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# Quantum Computing in Insurance Capital Modelling

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**Abstract:** This paper proposes a quantum computing approach for insurance capital modelling. Using an open-source software development kit, Qiskit, an algorithm for working on a superconducting type IBM quantum computer is developed and implemented to predict the capital of insurance companies in the classical surplus process. With the fundamental properties of quantum mechanics, Dirac notation and Feynman's path calculation are shown. Furthermore, custom quantum insurance premium and claim gates are investigated in order to build a quantum circuit with respect to initial reserve, premium and claim amounts. Some numerical results are presented and discussed at the end of the paper.

**Keywords:** insurance; quantum computing; risk theory; Qiskit; capital modelling

**MSC:** 58D30; 81P68; 91B30

## 1. Introduction

Despite continuing development in the field of classical computers, we are approaching a technological barrier. According to Moore's Law, the number of transistors, bits (0,1) doubles approximately every two years in an integrated circuit [1]. In 2005, Gordon Moore, co-founder of Intel, said "In terms of size (of transistors), you can see that we're approaching the size of atoms which is a fundamental barrier" [2]. Certain computational problems cannot be handled in a reasonable time in classical computers, including supercomputers. Solving such problems on quantum computers is called quantum advantage [3]. The technological barrier and difficulties in handling complex problems in conventional computers encourages us to focus on quantum computing and emerging technologies.

Another reason for quantum computing is cybersecurity, which is becoming more popular with the expansion of new technologies for secure communication. The cyber insurance industry is predicted to exceed USD 20 billion in gross written premium by 2025 [4]. In cybersecurity, application, network, and information securities are the main components. To reduce vulnerabilities in these areas, specifically, in eavesdropping, quantum cryptography plays a significant role, so to evaluate the exposure of cyber risk, quantum technologies should be observed. Although the main target of this paper is not cyber insurance or quantum cryptography, fundamental quantum properties, such as superposition, measurement, and entanglement used in quantum cryptography are explored.

A quantum computer leverages certain features of quantum mechanics to solve complex problems with high-speed. Thus, we should encompass quantum mechanics, quantum information theory and computer science in order to understand quantum computers [5]. The technology of today's quantum computers is produced via different approaches according to qubit types. Therefore, we have superconducting, photonics, topological, trapped-ion, and spin-qubits-based quantum computers today. To use quantum computers, several frameworks are available such as Qiskit, Cirq, and QDK, produced by IBM, Google, and Microsoft, respectively. In this research, Qiskit as an open-source software development kit [6] is used to predict the capital of an insurance company.

While the properties and representations of quantum mechanics are prominent in many disciplines, they are quite new in actuarial mathematics. Quantum mechanics,



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specifically Dirac Notation and Feyman's Path calculation, were used in the computation of ruin probability for insurance companies in 2018 [7], reinsurance optimisation in 2019 [8], analysis of insurance quantum data in 2018 [9] and CAT modelling in 2020 [10]. However, in these studies, quantum computers were not used. In this paper, the main aim is to show how to develop quantum algorithms, and use quantum computers in actuarial computations, which is also our primary novel contribution to the field.

## 2. Fundamentals and Notations of Quantum Mechanics

In quantum computing, unlike the classical bits $(0, 1)$, qubits are used as the fundamental computational unit. Qubits are represented using Dirac notation (or Bra-Ket notation) as $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. A ket vector $|x\rangle$ describes a quantum state. Its Hermitian conjugate is represented by bra vector $\langle x|$. Matrix and geometrical representation of qubits on a Bloch sphere [6] are displayed in Table 1. The definitions and notations needed in this paper are given in Table 2.

**Table 1.** Visualization of qubits on Qiskit.

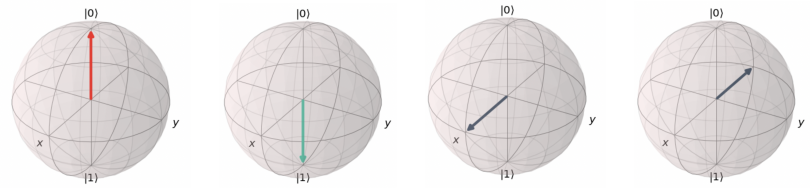
| | $\|\mathbf{0}\rangle$ | $\|\mathbf{1}\rangle$ | $\|+\rangle$ | $\|-\rangle$ |
|---|---|---|---|---|
| Bloch Sphere Visualization |  | | | |
| Matrix Representation | $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ | $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ | $\frac{1}{\sqrt{2}}(\|0\rangle + \|1\rangle)$ | $\frac{1}{\sqrt{2}}(\|0\rangle - \|1\rangle)$ |

**Table 2.** Fundamental notations.

| Notations | Definitions | Notations | Definitions |
|---|---|---|---|
| $\|\psi\rangle$ | Ket, represents a quantum state | $\mathcal{H}$ | Hilbert Space |
| $\langle\psi\|$ | Bra, transpose of Ket | $\otimes$ | Tensor Product |
| $\|+\rangle, \|-\rangle$ | Superposition State | $\langle x, y\rangle$ | Inner Product |
| $H$ | Hadamard Gate | $t$ | Time |
| $X$ | X Gate | $I$ | Identity Operator |
| $Y$ | Y Gate | $\mathbb{P}$ | Probability |
| $Z$ | Z Gate | $i$ | Complex unit |
| $PG$ | Premium Gate | $M$ | Measure Operator |
| $CG$ | Claim Gate | $\mathbb{C}$ | Complex Number |

Three properties of quantum mechanics play a significant role in quantum computing. They are superposition, entanglement, and interference. In Hilbert space, linear combination of state vectors creates another state vector.

$$|\psi\rangle = \alpha_1 |\psi_1\rangle + \alpha_2 |\psi_2\rangle + \dots \tag{1}$$

This linear combination is called superposition. Superposition tells us all states in the system are possible at the same time. For example, superposition of two states $|0\rangle, |1\rangle$ is

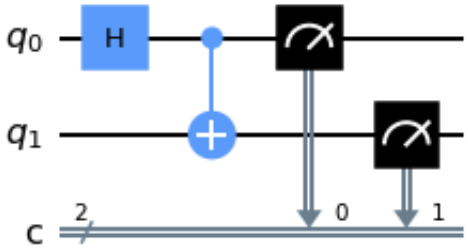
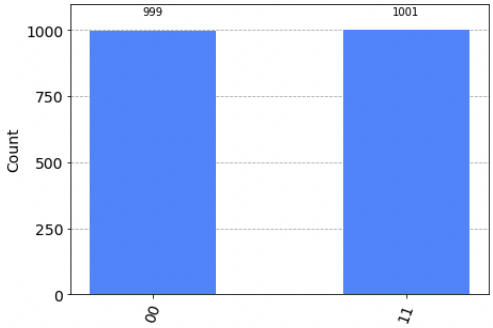
$$\alpha |0\rangle + \beta |1\rangle,$$

where coefficients of the qubits ($\alpha$ and $\beta$) are called probability amplitudes of the system, and modules squared of these amplitudes give probability of the state, so the sum of all probabilities should equal to 1. This is called the Born rule. For two states, the Born rule is

$$|\alpha|^2 + |\beta|^2 = 1. \tag{2}$$

Another principle of quantum mechanics is entanglement. According to the principle, quantum states in entangled situations cannot be described independently. In other words, states of an objects (electron or photon) are correlated as either $|00\rangle$ or $|11\rangle$. In 2022, the Nobel prize was given to researchers for their work on entangled photons [11]. To demonstrate the entanglement situation [12], superposition can be created by applying Hadamard gate onto the first qubit, and $CX$ gate (also known as CNOT gate) onto both qubits as seen in Table 3. In Tables 4 and 5, quantum gates for single and two qubits are displayed.

**Table 3.** A quantum circuit producing an entangled state.

| Quantum Circuit | Result |
|---|---|
|  |  |



Mathematical formulation of quantum entanglement in Table 3 is carried out as

$$\begin{aligned} CX(|0\rangle \otimes (H|0\rangle)) &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \left[ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \left[ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] \right] \\ &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} |00\rangle + \frac{1}{\sqrt{2}} |11\rangle, \end{aligned} \tag{3}$$

where $CX$ gate is chosen according to the qubit order ($|q_1 q_0\rangle$), and the tensor product of two vectors in $\mathbb{C}^2 \otimes \mathbb{C}^2$ form the following matrix in $\mathbb{C}^4$ as

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \otimes \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} v_1 \times w_1 \\ v_1 \times w_2 \\ v_2 \times w_1 \\ v_2 \times w_2 \end{pmatrix}. \tag{4}$$

As seen from Table 3 and Equation (3), we obtain a combined state that is a superposition of $|00\rangle$ state with 50% probability and $|11\rangle$ state with 50% probability. This is also known as a Bell state. Measuring one of the qubits tells us the state of the other qubit. This is also verified by the simulation on the Qiskit environment.

Superposition, entanglement, and quantum measurement are essential in encrypting and decrypting keys in cybersecurity to create the Quantum Key Distribution (QKD)

protocol among applications for secure communication. In quantum computing, when the system is measured, the superposition is collapsed and the entanglement is broken, which allows us to detect the presence of unwanted third parties (eavesdroppers) on the channel. This is one of the main advantages of quantum computing compared with classical computing. Even though many quantum algorithms cannot be used in a classical environment, QDK is compatible to classical computing and so can be used immediately. There are several different protocols based on measurement and entanglement types, such as B92, E91, and BB84 in quantum communication. However, this is beyond the scope of this paper.

In the interference principle, quantum states of a system can be cancelled or added to each other, which can be categorised as constructive interference and destructive interference [12]. The interference property in insurance modelling can be observed by the premium and claim gates, which change the quantum state representing the capital of the insurance company, in either up or down directions as long as premium and claim amounts are not the same.

**Table 4.** Quantum gates for a single qubit.

| ***X* Gate** | ***Y* Gate** | ***Z* Gate** | ***H* Gate** | ***S* Gate** | ***I* Gate** |
|---|---|---|---|---|---|
| $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ | $\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ | $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ | $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ | $\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$ | $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ |

**Table 5.** Quantum gates for two qubits.

| ***CX*, *CNOT* Gate** | ***CY* Gate** | ***CZ* Gate** | ***H* Gate** | ***I* Gate** |
|---|---|---|---|---|
| $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ | $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}$ | $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$ | $\frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$ | $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ |

Measurement operators are not unitary operators in quantum computing. When a Hermitian operator is measured in the state $|\psi\rangle = \sum_{i=1}^{N} \alpha_i |\psi_i\rangle$, the output spectrum is set of the eigenvalues $p_i$ of the Hermitian operator with probability $|\alpha_i|^2$. For a single qubit, the measurement operators are $M_0 = |0\rangle \langle 0|$ and $M_1 = |1\rangle \langle 1|$. The measurement operators satisfy the completeness equation [13] by

$$M_0^\dagger M_0 + M_1^\dagger M_1 = I, \tag{5}$$

where $I$ is the identity operator, and $M_0^\dagger$ and $M_1^\dagger$ are the adjoint operators associated with $M_0$ and $M_1$. When the system is measured by $M_0$ and $M_1$ projections, the result is either $|0\rangle$ with probability $|\alpha_1|^2$ or $|1\rangle$ with probability $|\alpha_2|^2$ correspondingly [14].

$$\langle\psi| M_0 |\psi\rangle = \begin{pmatrix} \alpha_1 & \alpha_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} \alpha_1 & \alpha_2 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ 0 \end{pmatrix} = |\alpha_1|^2, \tag{6}$$

$$\langle\psi| M_1 |\psi\rangle = \begin{pmatrix} \alpha_1 & \alpha_2 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} \alpha_1 & \alpha_2 \end{pmatrix} \begin{pmatrix} 0 \\ \alpha_2 \end{pmatrix} = |\alpha_2|^2. \tag{7}$$

Alternatively, for the Hermitian operators, measuring $|\psi\rangle$ with the help of density matrix and trace function is given by

$$\langle\psi| M_0 |\psi\rangle = tr(\rho M_0) \qquad \text{and} \qquad \langle\psi| M_1 |\psi\rangle = tr(\rho M_1),$$

where the trace function $tr(\rho M)$ is the sum of diagonal elements of $\rho M$, and the corresponding density matrix is defined by

$$\begin{aligned}\rho = |\psi\rangle\langle\psi| &= (\alpha_1|0\rangle + \alpha_2|1\rangle)(\alpha_1\langle 0| + \alpha_2\langle 1|) \\ &= \begin{pmatrix} \alpha_1^2 & \alpha_1\alpha_2 \\ \alpha_1\alpha_2 & \alpha_2^2 \end{pmatrix}. \end{aligned} \tag{8}$$

Then, $tr(\rho M_0) = tr(\begin{pmatrix} \alpha_1^2 & \alpha_1\alpha_2 \\ \alpha_1\alpha_2 & \alpha_2^2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}) = |\alpha_1|^2$. Similarly, $tr(\rho M_1) = |\alpha_2|^2$.

Physically, gates are transistors in a circuit, which are used to convert the inputs into outputs in the the form of electrical pulses. Mathematically, quantum logic gates are represented by unitary matrices in quantum computing, so they are reversible, unlike most of the classical logic gates. Quantum gates are used to manipulate qubits. For example, for a single-qubit system,

$$X|0\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |1\rangle, \qquad X|1\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle,$$

$$Z|0\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle, \qquad Z|1\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} = -|1\rangle.$$

In order to bring a qubit into a superposition state, Hadamard gate needs to be applied.

$$H|0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \quad H|1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle).$$

In the two-level quantum system, gates in Table 5 can be applied to the following computational basis states: $|00\rangle, |01\rangle, |10\rangle$, and $|11\rangle$, as shown in the following example.

$$CX|10\rangle = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = |11\rangle, \qquad CX|11\rangle = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = |10\rangle,$$

where $|10\rangle = |1\rangle \otimes |0\rangle$ and $|11\rangle = |1\rangle \otimes |1\rangle$.

### 3. Classical Surplus Process with Quantum Mechanics

The classical surplus process of an insurance company [15,16] with the initial reserve $(R(0) = u)$, insurance premium amount $(c)$ per unit time, and total claim amount $(S(t))$ at time $t$ is given by

$$R(t) = R(0) + ct - S(t), \tag{9}$$

where $S(t) = \sum_{i=1}^{N(t)} X_i, \quad t \geq 0$ is a compound Poisson process with sequence of positive integer-valued independent and identically distributed claim amounts $\{X_i\}$ and the number of insurance claims $N(t)$ until time $t$, which is a Poisson process with constant frequency rate $\lambda$. In this simple insurance random walk, the change in the capital of the insurance company during the unit time $\Delta t$ is written by

$$R(t+\Delta t) - R(t) = u + c(t+\Delta t) - S(t+\Delta t) - (u + ct - S(t)) = c\Delta t - S(\Delta t). \tag{10}$$

Suppose that the insurance company evaluates its reserve at fixed periodic time $(\Delta t)$, and the insurance premium is collected in advance. Even though claims arise at any time in $(t, t+\Delta t)$ but can only be registered at the end of the time period $(t+\Delta t)$.

The probability of the change with Poisson arriving process can be computed by

$$\begin{aligned}\mathbb{P}(R(t) \to R(t+\Delta t)) &= \mathbb{P}(R(t+\Delta t) - R(t) = c\Delta t - s) \\ &= \frac{e^{-\lambda\Delta t}\lambda\Delta t}{1!}\mathbb{P}(X_1 = s) + \frac{e^{-\lambda\Delta t}(\lambda\Delta t)^2}{2!}\mathbb{P}(X_1 + X_2 = s) + \ldots \quad \text{for} \quad s > 0, \end{aligned} \tag{11}$$

where $s = S(\Delta t)$ is the total claim amount at the interval $(t, t+\Delta t)$.

Let $x_i$ be level of a system at time $t_i, i = 0, 1, 2, \ldots$ Then, the transition probability of $\mathbb{P}(x_i \to x_{i+1})$ can be computed with a Markovian operator $P$ as $\langle x_i| P |x_{i+1}\rangle$ [17,18].

$\langle x_i| P |x_{i+1}\rangle$ is called a propagator that is used in calculating probability amplitude for the particle to travel in a given space time from point $(x_i, t_i)$ to point $(x_{i+1}, t_{i+1})$ in physics. In finance, the propagator helps us to calculate the probability of change in capital over a specific period of time.

When a path is taken into account [19], the probability is;

$$\mathbb{P}(x_0 \to x_1 \to x_2 \to \ldots \to x_n) = \langle x_0| P(t_1) |x_1\rangle \langle x_1| P(t_2 - t_1) |x_2\rangle \ldots \langle x_{n-1}| P(t - t_{n-1}) |x_n\rangle . \tag{12}$$

With Feynman path calculation and Dirac notations [20–22], probability of the reserve of insurance company at time $t$ can be modelled for all possible paths in the continuous space by

$$\begin{aligned}\mathbb{P}(R(t) = x_n | R(0) = u) = (1+o) \int_0^\infty \int_0^\infty \cdots \int_0^\infty dx_1 dx_2 \ldots dx_{n-1} \langle x_0| P(t_1) |x_1\rangle \langle x_1| P(t_2 - t_1) |x_2\rangle \\ \ldots \langle x_{n-1}| P(t - t_{n-1}) |x_n\rangle , \end{aligned} \tag{13}$$

where $x_i$ is the reserve of the insurance company at time $t_i$; $i = 0, 1, 2 \ldots$ $|x_i\rangle$ is a column vector that represents reserve states in quantum modelling; and $o$ is the error margin, which depends on the numerical approximation of the integrals. The numerical error is independent from the errors caused by environmental disruptors in quantum computers, which is one of the main challenges to manufacture more useful quantum computers, and this is beyond the scope of this research.

The continuity of space in quantum computing should be disregarded [23]. Since quantum objects are discrete, the probability for discrete reserves of the insurance company is carried out by

$$\begin{aligned}\mathbb{P}(R(t) = x_n | R(0) = u) = (1+o) \sum_{x_1} \langle u| P(t_1) |x_1\rangle \sum_{x_2} \langle x_1| P(t_2 - t_1) |x_2\rangle \sum_{x_3} \langle x_2| P(t_3 - t_2) |x_3\rangle \\ \cdots \sum_{x_{n-1}} \langle x_{n-2}| P(t_{n-1} - t_{n-2}) |x_{n-1}\rangle \langle x_{n-1}| P(t - t_{n-1}) |x_n\rangle , \end{aligned} \tag{14}$$

where $P$ is an operator with Hamiltonian operator $H$ providing $P(t_n - t_{n-1}) = e^{-(t_n - t_{n-1})H}$. In general, the Hamiltonian operator is equal to the minus generator operator in the Markovian process $H = -Q$ stated in [7,10]. The generator operator

$$P(0) = \lim_{t \to 0} P(t) = I \qquad \text{and} \qquad Q = \lim_{\Delta t \to 0} \frac{P(\Delta t) - I}{\Delta t}.$$

Throughout this paper, all quantum works is handled in Hilbert Space as a complete vector space, which is separable if its basis is countable regardless of its dimension [24].

With eigenvalue $K_p$ and eigenvector $|p\rangle$ of the Hamiltonian operator, a propagator with completeness equation is written by

$$\langle x_i| e^{-\Delta t H} |x_{i+1}\rangle = \int_0^{2\pi} \frac{dp}{2\pi} \langle x_i| e^{-\Delta t H} |p\rangle \langle p| x_{i+1}\rangle = \frac{1}{2\pi} \int_0^{2\pi} (e^{ix_i p} e^{-ix_{i+1} p}) e^{-\Delta t K_p} dp, \tag{15}$$

where $i$ is a complex unit. The completeness equation is $\int_0^{2\pi} \frac{dp}{2\pi} |p\rangle \langle p| = I$ with momentum basis $|p\rangle$. The inner products can be defined by

$$\langle x|p\rangle = e^{ipx} \qquad \text{and} \qquad \langle p|x\rangle = e^{-ipx}.$$

The relationship between the Hamiltonian operator, the eigenvector, and the eigenvalue can be pointed out as

$$H|p\rangle = K_p|p\rangle. \tag{16}$$

How to compute propagators according to different Hamiltonian operators and different claim distributions was investigated by Tamturk and Utev in [7,8]. Furthermore, the expected reserve of the insurance company and ruin probability were computed. To use quantum computers, initial quantum state should be defined, and quantum premium and claim gates as reversible unitary matrices need to be set out, which are handled in the next section.

## 4. Quantum Algorithm to Predict Insurance Capital

In computations on quantum computers, insurance premium and claim gates should be defined. All operators in quantum computing are reversible except the measurement operator [12]. In quantum computing, the insurance premium and claim gates are both reversible and unitary matrices. Matrix $U$ is unitary if $UU^\dagger = U^\dagger U = I$.

Determination of how many qubits the system needs is the first step in the quantum computing. In 2022, IBM released a new 433 qubit quantum processor, Osprey [25]. How many qubits we need in the quantum circuit depends on all possible states that the insurance company's capital can reach. If we ignore the negative scenarios, the qubits should be chosen based on the maximum capital value during the time and quantum state grid size $\varepsilon$. According to the classical surplus process mentioned in Section 3, the maximum capital at time $t$ is $max\{R(t)\} = u + ct$ with a no-claim case $S(t) = 0$, then we need $n$ qubits that satisfy the following condition

$$n \geqslant min\{\bar{n} | \frac{u+ct}{\varepsilon} + 1 \leqslant 2^{\bar{n}}\}, \tag{17}$$

where $\varepsilon = 1$ is taken since we work with integer reserve, $\frac{u+ct}{\varepsilon}$ represents the number of positive states, and 1 is added due to the zero-state. $n$ qubits, two-level system can handle $2^n$ states representing $\{0, 1, 2, \ldots, 2^n - 1\}$ capital levels. In Hilbert space, states of the qubits are represented as vectors. By default, qubits are initialised in zero state $|0\rangle$ in Qiskit. Therefore, the $n$ qubits should be modified according to the initial capital of the insurance company by using $X$ gate. As mentioned Section 2,

$$X|0\rangle = |1\rangle \qquad \text{and} \qquad X|1\rangle = |0\rangle.$$

The initial capital should be converted into binary form to create a ket vector as

$$u \to |q_n q_{n-1} \cdots q_0\rangle, \quad \text{where} \quad u = q_n 2^n + q_{n-1} 2^{n-1} + \ldots + q_0 2^0 \quad \text{and} \quad q_i \in \{0, 1\}. \tag{18}$$

To illustrate, the number of the qubits $(n)$ is 3 for $u = 3, c = 1$, and $t = 2$. Then, the initial reserve $(u)$ should be taken as $|011\rangle$. In Table 6, some initial capitals and their corresponding ket vectors for 8 qubits are listed.

**Table 6.** Initial capitals and corresponding basis vectors for 8 qubits.

| Initial Capital | Corresponding Basis Vector |
|---|---|
| 0 | $\|00000000\rangle$ |
| 1 | $\|00000001\rangle$ |
| 25 | $\|00011001\rangle$ |
| 65 | $\|01000001\rangle$ |
| 125 | $\|01111101\rangle$ |

After the creation of qubits to represent the initial reserve, the next step is to apply premium and claim gates for all qubits $\frac{t}{\Delta t}$ times in order. Let $PG$ and $CG$ be the insurance premium and claim gates correspondingly. Then,

$$CG(PG(\ldots(CG(PG(q_n \otimes (\ldots(q_2 \otimes ((Xq_1) \otimes (Xq_0)))))))))), \tag{19}$$

where $PG$ is fixed, not probabilistic. However, $CG$ is randomly generated each time. If the insurance claim is zero at the time interval, then the claim gate will be an identity matrix $(CG = I)$. For 3 qubits, $2^3 \times 2^3$ dimensional quantum premium and claim gates can be defined as in Tables 7 and 8.

**Table 7.** Quantum premium gates.

| $u = 2, t = 1, c = 1$ | $u = 2, t = 1, c = 2$ | $u = 2, t = 1, c = 3$ |
|---|---|---|
| $\begin{pmatrix} 0&0&0&0&0&0&0&1 \\ 1&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&1&0 \end{pmatrix}$ | $\begin{pmatrix} 0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&1 \\ 1&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&1&0&0 \end{pmatrix}$ | $\begin{pmatrix} 0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&1 \\ 1&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \end{pmatrix}$ |

**Table 8.** Quantum claim gates.

| $X = 1$ | $X = 2$ | $X = 3$ |
|---|---|---|
| $\begin{pmatrix} 0&1&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&1 \\ 1&0&0&0&0&0&0&0 \end{pmatrix}$ | $\begin{pmatrix} 0&0&1&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&1 \\ 1&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \end{pmatrix}$ | $\begin{pmatrix} 0&0&0&1&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&1 \\ 1&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \end{pmatrix}$ |

The premium and claim gates are created as a unitary matrix with $PGPG^\dagger = PG^\dagger PG = I$ and $CGCG^\dagger = CG^\dagger CG = I$.

As seen in Tables 7 and 8, the premium and claim gates for $n$ qubits can be generalised as

$$\text{Premium Gate} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ \vdots \\ c \\ c+1 \\ c+2 \\ \vdots \\ 2^n \end{array} \begin{array}{c} \begin{array}{ccccccccc} 2^n & 2^n-1 & \cdots & c+1 & c & c-1 & c-2 & \cdots & 1 \end{array} \\ \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \end{array}$$

and

$$\text{Claim Gate} = \begin{array}{c} \\ 2^n \\ 2^n-1 \\ 2^n-2 \\ \vdots \\ X+1 \\ X \\ X-1 \\ \vdots \\ 1 \end{array} \begin{array}{c} \begin{array}{ccccccccc} 1 & 2 & \cdots & X & X+1 & X+2 & X+3 & \cdots & 2^n \end{array} \\ \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \end{array}.$$

After applying all the gates, the system can be measured at time $t$. The measurement operator is not a reversible operator. After measuring of the system, all superpositions and entanglement situations are collapsed into one of the basis states 0 or 1. For the example: $u = 3, c = 1$, and $t = 2$, mathematical representation of the quantum circuit is given by

$$CG(PG(CG(PG(q_2 \otimes ((Xq_1) \otimes (Xq_0)))))). \tag{20}$$

After the measurement, the capital of the insurance company at time $t$ will be distributed in the following spectrum:

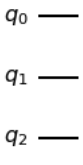
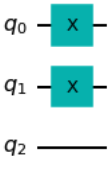
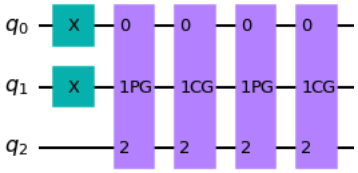
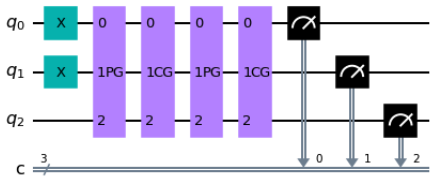
$$|000\rangle, |001\rangle, |010\rangle, |011\rangle, |100\rangle, |101\rangle, |110\rangle, |111\rangle$$

which corresponds to $R(t) \in \{0, 1, 2, 3, 4, 5, 6, 7\}$.

As a final step, a simulation of the quantum circuit is run to get random results. If the quantum circuit does not include any superposition, then the simulation produces the same results. Steps mentioned so far are displayed in Table 9.

To take changes in insurance risk behaviours, as a result of climate change, war, pandemics, and so on, into account, random noise approach using Hadamard gates can be considered. For instance, let's assume that the initial capital of the insurance company is 200, the premium is 20, and the claim mean is 15 with variance 4. We then compute the capital of the insurance company at time $t = 8$ by applying Hadamard gates into the first two qubits. In this circumstance, the quantum circuit diagram is produced as shown in Figure 1. Furthermore, potential quantum states of R(t) with their rate of occurrence, and their corresponding capital values with their probabilities are displayed in Figures 2 and 3, respectively.

**Table 9.** Quantum circuit in Qiskit.

| Initial Qubits at $\|0\rangle$ State | Apply X Gate to Obtain $u$ | Apply PG and CG $t/\Delta t$ Times | Measure All Qubits at Time $t$ |
|---|---|---|---|
|  |  |  |  |

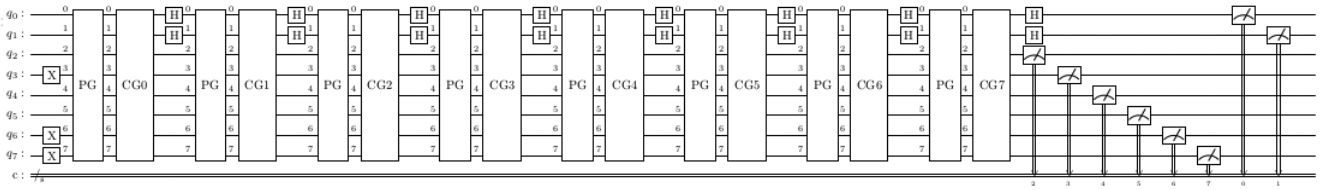


**Figure 1.** Quantum circuit diagram for $u = 200, c = 20, m = 15, \sigma^2 = 4, t = 8$.

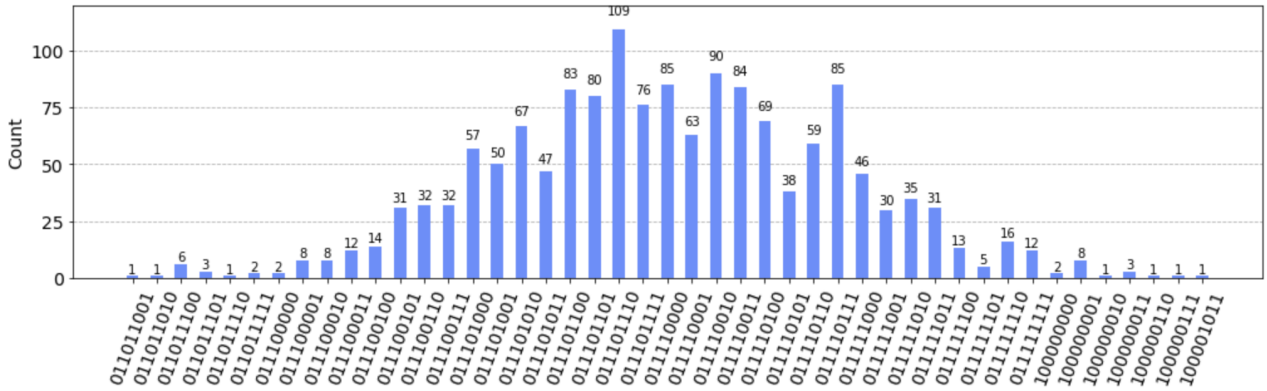


**Figure 2.** Output states of R(t) for $u = 200, c = 20, m = 15, \sigma^2 = 4, t = 8$.

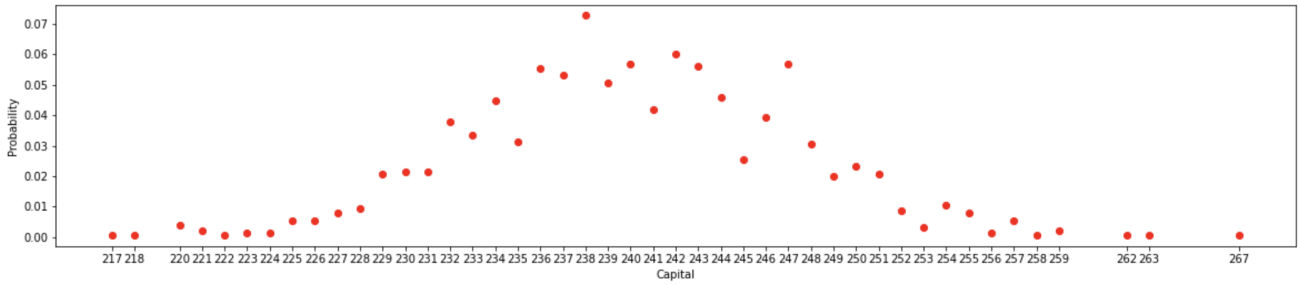


**Figure 3.** Probability of potential capital R(t) for $u = 200, c = 20, m = 15, \sigma^2 = 4, t = 8$.

This proposed algorithm works well in situations where the probability of the insurance company going bankrupt is low because quantum states for negative capitals have not been identified so far in this research. Therefore, with ruin time $T = min\{t : t > 0$ and $u + ct - S(t) < 0\}$ and ruin probability of the insurance company $\mathbb{P}(T \leqslant t) = 1 - \mathbb{P}(T > t)$, this algorithm can be used for the very small value of $\mathbb{P}(T \leqslant t)$.

If the ruin probability is not too small, then new quantum states for the negative capitals should be considered according to the minimum capital value. Let us denote the deficit amount at the time of ruin as $D(T) = -R(T)$. Then, we need $n$ qubits that comply with the following condition

$$n \geqslant min\{\bar{n} | \frac{\text{maximum capital+ maximum Deficit } +1}{\varepsilon} = \frac{u+ct+max\{D(t)\}+1}{\varepsilon} \leqslant 2^{\bar{n}}\}. \tag{21}$$

Since it is hard to estimate the possible deficit amount at the beginning, a sufficiently small value of $R(t)$ that maximises $D(t)$ should be chosen in order to take all possible negative states into account. Assume that $u = 100, c = 10, t = 20, m = 40, \lambda = 0.2, max\{D(t)\} = 300$ and maximum capital amount $u + ct = 300$, then the qubit number should be chosen as $n = 10$ due to $300 + 300 + 1 \leqslant 2^{10}$. In this circumstance, the some capitals and their modified corresponding basis vectors are listed in Table 10 by the following rule:

$$u \to |q_n q_{n-1} \cdots q_0\rangle, \tag{22}$$

where $u = q_n 2^n + q_{n-1} 2^{n-1} + \ldots + q_0 2^0 - max\{D(t)\}$ and $q_i \in \{0, 1\}$.

**Table 10.** Capitals and corresponding basis vectors for 10 qubits.

| Capital | Modified Corresponding Basis Vector |
|---|---|
| −300 | $\|0000000000\rangle$ |
| −299 | $\|0000000001\rangle$ |
| −288 | $\|0000000010\rangle$ |
| 0 | $\|0100101100\rangle$ |
| 55 | $\|0101100011\rangle$ |
| 300 | $\|1001011000\rangle$ |

The expected reserve of the insurance company at time $t$ in Figure 2 is computed by

$$\begin{aligned} E[R(t)] &= \sum_{x_n=0}^{\infty} \mathbb{P}(R(t) = x_n | R(0) = u) x_n \\ &= \sum_{x_n=0}^{\infty} \frac{\text{Count of the quantum state corresponding to } x_n}{\text{The number of the total iteration in the simulation}} x_n. \end{aligned} \tag{23}$$

For the example in Figures 2 and 3, the expected capital of the insurance company at time 8 is computed as $E[R(t)] = 240.04$ in the quantum computing, which is very close to the result of classical computation due to

$$E[R(t)] = u + ct - E[S(t)] = 200 + 20 \cdot 8 - 15 \cdot 8 = 240. \tag{24}$$

If there is a ruin risk, negative quantum states should be considered in the Formula (23) by starting $x_n$ from a sufficiently small value.

The surplus model and the quantum computing approach presented here have a number of limitations. For example, reinsurance cost, operational cost, capital injections, stock dividend, and inflation's effect are not taken into account in this research. For all capital increase and decrease activities, new reversible quantum gates should be created and added into the quantum circuit. Another significant issue is the order of the quantum gates. As mentioned in Section 3, while premiums are paid in advance, insurance claims are registered at the end of the time period, so $CG$ is applied after $PG$. In the quantum circuits, if the gates are applied in the following order: $CG, PG, CG, PG, \ldots$ instead of $PG, CG, PG, CG, \ldots$, the substitution of the quantum gates can increase the ruin probability despite of the same expected capitals in the two cases. For example, one of the following processes for $u = 2, c = 5, X = 3$ leads to the ruin.

First walk: $2 \to 7 \to 4 \to 9 \to 6$ for $PG, CG, PG, CG,$

Second walk: $2 \to$ **-1** $\to 4 \to 1 \to 6$ for $CG, PG, CG, PG.$

As seen above, even though we obtain same capital at the end of the process, the ruin happens after applying first claim gate in the second walk since the negative reserve $(R(1) = -1)$. Therefore, in this example, the commutative law holds for multiplication of the insurance premium and claim gates in terms of quantum computing due to

$$PG \cdot CG = CG \cdot PG \quad \text{and} \quad (PG \cdot CG)^k = (PG)^k \cdot (CG)^k = (CG)^k \cdot (PG)^k, \quad k \in Z^+.$$

However, the quantum gates are non-commutative from the perspective of actuarial science because of the stop-loss type approaches. Secondly, in case of implementation of controlled-X gate ($CNOT$ gate), the commutativity does not hold as seen in the following example in a two-level quantum system.

$$PG \cdot CNOT \cdot CG \neq PG \cdot CG \cdot CNOT \neq CG \cdot CNOT \cdot PG$$

for

$$CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad PG = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad CG = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

In some non-proportional reinsurance contracts [26], when the cedent's capital is below a specific retention level, a capital injection should be done by the reinsurance company to restore the loss of the primary insurance company. This case is studied in [8] by investigating commutative and non-commutative operators. As a result, even if the commutative property works for some quantum gates, it does not mean that it is applicable from the point of view of the insurance risk process. Therefore, for actuarial risk and capital modelling, both the creation of quantum gates and their application times are significant in the quantum circuit.

## 5. Conclusions

Even though it is not easy to answer the question of what the future holds for quantum computing, advances in this field continue to increase with government and private sector investments. A lack of experts in quantum mechanics and quantum computing in actuarial mathematics is one of the barriers to manufacturing more academic and industrial works. Using the principles of quantum mechanics in insurance researches is quite new. With this paper, actuarial researchers will become familiar with quantum computing. Quantum machine learning should be adapted to insurance risk and capital computations in further research, and quantum cryptography-based products should be considered in the exposure of cyber insurance risk and pricing.

Simulation of systemic risks is complex due to number of the parameters in the dependent risk models, so, in some cases, it is difficult to handle this in any feasible amount of time by classical computers. Implications of some systemic risks on insurance pricing encourage us to focus on new forecasting approaches and innovative technologies. Simulation of the systemic risks like the climate change, pandemics, and global financial crises for insurance industry via quantum computers is a potential future subject of study.

The code is available at the following Github repository: https://github.com/muhsintamturk/QuantumComputingCodes (accessed on 27 December 2022).

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## References


1. Katwala, A. *Quantum Computing: How It Works and How It Could Change the World*; Random House: London, UK, 2021.
2. Trotman, A.; Zhang, J. Future web growth and its consequences for web search architectures. *arXiv* **2013**, arXiv:1307.1179.
3. Elfving, V.E.; Broer, B.W.; Webber, M.; Gavartin, J.; Halls, M.D.; Lorton, K.P.; Bochevarov, A. How will quantum computers provide an industrially relevant computational advantage in quantum chemistry? *arXiv* **2020**, arXiv:2009.12472.
4. Insurance Times. Available online: https://www.insurancetimes.co.uk/news/cyber-insurance-industry-predicted-to-exceed-20bn-gwp-by-2025-globaldata/1438074.article (accessed on 27 December 2022).
5. Hidary, J.D.; Hidary, J.D. *Quantum Computing: An Applied Approach*; Springer: Cham, Switzerland, 2021; Volume 1.
6. IBM. Available online: https://quantum-computing.ibm.com/ (accessed on 27 December 2022).
7. Tamturk, M.; Utev, S. Ruin probability via quantum mechanics approach. *Insur. Math. Econ.* **2018**, *79*, 69–74. [CrossRef]
8. Tamturk, M.; Utev, S. Optimal reinsurance via Dirac-Feynman approach. *Methodol. Comput. Appl. Probab.* **2019**, *21*, 647–659. [CrossRef]
9. Lefèvre, C.; Loisel, S.; Tamturk, M.; Utev, S. A quantum-type approach to non-life insurance risk modelling. *Risks* **2018**, *6*, 99. [CrossRef]
10. Tamturk, M.; Cortis, D.; Farrell, M. Examining the effects of gradual catastrophes on capital modelling and the solvency of insurers: The case of COVID-19. *Risks* **2020**, *8*, 132. [CrossRef]
11. Nobel Prize. Available online: https://www.nobelprize.org/prizes/physics/2022/press-release/ (accessed on 27 December 2022).
12. Loredo, R. *Learn Quantum Computing with Python and IBM Quantum Experience: A Hands-on Introduction to Quantum Computing and Writing Your Own Quantum Programs with Python*; Packt Publishing Ltd.: Birmingham, UK, 2020.
13. Jacquier, A.; Kondratyev, O. *Quantum Machine Learning and Optimisation in Finance: On the Road to Quantum Advantage*; Packt Publishing Ltd.: Birmingham, UK, 2022.
14. Sutor, R.S. *Dancing with Qubits: How Quantum Computing Works and How It Can Change the World*; Packt Publishing Ltd.: Birmingham, UK, 2019.
15. Bowers, N.; Gerber, H.; Hickman, J.; Jones, D.; Nesbitt, C. *Actuarial Mathematics*; Society of Actuaries: Itasca, IL, USA, 1986.
16. Dickson, D.C. *Insurance Risk and Ruin*; Cambridge University Press: Cambridge, UK, 2016.
17. Baaquie B.E. *Quantum Finance: Path Integrals and Hamiltonians for Options and Interest Rates*; Cambridge University Press: Cambridge, UK, 2007.
18. Chang, K.L. *Mathematical Structures of Quantum Mechanics*; World Scientific: Singapore, 2012.
19. Hao, W.; Lefèvre, C.; Tamturk, M.; Utev, S. Quantum option pricing and data analysis. *Quant. Financ. Econ.* **2019**, *3*, 490–507. [CrossRef]
20. Dirac, P.A. *The Lagrangian in Quantum Mechanics*; World Scientific Publishing Co.: Singapore, 2005; pp. 111–119.
21. Feynman, R.P. Space-time approach to non-relativistic quantum mechanics. *Rev. Mod. Phys.* **1948**, *20*, 367. [CrossRef]
22. Feynman, R.P.; Hibbs, A.R.; Styer, D.F. *Quantum Mechanics and Path Integrals*; Courier Corporation: New York, NY, USA, 2010.
23. Feynman, R.P. Simulating physics with computers. In *Feynman and Computation*; CRC Press: Boca Raton, FL, USA, 2018; pp. 133–153.
24. Baaquie, B.E. *Interest Rates and Coupon Bonds in Quantum Finance*; Cambridge University Press: Cambridge, UK, 2009.
25. IBM. Available online: https://research.ibm.com/blog/next-wave-quantum-centric-supercomputing (accessed on 27 December 2022).
26. Nie, C.; Dickson, D.C.; Li, S. Minimizing the ruin probability through capital injections. *Ann. Actuar. Sci.* **2011**, *5*, 195–209. [CrossRef]