Hamiltonian equations admit a representation

$$\frac{dL_\lambda}{dt} = [M_\lambda, L_\lambda], L_\lambda = L_\mathfrak{p} - \lambda L_\mathfrak{k} - (\lambda^2 - s)A.$$

Since $L_\lambda = L_\mathfrak{p} - \lambda L_\mathfrak{k} - (\lambda^2 - s)A$ is a spectral curve for the canonical affine Hamiltonian $H = \frac{1}{2}\langle L_\mathfrak{k}, L_\mathfrak{k}\rangle - \langle A, L_\mathfrak{p}\rangle$ we have the following corollary.

Corollary 2. *The spectral invariants of* $L_\mathfrak{p} - \lambda L_\mathfrak{k} - (\lambda^2 - s)A$ *are common integrals of motion for both the canonical Hamiltonian* $H = \frac{1}{2}\langle L_\mathfrak{k}, L_\mathfrak{k}\rangle - \langle A, L_\mathfrak{p}\rangle$ *and the elliptic Hamiltonian* $H = \frac{1}{2}\langle A^{-1}L_\mathfrak{k}A^{-1}, L_\mathfrak{k}\rangle + \langle A^{-1}, L_\mathfrak{p}\rangle$.

As before, on coadjoint orbit through $X_0 = x_0 \otimes_\epsilon x_0, (x_0, x_0)_\epsilon = 1$, the Poisson equations of $\mathcal{H}$ (the semi-direct version) are given by

$$\begin{aligned} \frac{d}{dt}(x \wedge_\epsilon y) &= [A^{-1}(x \wedge_\epsilon y)A^{-1}, x \wedge_\epsilon y] + [A^{-1}, x \otimes_\epsilon x], \\ \frac{d}{dt}(x \otimes_\epsilon x) &= [A^{-1}(x \wedge_\epsilon y)A^{-1}), x \otimes_\epsilon x]. \end{aligned} \tag{72}$$

We then have

$$\langle A^{-1}L_\mathfrak{k}A^{-1}, L_\mathfrak{k}\rangle = \langle A^{-1}x \wedge yA^{-1}, x \wedge y\rangle = (A^{-1}x \cdot x)(A^{-1}y \cdot y) - (A^{-1}x \cdot y)^2,$$
$$\langle A^{-1}, L_\mathfrak{p}\rangle = -\frac{1}{2}(x \cdot A^{-1}x).$$

which shows that the Hamiltonian $H = \frac{1}{2}\langle A^{-1}L_{\mathfrak{k}}A^{-1}, L_{\mathfrak{k}}\rangle + \langle A^{-1}, L_{\mathfrak{p}}\rangle$ is given by

$$H = \frac{1}{2}((A^{-1}y \cdot y) - \frac{(A^{-1}x \cdot y)^2}{(A^{-1}x \cdot x)} - 1)(A^{-1}x \cdot x).$$

The correspondence $(x, y) \to x \wedge_\epsilon y + x \otimes_\epsilon x$ defines a symplectomorphism between the cotangent bundle of S^n_ϵ with its canonical Poisson bracket and the coadjoint orbit through X_0 (Proposition 5).

Proposition 7. *On energy level $H = 0$ Equation (72) correspond to*

$$\frac{dx}{dt} = (A^{-1}x \cdot x)(A^{-1}y - \frac{(A^{-1}x \cdot y)}{(A^{-1}x \cdot x)}A^{-1}x)$$
$$\frac{dy}{dt} = (A^{-1}x \cdot x)(\frac{(A^{-1}x \cdot y)}{(A^{-1}x \cdot x)}A^{-1}y - \frac{(A^{-1}x \cdot y)^2}{(A^{-1}x \cdot x)^2}A^{-1}x - x.)$$

under the correspondence $(x \otimes x, x \wedge y) \to (x, y)$.

These equations can be reparametrized by a parameter $s = \int (A^{-1}x(t) \cdot x(t))\, dt$ to read

$$\begin{aligned} \frac{dx}{ds} &= \frac{dx}{dt}\frac{dt}{ds} = A^{-1}y - \frac{(A^{-1}x \cdot y)}{(A^{-1}x \cdot x)}A^{-1}x \\ \frac{dy}{ds} &= \frac{dy}{dt}\frac{dt}{ds} = \frac{(A^{-1}x \cdot y)}{(A^{-1}x \cdot x)}(A^{-1}y - \frac{(A^{-1}x \cdot y)}{(A^{-1}x \cdot x)}A^{-1}x) - x. \end{aligned} \tag{73}$$

We will presently show that Equation (73) are Hamiltonian equations that correspond to the geodesic problem on the sphere relative to the elliptic metric $\frac{1}{2}(A\dot{x}, \dot{x})$.

As an intermediate step we will now derive the Hamiltonian equations associated with the geodesic problem on the quadric surface $(A^{-1}x, x) = 1$ induced by the scalar product $\frac{1}{2}(D\dot{x}, \dot{x})$. We will follow the procedure based on the version of the Maximum Principle for variational problems with constraints outlined in [7] (p. 218) and identify the quadric surface with the submanifold $N = \{x \in R^{n+1} : (x, A^{-1}x) = 1\}$. Then its cotangent bundle will be defined in terms of the constraints

$$G_1 = \{(x, A^{-1}x) - 1 = 0\}, \text{ and } G_2 = \{(x, A^{-1}p) = 0\}.$$

The Hamiltonian lift of a curve $\dot{x} = u(t)$ that belongs to $T_{x(t)}N$ is given by

$$h_{u(t)}(x, p) = -\frac{1}{2}(Du, u) + (p, u) + \lambda_1 G_1 + \lambda_2 G_2$$

for the multipliers λ_1 and λ_2 that satisfy $\{h_{u(t)}, G_1\} = \{h_{u(t)}, G_2\} = 0$. If $h^0_u = -\frac{1}{2}(Du, u) + (p, u)$, then

$$\{h_u, G_1\} = \{h^0_u, G_1\} + \lambda_2\{G_2, G_1\},\ \{h_u, G_2\} = \{h^0_u, G_2\} + \lambda_1\{G_1, G_2\}.$$

It follows that

$$\{h^0_u, G_1\} = -2u \cdot A^{-1}x,\ \{h^0_u, G_2\} = -u \cdot A^{-1}p,\ \{G_1, G_2\} = 2A^{-1}x \cdot A^{-1}x.$$

Hence,

$$\lambda_1 = -\frac{1}{\{G_1, G_2\}}\{h^0_u, G_2\} = \frac{1}{2}\frac{(u, A^{-1}p)}{(A^{-1}x, A^{-1}x)},\ \lambda_2 = -\frac{1}{\{G_2, G_1\}}\{h^0_u, G_1\} = -\frac{(u, A^{-1}x)}{(A^{-1}x, A^{-1}x)}.$$

According to the Maximum Principle an extremal control $u(t)$ must optimize

$$h^0_u = -\frac{1}{2}(u \cdot Du) + p \cdot u$$

on $G_1 = G_2 = 0$ over all controls that satisfy $u \cdot A^{-1}x = 0$. Hence extremal controls are the critical points of $-\frac{1}{2}(u, Du) + (p, u) - \alpha_0(u, A^{-1}x)$ for some multiplier α_0, that is, they are solutions of $-Du + p - \alpha_0 A^{-1}x = 0$. It follows that the extremal controls are of the form $u = D^{-1}(p - \alpha_0 A^{-1}x)$, But then $(u, A^{-1}x) = 0$ implies that $\alpha_0 = \frac{(A^{-1}x, D^{-1}p)}{(D^{-1}A^{-1}x, A^{-1}x)}$. For this choice of controls

$$h_u = H + \lambda_1 G_1 + \lambda_2 G_2,$$

where $H = \frac{1}{2}(D^{-1}(p - \alpha_0 A^{-1}x), p))$. An easy calculation shows that

$$H = \frac{1}{2}((D^{-1}p, p) - \frac{(D^{-1}p, A^{-1}x)^2}{(A^{-1}x, A^{-1}x)}).$$

Then the extremal curves are the solutions of the following differential equation:

$$\begin{aligned} \frac{dx}{dt} &= \frac{\partial H}{\partial p} = D^{-1}(p - \alpha_0 A^{-1}x), \\ \frac{dp}{dt} &= -\frac{\partial H}{\partial x} - 2\lambda_1 \frac{\partial G_1}{\partial x} = \alpha_0(A^{-1}(D^{-1}(p - \alpha_0 A^{-1}x)) - 2\lambda_1 A^{-1}x, \end{aligned} \tag{74}$$

which emanate from $H = \frac{1}{2}$, that is, satisfy

$$(D^{-1}p, p)(D^{-1}A^{-1}x, A^{-1}x) - (D^{-1}p, A^{-1}x)^2 = (D^{-1}A^{-1}x, A^{-1}x). \tag{75}$$

We will now single out the cases relevant for our earlier claims.

The geodesic problem on the ellipsoid. In this classic case initiated by C. Jacobi $D = I$ and $(A^{-1}x, p) = 0$. Hence

$$\alpha_0 = \frac{(A^{-1}x, p)}{(A^{-1}x, A^{-1}x)} = 0, \lambda_1 = \frac{1}{2}\frac{(p, A^{-1}p)}{(A^{-1}x, A^{-1}x)}.$$

Then Equation (74) reduce to

$$\frac{dx}{dt} = p, \frac{dp}{dt} = -\frac{(p, A^{-1}p)}{(A^{-1}x, A^{-1}x)}A^{-1}x. \tag{76}$$

The preceding equation agree with the equations in J. Moser [39].

The elliptic problem on the sphere. Here the ambient metric is defined by a positive-definite matrix D and $A = I$. In such a case Equation (74) are given by

$$(D^{-1}p, p) - \frac{(D^{-1}p, x)^2}{(D^{-1}x, x)} = 1.$$

Furthermore,

$$\lambda_1 = \frac{1}{2}D^{-1}(p - \alpha_0 x), p) = \frac{1}{2}(D^{-1}p, p) - \frac{(x, D^{-1}p)^2}{(D^{-1}x, x)}) = \frac{1}{2}, \alpha_0 = \frac{(D^{-1}x, p)}{(D^{-1}x, x)}.$$

The Hamiltonian equations are then given by

$$\begin{aligned} \frac{dx}{dt} &= D^{-1}p - \frac{(D^{-1}x, p)}{(D^{-1}x, x)}D^{-1}x, \\ \frac{dp}{dt} &= \frac{(D^{-1}x, p)}{(D^{-1}x, x)}(D^{-1}p - \frac{(D^{-1}x, p)}{(D^{-1}x, x)}D^{-1}x)) - x, \end{aligned} \tag{77}$$

which agrees with Equation (73) when $D = A$.

Proposition 8. *The Hamiltonian systems that correspond to the elliptic problem on the sphere and the geodesic problem on the ellipsoid are symplectomorphic.*

Proof. Let (x, y) denote the coordinates on the tangent bundle of the sphere and let (q, p) denote the coordinates on the tangent bundle of the ellipsoid $E = \{q \in R^{n+1} : (q, A^{-1}q) - 1 = 0\}$. In these coordinates the systems in question are given by

$$\frac{dx}{dt} = u, \ \frac{dy}{dt} = \alpha u - x, \text{ and } \frac{dq}{dt} = p, \ \frac{dp}{dt} = -\frac{(A^{-1}p, p)}{(A^{-1}q, A^{-1}q)} A^{-1}q, \tag{78}$$

where $u = A^{-1}(y - \alpha x)$ and $\alpha = \frac{(A^{-1}x,y)}{(A^{-1}x,A^{-1}x)}$.

Let Φ denote the mapping from the cotangent bundle of the sphere to the cotangent bundle of E defined by

$$q = A^{\frac{1}{2}}x, \ p = A^{-\frac{1}{2}}(y - \alpha x) = A^{\frac{1}{2}}u.$$

Let $\theta = \sum_{i=0}^{n} p_i dq_i = (p, dq), (dq, A^{-1}q) = 0$, denote the Liouville-Poincaré canonical form on T^*E. Then

$$\Phi^*\theta = (A^{-\frac{1}{2}}(y - \alpha x), A^{\frac{1}{2}}dx) = (y, dx) - \alpha(x, dx) = (y, dx),$$

because $0 = dq \cdot A^{-1}q = A^{\frac{1}{2}}dx \cdot A^{-1}A^{\frac{1}{2}}x = dx \cdot x$. Since Φ^* takes the Liouville form on T^*E to the Liouville form on T^*S^n, it also takes the canonical symplectic form on T^*E to the canonical symplectic for on T^*S_n and hence is a symplectomorphism.

It now follows from (78) that $\frac{du}{dt} = -(1 + \frac{d\alpha}{dt})A^{-1}x$ and that $1 + \frac{d\alpha}{dt} = \frac{(u,u)}{(A^{-1}x,x)}$. Then,

$$\frac{dq}{dt} = A^{\frac{1}{2}}\frac{dx}{dt} = A^{\frac{1}{2}}u = p,$$
$$\frac{dp}{dt} = A^{\frac{1}{2}}\frac{du}{dt} = -(\frac{(u,u)}{(A^{-1}x,x)})A^{-1}q = -\frac{(A^{-1}p,p)}{(A^{-1}q,A^{-1}q)}A^{-1}q,$$

and thus Φ_* takes the Hamiltonian flow on the sphere onto the Hamiltonian flow on the ellipsoid. □

Proposition 9. *Jacobi's problem on the ellipsoid is completely integrable. Functions*

$$G_k = p_k^2 + \sum_{j=1, j\neq k}^{n+1} \frac{(q_j p_k - q_k p_j)^2}{(a_k - a_j)}, k = 1, \dots, (n+1)$$

are constants of motion, all in involution with each other, for the Hamiltonian system $\frac{dq}{dt} = p$, $\frac{dp}{dt} = -\frac{(A^{-1}p,p)}{(A^{-1}q,A^{-1}q)}A^{-1}q$ *on the cotangent bundle of the ellipsoid.*

Proof. We have shown that

$$F_k = x_k^2 + \sum_{j=0, j\neq k}^{n} \frac{(x_j y_k - x_k y_j)^2}{(\alpha_k - \alpha_j)}, k = 0, \dots, n \tag{79}$$

are an involutive family of integrals of motion for the elliptic-geodesic problem on the sphere. We have also shown that the above integrals of motion are the residues of the function

$$F(z) = (R_z x, x) + (R_z x, x)(R_z y, y) - (R_z x, y)^2, R_z = (zI - A)^{-1}.$$

We will now show that functions (79) are the residues of the pull-back of F under the symplectomorphism Φ. First note that F remains unchanged if the variable y is replaced by $y + \alpha x$ with α an arbitrary number. Since $A^{\frac{1}{2}}p = y - \frac{(A^{-1}x,y)}{(A^{-1}x,x)}x$, we may replace y by

$A^{\frac{1}{2}}p$ and x by $A^{-\frac{1}{2}}q$. Also note that $(p,p) = (A^{-1}y,y) - \frac{(A^{-1}x,y)^2}{(A^{-1}x,x)} = 1$ (use Equation (75)). Then,

$$1 + (R_z y, y) = 1 + (R_z A p, p) = 1 + \sum_{k=0}^{n} \frac{a_k p_k^2}{z-a_k} =$$
$$\sum_{k=0}^{n} p_k^2 + \frac{a_k p_k^2}{z-a_k} = z\sum_{k=0}^{n} \frac{p_k^2}{z-a_k} = z(R_z p, p),$$
$$(R_z x, x) = (R_z A^{-1} q, q) = \sum_{k=0}^{n} \frac{q_k^2}{a_k(z-a_k)} = \frac{1}{z}\sum_{k=0}^{n} \frac{q_k}{a_k} + \frac{q_k^2}{z-a_k} = \frac{1}{z}(1 + (R_z q, q),$$
$$\text{and } (R_z x, y) = (R_z q, p).$$

It follows that

$$F(z) = (R_z p, p)(1 + R_z q, q) - (R_z q, p)^2$$

is constant along the solutions of Jacobi's equations.. A calculation identical to the one used for Neumann's system shows that

$$G_k = p_k^2 + \sum_{j=0, j\neq k}^{n} \frac{(q_j p_k - q_k p_j)^2}{(a_k - a_j)}, k = 1, \ldots, (n+1)$$

are the residues of F, and hence are integrals of motion for Jacobi's equations. □

Degenerate Case $A = 0$ and Kepler's Problem

Let us now return to the Hamiltonian equations generated by the canonical affine Hamiltonian $H = \frac{1}{2}(x,x)_\epsilon (y,y)_\epsilon - \frac{1}{2}(Ax,x)_\epsilon$ on the coadjoint orbit through $X_0 = a \otimes_\epsilon a$ for some $a \in R^{n+1}$ with

$$\frac{d}{dt}(x \wedge_\epsilon y) = [A, x \otimes_\epsilon x], \frac{d}{dt}(x \otimes_\epsilon x) = [x \wedge_\epsilon y, x \otimes_\epsilon x], \tag{80}$$

and their equivalent formulation on the tangent bundle of S_ϵ^n:

$$\dot{x} = (x,x)_\epsilon y, \dot{y} = Ax - (\frac{Ax,x)_\epsilon}{(x,x)_\epsilon} + (y,y)_\epsilon)x. \tag{81}$$

When $A = 0$ the Hamiltonian H reduces to $H = \frac{1}{2}(x,x)_\epsilon (y,y)_\epsilon$ and the corresponding equations reduce to

$$\frac{d}{dt}(x \wedge_\epsilon y) = 0, \frac{d}{dt}(x \otimes_\epsilon x) = [x \wedge_\epsilon y, x \otimes_\epsilon x]. \tag{82}$$

Then Equation (82) yield an integral of motion $x \wedge_\epsilon y = const$, and Equation (81) reduce to

$$\dot{x} = ||x||_\epsilon^2 y,\ \dot{y} = -||y||_\epsilon^2 x,\ \text{where } ||x||_\epsilon^2 = (x,x)_\epsilon, ||y||_\epsilon^2 = (y,y)_\epsilon.$$

Upon differentiating we get

$$\ddot{x} + ||x||_\epsilon^2 ||y||_\epsilon^2 x = 0. \tag{83}$$

We will now assume that $(a,a)_\epsilon = h^2$ so that S_{-1}^n is the hyperboloid $x_0^2 = h^2 + \sum_{i=1}^n x_i^2, x_0 \geq 0$. On energy level $H = \frac{\epsilon}{2h^2}$, $||x||_\epsilon^2 ||y||_\epsilon^2 = \frac{\epsilon}{h^2}$ the solutions of (83) are given by $x(t) = c_1 \cos\frac{t}{h}\sqrt{\epsilon} + c_2 \sin\frac{t}{h}\sqrt{\epsilon}$ where c_1 and c_2 are constant vectors (complex when $\epsilon = -1$) that satisfy $||c||_\epsilon^2 = ||c||_\epsilon^2 = h^2$, $(c_1, c_2)_\epsilon = 0$.

For $\epsilon = 1$ the above curves trace great circles on the sphere $||x||^2 = ||x_0||^2$ and for $\epsilon = -1$ the solutions trace great hyperbolas on the hyperboloid $||x||_{-1}^2 = ||x_0||_{-1}^2$ (an immediate consequence of the fact that $x(t) \wedge_\epsilon \dot{x}(t) = a \wedge_\epsilon b$). That is, solutions are the geodesics on spaces of constant non-zero curvature. The zero curvature case may be obtained by considering ϵ as a continuous parameter and then letting it tend to zero (as will be explained below).

We will now show that there exists a canonical change of coordinates $\{(x_0, \ldots, x_n, y_0, \ldots, y_n), (x,x)_\epsilon = h^2, (x,y)_\epsilon = 0\} \to (p_1, \ldots, p_n, q_1, \ldots, q_n)$ in which p is the stereographic projection through the point $x_0 = he_0$ given by

$$\lambda(x - he_0) + he_0 = (0, p) \text{ with } \lambda = \frac{h}{h - x_0}, x \in S^n_\epsilon \tag{84}$$

such that in the new coordinates the preceding geodesic differential system is transformed into the n-dimensional Kepler's system, an n-dimensional generalization of the Hamiltonian equations that describe the motion of a planet around an immovable planet in the presence of the gravitational force.

Equation (84) yields $p = \frac{h}{h-x_0}\bar{x}$, where $\bar{x} = (x_1, \ldots, x_n)$. Therefore the inverse map $x = \Phi_\epsilon(p)$ is given by

$$x_0 = \frac{h(||p||^2 - \varepsilon h^2)}{||p||^2 + \varepsilon h^2},\ \bar{x} = \frac{2\epsilon h^2}{||p||^2 + \varepsilon h^2} p. \tag{85}$$

Assume that the cotangent bundle of $\mathbb{R}^n$ is identified with its tangent bundle $\mathbb{R}^n \times \mathbb{R}^n$ via the Euclidean inner product $(\,\cdot\,)$, and let (p, q) denote the points of $\mathbb{R}^n \times \mathbb{R}^n$. We will next find $q = \Psi(x, y)$ such that $(dx, y)_\epsilon = (dp \cdot \Psi(x, y))$, for all (x, y) with $x \in S^n_\epsilon(h)$ and $(x, y)_\epsilon = 0$. For then the transformation $(p, q) \in \mathbb{R}^n \times \mathbb{R}^n \to (x, y) \in TS^n_\epsilon(h)$ is a symplectomorphism since it pulls back the Liouville form $(dx, y)_\epsilon$ on $TS^n_\epsilon(h)$ onto the Liouville form $(dp \cdot q)$ in $\mathbb{R}^n \times \mathbb{R}^n$ (The symplectic form is the exterior derivative of the Liouville-Poincaré form). It follows that

$$\begin{gathered}(y, dx)_\epsilon = \textstyle\sum_{j=1}^n y_0 \frac{\partial x_0}{\partial p_j} dp_j + \epsilon \sum_{i=1}^n \sum_{j=1}^n y_i \frac{\partial x_i}{\partial p_j} dp_j = \\ \textstyle\sum_{j=1}^n (y_0 \frac{\partial x_0}{\partial p_j} + \epsilon \sum_{i=1}^n y_i \frac{\partial x_i}{\partial p_j}) dp_j = \sum_{j=1}^n q_j dp_j = (dp \cdot q).\end{gathered}$$

Therefore,

$$q_j = \frac{\partial x_0}{\partial p_j} y_0 + \epsilon \sum_{i=1}^n y_i \frac{\partial x_i}{\partial p_j}, j = 1, \ldots, n.$$

After the appropriate differentiations in (85) we get

$$q = \frac{2h^2}{||p||^2 + \epsilon h^2}\left(\frac{2\epsilon h y_0}{||p||^2 + \epsilon h^2} p + \bar{y} - \frac{2(\bar{y} \cdot p)}{||p||^2 + \epsilon h^2} p\right), \bar{y} = (y_1, \ldots, y_n).$$

Hence,

$$\frac{||p||^2 + \epsilon h^2}{2h^2}(q \cdot p) = \frac{2\epsilon h y_0}{||p||^2|| + \epsilon h^2}||p||^2 - \frac{||p||^2 - \epsilon h^2}{||p||^2 + \epsilon h^2}(\bar{y} \cdot p).$$

Since y is orthogonal to x, $(\bar{y} \cdot p) = -\frac{y_0}{2h}(||p||^2 - \epsilon h^2)$. Therefore,

$$y_0 = \frac{1}{h} q \cdot p, \bar{y} = \frac{||p||^2 + \varepsilon h^2}{2h^2} q - \frac{q \cdot p}{h^2} p. \tag{86}$$

After the substitutions $||p||^2 = \frac{h^2}{h-x_0}||\bar{x}||^2$ into the preceding equation we get

$$q = \frac{2(h - x_0)}{||\bar{x}||^2 + (h - x_0)}((h - x_0)^2 \bar{y} + y_0 \bar{x}).$$

To pass to the problem of Kepler, write the Hamiltonian $H = \frac{1}{2}||x||^2_\varepsilon ||y||^2_\varepsilon$ in the variables (p, q). An easy calculation in (86) yields $(y, y)_\epsilon = \varepsilon \frac{(||p||^2 + \varepsilon h^2)^2}{4h^4} ||q||^2$. Therefore,

$$H = \frac{1}{2} h^2 \varepsilon \frac{(||p||^2 + \varepsilon h^2)^2}{4h^4} ||q||^2 = \frac{1}{2} \varepsilon \frac{(||p||^2 + \varepsilon h^2)^2}{4h^2} ||q||^2.$$

The corresponding flow is given by

$$\frac{dp}{ds} = \frac{\partial H}{\partial q} = \varepsilon \frac{(||p||^2 + \varepsilon h^2)^2}{4h^2} q, \; \frac{dq}{ds} = -\frac{\partial H}{\partial p} = -\varepsilon \frac{||p||^2 + \varepsilon h^2}{2h^2} ||q||^2 p$$

On energy level $H = \frac{\varepsilon}{2h^2}, \frac{(||p||^2 + \varepsilon h^2)^2}{4} ||q||^2 = 1$, and the preceding equations reduce to

$$\frac{dp}{ds} = \varepsilon \frac{q}{h^2 ||q||^2}, \; \frac{dq}{ds} = -\varepsilon \frac{||q||}{h^2} p.$$

After the reparametrization $t = -\frac{-\varepsilon}{h^2} \int_0^s ||q(\tau)|| d\tau$ Equation (23) become

$$\frac{dp}{dt} = \frac{dp}{ds} \frac{ds}{dt} = -\frac{q}{||q||^3}, \; \frac{dq}{dt} = \frac{dq}{ds} \frac{ds}{dt} = p.$$

On $H = \frac{\epsilon}{h^2}, \frac{(||p||^2 + \varepsilon h^2)^2}{4} ||q||^2 = 1$ and

$$E = \frac{1}{2} ||p||^2 - \frac{1}{||q||} = \frac{1}{2||q||} (||p||^2 ||q|| - 2) = \frac{1}{2||q||} (2 - \varepsilon h^2 ||q|| - 2) = -\frac{1}{2} \varepsilon h^2.$$

So $E < 0$ in the spherical case and $E > 0$ in the hyperbolic case.

The Euclidean case $E = 0$ can be obtained by a limiting argument in which ε is regarded as a continuous parameter which tends to zero.

To explain in more detail, let

$$w_0 = \lim_{\epsilon \to 0} = h, \; w = \lim_{\epsilon \to 0} \frac{1}{2\epsilon h^2} \bar{x} = \frac{1}{||p||^2}.$$

The transformation $p \to w$ with $w = \frac{1}{||p||^2} p$ is the inversion about the circle $||p||^2 = 1$ in the affine hyperplane $w_0 = h$, and $||dw||^2 = \frac{1}{||p||^4} ||dp||^2$ is the corresponding transformation of the Euclidean metric $||dp||^2$. The Hamiltonian H_0 associated with this metric is equal to $\frac{1}{2} \frac{||p||^4}{4} ||q||^2$. This Hamiltonian can be also obtained as the limit of $(\frac{h^2}{\epsilon}) \frac{1}{2} \frac{(||p||^2 + \epsilon h^2)^2}{4h^2} ||q||^2$ when $\epsilon \to 0$. On energy level $H = \frac{1}{2}$, $||p||^2 ||q|| = 2$ and therefore, $E = 0$. Of course, the solutions of (12) tend to the Euclidean geodesics as ϵ tends to zero. Consequently, $w(t) = \lim_{\epsilon \to 0} \frac{1}{2h^2 \varepsilon} (\bar{x}(t))$ is a solution of $\frac{d^2 w}{dt^2} = 0$, and hence, is a geodesic corresponding to the standard Euclidean metric.

Let us also note that the angular momentum $L = q \wedge p$ and the Laplace-Runge-Lenz vector $F = Lp - \frac{q}{||q||}$ for Kepler's problem have simple geometric interpretation on the coadjoint orbits according to the following proposition.

Proposition 10. *Let $x = x_0 e_0 + \bar{x}$ and $y = y_0 e_0 + \bar{y}$. On energy level $H = \frac{\epsilon}{2h^2}$,*

$$L = (\bar{y} \wedge_\epsilon \bar{x}) \text{ and } F = h(y_0 (e_0 \wedge \bar{x})_\varepsilon - x_{0(} e_0 \wedge \bar{y})_\varepsilon) e_0.$$

For a proof see [7].

This remarkable discovery that the solutions of Kepler's problem are intimately related to the geometry of spaces of constant curvature goes back to A.V. Fock's paper of 1935 [42] in which he reported that the symmetry group for the motions of the hydrogen atom is

$O_4(R)$ for negative energy, $E^3 \rtimes O_3(R)$ for zero energy and $O(1,3)$ for positive energy. It is then not altogether surprising that similar results apply to the problem of Kepler since the energy function for Kepler's problem is formally the same as the energy function for the hydrogen atom.

This connection between the problem of Kepler and the geodesics on the sphere was reported by J. Moser in 1970 [43], while Y. Osipov [44] reported similar results later for geodesics on spaces of negative constant curvature. In spite of their brilliance, these papers did not attempt any explanations in regard to this enigmatic connection between planetary motions and geodesics on space forms. This issue later inspired V. Guillemin and S. Sternberg to take up the problem of Kepler in a larger geometric context, with Moser's observation as the background, in a paper titled *Variations on a theme by Kepler* [45]. The introduction of Kepler's problem through the canonical affine-quadratic problem exemplifies, once again, this fascinating and recurrent interplay between mathematical physics, geometry and integrable systems.

5. Homogeneous Riemannian Manifolds and Rolling Geodesics

Our overview of integrable systems raises a natural question: what is the geometric origin behind the affine-quadratic problem that accounts for its ubiquitous presence in the theory of integrable systems? A partial answer to this question comes, somewhat unexpectedly, from a new class of variational problems, called rolling problems. We will take up this issue next. Since the underlying variational problems require new concepts and terminology, we will be obliged to make a slight detour into an earlier paper [9] in order to introduce the necessary ingredients.

The general setting is defined by a reductive pair (G, K) with G semi-simple and K compact. We assume that the Lie algebra decomposition $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$, with $\mathfrak{p}$ the orthogonal complement of $\mathfrak{k}$ relative to the Killing form on $\mathfrak{g}$ satisfies the strong Cartan conditions

$$[\mathfrak{p}, \mathfrak{k}] = \mathfrak{p}, [\mathfrak{p}, \mathfrak{p}] = \mathfrak{k}, [\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}. \tag{87}$$

We will also assume the Killing form is of definite sign on $\mathfrak{p}$ in which case $\langle\, , \,\rangle$ will denote a scalar multiple of the Killing form that is positive on $\mathfrak{p}$. We recall that the Killing form is invariant under any linear automorphism of $\mathfrak{g}$ and hence the quadratic form $\langle\, , \,\rangle$ is Ad_G invariant [15].

We consider G a semi-Riemannian manifold relative to the left-invariant metric $\langle\langle gX, gY\rangle\rangle_g = \langle X, Y\rangle, X, Y \in \mathfrak{g}$ induced by $\langle\, , \,\rangle$ (the Killing form is not necessarily positive on $\mathfrak{g}$, hence the metric is in general of indefinite sign, i.e., it is semi-Riemannian [46]). The left-invariant distributions $\mathcal{D}(g) = \{gX : X \in \mathfrak{p}\}$ and $\mathcal{V}(g) = \{gX) : X \in \mathfrak{k}\}$ are called horizontal and vertical respectively. Then curves that are tangent to $\mathcal{D}$, i.e., satisfy $\frac{dg}{dt} \in \mathcal{D}(g(t))$ are called horizontal. Likewise curves that are tangent to $\mathcal{V}$ are called vertical. It follows that

$$\mathcal{D}(g) \oplus \mathcal{V}(g) = T_g G, g \in G. \tag{88}$$

We will assume that $M = G/K$ consisting of the left coset gK is endowed with a manifold structure so that the natural projection $\pi(g) = gK$ is a smooth surjection [46]. A curve $g(t)$ in G is called a lift of a curve $p(t) \in M$ if $\pi(g(t)) = p(t)$. A lift is called horizontal when $g(t)$ is a horizontal curve. Every curve $p(t)$ in M is the projection of a horizontal curve $g(t)$. If a curve $g(t)$ is a solution of $\frac{dg}{dt} = g(t)U(t)$ for some curve $U(t) \in \mathfrak{p}$ then $d_{g(t)}\pi(g(t)U(t)) = \frac{dp}{dt}$. The correspondence $\mathcal{D}(g) \to T_{\pi(g)}M$ given by $d_g\pi(gU) = \frac{dp}{dt}$ is an isomorphism and induces a metric on M

$$(d_g\pi(gV), d_g\pi(gW)_{\pi(g)} = \langle gV, gW\rangle\rangle_g = \langle V, W\rangle, V, W \in \mathfrak{p}. \tag{89}$$

Let now $\{\tau_g : g \in G\}$ denote the group of diffeomorphisms defined by the left action

$$\pi(L_g(h)) = \tau_g(\pi(h)), h \in G, L_g(h) = gh.$$

We then have

Proposition 11. *The metric* (89) *is invariant under* $\{\tau_g : g \in G\}$, *that is,*

$$(d_o\tau_g(V(p), d_o\tau_g(W(p))_{\tau_g(p)} = (V(p), W(p))_p, \tag{90}$$

for any $g \in G$ *and any tangent vectors* $V(p)$ *and* $W(p)$ *in* T_pM.

For a proof see [47].

It follows that each τ_g is an isometry. Since G acts transitively on M, M can be represented by the orbit $\{\tau_g(o) : g \in G\}$ where $o = \pi(e)$ and e is the group identity in G. It follows that $\pi((\exp tU)g) = \tau_{\exp tU}(\pi(g))$ for any $U \in \mathfrak{g}$. Note that $g \to (\exp tU)g$ is the flow generated by a right-invariant vector field $U_r(g) = Ug$. Therefore the flow of U_r is π-related to the flow $\{\tau_{\exp tU} : t \in R\}$ in M. We will let $\vec{U}$ denote the infinitesimal generator of the flow $\{\tau_{\exp tU} : t \in R\}$.

It follows that each $\vec{U}$ is a Killing vector field on M. A vector field whose flow acts on M by isometries is called a Killing vector field (see [46] for additional details). The correspondence $U_r(g) \to \vec{U}(\pi(g))$ is one to one and onto $T_{\pi(g)}M$. Since the Lie brackets of vector fields related by a mapping F are also F-related, the Lie brackets $[U_r, V_r]$ are $d\pi$-related to $[\vec{U}, \vec{V}]$. Therefore the correspondence $U_r(g) \to \vec{U}(\pi(g))$ is a Lie algebra homomorphism, and hence $\mathcal{F} = \{\vec{U} : U \in \mathfrak{g}\}$ is a finite dimensional Lie algebra of Killing vector fields that satisfies $\mathcal{F}(p) = T_pM$ for each $p \in M$.

Note that $\pi(\exp tU) = \tau_{e^{tU}}(o) = \exp t\vec{U}(o)$. So if $U \in \mathfrak{k}$ then $\pi(\exp tU) = o$ and therefore $\vec{U}(o) = 0$. It then follows that $d_e\pi(U) = \vec{U}(o)$ is an isometric isomorphism from $\mathfrak{p}$ onto T_oM. More generally if $g(t)$ is any horizontal curve then $p(t) = \pi(g(t)) = \tau_g(t)\pi(e)$ implies that

$$\frac{dp}{dt} = d_{g(t)}(\pi(g(t)U(t))) = d_o\tau_{g(t)}d_e\pi(U(t)) = d_o\tau_{g(t)}\vec{U}(t)(o), \tag{91}$$

and

$$(d_o\tau_{g(t)}\vec{U}(t)(o), d_o\tau_{g(t)}\vec{V}(t)(o))_{p(t)} = (\vec{U}(o), \vec{V}(o))_o.$$

Therefore $d_o\tau_{g(t)}$ is an isometry that maps T_oM onto $T_{p(t)}M$.

A homogeneous manifold $M = G/K$ with a G-invariant metric defined by a reductive pair (G,K) with G semi-simple and K compact, will be referred to as semi-simple (it is defined by a semi-simple Lie group G, a compact subgroup K, and the metric induced by the Killing form). It can be shown that any symmetric Riemannian space with no Euclidean factors can be reduced to a semi-simple manifold (so that $[\mathfrak{p}, \mathfrak{p}] = \mathfrak{k}$ holds). Conversely, every semi-simple manifold is locally symmetric. It is symmetric when G is simply connected (see [48], Proposition 6.27). We will not pursue further proximities with symmetric spaces since the present exposition makes no use of geodesic symmetries.

We now come to the main topic of this section, rolling of semi-simple manifolds on their tangent spaces. We begin by recalling the basic definition.

Definition 1. *A curve* $\alpha(t)$ *on a Riemannian manifold* M *rolls on a curve* $\hat{\alpha}(t)$ *on another Riemannian manifold* $\hat{M}$ *if there exists an isometry* $A(t) : T_{\alpha(t)}M \to T_{\hat{\alpha}(t)}\hat{M}$ *that satisfies:*

$$\frac{d\hat{\alpha}}{dt} = A(t)\frac{d\alpha}{dt}, \tag{92}$$

and also satisfies the condition that $A(t)v(t)$ *is a parallel vector field in* $\hat{M}$ *along* $\hat{\alpha}(t)$ *for each parallel vector field* $v(t)$ *along* $\alpha(t)$ *in* M.

This intrinsic definition of rolling was introduced in [49], and later used in [50,51]. In this context the triple $(\alpha(t), \hat{\alpha}(t), A(t))$ is called a rolling curve. It is clear that rolling is reflexive in the sense that if $\alpha(t)$ is rolled on $\hat{\alpha}(t)$ by an isometry $A(t)$ then $\hat{\alpha}(t)$ is rolled on $\alpha(t)$ by the isometry $A^{-1}(t)$, and therefore $(\hat{\alpha}(t), \alpha(t), A^{-1}(t))$ is also a rolling curve.

We will now examine rollings of semi-simple manifolds on their tangent planes. It comes as a pleasant surprise that such rollings are essentially described by Equation (91) reinterpreted in terms of rolling. So the passage to rolling becomes largely a question of semantics, as demonstrated in the text below.

We will consider rollings of M on $\hat{M} = T_oM$ with its metric $(u, v)_o$ defined by (89). The rollings on other tangent spaces are conjugate to the rollings on T_oM [47]. Let $\alpha(t)$ be an arbitrary curve in M and let $\hat{\alpha}(t)$ be a curve in $\hat{M}$ that $\alpha(t)$ is rolled on. It follows that $\alpha(t) = \pi(g(t)) = \tau_{g(t)}(o)$ for some horizontal curve $g(t)$. If $g(t)$ is a solution of $\frac{dg}{dt} = g(t)U(t), U(t) \in \mathfrak{p}$ then according to (91)

$$\frac{d\alpha(t)}{dt} = d_{g(t)}\pi(g(t)U(t)) = d_o\tau_{g(t)}\vec{U}(t)(o),$$

If we now let $\hat{\alpha}(t)$ be any solution in $\hat{M}$ of $\frac{d\hat{\alpha}(t)}{dt} = \vec{U}(t)(o)$ then $A(t) = d_o\tau_{g(t)}$ is an isometry that rolls $\hat{\alpha}(t)$ on $\alpha(t)$ since the parallel transport condition is satisfied (for proofs see [47]). Of course, then $A^{-1}(t)$ rolls $\alpha(t)$ on $\hat{\alpha}(t)$.

It follows that each horizontal curve $g(t)$ in G defines a family of curves $\hat{\alpha}(t)$ in $\hat{M}$, each a solution of $\frac{d\hat{\alpha}}{dt} = \vec{U}(t)(o)$, with $\vec{U}(t)$ induced by $U(t) = g^{-1}(t)\frac{dg}{dt}$, that roll on $\alpha(t) = \pi(g(t))$. The converse is also true: every solution $(g(t), \hat{\alpha}(t))$ of the differential system

$$\frac{dg}{dt} = g(t)U(t), \frac{d\hat{\alpha}(t)}{dt} = \vec{U}(t)(o), U(t) \in \mathfrak{p} \tag{93}$$

defines a curve $\alpha(t) = \pi(g(t))$ in M on which $\hat{\alpha}(t)$ in $\hat{M}$ is rolled by the isometry $d_o\tau_{g(t)}$.

We will regard (93) as the fundamental object in rolling defined on $\mathbf{G} = G \times \hat{M}, \hat{M} = T_oM$, a Lie group with its group operation

$$\mathbf{gh} = (g, p)(h, q) = (gh, p + q), \mathbf{g} = (g, p), \mathbf{h} = (h, q).$$

Then $\mathcal{G} = \mathfrak{g} \times T_oM$ will denote the Lie algebra of $\mathbf{G}$ with the Lie bracket $[(X, \vec{U}(o)), (Y, \vec{V}(o)]$ $= ([X, Y], 0)$.

Let now $\mathcal{H}(g, p) = \{(gU, \vec{U}(o)) : U \in \mathfrak{p}\}, (g, p) \in \mathbf{G}$. We will view $\mathcal{H}$ as a left-invariant distribution on $\mathbf{G}$ defined by the left-translates of vector space $\Gamma = \{(U, \vec{U}(o)) : U \in \mathfrak{p}\}$ in $\mathcal{G}$. The distribution $\mathcal{H}$ is called the rolling distribution and its integral curves are called rolling motions. Any rolling motion $\mathbf{g}(t) = (g(t), p(t))$ is a solution of

$$\frac{dg}{dt} = g(t)U(t), \frac{dp}{dt} = \vec{U}(t)(o), \tag{94}$$

and can be associated with the rolling curve $(\hat{\alpha}(t), \alpha(t)), d_o\tau_{g(t)})$, where $\alpha(t) = \tau_{g(t)}(o)$ and $\hat{\alpha}(t) = p(t)$.

Since $\mathfrak{p}$ and $\mathfrak{k}$ satisfy strong Cartan conditions $[\mathfrak{p}, \mathfrak{k}] = \mathfrak{p}$ and $[\mathfrak{p}, \mathfrak{p}] = \mathfrak{k}$, Γ satisfies $[\Gamma, \Gamma] = (\mathfrak{k}, 0)$, and $[\Gamma, [\Gamma], \Gamma]] = (\mathfrak{p}, 0)$. Therefore,

$$\Gamma + [\Gamma, \Gamma] + [\Gamma, [\Gamma, \Gamma]] = \mathcal{G}, \tag{95}$$

Hence the Lie algebra generated by the left-invariant vector fields tangent to $\mathcal{H}$ is equal to $\mathcal{G}$, therefore any two points in $\mathbf{G}$ can be connected by a rolling motion, and each rolling motion inherits a natural length $\int_0^T \sqrt{\langle U(t), U(t) \rangle}\, dt$ from G. It is then known that any pair of points in $\mathbf{G}$ can be connected by an integral curve of $\mathcal{H}$ of minimal length because vector fields in $\mathcal{H}$ are complete [50]. The above shows that $\mathbf{G}$ with the above metric is a sub-Riemannian manifold. We will refer to the associated sub-Riemannian geodesics as the rolling geodesics.

We will now turn to the Maximum principle to find the necessary conditions that the rolling geodesics must satisfy. To put the matter in the control theoretic context, let $A_1, \dots, A_m$ be an orthonormal basis in $\mathfrak{p}$ so that $(A_i, \vec{A}_i(o))$ becomes an orthonormal basis in Γ. Then an absolutely continuous curve $\mathbf{g}(t) = (g(t), p(t))$ is a rolling motion if and only if

$$\frac{dg}{dt} = \sum_{i=1}^{m} u_i(t) g(t) A_i, \frac{dp}{dt} = \sum_{i=1}^{m} u_i(t) \vec{A}_i(o), \tag{96}$$

for some bounded and measurable control functions $u_1(t), \dots, u_m(t)$, in which case the length of $\mathbf{g}(t)$ is given by $\int_0^T \sqrt{u_1^2(t) + \cdots + u_m^2(t)}dt$. The rolling problem is an optimal control problem and consists of finding the solutions $\mathbf{g}(t) = (g(t), p(t))$ on a fixed time interval $[0, T]$ that satisfy the given boundary conditions $\mathbf{g}(0) = \mathbf{g}_0$ and $\mathbf{g}(T) = \mathbf{g}_1$ along which the energy of transfer $\frac{1}{2} \int_0^T \sum_{i=1}^m u_i^2(t)\, dt$ is minimal. It is known that each rolling geodesic is locally optimal and hence is a solution to the above control problem [7,50].

5.1. Rolling Hamiltonians

To emphasize the invariant properties of the problem we will rewrite (96) as

$$\frac{d\mathbf{g}}{dt} = \sum_{i=1}^{m} u_i(t) X_i(\mathbf{g}), \tag{97}$$

where each X_i a left-invariant vector field $X_i(\mathbf{g}) = (gA_i, \vec{A}_i(o))$, $\mathbf{g} = (g, p)$. If $\mathbf{g}(t)$ is an optimal trajectory then, according to the Maximum Principle, $\mathbf{g}(t)$ is the projection of an extremal curve $\xi(t)$ in $T^*\mathbf{G}$ along which the cost extended Hamiltonian

$$-\frac{\lambda}{2} \sum_{i=1}^{m} u_i^2(t) + \sum_{i=1}^{m} u_i(t) H_i(\xi(t)), \lambda = 0, 1$$

is maximal relative to all other control functions. Here H_i is the Hamiltonian lift of X_i, i.e., $H_i(\xi(t)) = \xi(t)(X_i(\mathbf{g}(t))$.

There are two kinds of extremal curves depending whether $\lambda = 0$ (abnormal case) or $\lambda = 1$ (normal case). In the abnormal case the Maximum principle results in the constraints

$$H_i(\xi(t)) = 0, i = 1, \dots, m, \tag{98}$$

and beyond that gives no further information about the optimal control in question. In the normal case, however, the above maximum yields $u_i(t) = H_i(\xi(t)), i = 1, \dots, m$, where $\xi(t)$ is a solution curve of a single Hamiltonian vector field corresponding to the Hamiltonian

$$H(\xi) = \frac{1}{2} \sum_{i=1}^{m} H_i^2(\xi). \tag{99}$$

Each optimal solution $\mathbf{g}(t)$ is either the projection of an abnormal or a normal extremal curve. If $\mathbf{g}(t)$ is the projection of a normal extremal curve $\xi(t)$ then $\xi(t)$ is an integral curve of $\vec{H}$ and the control $u(t)$ that generates $\mathbf{g}(t)$ is of the form $u_i(t) = H_i(\xi(t)), i = 1, \dots, m$.

We will not concern ourselves with the abnormal extremals. It is very likely that every optimal trajectory is the projection of a normal extremal curve, as in [52], in which case the abnormal extremals could be ignored. Instead we will turn to the normal Hamiltonian H and its Hamiltonian equations

$$\frac{d\mathbf{g}}{dt} = \sum_{i=1}^{n} H_i(\ell(t)) X_i(\mathbf{g}(t)), \frac{d\ell}{dt} = -ad^* dH(\ell(t))(\ell(t)).$$

Let us first consider the solutions of the associated Poisson equation

$$\frac{d\ell}{dt} = -ad^* dH(\ell(t))(\ell(t)) \tag{100}$$

and the structure of the coadjoint orbits.

Since $\hat{M}$ is a Euclidean vector space, its tangent space at the origin can be identified with $\hat{M}$. Then the Lie algebra $\mathcal{G}$ will be identified with $\mathfrak{g} \times \hat{M}$, and its dual with $\mathcal{G}^* = \mathfrak{g}^* \oplus \hat{M}^*$, where

$$\mathfrak{g}^* = \{\ell \in \mathcal{G}^* : \ell(\dot{p}) = 0, \dot{p} \in \hat{M}\}, \hat{M}^* = \{\ell \in \mathcal{G}^* : \ell(\mathfrak{g}) = 0\}.$$

It then follows that every $\ell \in \mathcal{G}^*$ can be written as $\ell = \ell_1 + \ell_2$ with $\ell_1 \in \mathfrak{g}^*$ and $\ell_2 \in \hat{M}^*$. Since $\hat{M}$ is an abelian algebra the projection ℓ_2 on $\hat{M}^*$ is constant on each coadjoint orbit of $\mathbf{G}$. The argument is straightforward:

$$Ad^*_{\mathbf{g}}(\ell)(X + \dot{p}) = \ell(Ad_{\mathbf{g}^{-1}}(X + \dot{p})) = \ell(Ad_{g^{-1}}(X) + \dot{p}) = \ell_1(Ad_{g^{-1}}(X)) + \ell_2(\dot{p}),$$

for any $\mathbf{g} = (g, p) \in \mathbf{G}$. It follows that the coadjoint orbits in $\mathcal{G}$ are of the form

$$\{Ad^*_g(\ell_1) : g \in G\} + \ell_2, \text{ for any } \ell = \ell_1 + \ell_2.$$

This fact can be also verified directly from Equation (100): we have

$$\frac{d\ell}{dt} V = -\ell[dH, V], \text{for any } V = X + \dot{p} \text{ in } \mathcal{G},$$

where $dH = \sum_{i=1}^m H_i(\ell)(A_i + \vec{A}_i(o))$ and $H_i(\ell) = \ell_1(A_i) + \ell_2(\vec{A}_i(o))$. Therefore,

$$\frac{d\ell_1}{dt}(X) + \frac{d\ell_2}{dt}(\dot{p}) = -(\ell_1 + \ell_2)([dH, X + \dot{x}]) = -\sum_{i=1}^m H_i(\ell_i)[A_i, X].$$

from which follows that

$$\frac{d\ell_1}{dt}(X) = -\sum_{i=1}^n H_i(\ell_i)[A_i, X], X \in \mathfrak{g}, \frac{d\ell_2}{dt}(\dot{p}) = 0.$$

Since $\dot{p}$ is arbitrary $\frac{d\ell_2}{dt} = 0$.

To uncover other constants of motion identify $\mathcal{G}^*$ with $\mathcal{G}$ via the natural quadratic forms on each of the factors, and then recast the preceding equations on $\mathcal{G}$. More precisely, identify each ℓ_2 in $\hat{M}^*$ with a tangent vector $l = \sum_{i=1}^m l_i \vec{A}_i(o)$ via the formula $\ell_2(\dot{p}) = (l, \dot{p}), \dot{p} \in \hat{M}$. Similarly, identify $\ell_1 \in \mathfrak{g}^*$ with $L \in \mathfrak{g}$ via the formula $\ell_1(X) = \langle L, X\rangle, X \in \mathfrak{g}$. Then decompose $L \in \mathfrak{g}$ into the sum $L = L_{\mathfrak{p}} + L_{\mathfrak{k}}$, $L_{\mathfrak{p}} \in \mathfrak{p}$ and $L_{\mathfrak{k}} \in \mathfrak{k}$. Relative to the basis $A_1, \dots, A_m$ in $\mathfrak{p}$, $L_{\mathfrak{p}} = \sum_{i=1}^m P_i A_i$ where $P_i = \ell_1(A_i) = \langle L, A_i\rangle$. It follows that

$$H_i(\xi) = \ell(A_i + \vec{A}_i(o)) = \ell_1(A_i) + \ell_2(\vec{A}_i(o)) = P_i + l_i,$$

and

$$\frac{d\ell_1}{dt}(X) = \langle \frac{dL}{dt}, X\rangle = -\langle L, [\textstyle\sum_{i=1}^m (l_i + P_i)A_i, X]\rangle = -\langle [L, \textstyle\sum_{i=1}^m (l_i + P_i)A_i], X\rangle,$$
$$(\frac{dl}{dt}, \dot{p}) = \frac{d\ell_2}{dt}(t)(\dot{p}) = 0$$

Since X and $\dot{p}$ are arbitrary,

$$\frac{dL}{dt} = [\sum_{i=1}^m (l_i + P_i)A_i, L] = [A + L_{\mathfrak{p}}, L], A = \sum_{i=1}^m l_i A_i, \frac{dl}{dt} = 0. \tag{101}$$

Coupled with

$$\frac{dg}{dt} = g(t)(A + L_{\mathfrak{p}}), \frac{dp}{dt} = \sum_{i=1}^{n}(l_i + P_i)\vec{A}_i(o), \tag{102}$$

Equation (101) constitute the Hamiltonian equations on $\mathbf{G} \times \mathcal{G}$ generated by the Hamiltonian $H = \frac{1}{2}\sum_{i=1}^{m} H_i^2 = \frac{1}{2}\sum_{i=1}^{m}(l_i + P_i)^2$.

Each extremal curve projects onto a geodesic $\mathbf{g}(t) = (g(t), p(t))$, and each geodesic further projects onto the pair of curves $\alpha(t) = \tau_{g(t)}(o)$ in M and $\beta(t) = p(t)$ in $\hat{M}$ that are rolled upon each other by $g(t)$.Note that in this identification of the Lie algebras with their duals, coadjoint orbits $\{Ad_g^*(\ell_1) + \ell_2 : g \in G\}$ are identified with the affine sets $\{Ad_g(L) + l : g \in G\}$.

Recall now the Hamiltonian equations associated with the canonical affine-quadratic problem (Equation (33)):

$$\frac{dg}{dt} = g(t)(A + L_{\mathfrak{k}}(t)), \frac{dL_{\mathfrak{k}}}{dt} = [A, L_{\mathfrak{p}}], \frac{dL_{\mathfrak{p}}}{dt} = [L_{\mathfrak{k}}, L_{\mathfrak{p}}] + s[A, L_{\mathfrak{k}}], s = 0, 1.$$

The propositions below reveal a remarkable fact that the Poisson equations of a canonical affine-quadratic Hamiltonian are subordinate to the Poisson equations associated with a rolling Hamiltonian. This connection identifies the drift term in the affine-quadratic system with a coadjoint invariant of the rolling Poisson system. To keep the systems apart we will use bold letters when referring to the variables in the rolling Hamiltonian in contrast to the variables in the affine-quadratic Hamiltonian which will remain the same.

Proposition 12. *Let $(\mathbf{g}(t), \mathbf{p}(t)), \mathbf{L}_{\mathfrak{p}}(t), \mathbf{L}_{\mathfrak{k}}(t)$ be an integral curve of the rolling Hamiltonian $\mathbf{H} = \frac{1}{2}||\mathbf{A} + \mathbf{L}_{\mathfrak{p}}||^2$, that is,*

$$\frac{d\mathbf{g}}{dt} = \mathbf{g}(t)(\mathbf{A} + \mathbf{L}_{\mathfrak{p}}(t)), \frac{d\mathbf{p}}{dt} = \sum_{i=1}^{m}(l_i + P_i)\vec{A}_i(o),$$
$$\frac{d\mathbf{L}_{\mathfrak{k}}}{dt} = [\mathbf{A}, \mathbf{L}_{\mathfrak{p}}], \frac{d\mathbf{L}_{\mathfrak{p}}}{dt} = [\mathbf{A} + \mathbf{L}_{\mathfrak{p}}, \mathbf{L}_{\mathfrak{k}}], \mathbf{A} = \sum_{i=1}^{m} l_i \mathbf{A}_i$$

Then

$$\tilde{g}(t) = \mathbf{g}(t)h(t), L_{\mathfrak{p}}(t) = Ad_{h^{-1}(t)}(\mathbf{L}_{\mathfrak{p}}(t)), L_{\mathfrak{k}} = Ad_{h^{-1}(t)}(\mathbf{L}_{\mathfrak{k}}(t)) \tag{103}$$

is an integral curve of the affine Hamiltonian $H = \frac{1}{2}\langle L_{\mathfrak{k}}, L_{\mathfrak{k}}\rangle + \langle A, L_{\mathfrak{p}}\rangle$, where $A = Ad_{h^{-1}(t)}(\mathbf{A} + \mathbf{L}_{\mathfrak{p}}(t))$, and $h(t)$ is the solution of $\frac{dh}{dt} = \mathbf{L}_{\mathfrak{k}}(t)h(t)$ with $h(0) = I$.

Moreover, if $x(t)$ a solution of $\frac{dx}{dt} = \mathbf{A} + \mathbf{L}_{\mathfrak{p}}(t)$ then $\tilde{g}(t) = (x(t), h(t))$ in $\mathfrak{p} \rtimes K$ is the projection of an extremal curve

$$L_{\mathfrak{k}}(t) = Ad_{h^{-1}(t)}\mathbf{L}_{\mathfrak{k}}(t), L_{\mathfrak{p}}(t) = Ad_{h^{-1}(t)}(\mathbf{L}_{\mathfrak{p}}(t)) - A, A = Ad_{h^{-1}(t)}(\mathbf{A} + \mathbf{L}_{\mathfrak{p}}(t))$$

associated with the shadow Hamiltonian $H = \frac{1}{2}\langle L_{\mathfrak{k}}, L_{\mathfrak{k}}\rangle + \langle A, L_{\mathfrak{p}}\rangle$.

The converse also holds according to the following proposition.

Proposition 13. *Suppose that $(\tilde{g}(t), L_{\mathfrak{p}}(t), L_{\mathfrak{k}}(t))$ is an extremal curve of the affine Hamiltonian $H = \frac{1}{2}\langle L_{\mathfrak{k}}, L_{\mathfrak{k}}\rangle + \langle A, L_{\mathfrak{p}}\rangle$. Let*

$$\mathbf{g}(t) = \tilde{g}(t)h^{-1}(t), \mathbf{L}_{\mathfrak{p}}(t) = Ad_{h(t)}(L_{\mathfrak{p}}(t)), \mathbf{L}_{\mathfrak{k}}(t) = Ad_{h(t)}(L_{\mathfrak{k}}(t)), \mathbf{A} = Ad_{h(t)}(A - L_{\mathfrak{p}}(t))$$

where $h(t)$ is a solution of $\frac{dh}{dt} = h(t)(L_{\mathfrak{k}}(t))$ and let $\mathbf{p}(t)$ be a solution of $\frac{d\mathbf{p}}{dt} = \vec{\mathbf{A}}(o) + \vec{\mathbf{L}}_{\mathfrak{p}}(t)(o)$. Then $(\mathbf{g}(t), \mathbf{p}(t))$ together with

$$\mathbf{L}_{\mathfrak{p}}(t) = Ad_{h(t)}(L_{\mathfrak{p}}(t)), \mathbf{L}_{\mathfrak{k}}(t) = Ad_{h(t)}(L_{\mathfrak{k}}(t)), \mathbf{A} = Ad_{h(t)}(A - L_{\mathfrak{p}}(t))$$

is an extremal curve of the rolling Hamiltonian $\mathbf{H} = \frac{1}{2}\langle \mathbf{A} + \mathbf{L}_{\mathfrak{p}}, \mathbf{A} + \mathbf{L}_{\mathfrak{p}} \rangle$.

However, if $\tilde{g}(t) = (x(t), R(t)), L_{\mathfrak{p}}(t) + L_{\mathfrak{k}}(t))$ *is an extremal curve of the shadow Hamiltonian* H*, then* $(\mathbf{g}(t), \mathbf{p}(t))$*, solutions of*

$$\frac{d\mathbf{g}}{dt} = \mathbf{g}(t)Ad_{R(t)}(A)), \frac{d\mathbf{p}}{dt} = \frac{d\vec{x}}{dt}(o),$$

together with

$$\mathbf{L}_{\mathfrak{p}}(t) = Ad_{R(t)}(A + L_{\mathfrak{p}}(t)), \mathbf{L}_{\mathfrak{k}}(t) = Ad_{R(t)}(L_{\mathfrak{k}}(t)), \mathbf{A} = -Ad_R L_{\mathfrak{p}}$$

define an extremal curve of the rolling Hamiltonian $\mathbf{H} = \frac{1}{2}\langle \mathbf{A} + \mathbf{L}_{\mathfrak{p}}, \mathbf{A} + \mathbf{L}_{\mathfrak{p}} \rangle$.

The proofs follow by straightforward calculations (also done in [9]).

Let us now come back to isospectral representations and Zimmerman's method [17,52]. For that purpose let $X_0(t) = \mathbf{A} + \mathbf{L}_{\mathfrak{p}}(t), X_1(t) = \mathbf{L}_{\mathfrak{k}}(t), X_2(t) = -\mathbf{A}, X_3 = 0$. Then Poisson's equations for the rolling problem can be written as

$$\frac{dX_i}{dt} = [X_0(t), X_{i+1}(t)], i = 0, 1, 2. \tag{104}$$

These equations are invariant under a dilational change of variables $X_i \to \lambda^{i-1} X_i$. It then follows that

$$\mathbf{L}_\lambda = \sum_{i=0}^{3} \lambda^i X_i = \mathbf{L}_{\mathfrak{p}}(t) + \lambda \mathbf{L}_{\mathfrak{k}}(t) + (1 - \lambda^2)\mathbf{A} \tag{105}$$

satisfies the equation

$$\frac{d\mathbf{L}_\lambda}{dt} = [\mathbf{M}_\lambda(t), \mathbf{L}_\lambda(t)], \mathbf{M}_\lambda(t) = \frac{1}{\lambda}(\mathbf{A} + \mathbf{L}_{\mathfrak{p}}(t)). \tag{106}$$

Therefore $\mathbf{L}_\lambda$ is the spectral curve for $\mathbf{H}$. But then the Poisson system associated with the affine-quadratic Hamiltonian also admits an isospectral representation after the substitutions

$$\mathbf{A} = Ad_{h(t)}(A - L_{\mathfrak{p}}), \mathbf{L}_{\mathfrak{k}} = Ad_{h(t)}(L_{\mathfrak{k}}), \mathbf{L}_{\mathfrak{p}} = Ad_{h(t)}(L_{\mathfrak{p}}), \frac{dh}{dt} = h(t)L_{\mathfrak{k}}(t).$$

For then $(L_{\mathfrak{p}}(t), L_{\mathfrak{k}}(t))$ are the extremal curves for the Poisson system associated with the affine-quadratic system (Proposition 13) and further satisfy

$$\mathbf{L}_\lambda = Ad_{h(t)}(L_{\mathfrak{p}}) + \lambda Ad_{h(t)}(L_{\mathfrak{k}}) + (1 - \lambda^2(Ad_{h(t)}(A - L_{\mathfrak{p}}) =$$
$$Ad_{h(t)}(\lambda^2 L_{\mathfrak{p}} + \lambda L_{\mathfrak{k}} + (1 - \lambda^2)A) = Ad_{h(t)} L_\lambda.$$

But then

$$Ad_{h(t)}[\tfrac{1}{\lambda}A, L_\lambda] = [\tfrac{1}{\lambda}(\mathbf{A} + \mathbf{L}_{\mathfrak{p}}), \mathbf{L}_\lambda] =$$
$$\tfrac{d\mathbf{L}_\lambda}{dt} = \tfrac{d}{dt}(Ad_{h(t)}(L_\lambda) = Ad_{h(t)}[L_\lambda, L_{\mathfrak{k}}] + Ad_{h(t)}\tfrac{dL_\lambda}{dt}$$

implies

$$\frac{dL_\lambda}{dt} = [L_{\mathfrak{k}}, L_\lambda] + [\frac{1}{\lambda}A, L_\lambda] = [\frac{1}{\lambda}A + L_{\mathfrak{k}}, L_\lambda].$$

To be consistent with my earlier publications, replace λ by $-\frac{1}{\lambda}$ to get

$$\frac{dL_\lambda}{dt} = [M_\lambda, L_\lambda], \tag{107}$$

where $M_\lambda = L_{\mathfrak{k}} - \lambda A$, and $L_\lambda = L_{\mathfrak{p}} - \lambda L_{\mathfrak{k}} + (\lambda^2 - 1)A$. Equation (107) agrees with the isospectral representation (34).

To get the spectral curve L_λ for the shadow Hamiltonian, use relations $\mathbf{L}_{\mathfrak{k}} = Ad_h(L_{\mathfrak{k}})$, $\mathbf{L}_{\mathfrak{p}} = Ad_h(L_{\mathfrak{p}} + A)$ and $\mathbf{A} = -Ad_h L_{\mathfrak{p}}$ from Proposition 13. Then

$$\mathbf{L}_\lambda = \mathbf{L}_{\mathfrak{p}} + \lambda \mathbf{L}_{\mathfrak{k}} + (1-\lambda^2)\mathbf{A} = Ad_h L_\lambda, L_\lambda = \lambda^2 L_{\mathfrak{p}} + \lambda L_{\mathfrak{k}} + A.$$

Then a calculation analogous to the one above gives $\frac{dL_\lambda}{dt} = [\frac{1}{\lambda}A + L_{\mathfrak{k}}, L_\lambda]$. After the rescaling $\lambda \to -\frac{1}{\lambda}$ we get a modified Lax pair

$$\frac{dL_\lambda}{dt} = [M_\lambda, L_\lambda], M_\lambda = L_{\mathfrak{k}} - \lambda A, L_\lambda = L_{\mathfrak{p}} - \lambda L_{\mathfrak{k}} + \lambda^2 A. \tag{108}$$

5.2. Rolling Problem on Spaces of Constant Curvature

We will now introduce another optimal problem intertwined with the rolling problem. It consists of finding a continuously differentiable curve $p(t)$ in M in an interval $[0, T]$, with its tangent vector $\dot{p}(t)$ of unit length and its covariant derivative bounded and measurable in $[0, T]$ that satisfies fixed tangential directions $\dot{p}(0) = v_0, v_0 \in T_{p(0)}M$ and $\dot{p}(T) = v_1, v_1 \in T_{p(T)}M$ along which the integral $\frac{1}{2}\int_0^T \kappa^2(s)\,ds$ minimal among all other curves that satisfy the same boundary conditions. Here $\kappa(t) = ||\frac{dD_{p(t)}}{dt}(\dot{p}(t))||$, where $\frac{dD_{p(t)}}{dt}$ denotes the covariant derivative along $p(t)$. The integral $\frac{1}{2}\int_0^T \kappa^2(s)\,ds$ is known as the elastic energy of the curve $p(t)$ [24]. Curves $p(t)$ defined on some interval $[0, T]$ are called elastic if for each $t \in (0, T)$ there exits an interval $[t_0, t_1] \subset [0, T]$ over which the elastic energy of $p(t)$ is minimal relative to the boundary conditions $\dot{p}(t_0)$ and $\dot{p}(t_1)$ [7].

On semi-simple manifolds the curvature problem can be lifted to the unit tangent bundle of G, and it is this lifted version of the problem that will be of interest for this paper. In this formulation of the problem the tangent bundle of G is realized as the product $G \times \mathfrak{g}$ with $(g, X) \in G \times \mathfrak{g}$ identified with $gX \in T_g G$. Then each tangent vector $v \in T_p M$ is the projection of a manifold $V = \{(gh, Ad_h(U)), h \in K\}$ in $G \times \mathfrak{g}$ where $p = \pi(g)$ and $v = d_g\pi(g)U, U \in \mathfrak{p}$. The lifted curvature problem consists of finding a curve $(g(t), \Lambda(t))$ in $G \times S_{\mathfrak{p}}$, $S_{\mathfrak{p}} = \{\Lambda \in \mathfrak{p} : \langle \Lambda, \Lambda \rangle = 1\}$, a solution of

$$\frac{dg}{dt} = g(t)\Lambda(t), \frac{d\Lambda}{dt} = U(t), \langle U(t), \Lambda(t) \rangle = 0, \tag{109}$$

that originates in the manifold $V_0 = \{(g_0 h, Ad_{h^{-1}}\Lambda_0), h \in K, \Lambda_0 \in \mathfrak{p}\}$ at $t = 0$ and terminates at the manifold $V_1 = \{(g_1 h, Ad_{h^{-1}}\Lambda_1) : h \in K, \Lambda_1 \in \mathfrak{p}\}$ at $t = T$ for which the energy of transfer $\frac{1}{2}\int_0^T ||U(s)||^2\,ds$ is minimal. If $p(t) = \pi(g(t)) = \tau_{g(t)}(o)$ is the projected curve, then $p(t)$ is the solution of

$$\dot{p}(t) = d_{g(t)}\pi(g(t))\Lambda(t) = d_o\tau_{g(t)}\vec{\Lambda}(t)(o).$$

that satisfies $||\dot{p}(t)|| = 1$ and the boundary conditions

$$p(0) = \pi(g(0)), \dot{p}(0) = d_{g(0)}\pi(V_0) = d_{g(0)}\pi(g(0)\Lambda_0),$$
$$p(T) = \pi(g(T)), \dot{p}(T) = d_{g(T)}\pi(V_1) = d_{g(T)}\pi(g(T)\Lambda_1).$$

It is a simple exercise to show that a curve $p(t)$ is elastic if and only if it is the projection of a solution of the lifted curvature problem on a fixed interval $[0, T]$.

The Hamiltonian system for the curvature problem (Equation (109)) can also be obtained through the Maximum principle properly modified to account for the constraints, as outlined in ([7], Chapter 11). To go into these details would take us away from the central theme of the paper, so instead, we will just quote the relevant equations from [7] (pp. 354–355).

The curvature Hamiltonian H is given by $H = \frac{1}{2}||X||^2 + \langle \Lambda, P\rangle$ together with the associated Hamiltonian equations

$$\begin{gathered}\frac{dg}{dt} = g\Lambda(t), \frac{dP}{dt} = [\Lambda, Q], \frac{dQ}{dt} = [\Lambda, P],\\ \frac{d\Lambda}{dt} = X(t), \frac{dX}{dt} = -P - (||X||^2 - \langle P, \Lambda\rangle)\Lambda,\end{gathered}$$

subject to the transversality condition $Q(t) + [\Lambda(t), X(t)] = 0$. The transversality condition can be incorporated into the above equations to yield an equivalent system

$$\begin{gathered}\frac{dg}{dt} = g\Lambda(t), \frac{d\Lambda}{dt} = X(t), \frac{dX}{dt} = -P - (||X||^2 - \langle P, \Lambda\rangle)\Lambda,\\ \frac{dP}{dt} = -[\Lambda, [\Lambda, X]], \frac{dQ}{dt} = [\Lambda, P].\end{gathered} \tag{110}$$

We will now confine our attention to spaces of constant curvature, with a particular interest on the connections between the rolling problems and the elastic curves reported in [52]. For those reasons let us return to the "spheres" $S^n_\epsilon(\rho) = \{x \in R^{n+1} : (x, x)_\epsilon = \rho^2, x_0 > 0 \text{ when } \epsilon = -1\}$ and their rollings on the isometry groups groups SO_ϵ, $\epsilon = \pm 1$ endowed with the quadratic form $\langle A, B\rangle_\epsilon = -\frac{1}{2}\epsilon\rho^2 Tr(AB)$. The rolling equations associated with the rollings of $S^n_\epsilon(\rho)$ on the tangent plane $\hat{M} = T_{\rho e_0} S^n_\epsilon(\rho)$ are given by

$$\frac{dg}{dt} = g(t)(u(t) \wedge_\epsilon e_0)), \frac{dp}{dt}(t) = \rho u(t). \tag{111}$$

In what follows we will make use of the following isospectral integrals of motion associated with the preceding rolling problem extracted from the functions $f_{2,\lambda} = Tr(L^2_\lambda)$ and $f_{4,\lambda} = Tr(L^4_\lambda)$

$$\begin{gathered}I_0 = 2H = ||A + L_{\mathfrak{p}}||^2, I_1 = ||L_{\mathfrak{p}}||^2 + \epsilon||L_{\mathfrak{k}}||^2\\ I_2 = |k|||L_{\mathfrak{k}}||^2||L_{\mathfrak{p}}||^2 - ||[L_{\mathfrak{p}}, L_{\mathfrak{k}}]||^2 + \frac{\epsilon}{2}(k||L_{\mathfrak{k}}||^4 - ||L^2_{\mathfrak{k}}||^2),\\ I_4 = |k|||L_{\mathfrak{k}}||^2||A + L_{\mathfrak{p}}||^2 - ||[A + L_{\mathfrak{p}}, L_{\mathfrak{k}}]||^2.\end{gathered} \tag{112}$$

These integrals of motion are rescaled variants of the integrals of motion in [52] after the metric is replaced by $\langle A, B\rangle_\rho = \rho\langle A, B\rangle_\epsilon$ (the metric in this paper is a scalar multiple of the metric used in [52]).

Recall that on spaces of constant Riemannian curvature the curvature k is defined by

$$[V, [V, X]] = -kX, \tag{113}$$

for any V and X in $\mathfrak{p}$ that satisfy $||V|| = 1$ and $\langle V, X\rangle = 0$. In particular k is equal to $\frac{\epsilon}{\rho^2}$ on $S^n_\epsilon(\rho)$. Note that $-\frac{1}{2}Tr(AB) = k\langle A, B\rangle$.

Proposition 14. *Rolling geodesics that are the projections of the extremal curves on $H = \frac{1}{2}$ and $I_4 = 0$ project on the elastic curves in $S^n_\epsilon(\rho)$. Conversely each elastic curve in $S^n_\epsilon(\rho)$ is the projection of such an extremal curve.*

Proof. Each elastic curve on $S^n_\epsilon(\rho)$ is the projection of an extremal curve corresponding to the curvature problem (Equation (110)). On spaces of constant Riemannian curvature

$$[\Lambda, [\Lambda, X]] = -kX. \tag{114}$$

Therefore, Equation (110) can be written as

$$\begin{gathered}\frac{dg}{dt} = g\Lambda(t), \frac{d\Lambda}{dt} = X(t), \frac{dX}{dt} = -P - (||X||^2 - \langle P, \Lambda\rangle)\Lambda,\\ \frac{dP}{dt} = kX, \frac{dQ}{dt} = [\Lambda, P].\end{gathered} \tag{115}$$

It follows that $k\frac{d\Lambda}{dt} - \frac{dP}{dt} = 0$, and therefore, $k\Lambda - P = kA$ for some constant element A in $\mathfrak{p}$. The transversality condition $Q + [\Lambda, X] = 0$ can be recast as $0 = [\Lambda, Q] + [\Lambda, [\Lambda, X]] = [\Lambda, Q] - kX$. These observations can be incorporated in the preceding equations to get

$$\begin{aligned}\frac{dg}{dt} &= g(t)\Lambda(t) = g(t)\tfrac{1}{k}(kA + P),\\ \frac{dP}{dt} &= kX = [\Lambda, Q] = \tfrac{1}{k}[kA + P, Q],\\ \frac{dQ}{dt} &= [\Lambda, P] = \tfrac{1}{k}[kA + P, P] = \tfrac{1}{k}[kA, P].\end{aligned} \tag{116}$$

If we now identify $\frac{1}{k}P$ with $L_{\mathfrak{p}}$, and $\frac{1}{k}Q$ with $L_{\mathfrak{k}}$, then the preceding equations reduce to the rolling Hamiltonian system. Moreover,

$$\Lambda = A + \frac{1}{k}P = A + L_{\mathfrak{p}}, \text{ and } L_{\mathfrak{k}} = \frac{1}{k}Q = \frac{1}{k}[A + L_{\mathfrak{p}}, X].$$

Hence $||A + L_{\mathfrak{p}}|| = 1$ so the first constraint is satisfied. To verify the second constraint note that $L_{\mathfrak{k}} = \frac{1}{k}[A + L_{\mathfrak{p}}, X]$, and therefore

$$||L_{\mathfrak{k}}||^2 = \frac{1}{k^2}||[A + L_{\mathfrak{p}}, X]||^2 = \frac{1}{k^2}\langle[[A + L_{\mathfrak{p}}, X], A + L_{\mathfrak{p}}], X\rangle = \frac{1}{|k|}||X||^2,$$

and $||[A + L_{\mathfrak{p}}, L_{\mathfrak{k}}]||^2 = \frac{1}{k^2}||[A + L_{\mathfrak{p}}, [A + L_{\mathfrak{p}}, X]]||^2 = ||X||^2$. Therefore,

$$I_4 = ||L_{\mathfrak{k}}||^2|k|||A + L_{\mathfrak{p}}||^2 - ||[A + L_{\mathfrak{p}}, L_{\mathfrak{k}}]||^2 = ||X||^2 - ||X||^2 = 0.$$

To prove the converse assume that $g(t), p(t), A, L_{\mathfrak{k}}(t), L_{\mathfrak{p}}(t)$ is a rolling extremal curve on $I_4 = 0$. As a geodesic it satisfies $H = \frac{1}{2}$, or $||A + L_{\mathfrak{p}}|| = 1$. We need to show that $L_{\mathfrak{k}}(t) = [A + L_{\mathfrak{p}}(t), X(t)]$ for some $X(t) \in \mathfrak{p}$ such that $\langle X(t), A + L_{\mathfrak{p}}(t)\rangle = 0$.

Let

$$\Lambda(t) = A + L_{\mathfrak{p}}(t), \mathfrak{p}^{\perp}_{\Lambda(t)} = \{X(t) \in \mathfrak{p} : \langle X(t), \Lambda(t)\rangle = 0\},$$
$$\mathfrak{k}_{\Lambda(t)} = \{Q(t) \in \mathfrak{k} : [Q(t), \Lambda(t)] = 0\}, \mathfrak{k}^{\perp}_{\Lambda(t)} = \{Q \in \mathfrak{k}; : \langle Q, \mathfrak{k}_{\Lambda}\rangle = 0\}.$$

Then $\Lambda(t) = \lambda(t) \wedge_{\epsilon} e_0, (\lambda(t), e_0)_{\epsilon} = 0$ for some vector $\lambda(t) \in R^{n+1}$. It then follows that $\mathfrak{p}^{\perp}_{\Lambda(t)} = \{u(t) \wedge_{\epsilon} e_0 : (u(t), e_0)_{\epsilon} = (\lambda(t), u(t))_{\epsilon} = 0\}$, and $\mathfrak{k}^{\perp}_{\Lambda(t)} = \{\lambda(t) \wedge_{\epsilon} u(t) : (u(t), \lambda(t))_{\epsilon} = (e_0, u(t))_{\epsilon} = 0\}$.

Hence, $\dim(\mathfrak{p}^{\perp}_{\Lambda(t)}) = \dim(\mathfrak{k}^{\perp}_{\Lambda})$. The mapping $F(t)X = ad\Lambda(t)(X), X \in \mathfrak{p}^{\perp}_{\Lambda(t)}$ satisfies $F(\mathfrak{p}^{\perp}_{\Lambda(t)}) \subseteq \mathfrak{k}^{\perp}_{\Lambda(t)}$ because $\langle[\Lambda, X], \mathfrak{k}_{\Lambda}\rangle = 0$. On spaces of non-zero constant curvature, the kernel of this mapping is zero because $ad\Lambda(t)X = 0$ implies that $0 = ad^2\Lambda(t)(X(t)) = -\epsilon\rho^2 X(t)$. Since $\mathfrak{p}^{\perp}_{\Lambda(t)}$ and $\mathfrak{k}^{\perp}_{\Lambda(t)}$ have the same dimension, F maps $\mathfrak{p}^{\perp}_{\Lambda(t)}$ onto $\mathfrak{k}^{\perp}_{\Lambda(t)}$. So every curve $L(t) \in \mathfrak{k}^{\perp}_{\Lambda(t)}$ is of the form $L(t) = [\Lambda(t), X(t)]$ for some $X(t) \in \mathfrak{p}$ perpendicular to $\Lambda(t)$.

It remains to show that $L_{\mathfrak{k}}(t)$ belongs to $\mathfrak{k}^{\perp}_{\Lambda(t)}$ when the rolling geodesic is on $I_4 = 0$, that is, when $||L_{\mathfrak{k}}||^2 = \frac{1}{k}||[\Lambda(t), L_{\mathfrak{k}}(t)||^2$. Now assume that $L_{\mathfrak{k}}(t) = U_1(t) + U_2(t), U_1(t) \in \mathfrak{k}_{\Lambda(t)}$ and $U_2(t) \in \mathfrak{k}^{\perp}_{\Lambda(t)}$. It follows from above that $U_2(t) = [\Lambda(t), X(t)]$, and therefore

$$||U_2(t)||^2 = ||[\Lambda(t), X(t)], [\Lambda(t), X(t)]||^2 = |\langle ad^2\Lambda(t)(X), X(t)\rangle| = |k|||X||^2$$

Hence,

$$\frac{1}{|k|}||[\Lambda(t), U_1(t) + U_2(t)]||^2 = \frac{1}{|k|}||[\Lambda(t), U_2(t)]||^2 = |k|||X||^2 = ||L_{\mathfrak{k}}||^2.$$

But $||L_{\mathfrak{k}}(t)||^2 = ||U_1||^2 + ||U_2(t)||^2 = ||U_1(t)||^2 + |k|||X||^2$, and therefore $U_1(t) = 0$. □

The following proposition characterizes elastic curves [7].

Proposition 15. *Let $\kappa(t)$ and $\tau(t)$ denote the geodesic curvature and the torsion of the projection curve $p(t)$ associated with an extremal curve of the curvature problem. Then $\xi(t) = \kappa^2(t)$ is the solution of the following equation*

$$(\frac{d\xi}{dt})^2 = -\xi^3 + 4(H-\epsilon)\xi^2 + 4(I_1 - H^2)\xi - 4I_2, \tag{117}$$

and $(\kappa^2(t)\tau(t))^2 = kI_2$. All other curvatures in the Serret-Frenet frame along $p(t)$ are zero.

I believe that the proof given below is more to the point than similar proofs given elsewhere [7,52].

Proof. We leave it to the reader to verify that $k||L_{\mathfrak{k}}||^4 - ||L_{\mathfrak{k}}^2||^2 = 0$ when $L_{\mathfrak{k}} = \frac{1}{k}[A + L_{\mathfrak{p}}, X]$. Let $P(t) = kL_{\mathfrak{p}}(t)$ and let $Q(t) = kL_{\mathfrak{k}}(t)$. Then

$$I_2k^2 = k^2(|k|||L_{\mathfrak{k}}||^2||L_{\mathfrak{p}}||^2 - ||[L_{\mathfrak{p}}, L_{\mathfrak{k}}]||^2) = ||P||^2||X||^2|| - ||[L_{\mathfrak{p}}, Q]||^2 =$$
$$||P||^2||X||^2 - ||[L_{\mathfrak{p}}, [A + L_{\mathfrak{p}}, X]]||^2 = ||P||^2||X||^2 - \langle A + L_{\mathfrak{p}}, L_{\mathfrak{p}}\rangle^2||X||^2 - \langle P, X\rangle^2.$$

Also

$$I_1k^2 = k^2(||L_{\mathfrak{p}}||^2 + \epsilon||L_{\mathfrak{k}}||^2) = ||P||^2 + \epsilon k^2||Q||^2 = ||P||^2 + k||X||^2.$$

Since $\kappa^2(t) = ||X||^2$, $\frac{d\xi}{dt} = 2\langle X, \dot{X}\rangle = 2\langle X, P\rangle$. Therefore,

$$(\tfrac{d\xi}{dt})^2 = 4\langle P, X\rangle^2 = 4(||P||^2||X||^2 - \langle A + L_{\mathfrak{p}}, P\rangle^2||X||^2) - 4k^2I_2$$
$$= 4((I_1k^2 - k||X||^2)||X||^2 - (H - \tfrac{1}{2}||X||^2)^2||X||^2) - 4k^2I_2$$
$$= 4(I_1k^2 - k\xi)\xi - 4(H - \tfrac{1}{2}\xi)^2\xi - 4k^2I_2 = -\xi^3 + 4(H-k)\xi^2 + (I_1k^2 - H^2)\xi - 4k^2I_2.$$

As to the second part, let $T = A + L_{\mathfrak{p}}(t)$. Since $||A + L_{\mathfrak{p}}(t)|| = 1$, $T(t)$ is a unit vector that projects onto the tangent vector $\dot{p}(t)$. Then

$$\frac{dT}{dt} = [A + L_{\mathfrak{p}}(t), L_{\mathfrak{k}}(t)] = [A + L_{\mathfrak{p}}(t), -[A + L_{\mathfrak{p}}(t), \frac{1}{k}X(t)]] = X(t).$$

Therefore $\frac{dT}{dt} = \kappa(t)N(t)$ where $N(t) = \frac{1}{||X(t)||}X(t)$ is a unit vector in $\mathfrak{p}$ that projects onto the unit normal $n(t)$ along $p(t)$. Continuing,

$$\tfrac{dN}{dt} = \tfrac{1}{||X(t)}||(-P - (||X||^2 - \langle\Lambda, P\rangle(A + L_{\mathfrak{p}})) - \tfrac{1}{||X||^2}\langle X, \dot{X}\rangle X) =$$
$$-||X||(A + L_{\mathfrak{p}}) + \tfrac{1}{||X||}(-P + \langle A + L_{\mathfrak{p}}, P\rangle(A + L_{\mathfrak{p}})) + \tfrac{1}{||X||^2}\langle P, X\rangle X) =$$
$$-\kappa(t)T(t) + Y(t),$$

where

$$Y(t) = \frac{1}{||X||}(-P(t) + \langle T(t), P(t)\rangle T(t)) + \frac{1}{||X||^2}\langle P(t), X(t)\rangle X.$$

Since $Y(t)$ is is orthogonal to $A + L_{\mathfrak{p}}$ and X, it is in the direction of the binormal vector $B(t)$. So if we define $\tau(t) = ||Y(t)||$ and $B(t) = \frac{1}{||Y||}Y$ then $\frac{dN}{dt} = \kappa(t)T(t) + \tau B(t)$ and $B(t)$ projects onto the binormal vector $b(t)$ along $p(t)$. Hence,

$$||X||^2\tau^2 = ||P||^2 - \langle A + L_{\mathfrak{p}}, P\rangle^2 - \frac{1}{||X||^2}\langle P, X\rangle^2,$$

or

$$|(\kappa^2\tau)^2 = ||X||^4\tau^2 = ||P||^2||X||^2 - \langle A + L_{\mathfrak{p}}, P\rangle^2 - \langle P, X\rangle^2 = k^2I_2,$$

Evidently $\frac{dB}{dt}$ is in the linear span of $T(t), N(t), B(t)$, hence the Serret-Frenet frame along $p(t)$ terminates. □

Corollary 3. *Elastic curves in $M_\epsilon = S^n_\epsilon(\rho)$ are rolled on the elastic curves in the tangent space $\hat{M} = T_{\mathbf{e}}M$.*

Proof. Since the geodesic curvature is preserved under rolling, the elastic curves in $S^n_\epsilon(\rho)$ are rolled on the elastic curves in $\hat{M}$ relative to the Euclidean metric inherited from the metric on $\mathfrak{p}$. So the statement follows from the rolling definition. □

This remarkable relation between the elastic curves and the rolling geodesics breaks down on spaces of non-constant Riemannian curvature, as it becomes evident when one compares Equation (110) for the curvature problem to the Equation (101) for the rolling problem. It is interesting to note that the solutions of either of these two Equations (101) and (110) are not known beyond the spaces of constant curvature. While the curvature equation seems particularly challenging beyond the spaces of constant curvature, the rolling geodesic equations remain integrable on all semi-simple spaces and should be "solvable" according to the general theory of integrable systems.

Apart from the above remarks, there is another spectacular property of elastic curves that makes them special: elastic curves appear as soliton solutions in the non-linear Schroedinger equation [53]. More generally it was shown in [53] that the space of periodic horizontal curves of fixed length L in the isometry group G over a three dimensional space of constant curvature can be given a structure of an infinite dimensional Poisson manifold relative to which some famous equations of mathematical physics appear as Poisson equations associated with geometric invariants of curves on the base space. In particular, Heisenberg's magnetic equation and Schroedinger's non-linear equation appear as Poisson equation associated with $f_0(g(s)) = \frac{1}{2}\int_0^L ||\frac{d\Lambda}{ds}(s)||^2\, ds$ where $\Lambda(s) = g^{-1}(s)\frac{dg}{ds}(s)$, $||\Lambda(s)|| = 1$. Since this function can be also expressed as $\frac{1}{2}\int_0^L \kappa^2(s)\, ds$ where $\kappa(s)$ is the geodesic curvature of the projected curve in the underlying symmetric space, elastic curves appear naturally in this setting (see also [54–57] for related results). This leap to infinite dimensional Hamiltonians and related hierarchies of commuting Hamiltonians further illustrates the relevance of Lie algebraic methods in the theory of integrable systems.

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