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A Unifying Principle in the Theory of Modular Relations †

Guodong Liu [1,*,‡], Kalyan Chakraborty [2,‡] and Shigeru Kanemitsu [2]

1 Department of Mathematics and Statistics, Huizhou University, Huizhou 516007, China
2 KSCSTE-Kerala School of Mathematics, Kozhikode 673571, India
* Correspondence: lgd@hzu.edu.cn
† Dedicated to Professor Dr. Yoshiyuki Kitaoka with great respect and friendship.
‡ These authors contributed equally to this work.

Abstract: The Voronoǐ summation formula is known to be equivalent to the functional equation for the square of the Riemann zeta function in case the function in question is the Mellin tranform of a suitable function. There are some other famous summation formulas which are treated as independent of the modular relation. In this paper, we shall establish a far-reaching principle which furnishes the following. Given a zeta function $Z(s)$ satisfying a suitable functional equation, one can generalize it to $Z_f(s)$ in the form of an integral involving the Mellin transform $F(s)$ of a certain suitable function $f(x)$ and process it further as $\tilde{Z}_f(s)$. Under the condition that $F(s)$ is expressed as an integral, and the order of two integrals is interchangeable, one can obtain a closed form for $\tilde{Z}_f(s)$. Ample examples are given: the Lipschitz summation formula, Koshlyakov's generalized Dedekind zeta function and the Plana summation formula. In the final section, we shall elucidate Hamburger's results in light of RHBM correspondence (i.e., through Fourier–Whittaker expansion).

Keywords: summation formulas; modular relation; Mellin tranform; Riemann zeta function; functional equation

MSC: 11F32; 11F20; 11A25

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1. The Principle and Statement of Results

We shall provide a new principle in the theory of modular relations—equivalent assertions to the functional equation—which enables us to establish a closed form for processed sums (Equation (5)) of $\sum_{n=1}^{\infty} a_n f(\lambda_n)$ in Equation (4), where f admits the Mellin or some integral transform F, for example. We shall refer to (the use of) Theorem 1 and its special cases in Corollary 1 and Example 1 as the *Principle*. This could be perceived by the argument of [1].

Let

$$f(x) = \frac{1}{2\pi i} \int_{(c)} x^{-s} F(s)\, \mathrm{d}s, \quad \operatorname{Re} x > 0, c > 0, \quad F(s) = \int_0^{\infty} \xi^s f(\xi) \frac{\mathrm{d}\xi}{\xi} \tag{1}$$

be the Mellin transform pair which satisfies the conditions of convergence necessitated in our discussion. (f, F) will always be used as the Mellin transform pair.

Let $\{\lambda_n\}$ be a strictly increasing sequence of real numbers with $\lambda_1 > 0$. Let

$$Z(s) = \sum_{n=1}^{\infty} \frac{a_n}{\lambda_n^s} \tag{2}$$

be absolutely convergent for $\operatorname{Re} s = \sigma > 1$. The abscissa of absolute convergence can be σ_a^*, but we assume it to be one for the sake of simplicity. Suppose it satisfies the functional equation of the following form (where we understand the right-hand side member may be a different Dirichlet series):

$$Z(s) = G(s) Z(1-s), \tag{3}$$

where $G(s)$ is a certain gamma factor to be specified in each occasion and the line of reflection is chosen to be $\frac{1}{2}$ instead of a more general $\frac{r}{2} \in \mathbb{R}$ (cf. *Convention* below).

For $c > 1$, we have

$$Z_f(s) = \sum_{n=1}^{\infty} a_n f(\lambda_n) = \frac{1}{2\pi i} \int_{(c)} F(z)Z(z)\,\mathrm{d}z \tag{4}$$

in the first instance. Here and in what follows, we use this type of suggestive notation to mean that the processing factors—f and F in this case—may also depend on the extraneous parameter s. This is similar for Equations (5), (7), etc. The following theorem gives the principle for obtaining a closed form for $\tilde{Z}_f(s)$:

Theorem 1. *We write $\tilde{Z}_f(s)$ for a modified form of $Z_f(s)$ by a certain process, and we assume that*

$$\tilde{Z}(s) = \tilde{Z}_f(s) = \frac{1}{2\pi i} \int_{(c)} F_f(z)\mathfrak{G}_f(z)Z(z)\,\mathrm{d}z \tag{5}$$

for some $c > 1$ and $\operatorname{Re} x \geq 0$ in Equation (1), where $F_f(z)$ is the processed Mellin inversion $F(s)$ in Equation (1). Here and throughout, $\mathfrak{G}_f(z)$, $\mathfrak{F}_F(w)$, etc. denote certain gamma factors, and $\mathfrak{G}_F(w,z)$ may depend only on z and may work as $\mathfrak{G}_f(z)$. $\mathfrak{F}_F(w)$ may be a function constructed from f. Suppose the integration path (c) of Equation (5) may be shifted to $(-d)$, $0 < d < 1$, such as with the resulting residual function $\mathrm{P}(s)$ (sum of the residues in the vertical strip $-d < \sigma < c$):

$$\tilde{Z}_f(s) = \frac{1}{2\pi i} \int_{(-d)} F_f(z)\mathfrak{G}_f(z)Z(z)\,\mathrm{d}z + \mathrm{P}(s) = J(s) + \mathrm{P}(s), \tag{6}$$

Say that

$$F_f(z) = \int \mathfrak{G}_F(w,z)\mathfrak{F}_F(w)\delta(w)\,\mathrm{d}w, \tag{7}$$

where the integral for $F_f(z)$ may be the infinite integral over $(0,\infty)$, the contour integral or may indicate the integrand itself, $\gamma(z) = \gamma(z,s), \delta(w) = \delta(w,z,s)$ are simple functions specified at each occasion and that the order of integration is interchangeable:

$$\frac{1}{2\pi i}\,\mathrm{d}z \int_{(-d)} \int \,\mathrm{d}w = \int \,\mathrm{d}w \frac{1}{2\pi i} \int_{(-d)} .\,\mathrm{d}z \tag{8}$$

We write

$$I(w) := \frac{1}{2\pi i} \int_{(-d)} \mathfrak{G}_F(w,z)M(z)Z(1-z)\gamma(z)\,\mathrm{d}z = \sum_{n=1}^{\infty} a_n K(w,n), \tag{9}$$

where

$$K(w,n) = \frac{1}{2\pi i} \int_{(-d)} \mathfrak{G}_F(w,z)M(z){\lambda_n}^{z-1}\gamma(z)\,\mathrm{d}z \tag{10}$$

and where

$$M(z) = G(z)\mathfrak{G}_f(z). \tag{11}$$

Then, if both $K(w,n)$ and

$$J(s) = \int I(w)\mathfrak{F}_F(w)\delta(w)\,\mathrm{d}w \tag{12}$$

admit a closed form, then Equation (12) gives a closed form for $\tilde{Z}_f(s)$.

For convenience of application, we extract an unprocessed case (Equation (4)) (i.e., the integral operator in Equation (7) for F_f is the identity operator as a corollary):

Corollary 1. *Consider the processed zeta function $Z_f(z)$ in Equation (4). Suppose the integration path (c) may be shifted to $(-d)$, $\frac{1}{2} < d < 1$, with the resulting residual function $\mathrm{P}(s)$ (sum of the residues in the vertical strip $-d < \sigma < c$):*

$$Z_f(s) = \frac{1}{2\pi i} \int_{(-d)} F(z)Z(z)\,\mathrm{d}z + \mathrm{P}(s) = J(s) + \mathrm{P}(s), \tag{13}$$

where

$$J(s) = I(s) = \frac{1}{2\pi i} \int_{(-d)} F(z)G(z)Z(1-z)\gamma(z.s)\,\mathrm{d}z = \sum_{n=1}^{\infty} a_n K(s,n), \tag{14}$$
$$K(s,n) = \frac{1}{2\pi i} \int_{(-d)} F(z)G(z){\lambda_n}^{z-1}\gamma(z,s)\,\mathrm{d}z.$$

Then, if Equation (14) is expressed in a closed form, it gives a closed form for $Z_f(z)$.

We note the specification

$$F_f(z) = \mathfrak{G}_F(w,z), \quad F_f(z)\mathfrak{G}_f(z) = \mathfrak{G}_F(w,z)\mathfrak{G}_f(z) = F(z), \quad F(z)M(z) = G(z),$$
$$\mathfrak{G}_f(z) = 1, \quad \mathfrak{F}_F(w)\delta(w) = 1.$$

We state the consequences of Theorem 1 in terms of the Dedekind zeta function $\zeta_\Omega(s)$ of an algebraic number field Ω of degree of two at most, which we let represent the case of the Riemann zeta function $\zeta(s)$, the zeta function $\Phi(s)$ associated with a modular form and $\zeta^2(s)$, which is a generating function of the divisor function according to $\Omega = \mathbb{Q}$, where Ω is an imaginary quadratic field or Ω is a real quadratic field. The authors of [2,3] generalized the divisor problem to the case of the mth power of the Dedekind zeta function and obtained the closed form for the partial sum for arbitrary degree. The authors of [4] established the identities of Hardy and Voronoĭ and those of the imaginary and real quadratic fields, while the authors of [5] contained the case of the Epstein zeta function with a positive definite binary quadratic form. These are unified as shown in Table 1 by viewing the zeta functions as represented by the Dedekind zeta function.

Table 1. Gamma factors in respective cases.

Section	$\mathfrak{F}_F(w)$	$\mathfrak{G}_F(w,z)$	$\mathfrak{G}_f(z)$	$K(\cdot,n),\gamma(z,s)$	$\delta(\cdot)$
Section 2	$\Gamma(-w)\Gamma(w+s)$	$\Gamma(w+z)$	$\cos\frac{\pi}{2}z$	(73)	$(-2\pi\alpha\omega)^{-w}$
Section 3	1	$\Gamma(w-z)$	$\Gamma(z)\cos\frac{\pi}{2}z$	(102)	$\gamma(z)$
Section 4	$\frac{f(iw)-f(-iw)}{2i}$	$\mathfrak{G}_f(z)$	$\frac{1}{\sin\frac{\pi}{2}z}$	(130)	1

Let Ω be an algebraic number field of a discriminant Δ and degree $\chi \le 2$:

$$\chi = r_1 + 2r_2, \tag{15}$$

where r_1 resp. $2r_2$ indicates the real resp. imaginary conjugates, and let

$$\zeta_\Omega(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}, \quad a_n = \sum_{N\mathfrak{a}=n} 1, \tag{16}$$

be the Dedekind zeta function, which is absolutely convergent for $\sigma = \mathrm{Re}\, s > 1$ on the grounds that

$$a_n = O(n^\eta), \tag{17}$$

for every $\eta > 0$.

The functional equation reads

$$A^{-s}\Gamma^{r_1}\left(\frac{s}{2}\right)\Gamma^{r_2}(s)\zeta_\Omega(s) = A^{-(1-s)}\Gamma^{r_1}\left(\frac{1-s}{2}\right)\Gamma^{r_2}(1-s)\zeta_\Omega(1-s), \tag{18}$$

where our A is the inverse of Koshlyakov's:

$$A = \frac{2^{r_2}\pi^{\frac{\chi}{2}}}{\sqrt{|\Delta|}}. \tag{19}$$

The Dedekind zeta function has a simple pole at $s = 1$ with residue

$$\rho = \frac{2^{r+1}\pi^{r_2}Rh}{W\sqrt{|\Delta|}}, \tag{20}$$

where $r = r_1 + r_2 - 1$ is the rank of the unit group, h is the class number, R is the regulator of Ω and W is the number of roots of unity in Ω. This may be expressed in the case where $\chi \leq 2$ as

$$\rho = -\frac{2^{r+1}\pi^{r_2}\zeta_\Omega^{(r)}(0)}{\sqrt{|\Delta|}}. \tag{21}$$

By implementing

$$G(s) = \frac{\Gamma^{r_1}\left(\frac{1-s}{2}\right)\Gamma^{r_2}(1-s)}{\Gamma^{r_1}\left(\frac{s}{2}\right)\Gamma^{r_2}(s)}, \tag{22}$$

then we can express Equation (18) as

$$\zeta_\Omega(s) = A^{2s-1}G(s)\zeta_\Omega(1-s). \tag{23}$$

Necessary information on algebraic numbers is available in many books, (see, for example, [6]). In applying Equation (23) to other cases, care must be taken regarding the constants. For example, in the case of the zeta function $\Phi(s)$ associated with a modular form of weight of $2k$, then $A = 2\pi$, the line of reflection is $2k$, and $G(s)$ is the same value as that of the imaginary quadratic case such that Equation (23) takes the following form (cf. Lemma 4 below):

$$\Phi(s) = (-1)^k G(s)\Phi(2k - s).$$

In such a case, we understand $(-1)^k\Phi(2k - s)$ is another Dirichlet series $\psi(2k - s)$ in the setting of Definition 1:

Definition 1. *Under the notation in Definition 2, we consider the functional equation*

$$A^{-s}\Gamma^{r_1}\left(\frac{s}{2}\right)\Gamma^{r_2}(s)\varphi(s) = A^{-(1-s)}\Gamma^{r_1}\left(\frac{1-s}{2}\right)\Gamma^{r_2}(1-s)\psi(1-s). \tag{24}$$

We call the cases $(r_1, r_2) = (1, 0)$, $(r_1, r_2) = (0, 1)$ and $(r_1, r_2) = (2, 0)$ for Equation (24) the Riemann, Hecke and Voronoĭ type functional equations, respectively. For the Hecke type, we understand Equation (24) to mean Equation (145) in Definition 2, with the line of reflection being $\sigma = r$ so as to include the weight aspect.

Convention. Throughout what follows, we form a convention where we let Equation (18) represent the general functional equation (Equation (24)), and for the Hecke type where $(r_1, r_2) = (0, 1)$, it is to be understood in Definition 2 with all the basic results in Theorem 6. We state the results with $r = 1$ elsewhere. The transition to the general r is simple while treating different Dirichlet series $\varphi(s)$, and $\psi(s)$ is more involved. In most applications, we proceed as in Example 1 below such that a_n represents the ideal function in Equation (16) or the coefficients of other allied zeta-functions. We denote the residual function with $\mathrm{P}(s)$, which may be different for different zeta functions (Tables 2 and 3).

Table 2. Dedekind zeta function and allied functional equation.

Field	Zeta	Covering	Gamma Factor
rational	Riemann	fundamental	$\Gamma\left(\frac{s}{2}\right)$
imaginary quadr.	Dedekind	modular form, Hecke	$\Gamma(s)$
real quadr.	Dedekind	divisor function, Voronoĭ	$\Gamma^2\left(\frac{s}{2}\right)$

Table 3. Values of constants corrected and modified from the table in [1] (I, p. 122).

Constant	A	$\zeta_\Omega(0)$	ρ (Residue)
Rational	$\sqrt{\pi}$	$-\frac{1}{2}$	1
Imaginary quadr.	$\frac{2\pi}{\sqrt{\lvert\Delta\rvert}}$	$-\frac{h}{W}$	$-\frac{2\pi\zeta_\Omega(0)}{\sqrt{\lvert\Delta\rvert}}$
Real quadr.	$\frac{\pi}{\sqrt{\lvert\Delta\rvert}}$	0	$-\frac{4\zeta'_\Omega(0)}{\sqrt{\Delta}}$

The Hurwitz–Lerch L-function $\Phi(z,s,\alpha;\chi)$ is defined by

$$\Phi(z,s,\alpha;\chi) = \sum_{n=0}^{\infty} \frac{\chi(n)z^n}{(n+\alpha)^s}$$

which is absolutely and uniformly convergent on any compact subset of

$$\{|z| \le 1\} \times \{\sigma > 1\} \times \{\alpha > -1\} \tag{25}$$

According to [7] (p. 30), the special case where χ is the trivial character mod 1 is the Hurwitz–Lerch zeta-function $\Phi(z,s,w)$, defined by

$$\Phi(z,s,\alpha) = \sum_{n=0}^{\infty} \frac{z^n}{(n+\alpha)^s} \tag{26}$$

which is absolutely and uniformly convergent on any compact subset of

$$\{|z| \le 1\} \times \{\sigma > 1\} \times \{\alpha \in \mathbb{C}, \alpha \ne 0, -1, -2, \cdots\}, \tag{27}$$

where a suitable branch is chosen of $(n+\alpha)^s$. Compare this with [8] (I, pp. 27–31) for $\alpha \in \mathbb{C}$ not being a non-positive integer (often restricted to $0 < \alpha \le 1$) and whether $|z| < 1$ or $|z| = 1$, $\sigma = \operatorname{Re} s > 1$.

The Lipschitz–Lerch transcendent $L(x,s,\alpha) = \Phi(e^{2\pi ix}, s, \alpha)$ was stated in [9] (pp. 121–123) as a special case with $x \in \mathbb{R}$:

$$L(x,s,\alpha) = \sum_{n=0}^{\infty} \frac{e^{2\pi inx}}{(n+\alpha)^s}. \tag{28}$$

For intended applications, we choose

$$\{\operatorname{Im} z \ge 0\} \times \{\sigma > 1\} \times \{\alpha \in \mathbb{C}, \operatorname{Re}\alpha \ge 0\}, \tag{29}$$

as the domain of absolute convergence. The convergence condition stated in [9] (p. 122, (11)) is mainly as a special case, especially for the second condition $x \in \mathbb{Z}$, and $\sigma > 1$ is the convergence condition for the Hurwitz zeta function. The first condition $x \in \mathbb{R}\backslash\mathbb{Z}$, $\sigma > 0$ (for uniform convergence and not absolute convergence) may be changed to $x \in \mathcal{H}$. The upper half-plane and $\sigma > 0$ correspond to the condition $|z| < 1$ for $\Phi(z,s,\alpha)$ above. The research in [10] is devoted to the theory of the Lipschitz–Lerch transcendent, which is referred to as the Lerch zeta function, and contains detailed proofs of the functional equation. Compare this with [9] (p. 121, (1)), in which the Hurwitz–Lerch zeta function part is verbatim to [8] (I, pp. 27–31).

Let $\ell_s(x)$ be the boundary Lerch zeta function defined by

$$\ell_s(x) = \sum_{n=1}^{\infty} e^{2\pi ixn} n^{-s}, \quad \sigma > 1 \quad \text{or} \quad \sigma > 0, x \notin \mathbb{Z}, \tag{30}$$

which has its counterpart in the Hurwitz zeta function:

$$\zeta(s,x) = \sum_{n=0}^{\infty} \frac{1}{(n+x)^s}, \quad \sigma > 1. \tag{31}$$

This is continued meromorphically over the whole plane with a simple pole at $s = 1$. Both of them reduce to the Riemann zeta function

$$\zeta(s,1) = \ell_s(1) = \zeta(s).$$

The distribution property of $\Phi(z,s,w)$ was studied in [11], and the Lipschitz summation formula was studied in [12] (pp. 128–132) and recently [13] interpreted as the functional equation for $L(x,s,w\alpha)$. In other words, Theorem 2 is the same as the functional equation for Corollary 2.

We define the Lipschitz–Lerch transcendent associated with the field Ω by

$$L_\Omega(x,s,\alpha) = \sum_{n=0}^{\infty} a_n \frac{e^{2\pi inx}}{(n+\alpha)^s}, \tag{32}$$

where a_0 is a constant, which is determined so that Equation (32) reduces to Equation (31) for $\Omega = \mathbb{Q}$. Hence, we implement $a_0 = -2\zeta_\Omega(0) = 1$. The authors of [1] (p. 241) introduced the Hurwitz-type Dedekind zeta function

$$\zeta_\Omega(s,w) = L_\Omega(0,s,w) = -2\frac{\zeta_\Omega(0)}{w^s} + \sum_{n=1}^{\infty} \frac{a_n}{(n+w)^s}, \quad \sigma > 1, \quad 0 < w < 1, \tag{33}$$

which will be studied in Section 3:

Example 1. *We assume the conditions in Theorem 4. We proceed almost verbatim to [12], and Equation (4) reads for $Z(s) = \zeta_\Omega(s)$ as follows:*

$$\zeta_{\Omega,f}(s) = \sum_{n=1}^{\infty} a_n f(n) = \frac{1}{2\pi i} \int_{(c)} F(s)\zeta_\Omega(s) \mathrm{d}s.$$

which is processed as $\tilde{Z}_f(s) = \tilde{\zeta}_{\Omega,f}(s)$ in Equation (5).

Then, in light of the estimate

$$\zeta_\Omega(s) = O(|t|^{\chi(1-\sigma)}), \quad \sigma \le 0 \tag{34}$$

in the strip $\sigma_1 \le \sigma \le \sigma_2$ and the Stirling formula (Equation (42)), we may uniformly shift the line of integration to $\sigma = -d$, $0 < d < 1$, passing through the poles in the strip $-d < \sigma < c$. Hence, we have Equation (13) with $Z(s) = \zeta_\Omega(s)$.

Then, by applying Equation (7), changing the order of integration and applying the functional in Equation (23), we arrive at Equation (12) with $I(w)$ in Equation (9).

Then, since $1 - d > 1$, we may use the Dirichlet series for $\zeta_\Omega(1-\zeta)$, and Equation (9) reads as follows:

$$I(w) = \sum_{n=1}^{\infty} a_n K(w,n), \quad K(w,n) = \frac{1}{2\pi i} \int_{(-d)} \mathfrak{G}_F(w,z) M(z) n^{z-1} \gamma(z)\, \mathrm{d}z. \tag{35}$$

The most essential part is justification of the change in order of integration. In Theorems 2 and 4, the exponential reduction is lost, and integration path(s) are to be taken such that the exponent of $|t|$ is < -1.

In the setting of Corollary 1, Equation (35) reads as follows:

$$J(s) = I(s) = \sum_{n=1}^{\infty} a_n K(s,n), \quad K(s,n) = \frac{1}{2\pi i} \int_{(-d)} F(z) M(z) n^{z-1} \gamma(z,s)\, \mathrm{d}z \tag{36}$$

while $\gamma(z,s)$ involves the factor A^{2z-1}.

Example 2. *Many transforms may be viewed as prototype applications of the Principle without processing, namely the Hecke gamma transform in Equation (39) (and the X-transform in Equation (165)) leading to the Bochner modular relation (Equation (173)), the beta transform (Equation (51)) leading to the Fourier–Bessel expansion (Equation (151)), the Hardy transform (and K-transform (Equation (165))) leading to the partial fraction expansion in Equation (174), the confluent hypergeometric transform in Equation (56) leading to Lerch's transformation formula, Theorem 2, etc.*

Notation and Terminologies

The gamma function is defined as the Mellin transform of $e^{-\xi}$ as in Equation (38). The extension of its validity plays an important role in our discussion. The extended right half-plane may be stated as follows:

$$x = |x| e^{i\theta}, \quad x \neq 0, \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}, \tag{37}$$

which is denoted by $\mathrm{Re}\, x \geq 0$:

Lemma 1. *The Mellin transform pair*

$$\Gamma(s) = \int_0^\infty \xi^s e^{-\xi} \frac{\mathrm{d}\xi}{\xi}, \quad e^{-x} = \frac{1}{2\pi i} \int_{(c)} x^{-z} \Gamma(z) \mathrm{d}z \tag{38}$$

which is valid for $\mathrm{Re}\, x > 0$ and $0 < c$ extends to the domain in Equation (37): $\mathrm{Re}\, x \geq 0$ for $0 < \sigma < 1$ resp. $0 < c < 1$.

Compare this with Lemma 3.

Most of the known Mellin transform pairs may be found in [14]. Compare this with [15] for its theory. The Mellin transform in Equation (38) is often applied as the Hecke gamma transform

$$x^{-s}\Gamma(s) = \int_0^\infty \xi^s e^{-x\xi} \frac{\mathrm{d}\xi}{\xi} \tag{39}$$

which holds for $x > 0, \sigma > 0$ or $\mathrm{Re}\, x > 0, \sigma > 0$. This is also true for $\mathrm{Re}\, x = 0, x \neq 0, 0 < \sigma < 1$. Compare this with [16] (Lemma 4.20, p. 169) for a very enlightening remark on prehomogeneous vector spaces.

We often make use of the following formulas, and we will not state the details at each occurrence. First, there is the duplication formula:

$$\Gamma(2s) = 2^{2s-1}\sqrt{\pi}^{-1}\Gamma(s)\Gamma\left(s+\frac{1}{2}\right). \tag{40}$$

Then, we have the reciprocity relation:

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}. \tag{41}$$

The Stirling formula [8] (p. 47, (6)) is

$$\Gamma(s) = \sqrt{2\pi}e^{-\frac{\pi}{2}|t|}|t|^{\sigma-\frac{1}{2}}, \quad |t| \to \infty. \tag{42}$$

We shall often use the Meijer G function, especially in Section 2 (partially because we would like to show the hierarchy of special functions) which is defined by

$$\begin{aligned} G^{m,n}_{p,q}\left(z \,\middle|\, \begin{matrix} a_r \\ b_s \end{matrix}\right) &= G^{m,n}_{p,q}\left(z \,\middle|\, \begin{matrix} a_1,\dots,a_n,a_{n+1},\dots,a_p \\ b_1,\dots,b_m,b_{m+1},\dots,b_q \end{matrix}\right) \\ &= H^{m,n}_{p,q}\left(z \,\middle|\, \begin{matrix} (a_1,1),\dots,(a_n,1),(a_{n+1},1),\dots,(a_p,1) \\ (b_1,1),\dots,(b_m,1),(b_{m+1},1),\dots,(b_q,1) \end{matrix}\right) \\ &= \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma\big(b_j+s\big)\prod_{j=1}^{n}\Gamma\big(1-a_j-s\big)}{\prod_{j=n+1}^{p}\Gamma\big(a_j+s\big)\prod_{m+1}^{q}\Gamma\big(1-b_j-s\big)} z^{-s}\mathrm{d}s. \end{aligned} \tag{43}$$

Compare this with, for example, [8] (I, p. 207), which contains most of the information needed in our argument. The integrals are absolutely convergent if $m+n > \frac{1}{2}(p+q)$, a condition which is satisfied in almost all the cases appearing below. Some delicate cases can be dealt with by other convergence conditions. We often state the G function expression for the special functions used such that it will yield a hierarchy.

The following special functions will often be used. We refer to Abramowitz and Stegun (1965), among others [8,17,18] for the J Bessel function [8] (II, p. 83, (36)), which reads as follows:

$$J_\nu(x) = \frac{1}{2\pi i}\int_{(c)} \frac{\Gamma(-z)}{\Gamma(1+\nu+z)}\left(\frac{x}{2}\right)^{\nu+2z}\mathrm{d}z = \frac{1}{2}G^{0,1}_{2,0}\left(\frac{x}{2} \,\middle|\, \begin{matrix} 1, 1+\nu \\ - \end{matrix}\right). \tag{44}$$

The Bessel function of the third kind or Basset's function, which we refer to as the K Bessel function, is introduced in many ways. The function considered by Voronoĭ [19] (p. 211)

$$2\int_1^\infty \frac{e^{-2\sqrt{x}t}}{\sqrt{t^2-1}}\,\mathrm{d}t, \quad \sqrt{x} > 0.$$

is a special case ($s = 0$) from [8] (II, p. 18, (15)), where

$$\Gamma\left(\frac{1}{2}-s\right)K_s(z) = \sqrt{\pi}\left(\frac{1}{2}z\right)^{-s}\int_1^\infty \frac{e^{-zt}}{(t^2-1)^{s+\frac{1}{2}}}\,\mathrm{d}t, \quad \sqrt{x} > 0. \tag{45}$$

According to [1], this is the theory of K Bessel functions, and it is often used as the inverse Heaviside integral

$$K_\nu(x) = \frac{1}{2\pi i}\int_L 2^{s-2}\Gamma\left(\frac{s+\nu}{2}\right)\Gamma\left(\frac{s-\nu}{2}\right)x^{-s}\,\mathrm{d}s, \tag{46}$$

which is the inversion of

$$\int_0^\infty x^s K_\nu(x)\,\frac{\mathrm{d}x}{x} = 2^{s-2}\Gamma\left(\frac{s+\nu}{2}\right)\Gamma\left(\frac{s-\nu}{2}\right). \tag{47}$$

Equation (46) is often expressed as

$$2K_\nu(2z) = \frac{1}{2\pi i}\int_L \Gamma\left(s+\frac{\nu}{2}\right)\Gamma\left(s-\frac{\nu}{2}\right)z^{-2s}\,\mathrm{d}s = G_{0,2}^{2,0}\left(z^2\,\middle|\,\begin{matrix}-\\ \frac{\nu}{2},-\frac{\nu}{2}\end{matrix}\right), \tag{48}$$

or

$$2K_\nu(2z) = \frac{1}{2\pi i}\int_L \Gamma(s)\Gamma(s-\nu)z^{\nu-2s}\,\mathrm{d}s = z^\nu G_{0,2}^{2,0}\left(z^2\,\middle|\,\begin{matrix}-\\ 0,-\nu\end{matrix}\right), \tag{49}$$

where L is a suitable Bromwich path, which we often abbreviate as (c). In the real quadratic case, the following particular case appears:

$$2K_0(2z) = \frac{1}{2\pi i}\int_L \Gamma(s)^2 z^{\nu-2s}\,\mathrm{d}s \tag{50}$$

The main ingredient in Koshlyakov's argument is the beta transform

$$\Gamma(s)(1+x)^{-s} = G_{1,1}^{1,1}\left(x\,\middle|\,\begin{matrix}1-s\\ 0\end{matrix}\right) \qquad = \frac{1}{2\pi i}\int_{(c)} \Gamma(z)\Gamma(s-z)x^{-z}\,\mathrm{d}z, \tag{51}$$

where $-\operatorname{Re} z < c < 0$ and $\operatorname{Re} x \geq 0$, or

$$G_{1,1}^{1,1}\left(x^{-1}\,\middle|\,\begin{matrix}1\\ s\end{matrix}\right) = \frac{1}{2\pi i}\int_{(c)} \Gamma(-w)\Gamma(w+s)x^{w}\,\mathrm{d}w.$$

This is a special case of

$$G_{1,1}^{1,1}\left(z\,\middle|\,\begin{matrix}a\\ b\end{matrix}\right) = \Gamma(1-a+b)z^b(1+z)^{a-b-1}, \tag{52}$$

which is not stated in [8].

The case in [8] (p. 256, (4)) reads as follows:

$$\begin{aligned}{}_1F_1(a,c;x) &= \frac{\Gamma(c)}{\Gamma(c)}G_{2,1}^{1,1}\left(\frac{1}{-x}\,\middle|\,\begin{matrix}1,c\\ a\end{matrix}\right) = \frac{\Gamma(c)}{\Gamma(a)}\frac{1}{2\pi i}\int_{(\gamma)}\frac{\Gamma(-z)\Gamma(a+z)}{\Gamma(c+z)}(-x)^s\,\mathrm{d}z\\ &= \frac{\Gamma(c)}{\Gamma(c)}G_{1,2}^{1,1}\left(-x\,\middle|\,\begin{matrix}1-a\\ 0,1-c\end{matrix}\right),\end{aligned}$$

where ${}_1F_1(a,c;x)$ and $U(a,c:x)$ in Equation (56) form the fundamental system of the confluent hypergeometric DE.

By letting $c=1$, $a=w+1$ and $z+1=s$, we have

$${}_1F_1(w+1,1;x) = \frac{1}{\Gamma(w+1)}\frac{1}{2\pi i}\int_{(\gamma+1)}\frac{\Gamma(1-s)\Gamma(w+s)}{\Gamma(z)}(-x)^{s-1}\,\mathrm{d}s \tag{53}$$

Lemma 2. *We have the Mellin transform pair $e^{-x}G_{p,q}^{m,n}(ax|\dot{)} \leftrightarrow G_{p+1,q}^{m,n+1}(a|\dot{)}$ [8] (I, (16), p. 338):*

$$\int_0^\infty x^{z-1}e^{-x}G_{p,q}^{m,n}\left(ax\,\middle|\,\begin{matrix}a_1,\cdots,a_p\\ b_1,\cdots,b_q\end{matrix}\right)\mathrm{d}x = G_{p+1,q}^{m,n+1}\left(a\,\middle|\,\begin{matrix}1-s,a_1,\cdots,a_p\\ b_1,\cdots,b_q\end{matrix}\right). \tag{54}$$

In addition, according to [8] (I, (9), p. 309), we have

$$G_{p,q}^{m,n}\left(z^{-1}\,\middle|\,\begin{matrix}a_r\\ b_s\end{matrix}\right) = G_{q,p}^{n,m}\left(z\,\middle|\,\begin{matrix}1-b_s\\ 1-a_r\end{matrix}\right). \tag{55}$$

2. General Lipschitz Summation Formula

Since in this section we shall be concerned with the confluent hypergeometric function, we state the basic knowledge of its theory. The confluent hypergeometric function $U(a,c;z)$, also denoted by $\Psi(a,c;z)$, was defined, for example, by [8] (I, p. 256, (5)):

$$\begin{aligned}\Gamma(a)\Gamma(1+a-c)z^a U(a,c;z) &= \frac{1}{2\pi i}\int_{(\gamma)} \Gamma(-w)\Gamma(w+a)\Gamma(w+1+a-c)z^{-w}\mathrm{d}w \\ &= G^{2,1}_{1,2}\left(z\,\middle|\, \begin{matrix} 1 \\ a, 1+a-c \end{matrix}\right)\end{aligned} \tag{56}$$

which is valid for $|\arg| < \frac{3\pi}{2}$, where (γ) signifies a vertical path $w = \gamma + iv,\ -\infty < v < \infty$ suitably indented to separate the poles of $\Gamma(-a)$ from those of $\Gamma(w+a)\Gamma(w+1+a-c)$.

The confluent hypergeometric function $U(a;b;z)$ is single-valued and analytic for $-\pi < \arg z < \pi$ and is one of two independent solutions to the confluent hypergeometric differential equation satisfying the boundary condition that $w(z) \to 0$ as $z \to \infty$ (compare with, for example, [8] (p. 278)). This verifies the formula in [8] (I, p. 255, (2), p. 260, (4)) and, most relevant to us, [20] (p. 505, 13.2.5):

$$\Gamma(a)U(a,c;\xi) = \int_0^\infty t^a e^{-\xi t}(1+t)^{c-a-1}\frac{\mathrm{d}t}{t} = \mathcal{M}[f(\cdot, -\cdot\log z, c-a-1, 1)](a) \tag{57}$$

for $\operatorname{Re} a > 0$.

The Mellin transform $F(s)$ of

$$f(x) = f(x,z,s,w) = \frac{z^x}{(x+w)^s}, \quad \sigma > 1. \tag{58}$$

is

$$F(z) = w^{z-s}\Gamma(z)U(z, z+1-s, -\log z^w). \tag{59}$$

The following [13] (Corollary 3) was deduced from the Ewald expansion [13] (Corollary 2), which in turn was a consequence of [13] (Theorem 4), where [13] (Theorem 4) is the modular relation corresponding to the ramified functional equation proven in [13] (Lemma 1). We shall treat the more general case (Equation (68)) in the proof, but only the rational case is tractable. We state the quadratic case as Proposition 1 at the end of the section:

Theorem 2. (Lerch's transformation formula) *For the Lipschitz–Kerch transcendent (Equation (28)), we have the transformation formula*

$$\begin{aligned}&L(x, 1-s, \alpha) \\ &= \frac{\Gamma(s)}{(2\pi)^s}\Big(e^{\frac{\pi i}{2}s - 2\pi x\alpha i}L\big(-\alpha, s, x\big) + e^{-\frac{\pi i}{2}s + 2\pi(1-x)\alpha i}L\big(\alpha, s, 1-x\big)\Big),\end{aligned} \tag{60}$$

where $0 < \alpha,\ x < 1$.

Proof. Equation (60) reads as follows

$$\begin{aligned}&L(i\omega, 1-s, \alpha) \\ &= \frac{\Gamma(s)}{(2\pi)^s}\Big(e^{\frac{\pi i}{2}s - 2\pi i\omega\alpha i}\sum_{n=0}^\infty \frac{e^{-2\pi i\alpha n}}{(n+i\omega)^s} + e^{-\frac{\pi i}{2}s - 2\pi i\omega\alpha i}\sum_{n=0}^\infty \frac{e^{2\pi i\alpha(n+1)}}{(n+1-i\omega)^s}\Big), \\ &= e^{2\pi\omega\alpha}\frac{\Gamma(s)}{(2\pi)^s}\Big(e^{\frac{\pi i}{2}s}\sum_{n=0}^\infty \frac{e^{-2\pi i\alpha n}}{(n+i\omega)^s} + e^{-\frac{\pi i}{2}s}\sum_{n=1}^\infty \frac{e^{2\pi i\alpha n}}{(n-i\omega)^s}\Big) \\ &= e^{2\pi\omega\alpha}\frac{\Gamma(s)}{(2\pi)^s}\Big(e^{\frac{\pi i}{2}s}L(-\alpha, s, i\omega) + e^{-\frac{\pi i}{2}s}L(\alpha, s, -i\omega) - \frac{1}{\omega^s}\Big),\end{aligned} \tag{61}$$

when $x = i\omega$. Hence, by recalling Equation (32) for $L_\Omega(x,s,\alpha)$, we are to consider

$$\frac{1}{2}e^{\frac{\pi i s}{2}}f_+(x) = \frac{1}{2}\frac{1}{2\pi i}\int_{(c)} x^{-z}F_+(z)\,\mathrm{d}z, \quad \frac{1}{2}e^{-\frac{\pi i s}{2}}f_-(x) = \frac{1}{2}\frac{1}{2\pi i}\int_{(c)} x^{-z}F_-(z)\,\mathrm{d}z \tag{62}$$

where $F_+(z)$ is the Mellin transform of $f_+(x)$, defined by

$$f_+(x) = f(x,-\alpha,s,\omega) = \frac{e^{-2\pi i\alpha x}}{(x+i\omega)^s}, \quad f_-(x) = f(x,\alpha,s,-\omega) = \frac{e^{2\pi i\alpha x}}{(x-i\omega)^s}. \tag{63}$$

By writing

$$\frac{1}{2}e^{\frac{\pi i s}{2}}f_+(x) = \frac{\omega^{-s}}{2}\frac{e^{-2\pi i\alpha x}}{(1+\frac{x}{i\omega})^s},$$

we then see that

$$F_\pm(z) = (\pm i)^z\omega^{z-s}\Gamma(z)U(z,z+1-s,-2\pi\alpha\omega). \tag{64}$$

Hence, Equation (56) gives

$$\begin{aligned}\omega^s\Gamma(s)(-2\pi\alpha e^{\mp\frac{\pi i}{2}})^z F_\pm(z) &= \Gamma(s)\Gamma(z)(-2\pi\alpha\omega)^z U(z,z+1-s;-2\pi\alpha\omega) \\ &= \frac{1}{2\pi i}\int_{(\gamma)}\Gamma(-w)\Gamma(w+z)\Gamma(w+s)(-2\pi\alpha\omega)^{-w}\,\mathrm{d}w,\end{aligned} \tag{65}$$

such that

$$\begin{aligned}F_f(z) = \frac{1}{2}(F_+(z)+F_-(z)) &= \frac{1}{\omega^s\Gamma(s)}(-2\pi\alpha)^{-z}\cos\frac{\pi}{2}z \\ &\times\left(\frac{1}{2\pi i}\int_{(\gamma)}\Gamma(-w)\Gamma(w+z)\Gamma(w+s)(-2\pi\alpha\omega)^{-w}\,\mathrm{d}w\right)\end{aligned} \tag{66}$$

Hence Equation (65) implies

$$\begin{aligned}&\frac{1}{2}e^{\frac{\pi i s}{2}}f_+(x) + \frac{1}{2}e^{-\frac{\pi i s}{2}}f_-(x) = \frac{1}{2}\frac{1}{2\pi i}\int_{(c)} x^{-z}(F_+(z)+F_-(z))\,\mathrm{d}z \\ &= \frac{1}{\omega^s\Gamma(s)}\frac{1}{2\pi i}\int_{(c)} x^{-z}(-2\pi\alpha)^{-z}\cos\frac{\pi}{2}z\,\mathrm{d}z \\ &\times\left(\frac{1}{2\pi i}\int_{(\gamma)}\Gamma(-w)\Gamma(w+z)\Gamma(w+s)(-2\pi\alpha\omega)^{-w}\,\mathrm{d}w\right)\end{aligned} \tag{67}$$

By letting $x = n$ and summing over $n = 1, 2, \cdots$ after multiplying by a_n, we have

$$\tilde{Z}_f(s) := \frac{1}{2}\left(e^{\frac{\pi i}{2}s}L_\Omega(-\alpha,s,i\omega) + e^{-\frac{\pi i}{2}s}L_\Omega(\alpha,s,-i\omega) - \frac{2}{\omega^s}\right) = \frac{1}{2\pi i}\int_{(c)} F_f(z)\zeta_\Omega(z)\,\mathrm{d}z \tag{68}$$

for $c > 1$. We work with Equation (68) up to Equation (75).

We may shift the integration path to $\operatorname{Re} z = -d$ at this stage or simply substitute Equation (65) to deduce that

$$\begin{aligned}J(s) = \tilde{Z}_f(s) &= \frac{1}{\omega^s\Gamma(s)}\frac{1}{2\pi i}\int_{(c)}\zeta_\Omega(z)(-2\pi\alpha)^{-z}\Gamma(w+z)\cos\frac{\pi}{2}z\,\mathrm{d}z \\ &\times\left(\frac{1}{2\pi i}\int_{(\gamma)}\Gamma(-w)\Gamma(w+s)(-2\pi\alpha\omega)^{-w}\,\mathrm{d}w\right)\end{aligned} \tag{69}$$

where $\gamma < 0$ is chosen to be so small that $\sigma = \gamma$ not only separates the poles of gamma factors but also satisfies $\gamma + c < -\frac{1}{2}$. Then, by the Stirling formula, we may change the order of integration so that

$$J(s) = \frac{1}{2\pi i} \int_{(\gamma)} I(w)\Gamma(-w)\Gamma(w+s)(-2\pi\alpha\omega)^{-w}\,\mathrm{d}w \tag{70}$$

where

$$I(w) = \frac{\omega^{-s}}{\Gamma(s)} \frac{1}{2\pi i} \int_{(c)} \zeta_\Omega(z)(-2\pi\alpha)^{-z}\Gamma(w+z)\cos\frac{\pi}{2}z\,\mathrm{d}z. \tag{71}$$

In the above process, we use Equation (33) and $\tilde{Z}_f(s)\ K(s)$, and in Equation (68), the correction term $\frac{2}{\omega^s}$ is to be replaced by $\frac{2\zeta_\Omega(0)}{\omega^s}$ in the general case. By moving the integration path to $\operatorname{Re} z = -d < 0$ and applying Equation (23), we obtain

$$\begin{aligned} I(w) &= \frac{\omega^{-s}}{\Gamma(s)} \frac{1}{2\pi i} \int_{(-d)} \Gamma(w+z)G(z)\cos\frac{\pi}{2}z\zeta_\Omega(1-z)A^{2z-1}(-2\pi\alpha)^{-z}\,\mathrm{d}z \\ &= \frac{1}{2\pi i} \int_{(-d)} \Gamma(w+z)M^{(1)}(z)\zeta_\Omega(1-z)\gamma(z)\,\mathrm{d}z = \sum_{n=1}^{\infty} a_n K(w,n), \end{aligned} \tag{72}$$

where

$$\begin{aligned} K(w,n) &= \frac{1}{2\pi i} \int_{(-d)} \Gamma(w+z)M^{(1)}(z)n^{z-1}\gamma(z)\,\mathrm{d}z, \\ \gamma(z) &= \frac{\omega^{-s}}{A\Gamma(s)}(-2A^{-2}\pi\alpha)^{-z} \end{aligned} \tag{73}$$

and

$$M^{(1)}(z) = M_\Omega^{(1)}(z) = G(z)\cos\frac{\pi}{2}z = \frac{1}{\Gamma(z)}M^{(2)}(z), \tag{74}$$

where $M^{(2)}(z)$ is defined by Equation (101) below. Hence, we have

$$\begin{aligned} M_{\mathbb{Q}}^{(1)}(z) &= 2^{z-1}\sqrt{\pi}\frac{1}{\Gamma(z)}, \\ M_{\mathbb{Q}(\sqrt{|\Delta|})}^{(1)}(z) &= \frac{\Gamma(1-z)}{\Gamma(z)}\cos\frac{\pi}{2}z = G(z)\cos\frac{\pi}{2}z, \quad \Delta < 0 \\ M_{\mathbb{Q}(\sqrt{\Delta})}^{(1)}(z) &= 2^{2z-1}\frac{\Gamma(1-z)}{\Gamma(z)}\sin\frac{\pi}{2}z, \quad \Delta > 0 \end{aligned} \tag{75}$$

For $\Omega = \mathbb{Q}$, it follows that

$$\begin{aligned} I(w) &= \sum_{n=1}^{\infty} \frac{a_n}{n} \frac{\omega^{-s}\sqrt{\pi}}{2A\Gamma(s)} \frac{1}{2\pi i} \int_{(-d)} \frac{\Gamma(w+z)}{\Gamma(z)}(-\alpha n^{-1})^{-z}\,\mathrm{d}z \\ &= \sum_{n=1}^{\infty} \frac{a_n}{n} \frac{\omega^{-s}}{2\Gamma(s)} G_{1,1}^{1,0}\left(-\alpha n^{-1} \middle| \begin{matrix} 0 \\ w \end{matrix}\right), \end{aligned} \tag{76}$$

where $a_n = 1$, but we keep them for other uses. According to [18] (p. 26) ([17] (p. 631, 8.4.2.(3))), we have

$$G_{1,1}^{1,0}\left(x \middle| \begin{matrix} a \\ b \end{matrix}\right) = \frac{\theta(1-|z|)}{\Gamma(a-b)}(1-x)^{a-b-1}x^b, \tag{77}$$

and thus

$$G_{1,1}^{1,0}\left(x \middle| \begin{matrix} 0 \\ w \end{matrix}\right) = x^w G_{1,1}^{1,0}\left(x \middle| \begin{matrix} -w \\ 0 \end{matrix}\right) = \frac{x^w}{\Gamma(-w)}(1-x)^{-w-1} = \frac{1}{\Gamma(-w)(1-x)}\left(\frac{1}{x}-1\right)^{-w}. \tag{78}$$

By substituting Equation (78) into Equation (76), we conclude that

$$I(w) = \sum_{n=1}^{\infty} \frac{a_n}{n} \frac{\omega^{-s}}{2\Gamma(s)\Gamma(-w)} \left(1 + \frac{\alpha}{n}\right)^{-1} \left(-1 - \frac{n}{\alpha}\right)^{-w}. \tag{79}$$

By substituting Equation (79) into Equation (70), we have

$$\begin{aligned} J(x) &= \sum_{n=1}^{\infty} a_n \frac{\omega^{-s}}{2\Gamma(s)} (n+\alpha)^{-1} \frac{1}{2\pi i} \int_{(\gamma)} \Gamma(w+s)(2\pi\omega(n+\alpha))^{-w}\, \mathrm{d}w \\ &= \sum_{n=1}^{\infty} a_n \frac{\omega^{-s}}{2\Gamma(s)} (n+\alpha)^{-1} G_{0,1}^{1,0}\left(2\pi\omega(n+\alpha) \middle| \begin{matrix} - \\ s \end{matrix}\right), \end{aligned} \tag{80}$$

Since

$$G_{0,1}^{1,0}\left(x \middle| \begin{matrix} - \\ s \end{matrix}\right) = x^e G_{0,1}^{1,0}\left(x \middle| \begin{matrix} - \\ 0 \end{matrix}\right) = x^s e^{-x}, \tag{81}$$

then Equation (80) leads to

$$\begin{aligned} J(s) &= \sum_{n=1}^{\infty} a_n \frac{\omega^{-s}}{2\Gamma(s)} (n+\alpha)^{-1} (2\pi\omega(n+\alpha))^s e^{-2\pi\omega(n+\alpha)} \\ &= \frac{e^{-2\pi\omega\alpha}}{2\Gamma(s)} \sum_{n=1}^{\infty} a_n (n+\alpha)^{s-1} e^{-2\pi\omega n} = \frac{e^{-2\pi\omega\alpha}}{2\Gamma(s)} L_{\mathbb{Q}}(i\omega, 1-s, \alpha), \end{aligned} \tag{82}$$

In other words, we have (61). □

Corollary 2. (Lipschitz summation formula) *For the complex variables* $z = x + iy,\ x > 0,$ $s = \sigma + it,\ \sigma > 1$ *and the real parameter* $0 < w \le 1$*, we have the Lipschitz summation formula*

$$\frac{(2\pi)^s}{\Gamma(s)} \sum_{n=0}^{\infty} (n+w)^{s-1} e^{-2\pi z(n+w)} = \sum_{n=-\infty}^{\infty} \frac{e^{2\pi i n w}}{(z+in)^s}. \tag{83}$$

Under the condition $0 < w < 1$*, this formula holds in the wider half-plane* $\sigma > 0$*.*

Indeed, changing s or x in Equation (60) by $1 - s$ or wi, respectively, leads to Equation (83) and vice versa.

The special case of Equation (83) leads to

$$\ell_{1-s}(x) = \frac{\Gamma(s)}{(2\pi)^s} \left(e^{-\frac{\pi i}{2}s} \zeta(s,x) + e^{\frac{\pi i}{2}s} \zeta(s, 1-x)\right) \tag{84}$$

which is inverse to the Hurwitz formula:

$$\zeta(1-s,x) = \frac{\Gamma(s)}{(2\pi)^s} \left(e^{-\frac{\pi i}{2}s} l_s(x) + e^{\frac{\pi i}{2}s} l_s(1-x)\right). \tag{85}$$

Equation (83) has sometimes been referred to as the Lipschitz summation formula for good reason ([21–23], etc.). Knopp and Robbins [24] in Remark 1 stated their view on the Lipschitz summation formula to the effect that it is conceptually simpler than Riemann's original method of using the theta series. However, at least the special case of the Lipschitz summation formula (Corollary 2) can be readily deduced from the partial fraction expansion for the cotangent function, which is known to be equivalent to the functional equation for the Riemann zeta function, so we may say these are all equivalent.

In the case where Ω is a quadratic field, the argument is similar, depending on Equation (53). In light of the form of Equation (75), it suffices to incorporate $e^{\pm\frac{\pi}{2}z}$. We rewrite $n^{z-1}\gamma(z)$ as

$$n^{z-1}\gamma(z) = -\frac{A^2 i}{2\pi\alpha}(-X)^{z-1}, \quad X = X(n) = \frac{1}{2\pi\alpha}A^2 ni,$$

where

$$K(w,n) = \frac{A\omega^{-s}}{\pi\alpha\Gamma(s)}\frac{1}{2\pi i}\int_{(-d)}\frac{\Gamma(w+z)\Gamma(1-z)}{\Gamma(z)}\frac{1}{2i}\left((-X)^{z-1} - (-\bar{X})^{z-1}\right)\mathrm{d}z, \tag{86}$$
$$= \frac{A\omega^{-s}\Gamma(w+1)}{\pi\alpha\Gamma(s)}\frac{1}{2i}({}_1F_1(w+1,1,X) - {}_1F_1(w+1,1,\bar{X}))$$

We state the intermediate results for want of a better treatment:

Proposition 1.

$$\tilde{Z}_f(s) = \frac{A\omega^{-s}}{\pi\alpha\Gamma(s)}\sum_{n=1}^{\infty} a_n \frac{1}{2\pi i}\int_{(-d)}\Gamma(-w)\Gamma(w+1)\Gamma(w+s) \tag{87}$$
$$\times \frac{1}{2i}({}_1F_1(w+1,1,X) - {}_1F_1(w+1,1,\bar{X}))\delta(w)\,\mathrm{d}w$$

for $\Delta < 0$ *and*

$$\tilde{Z}_f(s) = \frac{2A\omega^{-s}}{\pi\alpha\Gamma(s)}\sum_{n=1}^{\infty} a_n \frac{1}{2\pi i}\int_{(-d)}\Gamma(-w)\Gamma(w+1)\Gamma(w+s) \tag{88}$$
$$\times \frac{1}{2}({}_1F_1(w+1,1,4X) + {}_1F_1(w+1,1,4\bar{X}))\delta(w)\,\mathrm{d}w$$

for $\Delta > 0$, *where* $X = \frac{1}{2\pi\alpha}A^2 ni$ *and* $\delta(w) = (-2\pi\alpha\omega)^{-w}$

3. Koshlyakov's Generalized Dedekind Zeta Function

The authors of [1] (pp. 243–247) established the general Lipschitz summation formula for the perturbed Dedekind zeta function (Theorem 3) as a modular relation for Koshlyakov's slightly processed $\zeta_\Omega(s)$ ([1] (23.10)):

$$\tilde{Z}_f(s) := \frac{e^{\frac{\pi i}{2}s}}{2}\zeta_\Omega(s, i\omega) + \frac{e^{-\frac{\pi i}{2}s}}{2}\zeta_\Omega(s, -i\omega) + 2\frac{\zeta_\Omega(0)}{\omega^s} \tag{89}$$
$$= \frac{e^{\frac{\pi i}{2}s}}{2}\sum_{n=1}^{\infty}\frac{a_n}{(n+i\omega)^s} + \frac{e^{-\frac{\pi i}{2}s}}{2}\sum_{n=1}^{\infty}\frac{a_n}{(n-i\omega)^s}.$$

The authors of [25] (pp. 121–134) expounded upon Koshlyakov's results as applications of the Fourier–Bessel expansion (compare with Equation (151)).

It is shown that in the imaginary quadratic case, each sum may be expressed as the Fourier–Bessel expansion, which leads to Koshlyakov's results by addition.

Here, we prove Theorem 3 as an immediate consequence of Corollary 1 and Example 1. It is an analogue of the general Lipschitz summation formula in Section 2 as well as the Fourier–Bessel expansion in Equation (151): $G_{1,1}^{1,1} \leftrightarrow G_{0,2}^{2,0}$. In [1] (p. 244), there are two succeeding formulas for J which are incorrect. We state the corrected form (Equation (90)) for the second, which is a unified one:

Theorem 3.

$$\frac{e^{\frac{\pi i}{2}s}}{2}\zeta_\Omega(s,i\omega)+\frac{e^{-\frac{\pi i}{2}s}}{2}\zeta_\Omega(s,-i\omega)+2\frac{\zeta_\Omega(0)}{\omega^s}=\tilde{Z}_f(s)=\frac{\zeta_\Omega(0)}{\omega^s}+J(s), \tag{90}$$

$$J(s)=I(s)$$
$$=\frac{\pi^{1-\frac{\chi}{2}}\omega^{-s}}{A\Gamma(s)}\sum_{n=1}^{\infty}\frac{a_n}{n}\frac{1}{2\pi i}\int_{(-d)}2^{r_1(z-1)+r}\Gamma(s-z)\Gamma^{\chi-1}(1-z)\sin^r\frac{\pi}{2}z\cos^{r_2}\frac{\pi}{2}z(A^2\omega n)^z\,\mathrm{d}z.$$

I. *In the case where* $\Omega=\mathbb{Q}$*, this amounts to the inverse Hurwitz formula (Equation (84)).*
II. *In the case where* $\Omega=\mathbb{Q}(\sqrt{\Delta})$, $\Delta<0$*, each summand has the closed form*

$$\frac{e^{\frac{\pi i}{2}s}}{2}\sum_{n=1}^{\infty}\frac{a_n}{(n+i\omega)^s} \tag{91}$$
$$=\omega^{1-s}\rho i+\frac{\zeta_\Omega(0)}{2\omega^s}+A^s\frac{\omega^{\frac{1-s}{2}}}{\Gamma(s)}\varepsilon^{s+1}\sum_{n=1}^{\infty}\frac{a_n}{n^{\frac{1-s}{2}}}K_{s-1}(2\varepsilon A\sqrt{\omega n}),$$

where

$$\varepsilon=e^{\frac{\pi}{4}i}. \tag{92}$$

Together with its counterpart in Equation (111), this leads to the corrected version of [1] (23.15):

$$\tilde{Z}_f(s)=\frac{\zeta_\Omega(0)}{\omega^s}+A^s\frac{\omega^{\frac{1-s}{2}}}{\Gamma(s)}\Big(\varepsilon^{s+1}\sum_{n=1}^{\infty}\frac{a_n}{n^{\frac{1-s}{2}}}K_{s-1}(2\varepsilon A\sqrt{\omega n}) \tag{93}$$
$$+\bar{\varepsilon}^{s+1}\sum_{n=1}^{\infty}\frac{a_n}{n^{\frac{1-s}{2}}}K_{s-1}(2\bar{\varepsilon}A\sqrt{\omega n})\Big).$$

III. *In the case where* $\Omega=\mathbb{Q}(\sqrt{\Delta})$, $\Delta>0$*, we have*

$$\frac{e^{\frac{\pi i}{2}s}}{2}\zeta_\Omega(s,i\omega)+\frac{e^{-\frac{\pi i}{2}s}}{2}\zeta_\Omega(s,-i\omega) \tag{94}$$
$$=(2A)^s\frac{\omega^{\frac{1-s}{2}}}{\Gamma(s)}\Big(\frac{\varepsilon^{s+1}}{i}\sum_{n=1}^{\infty}\frac{a_n}{n^{\frac{1-s}{2}}}K_{s-1}(4\varepsilon A\sqrt{\omega n})-\frac{\bar{\varepsilon}^{s+1}}{i}\sum_{n=1}^{\infty}\frac{a_n}{n^{\frac{1-s}{2}}}K_{s-1}(4\bar{\varepsilon}A\sqrt{\omega n})\Big),$$

Proof. With the beta transform in Equation (51), we have

$$\frac{1}{2}e^{\frac{\pi is}{2}}\frac{1}{(x+i\omega)^s}=\frac{\omega^{-s}}{2}\frac{1}{(1+\frac{x}{i\omega})^s}=\frac{\omega^{-s}}{2}\frac{1}{2\pi i}\int_{(c)}\frac{\Gamma(z)\Gamma(s-z)}{\Gamma(s)}e^{\frac{\pi iz}{2}}\omega^z\frac{\mathrm{d}z}{x^z}. \tag{95}$$

Hence, we have

$$\frac{1}{2}e^{\frac{\pi is}{2}}\frac{1}{(x+i\omega)^s}+\frac{1}{2}e^{-\frac{\pi is}{2}}\frac{1}{(x-i\omega)^s}=\frac{\omega^{-s}}{\Gamma(s)}\frac{1}{2\pi i}\int_{(c)}\Gamma(z)\Gamma(s-z)\cos\frac{\pi}{2}z\omega^z\frac{\mathrm{d}z}{x^z}, \tag{96}$$

so that for $c>1$, we have

$$\tilde{Z}_f(s)=\frac{1}{2\pi i}\int_{(c)}F_f(z)\mathfrak{G}_f(z)Z(z)\,\mathrm{d}z=\frac{\omega^{-s}}{\Gamma(s)}\frac{1}{2\pi i}\int_{(c)}\Gamma(s-z)\Gamma(z)\cos\frac{\pi}{2}z\frac{\zeta_\Omega(z)}{\omega^{-z}}\,\mathrm{d}z. \tag{97}$$

where

$$F_f(w) = \frac{\omega^{-s}}{\Gamma(s)}\Gamma(s-w)\Gamma(w)\cos\frac{\pi}{2}w\frac{1}{\omega^{-w}} = \mathfrak{G}_F(w,z)\mathfrak{F}_F(w)\delta(w,s), \tag{98}$$
$$\mathfrak{G}_F(w,z) = 1, \mathfrak{F}_F(w) = \Gamma(s-w)\Gamma(w)\cos\frac{\pi}{2}w, \delta(w,s) = \frac{\omega^{-s}}{\Gamma(s)}\frac{1}{\omega^{-w}}$$

In other words, without integration, the case of Corollary 1 is

$$\tilde{Z}_f(s) = \mathrm{P}(s) + J(s), \quad \mathrm{P}(s) = \frac{\zeta_\Omega(0)}{\omega^s}, \tag{99}$$

where

$$\begin{aligned} J(s) = I(s) &= \frac{1}{2\pi i}\int_{(-d)} \Gamma(s-z)M^{(2)}(z)\zeta_\Omega(1-z)\gamma(z,s)\,\mathrm{d}z \\ &= \frac{\omega^{-s}}{A\Gamma(s)}\frac{1}{2\pi i}\int_{(-d)} \Gamma(s-z)M^{(2)}(z)\zeta_\Omega(1-z)\frac{\mathrm{d}z}{(A^2\omega)^{-z}}, \end{aligned} \tag{100}$$

In addition, $0 < d < 1$, and

$$M^{(2)}(z) = M^{(2)}_\Omega(z) = \Gamma(z)G(z)\cos\frac{\pi}{2}z. \tag{101}$$

Hence, Equation (36) reads as follows:

$$J(s) = \sum_{n=1}^{\infty} a_n K(s,n), \quad K(s,n) = \frac{1}{2\pi i}\int_{(-d)} \Gamma(s-z)M^{(2)}(z)n^{z-1}\gamma(z,s)\,\mathrm{d}z \tag{102}$$
$$\gamma(z,s) = \frac{\omega^{-s}}{A\Gamma(s)}(A^2\omega)^z.$$

and

$$\begin{aligned} &M^{(2)}_{\mathbb{Q}}(z) = 2^{z-1}\sqrt{\pi}, \\ &M^{(2)}_{\mathbb{Q}(\sqrt{|\Delta|})}(z) = \Gamma(1-z)\cos\frac{\pi}{2}z, \quad \Delta < 0 \\ &M^{(2)}_{\mathbb{Q}(\sqrt{\Delta})}(z) = 2^{2z-1}\Gamma(1-z)\sin\frac{\pi}{2}z, \quad \Delta > 0. \end{aligned} \tag{103}$$

I. The case where $\Omega = \mathbb{Q}$.

Since

$$K(s,n) = \frac{(2\pi)^s}{2\Gamma(s)}\frac{1}{n^{1-s}}e^{-2\pi wn},$$

then Equation (99) reads as follows:

$$\tilde{Z}_f(s) = -\frac{1}{2}\omega^{-s} + \frac{(2A^2)^s}{2\Gamma(s)}\sum_{n=1}^{\infty}\frac{e^{-2\pi wn}}{n^{1-s}} \tag{104}$$

or

$$\frac{e^{\frac{\pi i}{2}s}}{2}\zeta(s,i\omega) + \frac{e^{-\frac{\pi i}{2}s}}{2}\zeta(s,-i\omega) - \frac{1}{2}\omega^{-s} = \frac{(2\pi)^s}{2\Gamma(s)}\sum_{n=1}^{\infty}\frac{e^{-2\pi\omega n}}{n^{1-s}}, \tag{105}$$

which amounts to the inverse Hurwitz formula (Equation (84)).

II. The case where $\Omega = \mathbb{Q}(\sqrt{\Delta})$, $\Delta < 0$.

In this case, as can be seen from Equation (103), there is no effect of processing $Z_f(s)$ to form $\tilde{Z}_f(s)$ in Equation (89), and each term admits the Fourier–Bessel expansion. However, as noticed in deriving Equation (99), the cosine function cancels the pole of $\zeta_\Omega(z)$ at $z = 1$. Thus, in treating each term, we must compute the residue at $z = 1$. In place of Equation (97), we consider

$$\frac{e^{\frac{\pi i}{2}s}}{2}\sum_{n=1}^{\infty}\frac{a_n}{(n+i\omega)^s} = \frac{\omega^{-s}}{\Gamma(s)}\frac{1}{2\pi i}\int_{(c)}\Gamma(s-z)\Gamma(z)e^{\frac{\pi i}{2}z}\frac{\zeta_\Omega(z)}{\omega^{-z}}\,\mathrm{d}z. \tag{106}$$

The residue of the integrand at $z = 1$ is

$$\frac{\omega^{-s}}{\Gamma(s)}\operatorname*{Res}_{z=1}\Gamma(s-z)\Gamma(z)e^{\frac{\pi i}{2}z}\frac{\zeta_\Omega(z)}{\omega^{-z}}\,\mathrm{d}z = \omega^{1-s}\rho i, \tag{107}$$

Hence, we have

$$\frac{e^{\frac{\pi i}{2}s}}{2}\sum_{n=1}^{\infty}\frac{a_n}{(n+i\omega)^s} = \omega^{1-s}\rho i + \frac{\zeta_\Omega(0)}{2\omega^s} + \frac{1}{2}\sum_{n=1}^{\infty}a_n K^{(\varepsilon)}(s,n), \tag{108}$$

where

$$K^{(\varepsilon)}(s,n) = \frac{\omega^{-s}}{2nA\Gamma(s)}\frac{1}{2\pi i}\int_{(-d)}\Gamma(s-z)\Gamma(1-z)\xi^{2z}\,\mathrm{d}z. \tag{109}$$

with $\xi = e^{\frac{\pi}{4}i}A\sqrt{\omega n} = \varepsilon A\sqrt{\omega n}$, for example. By implementing $s - z = w$, the integral becomes

$$\begin{aligned}\xi^{s+1}\frac{1}{2\pi i}\int_{(s+d)}\Gamma(w)\Gamma(w-(s-1))\xi^{s-1-2w}\,\mathrm{d}w &= \xi^{s+1}\xi^{s-1}G^{2,0}_{0,2}\left(\xi^2\,\middle|\,\begin{matrix}-\\0,-(s-1)\end{matrix}\right)\\&= 2\xi^{s+1}K_{s-1}(2\xi)\end{aligned} \tag{110}$$

under Equation (49). When substituting Equation (110) into Equation (108), we deduce Equation (91).

The counterpart of Equation (91) (with $\bar{\varepsilon}$ in place of ε in Equation (109)) reads as follows:

$$\begin{aligned}&\frac{e^{\frac{-\pi i}{2}s}}{2}\sum_{n=1}^{\infty}\frac{a_n}{(n-i\omega)^s}\\&= -\omega^{1-s}\rho i + \frac{\zeta_\Omega(0)}{2\omega^s} + A^s\frac{\omega^{\frac{1-s}{2}}}{\Gamma(s)}\bar{\varepsilon}^{s+1}\sum_{n=1}^{\infty}\frac{a_n}{n^{\frac{1-s}{2}}}K_{s-1}(2\bar{\varepsilon}A\sqrt{\omega n})\end{aligned} \tag{111}$$

Hence, by addition, Equation (93) follows. Clearly, this also follows from Equation (102) under Euler's formula.

III. The case where $\Omega = \mathbb{Q}(\sqrt{\Delta})$, $\Delta > 0$.

Equation (90) reads as follows:

$$\tilde{Z}_f(s) = \frac{\zeta_\Omega(0)}{\omega^s} + J(s), \quad J(s) = \sum_{n=1}^{\infty}a_n K_1(s,n). \tag{112}$$

where

$$K_1(s,n) = \frac{1}{2\pi i}\int_{(d)}\Gamma(s-z)\Gamma(1-z)\sin\frac{\pi}{2}z(4A^2\omega n)^z\,\mathrm{d}z. \tag{113}$$

Under Euler's identity, we have

$$J(s) = \frac{1}{2i}\sum_{n=1}^{\infty} a_n \left(K_1^{(\varepsilon)}(s,n) - K_1^{(\bar{\varepsilon})}(s,n)\right) \tag{114}$$

where

$$K_1^{(\varepsilon)}(s,n) = \frac{1}{2\pi i}\int_{(d)} \Gamma(s-z)\Gamma(1-z){\xi_1}^z \, dz, \quad \xi_1 = 2\varepsilon A\sqrt{\omega n}. \tag{115}$$

Hence, similar to Equations (93) and (94) follows. □

The character analogue of the Lipschitz summation formula is known [26] (pp. 59–62), and thus in the case where the characters are Kronecker characters associated with a quadratic field, we can expect coincidence of the results. Treatment of a more general case will be conducted elsewhere.

4. Plana's Summation Formula à la Koshlyakov

In this section, we prove the general Plana summation formula in Theorem 4 (see also [27] (Section 3) [9] (p. 90) for the classical case). For a proof, we appeal to:

Lemma 3. *We extend the domain $\{\mathrm{Re}\, x > 0\}$ of $f(x)$ to Equation (37).*

Then, $f(x)$ is analytic in the domain of Equation (37), and we may speak of $f(\pm ix)$, $x > 0$. We have

$$F(s) = \frac{1}{\sin\frac{\pi s}{2}}\int_0^\infty \frac{f(ix) - f(-ix)}{2i} x^{s-1} \mathrm{d}x. \tag{116}$$

The double integral in Equation (127) is absolutely convergent, and the order of the integrals may be changed.

Proof. The first assertion was proven in [1] (II, pp. 215–216). By rotating the positive real axis by $\frac{\pi}{2}$, we obtain

$$F(s) = \int_0^\infty f(ix)(ix)^{s-1} i\mathrm{d}x = e^{\frac{\pi i}{2}s}\int_0^\infty f(ix)x^{s-1}\mathrm{d}x$$

and similarly

$$F(s) = e^{-\frac{\pi i}{2}s}\int_0^\infty f(-ix)x^{s-1}\mathrm{d}x.$$

Hence, we have

$$\sin\frac{\pi}{2}sF(s) = \frac{e^{\frac{\pi i}{2}s} - e^{-\frac{\pi i}{2}s}}{2i}F(s) = \int_0^\infty \frac{f(ix) - f(-ix)}{2i}x^{s-1}\mathrm{d}x, \tag{117}$$

which is Equation (116).

The second assertion was proven in [1] (II, p. 216) based on the estimate $\sigma(z) = o\left(|z|^{-\delta}\right)$ for every $\delta > 0$. In our case, we estimate the integral in Equation (117) by the left-hand side using the Stirling formula (Equation (118)). Since the integral is absolutely convergent and is estimated to be $O(|t|^{-\lambda})$, $\lambda > 1$, the outer integral $\int_{(-d)}$ is absolutely convergent. □

Theorem 4. *(Plana's summation formula [1] (p. 217))*

Suppose that the Mellin transform $F(s)$ of the function $f(x)$ $(x > 0)$

$$F(s) = \int_0^\infty f(x)x^{s-1}\mathrm{d}x, \ \sigma = \mathrm{Re}\, s > 0 \tag{118}$$

satisfies the growth condition

$$F(s) = O\left(e^{-\frac{\pi}{2}|t|}|t|^{m\sigma - n}\right), \tag{119}$$

uniformly in $-d \le \sigma \le c$*, where* $-1 < -d < 0$*,* $1 < c$ *and where* $m > 0$*,* $n > 0$ *are subject to the condition*

$$\frac{m}{2} + n \ge 1, \tag{120}$$

while $F(s)$ *is regular in the strip* $-d \le \sigma \le c$*, except for a simple pole at* $s = 0$ *with residue* $f(0)$*. Then, we have*

$$Z_f(s) = \sum_{n=1}^{\infty} a_n f(n) = J(s) + \mathrm{P}(s), \tag{121}$$

where

$$\mathrm{P}(s) = \zeta_\Omega(0) f(0) + \rho \int_0^\infty f(x) \mathrm{d}x, \quad J(s) = \int_0^\infty \frac{f(ix) - f(-ix)}{2i} I(x)\, \mathrm{d}x, \tag{122}$$

and where $\frac{1}{2} I(x) = \sigma(s)$ *is Koshlyakov's function (Equation (135)), while* ρ *is defined as in Equation (20). Equation (121) entails Plana's summation formula [9] (p. 90, (9))*

$$\sum_{n=1}^{\infty} f(n) = -\frac{1}{2} f(0) + \int_0^\infty f(x) \mathrm{d}x - 2 \int_0^\infty \frac{f(ix) - f(-ix)}{2i} \frac{1}{e^{2\pi x} - 1} \mathrm{d}x, \tag{123}$$

under the conditions in Lemma 3 and the infinite series and the integral in Equation (123) are (absolutely) convergent.

For the quadratic fields $\Omega = \mathbb{Q}(\sqrt{|\Delta|})$*, we have*

$$\begin{aligned} J(s) &= \int_0^\infty \frac{f(ix) - f(-ix)}{2i} I(x)\, \mathrm{d}x \\ &= \frac{2A}{\pi i} \sum_{n=1}^{\infty} a_n \big(K_0(2\bar{\varepsilon} A \sqrt{xn}) - K_0(2\varepsilon A \sqrt{xn})\big), \quad \Delta < 0 \\ &= \frac{4A}{\pi} \sum_{n=1}^{\infty} a_n \big(K_0(4\bar{\varepsilon} A \sqrt{xn}) + K_0(4\varepsilon A \sqrt{xn})\big), \quad \Delta > 0. \end{aligned} \tag{124}$$

Proof. We shift the integration path to $\sigma = -d$. In the strip, there are two simple poles at $s = 0$ and 1 with residue $R_j = \operatorname*{Res}_{s=j} F(s)\zeta_\Omega(s)$, $j = 0, 1$, which were found in [1] (I, pp. 214–1215) as follows:

$$R_0 = \zeta_\Omega(0) f(0) \tag{125}$$

and

$$R_1 = \rho \lim_{s \to 1} F(s) = \rho \int_0^\infty f(x) \mathrm{d}x.$$

Hence, $\mathrm{P}(s)$ is as in Equation (122) and

$$J(s) = \frac{1}{2\pi i} \int_{(-d)} F(z) \zeta_\Omega(z)\, \mathrm{d}z, \tag{126}$$

where $\frac{1}{2} < d < 1$.

By substituting Equations (23) and (116) into Equation (126), we deduce that

$$\begin{aligned} J(s) &= \frac{1}{2\pi i} \int_{(-d)} A^{2s-1} G(z) \zeta_\Omega(1 - z) \frac{1}{\sin \frac{\pi z}{2}}\, \mathrm{d}z \int_0^\infty \frac{f(ix) - f(-ix)}{2i}\, x^{z-1}\, \mathrm{d}x \\ &= \int_0^\infty \frac{f(ix) - f(-ix)}{2i} I(x)\, \mathrm{d}x = \int_0^\infty I(x) \mathfrak{F}_F(x) \delta(x)\, \mathrm{d}x \end{aligned} \tag{127}$$

upon changing the order of integration. Here, we have

$$\mathfrak{F}_F(x) = \frac{f(ix) - f(-ix)}{2i}, \quad \delta(x) = 1, \tag{128}$$

and

$$I(x) = \frac{1}{2\pi i} \int\limits_{(-d)} A^{2z-1} x^{s-1} G(z) \frac{1}{\sin \frac{\pi z}{2}} \zeta_\Omega(1-z)\,\mathrm{d}z = \frac{1}{2\pi i} \int\limits_{(-d)} G(z) \frac{1}{\sin \frac{\pi z}{2}} \zeta_\Omega(1-z) \gamma(z)\,\mathrm{d}z \tag{129}$$

$$= \sum_{n=1}^{\infty} a_n \frac{1}{2\pi i} \int\limits_{(-d)} M^{(3)}(z) n^{z-1} \gamma(z)\,\mathrm{d}z = \sum_{n=1}^{\infty} a_n K(x,n),$$

where

$$\gamma(z) = A^{2z-1} x^{s-1}, \quad K(x,n) = \frac{1}{2\pi i} \int\limits_{(-d)} M^{(3)}(z) n^{z-1} \gamma(z)\,\mathrm{d}z = \frac{2A}{\pi} K_{r_1,r_2}(A^2 xn), \tag{130}$$

and where

$$M^{(3)}(z) = G(z) \frac{1}{\sin \frac{\pi}{2} z} = \frac{2}{\pi} \Gamma(1-z) M^{(2)}(z) \tag{131}$$

while K_{r_1,r_2} is Koshlyakov's function, defined by Equation (136). Hence, under (103), we have

$$\begin{aligned} M^{(3)}_{\mathbb{Q}}(z) &= \frac{2^z}{\sqrt{\pi}} \Gamma(1-z), \\ M^{(3)}_{\mathbb{Q}(\sqrt{|\Delta|})}(z) &= \frac{2}{\pi} \Gamma(1-z)^2 \cos \frac{\pi}{2} z, \quad \Delta < 0 \\ M^{(3)}_{\mathbb{Q}(\sqrt{\Delta})}(z) &= \frac{2^{2z}}{\pi} \Gamma(1-z)^2 \sin \frac{\pi}{2} z, \quad \Delta > 0 \end{aligned} \tag{132}$$

Hence, in the rational case, we have

$$\begin{aligned} K(x,n) &= \frac{2}{2\pi i} \int\limits_{(-d)} \Gamma(1-z) (2A^2 xn)^{z-1}\,\mathrm{d}z \\ &= \frac{2}{2\pi i} \int\limits_{(1+d)} \Gamma(z) (2\pi xn)^{-z}\,\mathrm{d}z \end{aligned} \tag{133}$$

and thus $I(x)$ amounts to the left-hand side of Equation (178) (which, together with Equation (174), implies Equation (178)). Hence, when summing the geometric series, Equation (127) leads to

$$J(s) = -2 \int_0^\infty \frac{f(ix) - f(-ix)}{2i} \frac{1}{e^{2\pi x} - 1}\,\mathrm{d}x, \tag{134}$$

which proves Equation (123).

Equation (124) follows by substituting Equation (137) into Equation (130). □

Koshlyakov makes a change of variables in the last integral on the right side of Equation (126):

$$J(s) = \frac{1}{2\pi i} \int\limits_{(1+d)} F(1-s) \zeta(1-s) \mathrm{d}s = \int_0^\infty \frac{f(ix) - f(-ix)}{2i} \sigma(x) \mathrm{d}x,$$

which leads to the argument in the above proof. We collect data necessary for the proof.

$\sigma(x)$ is defined as in [1] (I, (2.9),(2.13)) by ($c > 1$):

$$\sigma(x) = \frac{A}{\pi} \sum_{n=1}^{\infty} a_n K_{r_1,r_2}(A^2 xn) = \frac{1}{2\pi i} \int\limits_{(c)} \frac{1}{2\cos \frac{\pi s}{2}} A^{1-2s} G(1-s) \zeta_\Omega(s) x^{-s}\,\mathrm{d}s, \tag{135}$$

where [1] (I, (2.8)) such that

$$K_{r_1,r_2}(x) = \frac{1}{2\pi i}\int_{(c)} \frac{\pi}{2\cos\frac{\pi s}{2}} G(1-s)x^{-s}\,\mathrm{d}s = \frac{1}{2\pi i}\int_{(c)} \frac{\pi}{2} M^{(3)}(1-s)x^{-s}\,\mathrm{d}s. \tag{136}$$

As in [1] (I, p. 114, (2.5), (2.6)), or as can be readily checked by Equation (50), we have the closed form for the K function

$$\begin{aligned} K_{1,0}(x) &= \sqrt{\pi}\, e^{-2x}, \\ K_{0,1}(x) &= -2\,\mathrm{kei}_0\left(4\sqrt{\frac{x}{4}}\right) = \frac{1}{i}\left(K_0(2\varepsilon\sqrt{x}) - K_0(2\bar{\varepsilon}\sqrt{x})\right), \\ K_{2,0}(x) &= 4\,\mathrm{ker}_0(4\sqrt{x}) = 2\left(K_0(4\varepsilon\sqrt{x}) + K_0(4\bar{\varepsilon}\sqrt{x})\right), \end{aligned} \tag{137}$$

where kei_0 and ker_0 are modified Kelvin functions ([8] (p. 6), [17]) and ε is defined by Equation (92).

Theorem 5. *For*

$$f(x) = f(x; z, s, a) = z^x (x+a)^{-s} \tag{138}$$

Theorem 4 reduces to the generalized Hermite formula

$$\begin{aligned} \Phi_\Omega(z,s,w) = &\frac{1}{2}w^{-s} + R_{r_1,r_2}\int_0^\infty \frac{z^x}{(x+w)^s}\mathrm{d}x \\ &- 2\int_0^\infty \left(x^2+w^2\right)^{-\frac{s}{2}} \sin\left(x\log z - s\arctan\frac{x}{w}\right)\sigma(t)\,\mathrm{d}t \end{aligned} \tag{139}$$

which is valid for $\mathrm{Re}\, w > 0$ *and is a consequence of the functional Equation (18) for the Dedekind zeta function.*

Proof. We note that

$$\begin{aligned} \frac{1}{2i}(f(ix) - f(-ix)) &= \left(a^2+x^2\right)^{-\frac{s}{2}} \mathrm{I_m} e^{i\left(x\log z - s\arctan\frac{x}{a}\right)} \\ &= \left(a^2+x^2\right)^{-\frac{s}{2}} \sin\left(x\log z - s\arctan\frac{x}{a}\right) \end{aligned} \tag{140}$$

for $z > 0$ in the first instance and then for $\mathrm{Re}\, z > 0$ by analytic continuation.

By substituting Equation (140) into Equation (123), we arrive at Equation (139). □

We assemble corollaries to the above theorem which were proven in [28] directly:

Corollary 3. *For* $\mathrm{Re}\, a > 0$, *we have*

$$\begin{aligned} \Phi(z,s,a) = &\frac{1}{2}a^{-s} + \int_0^\infty \frac{z^x}{(x+a)^s}\mathrm{d}x \\ &- 2\int_0^\infty \left(x^2+a^2\right)^{-\frac{s}{2}} \sin\left(x\log z - s\arctan\frac{x}{a}\right)\frac{\mathrm{d}x}{e^{2\pi x}-1}. \end{aligned} \tag{141}$$

Corollary 4. (Hermite's formula [9], p. 91, (12)). *For* $\mathrm{Re}\, a > 0$, *we have*

$$\zeta(s,a) = \frac{1}{2}a^{-s} + \frac{a^{1-s}}{s-1} + 2\int_0^\infty \left(a^2+x^2\right)^{-\frac{s}{2}} \sin\left(s\arctan\frac{x}{a}\right)\frac{\mathrm{d}x}{e^{2\pi x}-1}. \tag{142}$$

Corollary 5. (Binet's second expression [9], p. 17, (30)).
For $\operatorname{Re} a > 0$, *we have*

$$\log \frac{\Gamma(a)}{\sqrt{2\pi}} = \left(a - \frac{1}{2}\right) \log a + 2 \int_0^\infty \frac{\arctan \frac{x}{a}}{e^{2\pi x} - 1} \mathrm{d}x \tag{143}$$

5. Voronoǐ-Type Summation Formulas

Here, we shall elucidate the genesis of some identities (especially summation formulas) equivalent to the functional equations given in [29] (Corollary 7) and [30–32], with emphasis on Hamburger's results. The latter three papers of Soni et al. will be discussed elsewhere in light of [1] and the *Principle* (Theorem 8).

The research of [29] precedes [1] in the use of the partial fraction expansion (Equation (178)) for the cotangent function. The authors of [33] mentioned Hamburger's partial fraction expansion as a special case of Hardy's formula, and the authors of [30] only quoted this and the other two results as (10–12): the Fourier expansion (Equation (179)) for the second periodic Bernoulli polynomial $\bar{B}_2(y)$ and the Poisson summation formula (Equation (180)). The latter is valid for a function of bounded variation having an integral expression which is analytic in a vertical strip. We shall elucidate the meaning of Hamburger's results in the light of the *Riemann–Hecke–Bochner–Maass (RHBM) correspondence* [34], whose main body is the equivalence of the ramified functional equation and the Fourier–Whittaker expansion (covering the case where there is no q expansion). For the unramified functional equation in question, the expansion reduces to the Fourier–Bessel expansion (Equation (151)), perceived as the Chowla–Selberg integral formula in the theory of the Epstein zeta function of positive definite quadratic forms. A variant of the Fourier–Bessel expansion is the partial fraction expansion in Equation (174) as well as Hardy's formula (Equation (155)). Most of the subsequent results up to Theorem 7 are for elucidating Equation (178) in the framework of the RHBM correspondence. The research in [30] is an extract from [31] and centers around the extension of Koshlyakov's results [35–37], overlooking the more far-reaching research in [1]. The main results of [15,17,23,30–32,38] are summation formulas of Sierpiński (a special case of Hecke tyoe) and Voronoǐ in the case where the functions are a Mellin transform pair, which are essentially consequences of the theory of Koshlyakov's X functions (Equation (165)) and L functions (Equation (168)). (The X function aspect was explained by [25] (pp. 97–100). Here, the *Principle* applies without processing in the simple change in the Mellin transform and the interchange of integration (compare with Theorem 8). The novelty of [15,17,23,30,31,38] lies in the study of the functional equation of Hecke and Vononoǐ in analogy to Riemann's and establishing identities corresponding to the ones for the Riemann zeta function in the same sprit as Koshlyakov's study of Dedekind zeta functions, where the line of reflection is $r = 1$. The only difference occurs in the Vononoǐ case, and the difference is in the residual function. The authors of [32] gave proof of two more identities in [31] (p. 63) equivalent to the functional equation, which may be clarified through Equation (173) [1] (I, p. 119, (4.7)).

We state the case of the Hecke functional equation in its full generality:

Definition 2. *Let*

$$0 < \lambda_1 < \lambda_2 < \cdots \to \infty, 0 < \mu_1 < \mu_2 < \cdots \to \infty$$

be increasing sequences of real numbers. For complex sequences $\{a_n\}$, $\{b_n\}$ *form the Dirichlet series*

$$\varphi(s) = \sum_{n=1}^{\infty} \frac{a_n}{\lambda_n^s} \quad \text{and} \quad \psi(s) = \sum_{n=1}^{\infty} \frac{b_n}{\mu_n^s} \tag{144}$$

which we assume are absolutely convergent for $\sigma > \sigma_a^*$ and $\sigma > \sigma_b^*$, respectively. Then, $\varphi(s)$ and $\psi(s)$ are said to satisfy Hecke's functional equation

$$A^{-s}\Gamma(s)\varphi(s) = A^{-(r-s)}\Gamma(r-s)\psi(r-s), \tag{145}$$

where $A > 0$ is a constant if there exists a regular function $\chi(s)$ outside of a compact set $\mathcal{S}$ such that

$$\chi(s) = A^{-s}\Gamma(s)\varphi(s), \ \ \sigma > \alpha(\geq \sigma_a^*)$$

and

$$\chi(s) = A^{-(r-s)}\Gamma(r-s)\psi(r-s), \ \ \sigma < \beta\,(\leq r-\sigma_b^*)$$

such that $\chi(s)$ is convex in the sense that

$$e^{-\varepsilon|t|}\chi(\sigma+it) = O(1), \ \ 0 < \varepsilon < \frac{\pi}{2}, \tag{146}$$

uniformly in σ, $\sigma_1 \leq \sigma \leq \sigma_2$, $|t| \to \infty$.

Following Bochner [39], the residual function is defined as

$$\mathrm{P}(x) = \frac{1}{2\pi i}\int_C \chi(s)x^{-s}ds, \tag{147}$$

where C encircles all the singularities of $\chi(s)$ in $\mathcal{S}$. In the applications above, this is to be understood to mean the sum of the residues in the strip $r-\sigma_b^* < \sigma < \sigma_a^*$ (corresponding to $-d < \sigma < c$).

We introduce the modular-type functions corresponding to the Dirichlet series in Equation (144):

$$f(\tau) = \sum_{n=o}^{\infty} a_n e^{Ain\tau} \quad \text{and} \quad g(\tau) = \sum_{n=o}^{\infty} b_n e^{Ain\tau}, \quad \tau \in \mathcal{H} \tag{148}$$

which are absolutely convergent and satisfy the (theta) transformation formula

$$f(\tau) = C\Big(\frac{\tau}{i}\Big)^{-r} g\Big(-\frac{1}{\tau}\Big). \tag{149}$$

Lemma 4. (Hecke) *The Dirichlet series in Equation (144) satisfying the condition that $A^{-s}\Gamma(s)\varphi(s) + \frac{a_0}{s} + \frac{Cb_0}{r-s}$ is bounded in every vertical strip (BEV) and satisfying the functional Equation (145) with $\psi(r-s)$ replaced by $C\psi(r-s) = \sum_{n=1}^{\infty} \frac{Cb_n}{\mu_n^{r-s}}$ is equivalent to Equation (149).*

Lemma 4 is a slight modification of [38] (Theorem 1, p. I-5), which is a handy statement of Hecke's epoch-making discovery [40]. Ogg [38] (p. I-7) stated that "Theorem 1 was a great step forward 75 years after the functional equation for the zeta-function, for it reduces a question about Dirichlet series to one about modular forms, which are easier to work with." This is in contrast to Weil's interpretation that this is a revival of the theory of automorphic forms [41].

The following theorem constitutes the basis of the results related to Hecke's functional equation (compare with [22,42], [25] ([p.10])).

Theorem 6. *The functional Equation (145), the theta transformation formula (Equation (149)), the Bochner modular relation (Equation (150)), the Fourier–Bessel expansion (Equation (151)), the Riesz sum (Equation (153)), and the Ewald expansion (Equation (154)) below are all equivalent:*

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n x} = \Big(\frac{A}{x}\Big)^r \sum_{n=1}^{\infty} b_n e^{-\mu_n \frac{A^2}{x}} + \mathrm{P}\Big(\frac{x}{A}\Big) \tag{150}$$

with $\operatorname{Re} x > 0$*, and we have*

$$A^{-s}\Gamma(s)\varphi(s,\alpha) = 2\alpha^{\frac{r-s}{2}} \sum_{n=1}^{\infty} b_n \mu_n^{-\frac{r-s}{2}} K_{s-r}(2A\sqrt{\alpha\mu_n}) + A^{-s} \int_0^{\infty} e^{-\alpha u} u^{s-1} \mathrm{P}\left(\frac{u}{A}\right) \mathrm{d}u \tag{151}$$

with $\alpha > 0, \sigma > \max\{r - \frac{1}{2}, -1\}, s \neq 0$*, where*

$$\varphi(s,\alpha) = \sum_{n=1}^{\infty} \frac{a_n}{(\lambda_n + \alpha)^s} \tag{152}$$

denotes the (Hurwitz-type) perturbed Dirichlet series associated with φ*.*

The Riesz sum is

$$\frac{1}{\Gamma(\varkappa+1)} \sideset{}{'}\sum_{\lambda_n \le x} a_n (x-\lambda_n)^{\varkappa} = A^{-\varkappa} \sum_{n=1}^{\infty} \left(\frac{x}{\mu_n}\right)^{\frac{1}{2}(\varkappa+r)} b_n J_{\varkappa+r}(2A\sqrt{\mu_n x}) + \mathrm{P}_{\varkappa}(x), \tag{153}$$

$$\mathrm{P}_{\varkappa}(x) = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{\Gamma(s)\varphi(s)x^{s+\varkappa}}{\Gamma(s+\varkappa+1)} \mathrm{d}s,$$

where $\mathcal{C}$ *is as depicted in Equation (147).*

The Ewald expansion is

$$A^{-s}\Gamma(s)\varphi(s) = A^{-s} \sum_{n=1}^{\infty} \frac{a_n}{\lambda_n^s} \Gamma(s, Aw\lambda_n) + A^{-(\delta-s)} \sum_{n=1}^{\infty} \frac{b_n}{\mu_n^{\delta-s}} \Gamma(\delta - s, Aw^{-1}\mu_n) + \int_0^{w} \mathrm{P}(x) x^{s-1} dx \tag{154}$$

with $\operatorname{Re} w > 0$*.*

In [42] (Lemma 6), the first three are equivalent assertions, and the Fourier–Bessel expansion (Equation (151)) is replaced by Hardy's formula:

$$\begin{aligned} &\left(-\frac{1}{s}\frac{\mathrm{d}}{\mathrm{d}s}\right)^{\varkappa} \sum_{n=1}^{\infty} a_n \frac{1}{s} e^{-s\sqrt{\lambda_n}} \\ &= 2^{3r+\varkappa}\Gamma\left(r+\varkappa+\frac{1}{2}\right)\pi^{r-\frac{1}{2}} \sum_{n=1}^{\infty} \frac{b_n}{(s^2+16\pi^2\mu_n)^{r+\varkappa+\frac{1}{2}}} \\ &+ \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{\Gamma(z)\varphi(z)\Gamma(2z+2\varkappa+1)2^{-\varkappa}}{\Gamma(z+\varkappa+1)} s^{-2z-2\varkappa-1} \mathrm{d}z. \end{aligned} \tag{155}$$

This is first derived by the Hardy transform in [43] without differentiation. It is remarked in [44] (Theorem 8.1, p.342) and [34] that the Hardy transform and the beta transform are almost reverse processes leading to the partial fraction expansion and the Fourier–Bessel expansion, respectively.

The Riesz sum of the order $\varkappa$ is defined by

$$A_{\lambda}^{\varkappa}(x) = \frac{1}{\Gamma(\varkappa+1)} \sideset{}{'}\sum_{\lambda_n \le x} a_n (x-\lambda_n)^{\varkappa} = \frac{1}{2\pi i} \int_{c} \frac{\Gamma(s)\varphi(s)x^{s+\varkappa}}{\Gamma(s+\varkappa+1)} \mathrm{d}s, \tag{156}$$

where $\varphi(s) = \sum_{n=1}^{\infty} \frac{a_k}{\lambda_n^s}$ and the prime on the summation means that the term corresponding to $x = [x]$ is to be halved [25] (p. 171, (G-8-1)).

The following lemma reveals the situation surrounding equating the Riemann-type (Hecke-type) as the Hecke-type (Riemann-type) functional equation:

Lemma 5. ([25] (p. 119, Corollary 4.1)) *Under similar assumptions to those in Definition 2, the functional equation*

$$\Gamma(A_1 s)\varphi(s) = \Gamma(A_1(r-s))\psi(r-s) \tag{157}$$

is equivalent to

$$\begin{aligned}&\frac{1}{A_1}\Gamma(A_1 s)\sum_{k=1}^{\infty}\frac{\alpha_k}{\left(\lambda_k^{\frac{1}{A_1}}+z^{\frac{1}{A_1}}\right)^{A_1 s}} \\ &= 2{A_1}^{-1}z^{\frac{1}{2}A_1(r-s)}\sum_{k=1}^{\infty}\frac{\beta_k}{{\mu_k}^{\frac{1}{2}A_1(r-s)}}K_{A_1(r-s)}\left(2(\mu_k z)^{\frac{1}{2A_1}}\right)+\mathrm{P}(s),\end{aligned} \tag{158}$$

where $\mathrm{P}(s) = \sum_{k=1}^{L}\mathrm{Res}\Big(\Gamma(A_1(s-w))\Gamma(A_1 s)\varphi(s)\,z^{w-s}, w = s_k\Big)$ *and where* s_k *represents all the poles of* $\Gamma(A_1(s-w))\Gamma(A_1 s)\varphi(s)\,z^{w-s}$ *in* $\mathcal{S}$.

The next corollary is a consequence of Lemma 5 with $A_1 = \frac{1}{2}$ and Theorem 6, and it elucidates Hamburger's Equation (178) (compare with Theorem 7):

Corollary 6. *The Riemann functional Equation (176), viewed as the Hecke type*

$$\pi^{-s}\Gamma(s)\zeta(2s) = \pi^{-(\frac{1}{2}-s)}\Gamma\left(\frac{1-2s}{2}\right)\zeta(1-2s) \tag{159}$$

is equivalent to

$$\sum_{n=1}^{\infty}e^{-\pi n^2 x} = x^{-\frac{1}{2}}\sum_{n=1}^{\infty}e^{-\frac{\pi n^2}{x}}+\frac{1}{2}\left(x^{-\frac{1}{2}}-1\right), \quad \mathrm{Re}\, x > 0, \tag{160}$$

and Watson's formula

$$\pi^{-s}\Gamma(s)\sum_{n=1}^{\infty}\frac{1}{(n^2+\alpha^2)^s} = 2\alpha^{\frac{1}{2}-s}\sum_{n=-\infty}^{\infty}|n|^{s-\frac{1}{2}}K_{s-\frac{1}{2}}(2\pi\alpha|n|), \tag{161}$$

where the term corresponding to $n = 0$ *on the right-hand side is to be understood to mean*

$$\lim_{u\to 0}u^{s-\frac{1}{2}}K_{s-\frac{1}{2}}(2\pi\alpha u) = \frac{\sqrt{\pi}}{2}\alpha^{\frac{1}{2}-s}\Gamma\left(s-\frac{1}{2}\right). \tag{162}$$

Hamburger's Results

Although the authors of [42] (p. 2) stated that the equivalence "established constitutes an analogue, for Hecke's functional equation, of a result of Hamburger [29] on Riemann's equation," the analogous part is limited to the Bochner modular relation and Hardy's formula (Equation (155)) as a counterpart of the partial fraction expansion (Equation (178)) (compare with Theorem 6 and Remark 1). The third equivalence is for the Riesz sum in Equation (153), but it does not reduce to Equation (179).

In restoring Hamburger's results in a wider framework of modular relations for the Riemann, Hecke and Voronoĭ functional equation, we interpret Equation (179) as a Riesz sum (Equation (153)) in addition to elucidating Equation (178) as (a variant of) the Fourier–Bessel expansion. In the proof of equivalence of Equations (179) and (176), we encounter completing the integral from $[1,\infty]$ to $[0,\infty]$, which is the genesis of the Ewald expansion (Equation (154)): $\Gamma(s,a)+\gamma(s,a)0\Gamma(s)$.

In its original form, the Fourier expansion (Equation (179)) looks rather foreign compared with others. Note that the left-hand side of Equation (179) is equal to

$$\frac{1}{2}x(x-1) - \frac{1}{2}\bar{B}_2(x) + \frac{1}{12} = \frac{1}{2}(B_2(x) - \bar{B}_2(x)),$$

where $\bar{B}_2(x) = B_2(x - [x])$, with $[x]$ denoting the integer part of x, is the second periodic Bernoulli polynomial. Hence, as remarked in [25] (p. 170), the left-hand side is the Riesz sum of the first order

$$\frac{1}{2}(B_2(x) - \bar{B}_2(x)) = \sum_{n\le x}(x-n) = \frac{1}{2\pi^i}\int_{(c)} \frac{1}{s(s+1)}\zeta(s)x^{s+1}\,\mathrm{d}s \tag{163}$$

for $c > 1$. Hence, Equation (179) amounts to the familiar *absolutely convergent Fourier series*

$$\bar{B}_2(x) = B_2(x-[x]) = \frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos 2\pi n x}{n^2}. \tag{164}$$

Around the same time, the authors of [2,3] remarked that the Fourier series for the first periodic Bernoulli polynomials is a consequence of the functional equation.

Following Koshlyakov, we define the X function as

$X(x) = X_{r_1,r_2}(x)$, to be $H^{\chi,0}_{0,\chi}\left(x \,\middle|\, \begin{matrix} - \\ \{(0,\frac{1}{2})\}_{j=1}^{r_1}, \{(0,1)\}_{j=r_1+1}^{r_2} \end{matrix}\right)$,

In other words, we have

$$X_{r_1,r_2}(x) = \frac{1}{2\pi i}\int_{(c)} \Gamma^{r_1}\left(\frac{s}{2}\right)\Gamma^{r_2}(s)x^{-s}\,\mathrm{d}s, \quad c > 0, \tag{165}$$

for $x > 0$ ($\mathrm{Re}(x) > 0$). It can be easily verified that

$$X_{1,0}(x) = 2e^{-x^2}, \quad X_{0,1}(x) = e^{-x}, \quad X_{2,0}(x) = 4K_0(2x). \tag{166}$$

The Koshlyakov L function is a relative of the K functions and is related through

$$L_{r_1,r_2}(x) = \frac{1}{2}\left(K_{r_1,r_2}(-ix) + K_{r_1,r_2}(ix)\right) \tag{167}$$

([1] (7.11)).

It is defined by [1] (8.9)

$$\begin{aligned} L_{r_1,r_2}(x) &= \frac{1}{2\pi i}\int_{(c)} \frac{\pi}{2}\,G(1-s)\,x^{-s}\mathrm{d}s \\ &= \frac{1}{2\pi i}\int_{(c)} \frac{\pi}{2}\,\frac{\Gamma^{r_1}(\frac{s}{2})\Gamma^{r_2}(s)}{\Gamma^{r_1}\left(\frac{1-s}{2}\right)\Gamma^{r_2}(1-s)}\,x^{-s}\mathrm{d}s, \quad c > 0,\ x > 0 \end{aligned} \tag{168}$$

We have

$$\begin{aligned} &L_{1,0}(x) = \pi x^{\frac{1}{2}} J_{\frac{1}{2}}(2x), \quad L_{0,1}(x) = \frac{\sin(x)}{x} = \mathrm{si}(x), \\ &L_{2,0}(x) = \pi\left(\frac{2}{\pi}K_0(4\sqrt{x}) - Y_0(4\sqrt{x})\right) \end{aligned} \tag{169}$$

where $\mathrm{si}(x)$ is the sinus cardinalis function. Proof is given in [25] (p. 105, Example 3.8).

We use the data on the Dedekind zeta function (Equation (16)). We let $\mathrm{P}(x) = R_0 + R_1$ be the residual function, where

$$R_j = \mathrm{Res}\big(\chi_{\mathfrak{K}}(s)z^{-s}, s = j\big), \quad j = 0, 1, \tag{170}$$

and where

$$\chi_{\mathfrak{K}}(s) = \frac{\pi}{2\cos\frac{\pi s}{2}} G(1-s)x^{-s} \quad \text{for} \quad \mathfrak{K} = K_{r_1,r_2} \tag{171}$$

$$\chi_{\mathfrak{K}}(s) = \Gamma^{r_1}\Big(\frac{s}{2}\Big)\Gamma^{r_2}(s)x^{-s} \quad \text{for} \quad \mathfrak{K} = X_{r_1,r_2}$$

$$\chi_{\mathfrak{K}}(s) = \frac{\pi}{2}\frac{\Gamma^{r_1}(\frac{s}{2})\Gamma^{r_2}(s)}{\Gamma^{r_1}\Big(\frac{1-s}{2}\Big)\Gamma^{r_2}(1-s)} x^{-s} \quad \text{for} \quad \mathfrak{K} = L_{r_1,r_2}$$

For $\mathfrak{K} = X_{r_1,r_2}$ we have

$$\mathrm{P}(x) = 2^{r_1}\zeta_{\Omega}^{(r)}(0) - 2^{r_1}\zeta_{\Omega}^{(r)}(0)x^{-1} \quad \text{for} \quad \zeta_{\Omega}(s) \tag{172}$$

$$\mathrm{P}(x) = \frac{1}{4}\Big(\gamma - \log\frac{4\pi}{x} - \frac{1}{x}(\gamma - \log 4\pi x)\Big) \quad \text{for} \quad \zeta(s)^2$$

Theorem 7. (Koshlyakov) *The functional Equation* (18) *(which we apply in the form of Equation* (23)*) is equivalent to each of the following*

The Bochner modular relation, which is

$$\sum_{n=1}^{\infty} a_n X_{r_1,r_2}(Axn) = \mathrm{P}(x) + x^{-1}\sum_{n=1}^{\infty} a_n X_{r_1,r_2}(Ax^{-1}n), \quad \mathrm{Re}\, x > 0. \tag{173}$$

and the partial fraction expansion, expressed as

$$\frac{A}{\pi}\sum_{n=1}^{\infty} a_n K_{r_1,r_2}(A^2xn) = \sigma(x) = -\frac{1}{2}\rho - \frac{\zeta_{\Omega}(0)}{\pi}\frac{1}{x} + \frac{x}{\pi}\sum_{n=1}^{\infty}\frac{a_n}{n^2+x^2}, \tag{174}$$

where this gives Soni's result [32] ((5)) if the residual function is replaced by $-\frac{1}{4\pi x} - \frac{1}{2}\log x - \gamma$.

Deduction of Equation (18) from Equation (173) was performed in [1] (I, p. 120, (3.4)). For Equation (174) compare with the results in [1] (I, p. 117, (3.4)) as modified in [25] (p. 129, Theorem 4.7). Deduction of Equation (18) from Equation (174) was performed in [1] (II, pp. 225–226) based on a version of the Plana summation formula [1] (II, p. 223, IV). The main ingredient is from [1] (II, p. 225, (16.1)) in its limit case as $\alpha \to 0$, which should read (supplementing the missing second integral on the right) as follows:

$$\zeta_{\Omega}(s) = \rho\frac{\alpha^{1-s}}{s-1} + \int_{\alpha}^{\alpha+i\infty}\sigma(-iz)z^{-s}\,\mathrm{d}z + \int_{\alpha}^{\alpha-i\infty}\sigma(iz)z^{-s}\,\mathrm{d}z, \quad 0<\alpha<1. \tag{175}$$

Formulas similar to this appeared in [29] (p. 137) and [32] (p. 547), used by Hamburger and Soni to deduce Equation (180) from Equation (178) and Equation (174) from Equation (173), respectively. It would turn out that Equation (175) is a version of the Plana summation formula. More details on this aspect and Theorem 8 below will be given in a forthcoming paper:

Corollary 7. ([29])

The functional equation for the Riemann zeta function

$$\pi^{-\frac{s}{2}}\Gamma\Big(\frac{s}{2}\Big)\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma\Big(\frac{1-s}{2}\Big)\zeta(1-s) \tag{176}$$

is equivalent to the following equalities: the theta transformation formula (the Bochner modular relation), expressed as

$$\sum_{n=1}^{\infty} e^{-\pi n^2 x} = x^{-\frac{1}{2}}\sum_{n=1}^{\infty} e^{-\frac{\pi n^2}{x}} + \frac{1}{2}\Big(x^{-\frac{1}{2}} - 1\Big), \quad \mathrm{Re}\, x > 0. \tag{177}$$

the partial fraction expansion for the cotangent function $i\cot \pi i z$, *which is*

$$1+2\sum_{n=1}^{\infty} e^{-2\pi n z}=\frac{1}{\pi z}+\frac{2z}{\pi}\sum_{n=1}^{\infty}\frac{1}{z^2+n^2} \tag{178}$$

the Fourier expansion, expressed as

$$\sum_{n=1}^{[x]}(x-n)=\frac{x(x-1)}{2}-\frac{1}{2\pi^2}\sum_{n=1}^{\infty} n^{-2}(\cos 2\pi n x-1) \tag{179}$$

and the Poisson summation formula ($T>0$), *where*

$$T\sum_{n=-\infty}^{\infty}\varphi(Tn)=\sum_{n=-\infty}^{\infty}\Phi\left(\frac{T}{2\pi}n\right) \tag{180}$$

is valid for a function $\varphi(u)$ *which is of bounded variation in any finite interval satisfying the convergence conditions at infinity (p. 136) and such that*

$$\Phi(z):=\int_{-\infty}^{\infty}\varphi(u)e^{zu}\,\mathrm{d}u \tag{181}$$

is analytic in a certain (vertical) strip.

Proof. The first two are contained in Theorem 7. We give an independent proof of equivalence of Equations (179) and (176). We first assume Equation (179) and recall the special case in [45] ((3.50), p. 72):

$$\zeta(s)=\frac{1}{s-1}+\frac{1}{2}+\frac{1}{12}s-\frac{s(s+1)}{2}\int_1^{\infty}\bar{B}_2(t)t^{-s-2}\,\mathrm{d}t \tag{182}$$

which is valid for $\sigma>-1$. Noting that

$$-\frac{s(s+1)}{2}\int_0^{1}\bar{B}_2(t)t^{-s-2}\,\mathrm{d}t=\frac{1}{s-1}+\frac{1}{2}+\frac{1}{12}s. \tag{183}$$

we may then *complete the integral* in Equation (182) with Equation (183) to obtain

$$\zeta(s)=-\frac{s(s+1)}{2}\int_0^{\infty}\bar{B}_2(t)t^{-s-2}\,\mathrm{d}t. \tag{184}$$

By substituting Equation (164) into Equation (184) and changing the order of integration and summation (by the absolute convergence of both), we arrive at

$$\begin{aligned}\zeta(s)&=-\frac{s(s+1)}{2\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\int_0^{\infty}t^{-s-1}\cos 2\pi n t\,\frac{\mathrm{d}t}{t}\\&=-\frac{s(s+1)}{2\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}(2\pi n)^{s+1}\int_0^{\infty}u^{-s-1}\cos u\,\frac{\mathrm{d}u}{u}\end{aligned} \tag{185}$$

after transformation. We appeal to a generalized Euler's formula valid for $|\alpha|\le\frac{\pi}{2}$, $\sigma>0$

$$\Gamma(s)\cos\alpha s=\int_0^{\infty}t^s e^{-t\cos\alpha}\cos(t\sin\alpha)\,\frac{\mathrm{d}t}{t} \tag{186}$$

with $\alpha=\frac{\pi}{2}$. In other words, we substitute

$$\Gamma(s)\cos\frac{\pi}{2}s=\int_0^{\infty}t^s\cos t\,\frac{\mathrm{d}t}{t} \tag{187}$$

into Equation (185) to deduce that

$$\zeta(s) = 2^s \pi^{s-1} \sin\frac{\pi}{2}\Gamma(1-s)\zeta(1-s) \tag{188}$$

after simplification, which is equivalent to Equation (176).

Now, we deduce Equation (179) from Equation (176). We generalize Equation (163) as the Riesz sum of the first order of a_n:

$$Z_f(x) := \sum_{n\le x}(x-n)a_n = \sum_{n=1}^{\infty}(x-n)a_n f\left(\frac{n}{x}\right) = \frac{1}{2\pi i}\int_{(c)} \frac{1}{s(s+1)}\zeta_\Omega(s)x^{s+1}\,\mathrm{d}s, \tag{189}$$

where $f(y) = f_1(y)$ and

$$f_\varkappa(y) = \begin{cases} \frac{1}{\Gamma(\varkappa+1)}(1-y)^\varkappa, & (|y|<1) \\ \frac{1}{2}, & (y=1) \\ 0, & (|y|>1). \end{cases}$$

The Mellin inversion of $f(y)$ is

$$f(x) = \frac{1}{2\pi i}\int_c \frac{\Gamma(s)}{\Gamma(s+2)}x^{-s}\,\mathrm{d}s = G_{1,1}^{1,0}\left(x \,\middle|\, \begin{matrix} 2 \\ 0 \end{matrix}\right). \tag{190}$$

Hence, we have

$$F(z) = F_f(z) = \mathfrak{G}_F(s,z) = \frac{\Gamma(s)}{\Gamma(s+2)} = \frac{1}{s(s+1)}.$$

Now, we apply Corollary 1. Equation (36) reads as follows:

$$K(x,n) = \frac{1}{2\pi i}\int_{(-d)} F(z)G(z)A^{2z-1}n^{z-1}x^{z+1}\,\mathrm{d}z = \frac{1}{2\pi i}\int_{(-d)} \frac{1}{z(z+1)}G(z)\gamma(z,x)\,\mathrm{d}z, \tag{191}$$

where $\gamma(z,x) = \frac{1}{A(An)^2}(A^2xn)^{z+1}$.

For the rational case, $\mathrm{P}(x) = B_2(x)$, $A = \sqrt{\pi}$ and

$$\frac{1}{z(z+1)}G(z) = \frac{1}{\sqrt{\pi}}2^z\sin\frac{\pi}{2}z\Gamma(-z-1) = -\frac{1}{2i\sqrt{\pi}}2^z\left(e^{\frac{\pi}{2}z} - e^{-\frac{\pi}{2}z}\right)\Gamma(-z-1)$$

Hence, we have

$$\begin{aligned} K(x,n) &= -\frac{1}{4i\pi^2n^2}\frac{1}{2\pi i}\int_{(-d)}\left(e^{\frac{\pi}{2}z} - e^{-\frac{\pi}{2}z}\right)\Gamma(-z-1)(2\pi xn)^{z+1}\,\mathrm{d}z \\ &= \frac{1}{4\pi^2n^2}\frac{1}{2\pi i}\int_{(1-d)}\left(e^{\frac{\pi}{2}w} + e^{-\frac{\pi}{2}w}\right)\Gamma(-w)(2\pi xn)^{w}\,\mathrm{d}w \end{aligned} \tag{192}$$

by the change in variable. Hence, by Lemma 1, we have

$$J(x) = I(x) = \sum_{n=1}^{\infty}K(x,n) = \frac{1}{2\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left(e^{2\pi ixn} + e^{-2\pi ixn}\right),$$

which leads to the Fourier series in Equation (164).

In the quadratic case, we state the results on the Riesz sum of a zero order in Remark 1, and the corresponding results are the integrals thereof. □

Remark 1. *(1) We remark that the partial fraction expansion for the cotangent function in Equation (174) is the special case of Watson's formula (Equation (161)) (i.e., by viewing the Riemann-type functional equation as being of the Hecke type). On the other hand, as elucidated in [25] (pp. 241–243), Equation (155) is a special case of Equation (151) and follows from Lemma 5 with $A_1 = \frac{1}{2}$ (i.e., by viewing the Hecke-type functional equation as being of the Riemann type). Since the partial fraction expansion in Equation (174) is essential in Hamburger's determination of the Riemann zeta function, we see a plausible hint to Hecke's theory.*

(2) The above deduction of Equation (176) from Equation (179) is a simplified version of the first proof given in [46] (pp. 13–17). The proof is given for $\bar{B}_1(x)$, and term-by-term integration is checked as in [46] (p. 13) by integration by parts. It is seen that this corresponds with the integration of $\bar{B}_1(x)$ to arrive at $\bar{B}_2(x)$.

In the case of an imaginary quadratic field, we have

$$K(x,n) = \sqrt{\frac{x}{n}} J_1(2A\sqrt{xn})$$

through Equation (44) such that

$$Z_f(x) = \rho x + \zeta_\Omega(0) + \sqrt{x} \sum_{n=1}^{\infty} \frac{a_n}{\sqrt{n}} J_1\left(4\pi\sqrt{\frac{xn}{\sqrt{|\Delta|}}}\right),$$

according to [4] (p. 19), which reduces to the form in [4] (p. 14) in the case of $\Delta = -4$.

In the case of a real quadratic field, we have

$$K(x,n) = -\sqrt{\frac{x}{n}}\left(Y_1(4A\sqrt{xn}) + \frac{2}{\pi}K_1(4A\sqrt{xn})\right)$$

according to [2], modified so that $J_{2,0}(w) = -Y_1(w) - \frac{2}{\pi}K_1(w)$ in the notation of Watson. Hence, we have

$$Z_f(x) = \rho x + \zeta_\Omega(0) - \sqrt{x} \sum_{n=1}^{\infty} \frac{a_n}{\sqrt{n}} \left(Y_1\left(4\pi\sqrt{\frac{xn}{\sqrt{|\Delta|}}}\right) + \frac{2}{\pi}K_1\left(4\pi\sqrt{\frac{xn}{\sqrt{|\Delta|}}}\right)\right),$$

according to [4] (p. 18), which reduces to the series in [4] (p. 16) in the case of $\Delta = 1$. The residual function is the well-known $x\log x + (2\gamma - 1)x + \frac{1}{4}$ coming from the double pole at $s = 1$, and the coefficients are $d(n)$.

As mentioned above, the author of [2] (p. 175) stated that his main theorem leads to the Fourier expansion

$$\bar{B}_1(x) = x - [x] - \frac{1}{2} = -\frac{1}{\pi}\sum_{n=1}^{\infty} \frac{\sin 2\pi n x}{n}, \quad x \notin \mathbb{Z}. \tag{193}$$

In Corollary 7, the Fourier expansion (Equation (179)) may be replaced with

$$\sum_{n\le x}{}' 1 = B_1(s) + \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{\sin 2\pi n x}{n}, \quad x \notin \mathbb{Z}. \tag{194}$$

Some more analysis revealed the following:

Theorem 8. *Hamburger's equation (Equation (180)) is manifestation of the Principle with the Fourier transform pair. The Soni–Oberhettinger formula of the form*

$$\sum_{n=1}^{\infty} a_n f(\lambda_n) = \mathrm{P}(\cdot) + \sum_{n=1}^{\infty} b_n \hat{f}(\mu_n), \quad \hat{f}(y) = \int_0^{\infty} f(x)K(xy)\,\mathrm{d}x \tag{195}$$

is a consequence of Corollary 1, with $F(z)$ replaced by the Mellin transform in Equation (1).

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