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On Some Expansion Formulas for Products of Jacobi's Theta Functions

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Abstract: In this paper, we establish several expansion formulas for products of the Jacobi theta functions. As applications, we derive some expressions of the powers of $(q;q)_\infty$ by using these expansion formulas.

Keywords: theta function; Dedekind's eta-function; triple product identity; the power of $(q;q)_\infty$

MSC: 33E05; 11F11; 11F20; 11F27

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1. Introduction

Throughout this paper, we suppose that $q = \exp(\pi i \tau)$, where τ has a positive imaginary part and $i = \sqrt{-1}$.

The Jacobi theta functions $\theta_1(z|\tau)$, $\theta_2(z|\tau)$, $\theta_3(z|\tau)$ and $\theta_4(z|\tau)$ are defined by [1–3]

$$\theta_1(z|\tau) = -iq^{\frac{1}{4}} \sum_{n=-\infty}^{\infty} (-1)^n q^{n(n+1)} e^{(2n+1)zi}, \quad \theta_3(z|\tau) = \sum_{n=-\infty}^{\infty} q^{n^2} e^{2nzi},$$
$$\theta_2(z|\tau) = q^{\frac{1}{4}} \sum_{n=-\infty}^{\infty} q^{n(n+1)} e^{(2n+1)zi}, \qquad \theta_4(z|\tau) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} e^{2nzi}.$$

For convenience, we use the following abbreviated multiple parameter notation:

$$(a_1, a_2, \cdots, a_l; q)_\infty = (a_1; q)_\infty (a_2; q)_\infty \cdots (a_l; q)_\infty$$

with

$$(a;q)_\infty = \prod_{n=0}^{\infty} (1 - aq^n).$$

With this notation, the well-known Jacobi triple product identity can be written as [2]

$$(q, z, q/z; q)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(n-1)/2} z^n, \ z \neq 0.$$

Using the Jacobi triple product identity, we deduce the infinite product representations for the Jacobi theta functions [2,4]:

$$\begin{aligned}
\theta_1(z|\tau) &= 2q^{\frac{1}{4}} (\sin z)(q^2, q^2 e^{2iz}, q^2 e^{-2iz}; q^2)_\infty, \\
\theta_2(z|\tau) &= 2q^{\frac{1}{4}} (\cos z)(q^2, -q^2 e^{2iz}, -q^2 e^{-2iz}; q^2)_\infty, \\
\theta_3(z|\tau) &= (q^2, -qe^{2iz}, -qe^{-2iz}; q^2)_\infty, \\
\theta_4(z|\tau) &= (q^2, qe^{2iz}, qe^{-2iz}; q^2)_\infty.
\end{aligned}$$

With respect to the (quasi) periods π and $\pi\tau$, we have [4]

$$\theta_1(z+\pi|\tau) = -\theta_1(z|\tau), \qquad \theta_1(z+\pi\tau|\tau) = -q^{-1}e^{-2iz}\theta_1(z|\tau), \tag{1}$$
$$\theta_2(z+\pi|\tau) = -\theta_2(z|\tau), \qquad \theta_2(z+\pi\tau|\tau) = q^{-1}e^{-2iz}\theta_2(z|\tau), \tag{2}$$
$$\theta_3(z+\pi|\tau) = \theta_3(z|\tau), \qquad \theta_3(z+\pi\tau|\tau) = q^{-1}e^{-2iz}\theta_3(z|\tau), \tag{3}$$
$$\theta_4(z+\pi|\tau) = \theta_4(z|\tau), \qquad \theta_4(z+\pi\tau|\tau) = -q^{-1}e^{-2iz}\theta_4(z|\tau). \tag{4}$$

For brevity, we use $\theta_2(\tau)$, $\theta_3(\tau)$ and $\theta_4(\tau)$ to represent $\theta_2(0|\tau)$, $\theta_3(0|\tau)$ and $\theta_4(0|\tau)$, respectively.

There are many experts studying theta functions. Schiefermayr [5] proved a monotonicity property for the quotient of two Jacobi theta functions with respect to the modulus k. Liu [6] derived many nontrivial identities from a single identity and also derived four Ramanujan-type modular equations. Tsumura [7,8] deduced some series identities arising from Jacobi's identity of the theta function, which were a certain finite combination of the Riemann zeta-function, Dirichlet L-function with character modulo 4, and the Eisenstein series. Schneider [9] made an interesting connection between the Jacobi triple product and the universal mock theta function. Singh and Yadav [10] determined certain properties of Jacobi's theta functions. Berndt, Chan and Liu [11] studied many important identities involving Eisenstein series and eta functions. Chan, Cooper and Toh [12] researched the expression of theta functions, or, rather, very close relatives of theta functions, as polynomials in Ramanujan's Eisenstein series, multiplied by powers of Dedekind's eta function. Chu [13] gave a new proof of the theta function identity by specializing the well-known Bailey summation formula. The authors of [14,15] utilized the classical theory of elliptic functions to prove a theta function identity and deduced some nontrivial identities on circular summation of theta functions. For more information, please refer to above references.

In particular, Liu [16] first established a general identity involving an entire function $f(z)$ satisfying two functional equations and presented several interesting applications of these theta function identities, involving a one identity for $(q;q)_\infty^{10}$. Motivated by Liu [16,17] and the above references, we deduce some expansion formulas for products of Jacobi's theta functions in this paper, as applications, and we give some expressions of the powers of $(q;q)_\infty$ by using these expansion formulas.

This article is organized as follows: In Section 2, we deduce some expansion formulas for products of the Jacobi theta functions. In Section 3, as applications, we derive some expressions of the powers of $(q;q)_\infty$ by using the formulas obtained in Section 2.

2. Main Results

In this section, we first recall some identities on the Jacobi theta functions and then deduce some expansion formulas for products of the Jacobi theta functions. See [14,15,17] for examples of the Jacobi theta function identities and their applications.

Lemma 1 (See [18]). *We have:*

$$\theta_1(x|\tau)\theta_1(y|\tau) = \theta_2(x-y|2\tau)\theta_3(x+y|2\tau) - \theta_2(x+y|2\tau)\theta_3(x-y|2\tau). \tag{5}$$

In order to prove Lemma 1, we need the following Lemma will be needed.

Lemma 2 (See [18]). *If the elliptic function f has no poles, it is a constant.*

Proof. Let $f(x,y)$ be the function defined as [18]

$$f(x,y) = \frac{\theta_2(x-y|2\tau)\theta_3(x+y|2\tau) - \theta_2(x+y|2\tau)\theta_3(x-y|2\tau)}{\theta_1(x|\tau)\theta_1(y|\tau)}. \tag{6}$$

Next, replacing $(x, y) = (x + \pi, y + \pi)$ and $(x, y) = (x + 2\pi\tau, y + 2\pi\tau)$ and using (1) to (4), we get:

$$f(x+\pi, y) = f(x, y), \qquad f(x, y+\pi) = f(x, y) \tag{7}$$

$$f(x+2\pi\tau, y) = f(x, y), \qquad f(x, y+2\pi\tau) = f(x, y). \tag{8}$$

Hence, the function $f(x, y)$ in an elliptic function with periods π and $2\pi\tau$. If we fixed y, then $f(x, y)$ is a function of x. From the definition of theta functions, we know that $x = 0$ and $x = \pi\tau$ are likely poles. Furthermore, they are simple poles. However, when $x = 0$ the numerator of $f(x, y)$ is reduced that

$$\theta_2(-y|2\tau)\theta_3(y|2\tau) - \theta_2(y|2\tau)\theta_3(-y|2\tau) = 0.$$

Therefore, $x = 0$ is not a simple pole. We can see a similar case when $x = \pi\tau$ and for y. Then, we know $f(x, y)$ is an elliptic function. By Lemma 2, it is a constant independent x and y. Let $x = y = \pi/4$ in $f(x, y)$, easily know $f(x, y) = 1$. This completes the proof. □

Lemma 3 (See [19])**.** *We have:*

$$\theta_1(x|\tau)\theta_3(x|\tau)\theta_4(x|\tau) = 2q^{1/4}(q^2; q^2)_\infty^2 \sum_{n=-\infty}^{\infty} q^{n(3n+1)} \sin(6n+1)x, \tag{9}$$

$$\theta_1(x|\tau)\theta_2(x|\tau)\theta_4(x|\tau) = -2q^{3/2}(q^2; q^2)_\infty^2 \sum_{n=-\infty}^{\infty} q^{n(3n+4)} \sin(6n+4)x. \tag{10}$$

Using the infinite product representation for Jacobi theta functions, we can derive the following identity easily.

Lemma 4. *We have*

$$\theta_1\left(x|\frac{\tau}{2}\right) = q^{-1/8}\theta_1(x|\tau)\theta_4(x|\tau)\frac{(q;q)_\infty}{(q^2;q^2)_\infty^2}. \tag{11}$$

Our main results are as follows.

Theorem 1. *We have*

$$\frac{1}{(q;q)_\infty}\theta_1\left(x|\frac{\tau}{2}\right)\theta_1\left(\frac{x+y}{2}|\frac{\tau}{2}\right)\theta_1\left(\frac{x-y}{2}|\frac{\tau}{2}\right) \tag{12}$$
$$= 2q^{1/8}\theta_2(y|\tau)\sum_{n=-\infty}^{\infty} q^{n(3n+1)}\sin(6n+1)x + 2q^{11/8}\theta_3(y|\tau)\sum_{n=-\infty}^{\infty} q^{n(3n+4)}\sin(6n+4)x.$$

Proof. We first replace τ by $\frac{\tau}{2}$ and then replace x and y by $\frac{x+y}{2}$ and $\frac{x-y}{2}$, respectively, in the identity of Lemma 1 to get

$$\theta_1\left(\tfrac{x+y}{2}|\tfrac{\tau}{2}\right)\theta_1\left(\tfrac{x-y}{2}|\tfrac{\tau}{2}\right) = \theta_2(y|\tau)\theta_3(x|\tau) - \theta_2(x|\tau)\theta_3(y|\tau). \tag{13}$$

Next, multiplying the identity (13) by the identity (11) gives

$$\theta_1\left(x|\frac{\tau}{2}\right)\theta_1\left(\frac{x+y}{2}|\frac{\tau}{2}\right)\theta_1\left(\frac{x-y}{2}|\frac{\tau}{2}\right)$$
$$= \frac{q^{-1/8}(q;q)_\infty}{(q^2;q^2)_\infty^2}\{\theta_1(x|\tau)\theta_2(y|\tau)\theta_3(x|\tau)\theta_4(x|\tau) - \theta_1(x|\tau)\theta_2(x|\tau)\theta_3(y|\tau)\theta_4(x|\tau)\}.$$

Inserting the Equations (9) and (10) into the right side of the above equation, we obtain the required identity. □

Theorem 2. *We have*

$$\begin{aligned}
&\frac{1}{4q^{3/2}(q;q)_\infty^2}\theta_1\left(x|\frac{\tau}{2}\right)\theta_1\left(y|\frac{\tau}{2}\right)\theta_1\left(\frac{x+y}{2}|\frac{\tau}{2}\right)\theta_1\left(\frac{x-y}{2}|\frac{\tau}{2}\right)\\
&=\sum_{n=-\infty}^{\infty}q^{3n^2+4n}\sin(6n+4)x\sum_{n=-\infty}^{\infty}q^{3n^2+n}\sin(6n+1)y\\
&-\sum_{n=-\infty}^{\infty}q^{3n^2+n}\sin(6n+1)x\sum_{n=-\infty}^{\infty}q^{3n^2+4n}\sin(6n+4)y.
\end{aligned}$$

Proof. It follows from the identity in Lemma 4 that:

$$\theta_1\left(x|\frac{\tau}{2}\right)\theta_1\left(y|\frac{\tau}{2}\right)=q^{-1/4}\theta_1(x|\tau)\theta_4(x|\tau)\theta_1(y|\tau)\theta_4(y|\tau)\frac{(q;q)_\infty^2}{(q^2;q^2)_\infty^4}. \tag{14}$$

Multiplying the identity (13) by (14), we obtain

$$\begin{aligned}
&q^{-1/4}\frac{(q;q)_\infty^2}{(q^2;q^2)_\infty^4}\theta_1(x|\tau)\theta_4(x|\tau)\theta_1(y|\tau)\theta_4(y|\tau)\{\theta_2(y|\tau)\theta_3(x|\tau)-\theta_2(x|\tau)\theta_3(y|\tau)\}\\
&=\theta_1\left(x|\frac{\tau}{2}\right)\theta_1\left(y|\frac{\tau}{2}\right)\theta_1\left(\frac{x+y}{2}|\frac{\tau}{2}\right)\theta_1\left(\frac{x-y}{2}|\frac{\tau}{2}\right).
\end{aligned}$$

Plugging (9) and (10) into the above equation and then simplifying, we obtain the required result. □

3. Powers of $(q;q)_\infty$

In this section, we deduce some expressions of the powers of $(q;q)_\infty$ by using the identities in Theorems 1 and 2. These formulas can also be written as Dedekind's eta-functions

$$\eta(\tau)=q^{1/24}\prod_{j=1}^{\infty}(1-q^j), \tag{15}$$

where $q=e^{2\pi i\tau}$ and $\Im m(\tau)>0$.

Theorem 3. We have

$$(q;q)_\infty^2=\sum_{n=0}^{\infty}q^{n(n+1)/3}\sum_{n=-\infty}^{\infty}q^{3n^2}-\sum_{n=0}^{\infty}q^{3n(n+1)}\sum_{n=-\infty}^{\infty}q^{(n^2+6)/3}. \tag{16}$$

Proof. Let $x=\frac{\pi}{3}, y=\frac{2\pi}{3}$ in the identity of Lemma 1. We have

$$\theta_1\left(\frac{\pi}{3}|\tau\right)\theta_1\left(\frac{2\pi}{3}|\tau\right)=\theta_2\left(\frac{\pi}{3}|2\tau\right)\theta_3(\pi|2\tau)-\theta_2(\pi|2\tau)\theta_3\left(\frac{\pi}{3}|2\tau\right).$$

Noticing the definitions of the Jacobi theta functions and the identity $\theta_1\left(\frac{\pi}{3}|\tau\right)=\theta_1\left(\frac{2\pi}{3}|\tau\right)=\sqrt{3}q^{1/4}(q^6;q^6)_\infty$, we have:

$$3(q^6;q^6)_\infty^2=\sum_{n=0}^{\infty}q^{2n(n+1)}cos\frac{2n+1}{3}\pi\sum_{n=-\infty}^{\infty}q^{2n^2}+\sum_{n=0}^{\infty}q^{2n(n+1)}\sum_{n=-\infty}^{\infty}q^{2n^2}cos\frac{2n}{3}\pi.$$

Namely,

$$3(q^6;q^6)_\infty^2=3\sum_{n=0}^{\infty}q^{2n(n+1)}\sum_{n=-\infty}^{\infty}q^{18n^2}-3\sum_{n=0}^{\infty}q^{18n(n+1)}\sum_{n=-\infty}^{\infty}q^{2(n^2+6)}.$$

Replacing q^6 by q in the above identity gives

$$(q;q)_\infty^2 = \sum_{n=0}^{\infty} q^{n(n+1)/3} \sum_{n=-\infty}^{\infty} q^{3n^2} - \sum_{n=0}^{\infty} q^{3n(n+1)} \sum_{n=-\infty}^{\infty} q^{(n^2+6)/3}. \tag{17}$$

Namely,

$$\eta(\tau)_\infty^2 = \sum_{n=0}^{\infty} q^{(2n+1)^2/12} \sum_{n=-\infty}^{\infty} q^{3n^2} - \sum_{n=0}^{\infty} q^{3n(n+1)} \sum_{n=-\infty}^{\infty} q^{(4n^2+25)/12}. \tag{18}$$

This completes the proof. □

Theorem 4. *We have*

$$\begin{aligned} 4(q;q)_\infty^4 = &\sum_{n=-\infty}^{\infty} (6n+1) q^{n(3n+1)/2} \sum_{n=-\infty}^{\infty} q^{n(n+1)/2} \\ &- \sum_{n=-\infty}^{\infty} (6n+4) q^{n(3n+4)/2} \sum_{n=-\infty}^{\infty} q^{(n^2+1)/2}. \end{aligned}$$

Proof. In Theorem 1, take derivative on both sides with respect to x, and then set $x = 0$. We get

$$\begin{aligned} \frac{1}{(q;q)_\infty} \theta_1'\Big(0\Big|\frac{\tau}{2}\Big) \theta_1^2\Big(\frac{y}{2}\Big|\frac{\tau}{2}\Big) = &\, 2q^{1/8}\theta_2(y|\tau) \sum_{n=-\infty}^{\infty} (6n+1) q^{n(3n+1)} \\ &+ 2q^{11/8}\theta_3(y|\tau) \sum_{n=-\infty}^{\infty} (6n+4) q^{n(3n+4)}. \end{aligned}$$

Notice that:

$$\begin{cases} \theta_1'(0|\tau) &= 2q^{1/4}(q^2;q^2)_\infty^3, \\ \theta_1(\frac{\pi}{2}|\tau) &= \theta_2(\tau), \\ \theta_3(\pi|\tau) &= \theta_3(\tau), \\ \theta_2(\pi|\tau) &= -\theta_2(\tau). \end{cases}$$

Letting $y = \pi$ in the above equation, and after some simplifications, we obtain

$$4(q^2;q^2)_\infty^4 = \sum_{n=-\infty}^{\infty} (6n+1) q^{n(3n+1)} \sum_{n=-\infty}^{\infty} q^{n(n+1)} - q \sum_{n=-\infty}^{\infty} (6n+4) q^{n(3n+4)} \sum_{n=-\infty}^{\infty} q^{n^2}.$$

Replacing q^2 with q in the above equation, we can get the required conclusion. □

Theorem 5. *We have*

$$(q;q)_\infty^4 = \sum_{n=0}^{\infty} q^{n(n+1)/2} \left[\sum_{n=0}^{\infty} q^{n(3n+1)/2} - \sum_{n=0}^{\infty} q^{(n-1)(3n-4)/2} \right]. \tag{19}$$

Proof. Setting $x = \dfrac{3\pi}{4}$ and $y = \dfrac{\pi}{4}$ in the identity of Theorem 1, we are able to obtain that

$$\begin{aligned} \frac{1}{(q;q)_\infty} \theta_1\Big(\frac{3\pi}{4}\Big|\frac{\tau}{2}\Big) \theta_1\Big(\frac{\pi}{2}\Big|\frac{\tau}{2}\Big) \theta_1\Big(\frac{\pi}{4}\Big|\frac{\tau}{2}\Big) = &\, 2q^{1/8}\theta_2\Big(\frac{\pi}{4}\Big|\tau\Big) \sum_{n=-\infty}^{\infty} q^{n(3n+1)} \sin(6n+1)\frac{3\pi}{4} \\ &+ 2q^{11/8}\theta_3\Big(\frac{\pi}{4}\Big|\tau\Big) \sum_{n=-\infty}^{\infty} q^{n(3n+4)} \sin(6n+4)\frac{3\pi}{4}. \end{aligned}$$

It is obvious that $\theta_1\left(\frac{\pi}{4}|\tau\right) = \theta_1\left(\frac{3\pi}{4}|\tau\right) = \sqrt{2}q^{1/4}(q^2;q^2)_\infty(-q^4;q^4)_\infty$. Then:

$$2(q^2;q^2)_\infty^4 = 2\sum_{n=0}^{\infty} q^{n(n+1)} cos\frac{2n+1}{4}\pi \sum_{n=-\infty}^{\infty} q^{n(3n+1)} sin\frac{2n+3}{4}\pi$$
$$+ q\sum_{n=-\infty}^{\infty} q^{n^2} cos\frac{n}{2}\pi \sum_{n=-\infty}^{\infty} q^{n(3n+4)} sin\frac{n+2}{2}\pi.$$

In addition, when n is odd, $\cos\frac{n}{2}\pi = 0$ and when n is even, $\sin\frac{n+2}{2}\pi = 0$. Thus, after some simplifications, the above formula can be transformed into the following formula:

$$(q^2;q^2)_\infty^4 = \sum_{n=0}^{\infty} q^{n(n+1)} \left[\sum_{n=0}^{\infty} q^{n(3n+1)} - \sum_{n=0}^{\infty} q^{(n-1)(3n-4)}\right].$$

In the above equation, replacing q^2 with q, we can get the required conclusion. □

Remark 1. *Using the expression of* $(q;q)_\infty^4$ *, a new proof of the partition congruence* $p(5m+4) \equiv 0(mod5)$ *can be given.*

Theorem 6. *We have*

$$(q;q)_\infty^6 = \sum_{n=0}^{\infty} (2n+1)^2 q^{n(n+1)} \sum_{n=-\infty}^{\infty} q^{n^2} - \sum_{n=0}^{\infty} q^{n(n+1)} \sum_{n=-\infty}^{\infty} (2n)^2 q^{n^2}. \tag{20}$$

Proof. In the identity of Lemma 1, take derivative with respect to x and y respectively, and then let $x = y = 0$. After simplifications, we obtain the required equation. □

Theorem 7. *We have*

$$4(q;q)_\infty^8 = -\sum_{n=-\infty}^{\infty} q^{n(n+1)} \sum_{n=-\infty}^{\infty} (6n+1)^3 q^{n(3n+1)}$$
$$-\sum_{n=-\infty}^{\infty} q^{n^2+1} \sum_{n=-\infty}^{\infty} (6n+4)^3 q^{n(3n+4)}. \tag{21}$$

Proof. In the equation of Theorem 1, take the third derivative with respect to x, and then let $x = y = 0$. We get

$$\frac{1}{(q;q)_\infty}\left[\theta_1'\left(0|\frac{\tau}{2}\right)\right]^3 = -q^{1/8}\theta_2(\tau) \sum_{n=-\infty}^{\infty} (6n+1)^3 q^{n(3n+1)}$$
$$- q^{11/8}\theta_3(\tau) \sum_{n=-\infty}^{\infty} (6n+4)^3 q^{n(3n+4)}.$$

Using the following identities

$$\theta_1'(0|\tau) = 2q^{1/4}(q^2;q^2)_\infty^3, \quad \theta_2(\tau) = q^{1/4}\sum_{n=-\infty}^{\infty} q^{n(n+1)}, \quad \theta_3(\tau) = \sum_{n=-\infty}^{\infty} q^{n^2}$$

in the above formula and after some simplifications, we can obtain the required equation. □

Remark 2. *The identity in Theorem 7 appeared in [12]. In [20] Winquist gave without proof a formula for* $\eta^8(\tau)$.

Theorem 8. *We have*

$$(q;q)_\infty^{10} = q \sum_{n=-\infty}^{\infty} (6n+1)^3 q^{n(3n+1)} \sum_{n=-\infty}^{\infty} (6n+4) q^{n(3n+4)} - q \sum_{n=-\infty}^{\infty} (6n+4)^3 q^{n(3n+4)} \sum_{n=-\infty}^{\infty} (6n+1) q^{n(3n+1)}. \quad (22)$$

Proof. In the identity of Theorem 2, first take the third derivative with respect to x, and then take the first derivative with respect to y. Finally, set $x = y = 0$ and after some simplifications, we obtain the required conclusion. □

Remark 3. *The identity of Theorem 8 appeared in [12]. Expressions for $(q;q)_\infty^{10}$ has been discussed in [11,13,16]. With the expression for $(q;q)_\infty^{10}$, the partition congruence $p(11m+6) \equiv 0 (mod\, 11)$ can be re-proved.*

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References

1. Andrews, G.E.; Berndt, B.C. *Ramanujan's Lost Notebooks, Part I*; Springer Science+Business Media, Inc.: Berlin/Heidelberg, Germany, 2005.
2. Berndt, B.C. *Ramanujan's Notebooks, Part III*; Springer: New York, NY, USA, 1991.
3. Jacobi, C.G.J. *Fundamenta Nova Theoriae Functionum Ellipticarum*; Borntrager: Stuttgart, Germany, 1829.
4. Whittaker, E.T.; Watson, G.N. *A Course of Modern Analysis*, 4th ed.; Cambridge University Press: Cambridge, UK, 1966.
5. Schiefermayr, K. Some new properties of Jacobi's theta functions. *J. Comput. Appl. Math.* **2005**, *178*, 419–424. [CrossRef]
6. Liu, Z.-G. Addition formulas for Jacobi theta functions, Dedekind's eta function, and Ramanujan's congruences. *Pacific J. Math.* **2009**, *240*, 135–150. [CrossRef]
7. Tsumura, H. Double series identities arising from Jacobi's identity of the theta function. *Results Math.* **2018**, *73*, 10–12. [CrossRef]
8. Tsumura, H. On series identities arising from Jacobi's identity of the theta function. *Int. J. Number Theory* **2018**, *14*, 1317–1327. [CrossRef]
9. Schneider, R. Jacobi's triple product, mock theta functions, unimodal sequences and the q-bracket. *Int. J. Number Theory* **2018**, *14*, 1961–1981. [CrossRef]
10. Singh, S.P.; Yadav, R.K.; Yadav, V. Certain properties of Jacobi's theta functions. *South East Asian J. Math. Math. Sci.* **2021**, *17*, 119–130.
11. Berndt, B.C.; Chan, S.H.; Liu, Z.-G.; Yesilyurt, H. A new identity for $(q;q)_\infty^{10}$ with an application to Ramanujan's partion congruence modulo 11. *Quart. J. Math.* **2004**, *55*, 13–30. [CrossRef]
12. Chan, H.H.; Cooper, S.; Toh, P.C. Ramanujan's Eisenstein series and powers of Dedekind's eta-function. *J. Lond. Math. Soc.* **2007**, *75*, 225–242. [CrossRef]
13. Chu, W. Theta function identites and Ramanujan's congruence on partion function. *Quart. J. Math.* **2005**, *56*, 491–506. [CrossRef]
14. He, B. A theta function identity and its applications. *Ramanujan J.* **2015**, *38*, 423–433. [CrossRef]
15. Ma, H.-N.; He, B. On a theta function identity. *Integral Transform. Spec. Funct.* **2016**, *27*, 365–370. [CrossRef]
16. Liu, Z.-G. A theta function identity and its implications. *Trans. Am. Math. Soc.* **2005**, *357*, 825–835. [CrossRef]
17. Liu, Z.-G. A three-term theta function identity and its applications. *Adv. Math.* **2005**, *195*, 1–23. [CrossRef]
18. Zhai, H.-C. Additive formulae of theta functions with applications in modular equations of degree three and five. *Integral Transform. Spec. Funct.* **2009**, *20*, 769–773. [CrossRef]

19. Shen, L.-C. On the products of three theta functions. *Ramanujan J.* **1999**, *3*, 343–357. [CrossRef]
20. Winquist, L. An elementary proof of $p(11m + 6) \equiv 0(mod 11)$. *J. Comb. Theory* **1969**, *6*, 56–59. [CrossRef]