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Article

Trans-Planckian Censorship and Spacetime Singularities

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Abstract: We study the effects of trans-Planckian censorship conjecture (TCC) bounds on geodesic completeness of spacetime and the associated existence for an infinite proper time. Using Gronwall's lemma, TCC bounds can be derived directly, leading to a result about the absence of blowup solutions. We show that the TCC provides part of the required criteria for geodesic completeness, and we then provide the remaining ones, the norm of the extrinsic curvature being bounded away from zero. We also discuss the importance of these results for the classical evolution of Friedmann universes under the assumptions of global and regular hyperbolicity.

Keywords: trans-planckian censorship; spacetime singularities

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1. Introduction

It is well-known that the Hawking–Penrose theorems provide sufficient conditions for the existence of singularities in spacetime [1], while completeness theorems associated with the work of Y. Choquet-Bruhat give sufficient conditions for the possible geodesic completeness of spacetimes [2]. In the first case we have geometric and causality conditions leading to geodesic incompleteness, while in the second case completeness of geodesics is established under various analytic criteria. In both cases, such conditions may be realized in effective theories and, as it has been repeatedly emphasized, such theories may not be consistent with modern unification ideas, cf., e.g., [3].

In fact, according to the trans-Planckian censorship conjecture, initial fluctuations can never exit the Hubble radius, and in this sense such information can never classicalize and become 'visible' to classical evolution [4,5]. This is like having a cosmological censor that, in an analogous way to that in cosmic censorship, hides any trans-Planckian information (see, e.g., [6] for more recent work on lower bounds on black hole masses, refs. [7,8] for related work on inflation and dark energy, and [9] on the influence of negative potentials). Since a central question in studies of the early structure and evolution of the universe is the possible presence of singularities, it is important to understand how the trans-Planckian censorship conjecture relates to the possible resolution of cosmological singularities.

The structure of this paper is as follows. In Section 2, we introduce three different forms of trans-Planckian bounds, and then provide sufficient conditions in the form of integrability assumptions of the Hubble parameter (i.e., extrinsic curvature) that lead to two of them. In Section 3, we show how trans-Planckian bounds lead to the absence of a blowup in the classical solutions, and discuss why such bounds alone cannot provide an overall criterion for the possible geodesic completeness of spacetime. We then show how one can obtain such criteria by introducing a further condition that we call the 'anti-Gronwall assumption', that together with the trans-Planckian bounds may lead to a total bound on the norm of the Hubble parameter. We further discuss these results in Section 5.

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2. Trans-Planckian Bounds

In this Section, we introduce a new method to derive trans-Planckian bounds based on the Gronwall's lemma.

We start with the 'Gronwall hypothesis', which is contained in the following differential inequality:

$$\frac{\dot{a}(t)}{a(t)} \leq H_0(t), \tag{1}$$

for the two functions a, H_0 defined for all t in the interval $[t_i, t_f]$ and assumed to be differentiable and nonnegative (weaker assumptions are possible). Using Gronwall's lemma (see [10] for a discussion closer to its usage in the present work, and [11,12] for wider applications and references on inequalities), we find:

$$\frac{a(t_f)}{a(t_i)} \leq e^{\int_{t_i}^{t_f} H_0(s)ds}. \tag{2}$$

Let us first consider the case that $H_0 = \text{const}$. For each finite t_f there is a nonzero constant H_f such that the right-hand side of (2) is pointwise bounded, namely,

$$H_0(t_f - t_i) < \ln \frac{M_P}{H_f}. \tag{3}$$

Then it follows from the conclusion of the Gronwall's lemma (2) that:

$$\frac{a(t_f)}{a(t_i)} l_P < H_f^{-1}, \tag{4}$$

with $l_P = M_P^{-1}$ (in other notation, setting $N = H_0(t_f - t_i)$ for the number of 'e-folds', if we assume $e^N < M_P/H_f$ as in (3), then (4) follows.) We note that the trans-Planckian bound in the form stated in Ref. [4] does not hold in the interval $[t_i, \infty)$ for each finite t_i, because when the upper endpoint $t_f \to \infty$, the left-hand side of (3) is infinite.

We move on to the second case that is when H_0 is *not* assumed constant. We suppose that H_0 is an integrable function on $[t_i, \infty)$, and replace the left-hand side of inequality (3) with the expression $\int_{t_i}^{t_f} H_0(s)ds$. We then end up with the pointwise assumption that for each t_f, we have:

$$\int_{t_i}^{t_f} H_0(s)ds < \ln \frac{M_P}{H_f}. \tag{5}$$

This implies that the statement of the trans-Planckian censorship conjecture as formulated in [4] now becomes a trans-Planckian censorship *theorem* provided H_0 is integrable: for any integrable function $H_0(t)$ the integral $\int_{t_i}^{t_f} H_0(s)ds$ is bounded, and we have:

$$\frac{a(t)}{a(t_i)} l_P < H_\infty, \quad t \in [t_i, \infty), \tag{6}$$

where H_∞ is a suitable constant that provides a uniform bound for the left-hand side of (6). Hence, the integrability of H_0 provides a sufficient condition for the validity of the trans-Planckian censorship conjecture.

In other words, under assumption (1), inequality (3) (and similarly (5)) *implies* (4) (or (6)), but *not* vice-versa. Sometimes a stronger version of the trans-Planckian censorship conjecture is stated in the form of a double implication, which, however, assumes more than just the integrability of H_0. The following *equivalence*,

$$\frac{a(t_f)}{a(t_i)} l_P < H_f^{-1} \quad \text{if and only if} \quad H_0(t_f - t_i) < \ln \frac{M_P}{H_f}, \tag{7}$$

is true (not just as a one-way implication), provided that the *equality* $\dot{a}/a = H_0$ is assumed instead of the differential inequality (1).

Another possible form is to take the trans-Planckian censorship conjecture to mean the reverse statement, namely that (4) $\Rightarrow$ (3) for any integrable H_0; namely, that for any t_f and any nonzero H_f, we have [5]:

$$\frac{a(t_f)}{a(t_i)}\, l_P < H_f^{-1} \quad \text{implies} \quad \int_{t_i}^{t_f} H_0(s)ds < \ln \frac{M_P}{H_f}. \tag{8}$$

This statement is different in meaning from Equations (4), (6), or (7), and is true provided again that H_0 is an integrable function.

3. A Breakdown Criterion

In this Section we show that a trans-Planckian bound together with the additional assumption of the existence of a lower bound for the scale factor are sufficient conditions for producing singularity-free universes.

First we show that since any of the trans-Planckian bounds discussed in the previous section provide an upper bound for a, we can obtain a criterion about the possible absence of blowup solutions for the scale factor a in any interval of the form $[t_i, t_f]$.

For an initial time t_i, we take the the 'initial datum' to be $a(t_i) = a_i$, and consider the maximal interval of existence of solutions $a(t)$ to be $I = (T_-, T_+)$ where $-\infty \leq T_- < t_i < T_+ \leq \infty$. Any trans-Planckian bound provides a suitable upper bound for a, and therefore by the Picard existence and uniqueness theorem (cf. e.g., [10], p. 14) we have a global solution, which is $T_+ = \infty$, that does not go to infinity in a finite time in the future.

A physical interpretation of this result is that singularities of the finite-time blow-up type for $a(t)$ are strictly prohibited when (2) holds and H_0 is integrable.

However, in general relativity a singularity is defined as geodesic incompleteness [1]. The previous discussion does not of course prove geodesic completeness, and so cannot provide an argument for a resolution of singularities of spacetime under the above assumptions. The physical problem is to prove the existence for an infinite proper time, and in this respect the work in Ref. [2] becomes relevant.

In [2], a theorem was proven giving sufficient conditions for geodesic completeness in the following sense. We assume the standard $(3+1)$-splitting of a globally hyperbolic spacetime where the lapse function, shift vector field and spatial metric are all bounded (regular hyperbolicity). If we further take the norms of the spatial gradient of the lapse function as well as that of the extrinsic curvature to be bounded by integrable functions on the interval $[t_i, \infty)$, then it follows that the spacetime is future timelike and null geodesically complete.

For example, in the case of an FRW universe with scale factor a, the lapse $N = 1$, the shift $\beta = 0$, and thus the gradient of the lapse vanishes, while the norm of the extrinsic curvature is given by $|K|_g{}^2 = 3(\dot{a}/a)^2 = 3H^2$. Hence, this result tells us that in FRW universes that have their scale factor bounded below will be singular only if there is a finite time $t_1 \in [t_i, \infty)$ such that the Hubble parameter H is not integrable on the corresponding interval $[t_1, \infty)$.

Previously we assumed the Gronwall bound (2) for H, where H_0 could also be negative, and we discussed its importance in the formulation of trans-Planckian bounds. That discussion provides only half of the conditions needed for a complete singularity resolution, however, and we will now discuss the other half.

Let us introduce the following 'anti-Gronwall' assumption, namely,

$$H(t) \geq b > 0, \tag{9}$$

with $t \in [t_i, t_f]$, for some constant b, so that $0 < b \leq \dot{a}/a$. Integrating on $[t_i, t_f]$ we find that,

$$a(t_f) \geq a(t_i)\, e^{b(t_f - t_i)}, \tag{10}$$

i.e., the scale factor a is bounded from below. This is a way to circumvent the singularity at $a(t) = 0$ for some t earlier than t_i that is expected from the Raychaudhuri equation, because the anti-Gronwall condition (9) is the opposite of the usual one, i.e., negative expansion (or positive convergence) assumed in the singularity theorems (cf. [1], Thm. 3, p. 271).

The question is then whether the interval $I = (T_-, T_+)$, where the scale factor a is bounded, is finite or infinite. From the results above it follows that using the anti-Gronwall condition (9) (i.e., a is bounded below) together with the trans-Planckian bound, we find that the *norm* $|H(t)|$ will be bounded for all time, not just H, so that the interval I can be infinite (to the left, right, or both). This is so because according to the completeness theorem of [2] mentioned above, the integrability of $|H|$, i.e., $|H|$ is bounded by the integrable function H_0 as in (1), and is also a *sufficient* condition for geodesic completeness (the others being that spacetime is globally and regularly hyperbolic) to the past, future, or both.

We note that this argument is independent of the the usual assumption on the Ricci tensor, because the positive convergence condition is an independent hypothesis (i.e., $R^{\mu\nu}X_\mu X_\nu \geq 0$ for non-spacelike vector fields), and leads to the absence of past or future singularities.

For a Friedman universe, in particular, geodesic completeness can only fail if there exists a time t_i such that the norm of the Hubble parameter $|H|$ becomes non-integrable in the interval $[t_i, \infty)$. The non-integrability of $|H|$ provides the only necessary condition for a Friedman universe to be singular. There are different ways for this non-integrability to arise, and an exhaustive classification of the nature of possible singularities that occur this way was presented in [13,14].

Therefore we are led to conclude that using the completeness theorem, the trans-Planckian bounds, and the anti-Gronwall assumption, there is a way out of the inevitability of the singular nature of Friedman universes either in the past or future, by providing conditions for the norm of the Hubble parameter to be bounded and hence be integrable.

This argument also explains why the trans-Planckian censorship conjecture favors scenarios such as the ekpyrotic universe where the scale factor is bounded below, or the emergent universe scenarios [15] where H is not only integrable but in fact is asymptotically vanishing [13], rather than an inflationary universe where there is a singularity with a *finite* H, cf. [13,16].

Therefore, we conclude that future (or past) geodesic completeness and the associated absence of future (past) singularities is a necessary consequence of trans-Planckian bounds in any scenario in which the universe satisfies the anti-Gronwall assumption.

4. Examples

As an application of the previous results, we consider here a few representative examples that illustrate some of the features of the use of trans-Planckian bounds in proving geodesic completeness.

A a first example, let us consider the emergent universe scenario of [15]. For this model, the Gronwall hypothesis, namely that the expansion is sub-Hubblian, together with the trans-Planckian bound (4), implies that the initial (Einstein static universe) scale factor $a(t_i)$ is bounded from below, avoiding the usual fine-tuning issues associated with the emergent scenario. In addition, the anti-Gronwall bound on the Hubble parameter (9) implies a large classical expanding universe with a scale factor given by (10) at late times. This universe is also future geodesically complete because the Hubble parameter is not only bounded by asymptotically vanishing, cf. [13].

In fact it is not difficult to devise universes with an asymptotically vanishing Hubble parameter, thus signaling future geodesic completeness. As an example, in any flat or negatively curved FRW model filled with a perfect fluid and scalar field with a positive, bounded potential, one can show that not only H but also the fluid density are future asymptotically vanishing, cf. [17], Proposition 2. Hence, in any model with logarithmic or generalized potentials, e.g., of the form studied in [18,19], the trans-Planckian bound together with the anti-Gronwall hypothesis imply a singularity-free evolution.

5. Discussion

In this paper we have discussed the role of trans-Planckian bounds in relation to the formation of singularities. We have first shown that such bounds can be naturally deduced from the Gronwall hypothesis, which provides upper bounds to the Hubble parameter.

This leads to a new criterion for the absence of diverging cosmological solutions either at a finite time or at infinity.

Furthermore, we have shown that trans-Planckian bounds, when combined with the condition that the Hubble parameter is bounded away from zero, lead to geodesically complete universes satisfying the usual causality assumptions. We therefore conclude that trans-Planckian bounds provide a way to singularity-free universes if the Hubble parameter is integrable.

This result opens the way to constructing singularity-free cosmologies starting from a trans-Planckian bound and examining the integrability of the expansion parameter. This in turn depends on the type of matter content of the universe, and may lead to selection rules for non-singular cosmologies from suitable restrictions on the fluid or other parameters of the matter fields. Due to the generality of our criteria, we believe that our present results may also be extended to more general anisotropic or inhomogeneous cosmologies.

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