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Topological Structure and Existence of Solutions Set for q-Fractional Differential Inclusion in Banach Space

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Abstract: In this work, we concentrate on the existence of the solutions set of the following problem ${}^cD_q^{\alpha}\sigma(t) \in F(t, \sigma(t), {}^cD_q^{\alpha}\sigma(t)), t \in I = [0, T]$ $\sigma(0) = \sigma_0 \in E$, as well as its topological structure in Banach space E. By transforming the problem posed into a fixed point problem, we provide the necessary conditions for the existence and compactness of solutions set. Finally, we present an example as an illustration of main results.

Keywords: Caputo fractional q-difference inclusion; measure of non-compactness; Darbo point theorem; selection theory

MSC: 34A60; 34K30; 34A08

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1. Introduction

One of the most important branches of modern mathematics is the study of the fractional differential equations and inclusion, which are considered as powerful and effective tools for studying many problems in science and engineering, thermodynamics, finance, astrophysics, bioengineering, hydrology, mathematical physics, biophysics, statistical mechanics, control theory, and cosmology, see [1–5] and its references mentioned.

Recently, many authors have been attracted by the study of fractional q-difference boundary value problems in Banach Spaces, for recent contributions are included in [6–13].

During the year 2020, the authors in [8], through the use of multi-valued analysis, Kuratowski measure of non-compactness and fixed-point theory on Banach space, they discussed the existence of solutions for the fractional q-differential inclusion of the form

$$ {}^cD_q^{\alpha}\sigma(t) \in F(t, \sigma(t)), t \in I = [0, T] $$

with

$$ \sigma(0) = \sigma_0 \in E, $$

where α, q are constants with $\alpha \in (0, 1], q \in (0, 1), T > 0, F : I \times E \to \mathcal{P}(E)$ is a is a multi-valued map, and ${}^cD_q^{\alpha}$ is the Caputo fractional q-difference derivative of order α. By employing some fixed point theorems in Banach spaces, the authors proved the existence of solution set defined on I.

During the following year, in [14], the author was given some conditions for the existence solution set and Filippov-type results for the fractional q-differential equation

$$ {}^cD_q^{\alpha}\sigma(t) \in F(t, \sigma(t), {}^cD_q^{\alpha}\sigma(t)), t \in [0, T] $$

with

$$ \sigma(0) = \sigma_0, $$

where $q \in (0,1)$ and $\alpha \in (0,1], T > 0, F : [0,T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multi-valued map, and $\mathcal{P}(\mathbb{R})$ is the family of all subsets of $\mathbb{R}$, and ${}^cD_q^\alpha$ is the Caputo fractional q-difference derivative of order α.

The purpose of this article is to study the q-fractional differential inclusion of the form:

$$ {}^cD_q^\alpha \sigma(t) \in F(t, \sigma(t), {}^c D_q^\alpha \sigma(t)), t \in I = [0,T] \quad (1)$$

$$\sigma(0) = \sigma_0 \in E, \quad (2)$$

where $(E, \|.\|)$ is a real or complex Banach space, $\alpha \in (0,1], q \in (0,1), T > 0, F : I \times E \times E \to \mathcal{P}(E)$ is a multi-valued map, and ${}^cD_q^\alpha$ is the Caputo fractional q-difference derivative of order α. By using the set-valued analysis, *Kuratowski* measure of non-compactness and *Darbo* fixed point theorem, we concentrate on the existence and the topological structure of the solutions set for the problem (1) and (2).

This work is structured as follows: in Section 2, we mention some theorems and lemmas which play an important role in our proofs. In Section 3, we present two results, the first obtained by combining the selection theory with *Kuratowski* measure of non-compactness, and the Darbo fixed-point theorem. For the second result, we study the compactness of the solution set for the problem (1) and (2). The last section is for an example as an illustration of our results.

2. Preliminaries

Firstly, we introduce some useful spaces. The classical Banach spaces $C(I,E) = \{\sigma : I \to E, \ \sigma \text{ is continuous functions}\}$, with the norm $\|\sigma\|_\infty = \sup_{t\in I}\{\|\sigma(t)\|, t \in I\}$, where $(E, \|.\|)$ is a separable Banach spaces. The space $L^1(I,E)$ of measurable functions $\varphi : I \to E$ which are Bochner integrable, normed by $\|\varphi\|_{L^1} = \int_I \|\varphi(t)\|dt$. We also use the Banach space $C_q^\alpha(I,E)$ defined by

$$C_q^\alpha(I,E) = \left\{\sigma : \sigma \in C(I,E), {}^c D_q^\alpha \sigma \in C(I,E)\right\},$$

equipped with the norm $\|\sigma\|_q = \max\left\{\|\sigma\|_\infty, \left\|{}^cD_q^\alpha \sigma\right\|_\infty\right\}$.

Now we mention some basic definitions, lemmas, and theorems related to multi-valued analysis that we need. Let (E,e) be a metric space generated by the normed space $(E, \|.\|)$. We denote by $\mathcal{P}_0(E) = \{S \in \mathcal{P}(E), S \neq \emptyset\}$, $\mathcal{P}_{cl}(E) = \{S \in \mathcal{P}_0(E) : S \text{ is closed}\}$, $\mathcal{P}_b(E) = \{S \in \mathcal{P}_0(E) : S \text{ is bounded}\}$, $\mathcal{P}_c(E) = \{S \in \mathcal{P}_0(E) : S \text{ is compact}\}$, $\mathcal{P}_v(E) = \{S \in \mathcal{P}_0(E) : S \text{ is convex}\}$, $\mathcal{P}_{cl,b}(E) = \mathcal{P}_{cl}(E) \cap \mathcal{P}_b(E)$.

Let the distance $H_e : \mathcal{P}(E) \times \mathcal{P}(E) \to \mathbb{R} \cup \{\infty\}$ given by

$$H_e(C,D) = \max\left\{\sup_{c\in C} e(c,D), \sup_{d\in D} e(d,C)\right\},$$

where $e(c,D) = \inf_{d\in D} e(c,d)$ and $e(d,C) = \inf_{c\in C} e(d,c)$, then $(\mathcal{P}_{b,cl}(E), H_e)$ is a metric space see [15].

Let E be a separable Banach space, $C \in \mathcal{P}_{cl}(E)$ and $F : C \to \mathcal{P}_{cl}(E)$ a multi-valued operator. F has convex (closed) values if $F(x)$ is convex (closed) for all $x \in E$. F is said to be upper semi-continuous $(u.s.c)$ at a point $c_0 \in C$ if for every open $O \subseteq C$, such that $F(c_0) \subset O$ there exists a neighborhood N of c_0, such that $F(N) \subset O$. F has a closed graph, that is, $x_n \to x, y_n \to y, y_n \in F(x_n)$ imply that $y \in F(x)$. We say that F is bounded on bounded sets if $F(\Omega)$ is bounded in E for each bounded set Ω of E (i.e., $\sup_{x\in\Omega}\{\sup\{\|\overline{x}\| : \overline{x} \in F(x)\}\} < +\infty$). F is completely continuous if $F(\Omega)$ is relatively compact for every $\Omega \in \mathcal{P}_b(X)$. Suppose that $F : C \to \mathcal{P}_c(E)$ is completely continuous, then F is upper semi-continuous (u,s,c), is equivalent to F has a closed graph. If $x \in F(x)$, we say that F has a fixed point in E. F is said to be measurable if the function $f : I \to \mathbb{R}$ defined by $f(t) = e(x, F(t)) = \inf\{\|x - y\| : y \in F(t)\}$ is measurable.

Lemma 1 ([16], *Thm*19, 7)**.** *Let E be a separable metric space and F a multi-valued map with non-empty closed values. Then F has a measurable selection.*

Definition 1. *$F : I \times E \times E \to \mathcal{P}(E)$ is Caratheodory multi-valued map, if:*
(1) $t \to F(t, x_1, x_2)$ *is measurable for each* $x_1, x_2 \in E$,
(2) $(x_1, x_2) \to F(t, x_1, x_2)$ *is upper semi-continuous for almost all* $t \in I$,
F is called L^1-Caratheodory if F is Caratheodory and,
(3) for each $r > 0$*, there exists* $\varphi_r \in L^1(I, \mathbb{R}^+)$*, such that*

$$\|F(t, x_1, x_2)\| = \sup\{\|x\|_\infty : x \in F(t, x_1, x_2)\} \leq \varphi_r(t).$$

For more details of the multi-valued analysis, we refer the reader to the following books [15,17–20].

Definition 2. *A function $\kappa : \mathcal{P}_b(E) \to \mathbb{R}^+$ is called a measure of non-compactness on E, if for each subsets $C, C_1, C_2 \in \mathcal{P}_b(E)$, the following conditions are hold :*
(1) $\kappa(C) = 0$ *if and only if C is precompact,*
(2) $\kappa(C) = \kappa(\overline{C})$,
(3) $\kappa(C_1 \cup C_2) = max\{\kappa(C_1), \kappa(C_2)\}$.

Let $\mathcal{B}_E$ the family of bounded subsets of a Banach space E.

Definition 3 ([21,22])**.** *The Kuratowski measure of non-compactness is defined as $\kappa : \mathcal{B}_E \to \mathbb{R}^+$, such that, $\kappa(C) = \inf\{\varepsilon > 0 | C \subset \cup_{i=1}^n C_i, diam(C_i) \leq \varepsilon\}, C \in \mathcal{B}_E$.*

Definition 4. *A multi-valued mapping $\Phi : E \to \mathcal{P}_{cl,b}(E)$ is said to be γ-Lipschitz, if there exists a constant $\gamma > 0$, such that $\kappa(\Phi(\Omega)) \leq \gamma\kappa(\Omega)$ for all closed bounded set Ω in E with $\Phi(\Omega)$ is a closed bounded set in E.*
If $\gamma < 1$, then Φ is called a γ-contraction on E.

Let us recall some definitions and properties of fractional q-calculus [23–27]. For $x \in \mathbb{R}$, let $q \in (0, 1)$

$$[x]_q = \frac{q^x - 1}{q - 1} = 1 + q + q^2 + ... + q^{x-1}, x \in \mathbb{R}.$$

The q-analogue of the power function $(x - y)^n$, $n \in \mathbb{N}$ is

$$(x - y)^0 = 1, (x - y)^n = \prod_{k=0}^{n-1} \left(x - yq^k\right), x, y \in \mathbb{R}, n \in \mathbb{N}.$$

If $\alpha \in \mathbb{R}$, then

$$(x - y)^{(\alpha)} = x^\alpha \prod_{k=0}^{+\infty} \frac{x - yq^i}{x - yq^{\alpha+i}}.$$

When $y = 0$, then

$$x^{(\alpha)} = x^\alpha.$$

The q-gamma function is given by:

$$\Gamma_q(\alpha) = \frac{(1 - q)^{(\alpha-1)}}{(1 - q)^{\alpha-1}}, \alpha \in \mathbb{R} \backslash \{..., -2, -1, 0, 1, 2, ...\}, 0 < q < 1$$

and verifies that

$$\Gamma_q(\alpha+1) = [\alpha]_q \Gamma_q(\alpha).$$

The q-derivative of a function $h(x)$ is defined by

$$D_q h(x) = \frac{d_q h(x)}{d_q x} = \frac{h(qx) - h(x)}{(q-1)x}, x \neq 0.$$

The higher order q-derivative of $h(x)$ is given as the following formula

$$D_q^n h(x) = \begin{cases} h(x), & \text{if } n = 0, \\ D_q D_q^{n-1} h(x), & \text{if } n \in \mathbb{N}. \end{cases}$$

Let h a function defined on $[0,b]$,the q-integral of is given by

$$\int_0^t h(x) d_q x = t(1-q) \sum_{n \geq 0} h(tq^n) q^n, 0 \leq |q| < 1,\ t \in [0,b].$$

If $a \in [0,b]$, then

$$\int_a^b h(x) d_q x = \int_0^b h(x) d_q x - \int_0^a h(x) d_q x.$$

Similarly as performed for derivatives, it can be defined an operator I_q^n, namely,

$$\left(I_q^0 h\right)(x) = h(x) \text{ and } \left(I_q^n h\right)(x) = I_q\left(I_q^{n-1} h\right)(x),\ n \in \mathbb{N}.$$

The fundamental theorem of calculus applies to these operators I_q and D_q, i.e

$$D_q\big(I_q h\big)(x) = h(x),$$

if h is continuous at $x = 0$, then

$$I_q\big(D_q h\big)(x) = h(x) - h(0).$$

For more information and basic properties of these operators, we recommend [28] to the reader.

Definition 5. *Let $\alpha \geq 0$ and h be a function defined on I. The fractional q-integral of the Riemann–Liouville type is*

$$\left(I_q^\alpha h\right)(x) = \begin{cases} h(x), & \text{if } \alpha = 0 \\ \frac{1}{\Gamma_q(\alpha)} \int_0^x (x - qs)^{(\alpha-1)} h(s) d_q s, & \text{if } \alpha > 0 \end{cases},\ x \in I.$$

Definition 6. *The fractional q-derivative of the Riemann–Liouville with order $\alpha \geq 0$ is defined by*

$$\left(D_q^\alpha h\right)(x) = \begin{cases} h(x), & \text{if } \alpha = 0 \\ \left(D_q^{[\alpha]}\left(I_q^{[\alpha]-\alpha} h\right)\right)(x), & \text{if } \alpha > 0 \end{cases},\ x \in I,$$

where $[.]$ is the smallest integer greater than or equal to α.

Definition 7. *The fractional q-derivative of Caputo with order $\alpha \geq 0$ is defined by*

$$\left({}^c D_q^\alpha h\right)(x) = \begin{cases} h(x), & \text{if } \alpha = 0 \\ \left(I_q^{[\alpha]-\alpha}\left(D_q^{[\alpha]} h\right)\right)(x), & \text{if } \alpha > 0 \end{cases},\ x \in I.$$

Lemma 2. *Let $\alpha \geq 0$. Then we have*

$$I_q^{\alpha}\left({}^{c}D_q^{\alpha}h\right)(x) = h(x) - \sum_{n=0}^{[\alpha]-\alpha} h(tq^n)\frac{t^n}{\Gamma_q(n+1)}\left(D_q^{\alpha}h\right)(0),$$

and if $\alpha \in (0,1)$, then

$$I_q^{\alpha}\left({}^{c}D_q^{\alpha}h\right)(x) = h(x) - h(0).$$

Now we give the Darbo fixed point theorems, which our results will be based on:

Theorem 1 ([29]). *Let E be a Banach space, a set $\mathcal{C} \in \mathcal{P}_{cl,b}(E) \cap \mathcal{P}_v(E)$ and let $\psi : \mathcal{C} \to \mathcal{P}_{cl,b}(\mathcal{C})$ be a closed and γ-contraction. Then ψ has a fixed point.*

3. Existence Results

In this section, by applying Darbo fixed point theorem [29] for multi-valued map, we prove the existence of solutions for the problem (1) and (2).

First, we introduce the definition of the solution of the problem (1) and (2).

Definition 8. *A function $\sigma \in C_q^{\alpha}(I,E)$ is called a solution of problem (1) and (2) if there exists a function $h \in L^1(I,\mathbb{R})$ with $h \in F(t,\sigma(t),{}^{c}D_q^{\alpha}\sigma(t))$, a.e.t $\in I$, such that ${}^{c}D_q^{\alpha}\sigma(t) = h(t)$, a.e.t $\in I$ and condition (2) is satisfied.*

Now, we assume the following assumptions:

(H_1) $F : I \times E \times E \to \mathcal{P}_c(E)$ be a L^1-Caratheodory multi-valued mapping.

(H_2) There exists a function $\phi \in L^1(I,\mathbb{R}^+)$, such that, for each set, $B_1, B_2 \in \mathcal{P}_{cl,b}(C(I,E))$ and $t \in I$, we have

$$\kappa(F(t,B_1(t),B_2(t))) \leq \phi(t)\max(\kappa(B_1(t)),\kappa(B_2(t))).$$

Theorem 2. *Assume that (H_1), (H_2), $\max\left\{\|\sigma_0\|_{\infty} + \varphi_r^* \frac{T^{(\alpha)}}{\Gamma_q(\alpha+1)}, \varphi_r^*\right\} \leq r$, and $\phi(t) \leq 1$, for each $t \in I$, hold, then the problem (1) and (2) has at least one solution in $C_q^{\alpha}(I,E)$, for all $t \in I$.*

Proof. For each $x \in C(I,E)$, define the set of selections of F by

$$S_{F,\sigma} = \left\{\varsigma \in L^1(I,E) : \varsigma(t) \in F(t,\sigma(t),{}^{c}D_q^{\alpha}\sigma(t)),\ \text{for all } t \in I\right\}.$$

Let for $r \in \mathbb{R}^+$, the set $\mathcal{C}_r \in \mathcal{P}_{cl,b}(C_q^{\alpha}(I,E)) \cap \mathcal{P}_v(C_q^{\alpha}(I,E))$, defined by

$$\mathcal{C}_r = \left\{\sigma \in C_q^{\alpha}(I,E),\ \|\sigma\|_q \leq r\right\}.$$

Now, we consider the multi-valued operator $\psi : C_q^{\alpha}(I,E) \to \mathcal{P}_{cl,b}\left(C_q^{\alpha}(I,E)\right)$ defined by

$$\psi(\sigma) = \left\{\rho \in C_q^{\alpha}(I,E) : \rho(t) = \sigma_0 + \left(I_q^{\alpha}\varsigma\right)(t),\ \text{for } \varsigma \in S_{F,\sigma}\right\}.$$

Observe that, for each $\sigma \in C_q^{\alpha}(I,E)$ then the set $S_{F,\sigma} \neq \emptyset$, by the hypothesis H_1, the multi-valued function F has a measurable selection. We shall prove that the operator ψ fulfills the conditions of Darbo fixed point theorem.

Step 1. We prove that $\psi(\sigma) \in \mathcal{P}_b(\mathcal{C}_r)$.

Let $\sigma \in \mathcal{C}_r$ and $\rho \in \psi(\sigma)$, then there exists $\varsigma \in S_{F,\sigma}$, such that for each $t \in I$, we have

$$\rho(t) = \sigma_0 + \left(I_q^{\alpha}\varsigma\right)(t).$$

then

$$\begin{aligned} \|\rho(t)\| &\leq \|\sigma\| + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \|\varsigma(s)\| d_q s \\ &\leq \|\sigma_0\| + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \varphi_r(s) d_q s \\ &\leq \|\sigma_0\| + \int_0^T \frac{(T-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \varphi_r(s) d_q s. \end{aligned}$$

Let $ess \sup \varphi_r = \varphi_r^*$, then

$$\|\rho\|_\infty \leq \|\sigma_0\|_\infty + \varphi_r^* \frac{T^{(\alpha)}}{\Gamma_q(\alpha+1)},$$

and

$$\begin{aligned} \left\| {}^c D_q^\alpha \rho(t) \right\| &= \|\varsigma(t)\|, \\ &\leq \varphi_r(t), \\ &\leq \varphi_r^*. \end{aligned}$$

Then

$$\left\| {}^c D_q^\alpha \rho \right\|_\infty \leq \varphi_r^*,$$

So,

$$\|\rho\|_q \leq \max\left\{ \|\sigma_0\|_\infty + \varphi_r^* \frac{T^{(\alpha)}}{\Gamma_q(\alpha+1)}, \varphi_r^* \right\} \leq r.$$

Step 2. We show that $\psi(\sigma) \in \mathcal{P}_{cl}(\mathcal{C}_r)$.

Let $\{\rho_n\}_{n\in\mathbb{N}}$ a sequence in $\psi(\sigma)$, such that $\rho_n \to \rho$ $(n \to \infty)$ in $C_q^\alpha(I, E)$. Then for each $t \in I$ there exists $\varsigma_n \in S_{F,\sigma}$ such that

$$\rho_n(t) = \sigma_0 + I_q^\alpha \varsigma_n(t),$$

As F has compact values, we pass on to a subsequence to get that ς_n converge to ς in $L^1(I \times E)$. Thus $\varsigma \in S_{F,\sigma}$ and for each $t \in I$,

$$\rho_n(t) \to \rho(t)$$

with $\rho(t) = \sigma_0 + I_q^\alpha \varsigma(s)$. Hence $\rho \in \psi(\sigma)$, then $\psi(\sigma)$ is closed in $\mathcal{C}_r$ for each $\sigma \in \mathcal{C}_r$.

Step 3. We prove that ψ a is γ-contraction.

Let $B \in \mathcal{P}_{cl,b}(\mathcal{C}_r)$, then for each $t \in I$, we have

$$\kappa(\psi(B)) = \kappa\{\psi(\sigma) : \sigma \in B\}.$$

Let $\rho \in \psi(\sigma)$ Then there exists $\varsigma \in S_{F,\sigma}$ such that, for each $t \in I$,

$$\rho(t) = \sigma_0 + I_q^\alpha \varsigma(t).$$

For each x and ${}^c D_q^\alpha x \in B$, we have

$$\begin{aligned} \kappa(\psi(B)(t)) &= \kappa\Big(\rho \in C_q^\alpha(I, E) : \rho(t) \in F(t, \sigma(t), {}^c D_q^\alpha \sigma(t))\Big) \\ &\leq \kappa\Big(F(t, \sigma(t), {}^c D_q^\alpha \sigma(t))\Big) \\ &\leq \phi(t)\kappa(B), \end{aligned}$$

so the operator ψ is a γ-contraction. By the Theorem 1, we deduce that ψ has a fixed point that is a solution of the problem (1) and (2). $\square$

Now, we give some conditions that guarantee the compactness of solutions set for our problem.

Theorem 3. *Let* (H_1) *holds. Then the set* $\mathcal{S} = \{\sigma \in C_q^\alpha(I, E) : \sigma$ *is solution of the problem* (1) *and* (2)$\}$ *is an element of* $\mathcal{P}_c(C_q^\alpha(I, E))$.

Proof. From Theorem 2, the set $\mathcal{S}$ is not empty. Now, we prove that $\mathcal{S} \in \mathcal{P}_c(C_q^\alpha(I, E))$. Let $(\sigma_n)_{n\in\mathbb{N}} \in \mathcal{S}$, then there exist $\varsigma_n \in S_{F,\sigma_n}$ such that

$$\sigma_n(t) = \sigma_0 + I_q^\alpha \varsigma_n(t).$$

Step 1. We show that the set $\{\sigma_n, n \in \mathbb{N}\}$ is equicontinuous in $C_q^\alpha(I, E)$.

Let $t_1, t_2 \in I$, with $t_1 < t_2$, we obtain

$$\begin{aligned}
\|\sigma_n(t_2) - \sigma_n(t_1)\| &= \left\| I_q^\alpha \varsigma_n(t_2) - I_q^\alpha \varsigma_n(t_1) \right\| \\
&\leq \frac{1}{\Gamma_q(\alpha)} \left(\int_0^{t_1} \left| (t_2 - qs)^{(\alpha-1)} - (t_2 - qs)^{(\alpha-1)} \right| \|\varsigma_n(s)\| d_q s + \right. \\
&\quad \left. \int_{t_1}^{t_2} \left| (t_2 - qs)^{(\alpha-1)} \right| \|\varsigma_n(s)\| d_q s \right) \\
&\leq \frac{1}{\Gamma_q(\alpha)} \left(\int_0^{t_1} \left| (t_2 - qs)^{(\alpha-1)} - (t_2 - qs)^{(\alpha-1)} \right| \varphi_r(s) d_q s + \right. \\
&\quad \left. \int_{t_1}^{t_2} \left| (t_2 - qs)^{(\alpha-1)} \right| \varphi_r(s) d_q s \right)
\end{aligned}$$

and

$$\left\| {}^c D_q^\alpha \sigma_n(t_2) - {}^c D_q^\alpha \sigma_n(t_1) \right\| = \|\varsigma_n(t_2) - \varsigma_n(t_1)\|.$$

Then, when $t_2 \to t_1$, we get

$$\|\sigma_n(t_2) - \sigma_n(t_1)\|_q \to 0.$$

With the theorem of Arzela–Ascoli, we conclude that, there exists a subsequence $\{\sigma_{n_k}\}$, such that σ_{n_k} converges to some σ in $C_q^\alpha(I, E)$. Now we prove that there exists $\varsigma(.) \in F(., \sigma(.), {}^c D_q^\alpha \sigma(.))$, such that

$$\sigma(t) = \sigma_0 + I_q^\alpha \varsigma(t).$$

Since $F(t, ., .)$ is upper semi-continuous, then for every $\varepsilon > 0$, there exists $n_0(\varepsilon)$, such that for every $n \geq n_0$, we have

$$\varsigma_n(t) \in F(t, \sigma_n(t), {}^c D_q^\alpha \sigma_n(t)) \subset F(t, \sigma(t), {}^c D_q^\alpha \sigma(t)) + B(0, \varepsilon), \ a.e.t \in I.$$

As $F(., ., .) \in \mathcal{P}_c(C_q^\alpha(I, E))$ then there exists a subsequence ς_{n_m}, such that

$$\varsigma_{n_m}(.) \to \varsigma(.) \text{ as } m \to +\infty$$

and

$$\varsigma(t) \in F(t, \sigma(t), {}^c D_q^\alpha \sigma(t)), \ a.e.t \in I \text{ for all } m \in \mathbb{N}.$$

Since, $\varsigma_{n_m}(t) \leq \varphi_r(t)$, $a.e.t \in I$, Lebesgue's Dominated Convergence Theorem give us that $\varsigma(t) \in L^1(I \times E)$ implies $\varsigma \in S_{F,\sigma}$. Therefore, $\sigma(t) = \sigma_0 + I_q^\alpha \varsigma(t)$. So $\mathcal{S} \in \mathcal{P}_c\left(C_q^\alpha(I, E)\right)$. $\square$

4. An Example

Now, we give an example as an illustration of the results obtained in Theorem 2 and Theorem 3.

Example 1. *Let $E = C([0,1])$ be the Banach space of all real continuous function on $[0,1]$ equipped with the norm*

$$\|f\| = \sup_{t\in[0,1]} \{|f(t)|\}.$$

Now we consider the q-fractional differential inclusion, given by:

$$^cD_{0.5}^{0.2}\sigma(t) \in F(t,\sigma(t),^c D_{0.5}^{0.2}\sigma(t)), t \in I = [0,1] \tag{3}$$

$$\sigma(0) = t\cosh(t). \tag{4}$$

where $\alpha = 0.2$, $q = 0.5$, $T = 1$, and

$$F(t,\sigma(t),^c D_{0.5}^{0.2}\sigma(t)) = \frac{1}{\|\sigma\|_q}\left(\frac{1}{1+t+e^t}\right).\left\{f \in C([0,1]) : \|f\| \le \|\sigma\|_q\right\}.$$

Let

$$\mathcal{C}_3 = \left\{\sigma \in C_{0.5}^{0.2}(I,E) : \|\sigma\|_q \le 3\right\}.$$

For each $\sigma \in E$ and $t \in I$, we have

$$\|F(t,\sigma,\sigma_1)\| \le \frac{1}{1+t+e^t} = \varphi_3(t) \textit{ implies } \varphi_3^* = 0.5$$

and for each $B \in \mathcal{P}_{cl,b}(\mathcal{C}_3)$, we get

$$\kappa(\psi(B)(t)) \le e^{t-2}\kappa(B) = \phi(t)\kappa(B),$$

$$\max\left\{\|\sigma_0\|_\infty + \varphi_3^*\frac{T^{(0.2)}}{\Gamma_{0.5}(0.2+1)}, \varphi_3^*\right\} \approx 2.07 < 3,$$

and

$$\phi(t) \le 1, \textit{for all } t \in I.$$

Then Theorem 2 and Theorem 3 guarantee that the set of solutions of the problem (3) and (4) is not empty and also is compact.

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