mathematics

Article

Weighted Fractional Hermite–Hadamard Integral Inequalities for up and down Ɉ-Convex Fuzzy Mappings over Coordinates

Muhammad Bilal Khan [1,*], Eze R. Nwaeze [2], Cheng-Chi Lee [3,4,*], Hatim Ghazi Zaini [5], Der-Chyuan Lou [6,*] and Khalil Hadi Hakami [7]

1 Department of Mathematics and Computer Science, Transilvania University of Brasov, 29 Eroilor Boulevard, 500036 Brasov, Romania
2 Department of Mathematics and Computer Science, Alabama State University, Montgomery, AL 36101, USA; enwaeze@alasu.edu
3 Department of Library and Information Science, Fu Jen Catholic University, New Taipei City 24205, Taiwan
4 Department of Computer Science and Information Engineering, Fintech and Blockchain Research Center, Asia University, Taichung City 41354, Taiwan
5 Department of Computer Science, College of Computers and Information Technology, Taif University, P.O. Box 11099, Taif 21944, Saudi Arabia; h.zaini@tu.edu.sa
6 Department of Computer Science and Information Engineering, Chang Gung University, Stroke Center and Department of Neurology, Chang Gung Memorial Hospital at Linkou, Taoyuan 33302, Taiwan
7 Department of Mathematics, College of Science, Jazan University, P.O. Box. 114, Jazan 45142, Saudi Arabia; khakami@jazanu.edu.sa
* Correspondence: muhammad.bilal@unitbv.ro or bilal42742@gmail.com (M.B.K.); cclee@mail.fju.edu.tw (C.-C.L.); dclou@mail.cgu.edu.tw or dclouprof@gmail.com (D.-C.L.)

Abstract: Due to its significant influence on numerous areas of mathematics and practical sciences, the theory of integral inequality has attracted a lot of interest. Convexity has undergone several improvements, generalizations, and extensions over time in an effort to produce more accurate variations of known findings. This article's main goal is to introduce a new class of convexity as well as to prove several Hermite–Hadamard type interval-valued integral inequalities in the fractional domain. First, we put forth the new notion of generalized convexity mappings, which is defined as *UD*-Ɉ-convexity on coordinates with regard to fuzzy-number-valued mappings and the up and down (*UD*) fuzzy relation. The generic qualities of this class make it novel. By taking into account different values for Ɉ, we produce several known classes of convexity. Additionally, we create some new fractional variations of the Hermite–Hadamard (*HH*) and Pachpatte types of inequalities using the concepts of coordinated *UD*-Ɉ-convexity and double Riemann–Liouville fractional operators. The results attained here are the most cohesive versions of previous findings. To demonstrate the importance of the key findings, we offer a number of concrete examples.

Keywords: fuzzy-number valued mappings; fractional integral; coordinated *UD*-Ɉ-convexity; fractional Hermite–Hadamard inequalities

MSC: 26A33; 26A51; 26D10

Citation: Khan, M.B.; Nwaeze, E.R.; Lee, C.-C.; Zaini, H.G.; Lou, D.-C.; Hakami, K.H. Weighted Fractional Hermite–Hadamard Integral Inequalities for up and down Ɉ-Convex Fuzzy Mappings over Coordinates. *Mathematics* **2023**, *11*, 4974. https://doi.org/10.3390/math11244974

Academic Editors: Lijuan Zha, Jian Liu and Jinliang Liu

Received: 17 November 2023
Revised: 6 December 2023
Accepted: 13 December 2023
Published: 16 December 2023

1. Introduction

The most fundamental area of mathematical analysis is known as convex analysis; see [1,2]. Due to its significant contribution to the advancement of both pure and applied mathematics, it has attracted considerable attention. A problem can be solved geometrically and analytically using convexity and its effects. Convexity plays a crucial role in topology, functional analysis, specifically separation axioms, fixed-point theory, engineering, and economics. First, by proposing the idea of convex mappings based on a convex set in 1905, Jensen greatly increased the appeal of the theory of convex functions. Since a positive second derivative denotes the convexity of functions, one would wish to describe it in terms of functions and their derivatives. It has a close relationship with optimization theory, particularly with linear programming. Convex mappings frequently offer distinctive minima and are used to derive a workable solution; see [3–5].

The theory of mathematical inequalities has various applications in many branches of physics and engineering. This theory is closely connected to fields such as approximation theory, probability theory, and information theory. The importance of this topic will increase in the future due to its impact on applied mathematics. The theory of inequalities has greatly benefited from the study of the theory of convex functions. By using the idea of convexity, it is possible to directly obtain many inequalities, like Jensen's inequality, the *HH* inequality, Young's inequality, etc. In this regard, we are reminded of the well-known inequality resulting from Hermite and Hadamard acting independently.

Theorem 1. *Assume that the convex mapping* $G : [\mathfrak{e}, \mathfrak{g}\,] \to \mathfrak{R}$. *Then, the following double-inequality holds*:

$$G\left(\frac{\mathfrak{e}+\mathfrak{g}}{2}\right) \leq \frac{1}{\mathfrak{g}-\mathfrak{e}}\int_{\mathfrak{e}}^{\mathfrak{g}} G(x)dx \leq \frac{G(\mathfrak{g})+G(\mathfrak{e})}{2}, \tag{1}$$

where $\mathfrak{R}$ *is set of real numbers.*

One can check the concavity of the mappings by using the aforementioned inequality. See [6,7] for further information about this. For more information, related to different inequalities, see [8–16] and the references therein.

With the help of new and creative ideas, particularly the use of weighted means, the concept of convexity has recently been improved and expanded. Examples include harmonic, geometric, and *P*-convexity, which are based on the weighted harmonic mean, the weighted geometric mean, and the generalized weighted p mean, respectively. In 2019, Wu et al. [17,18] used the quasi-arithmetic mean to investigate a new class of convexity.

We work with multi-valued functions in a set-valued analysis, and interval-valued (*IV*) analysis is a branch of this field. The initial method for calculating the error estimates of finite machines was an interval analysis. If we assign a single value to any variable, just like in everyday life operations, the likelihood of inaccuracy rises; to address this shortcoming, interval numbers are used in place of single numerical values. Moore authored some fascinating works on interval analysis that offered fresh approaches to putting this theory into practice and suggested some uses for it in computer programming and error analysis; see [19–21].

Since Moore's outstanding and useful work, several authors have expressed interest in the area and exploited it in various ways. Investigations into the dynamic systems of differential equations, fluid mechanics, combinatorics, neural networking, and inequalities are carried out using *IV* approaches (see [22]). Breckner [23] continued by advancing the concept of convexity from the standpoint of set-valued mappings.

By using the ordering relations and *IV* mappings defined over interval numbers, certain inequalities have recently been improved and extended; see [24–30]. Regarding this, Chalco Cano et al. [31,32] computed the well-known Ostrowski's integral inequality using *IV* mappings and Hukuhara derivatives and came to the conclusion that the primary results were useful in numerical analyses. In 2017, Costa et al. [33] applied the mappings established over fuzzy numbers to investigate fresh integral inequalities. In the follow-up, Flores et al. [34] calculated new integral inequality variations related to *IV* mappings. In [35], the authors looked into the preinvex-*IV*-mappings-related integral inequalities of *HH*. Jensen's and *HH*-type containments involving a general class of *IV* convexity, also known as the h-*IV* mapping and Chebyshev-type inequality, respectively, were established by Zhao et al. [36,37]. Extremely significant contributions to the growth of integral inequality have come from fractional calculus. The first successful attempt to build fractional equivalents of *HH*-type inequalities was carried out in 2012 by Sarikaya et al. [38], who essentially took integral fractional operators into consideration. Following this, other inequalities have been reduced utilizing fractional methods, and this area of study is still quite active. Mohammad et al. explored the novel tempered Hermite–Hadamard-like inequalities and offered several applications in [39–44], along with fractional mid-point-like inequalities within the framework of fractional calculus. In [45], Akdemir et al. used unified fractional operators to evaluate the Chebyshev-like inequality. Inequalities involving AB-fractional integral operators and the differentiability of convex mappings were the conclusions of Set et al.'s [46] work. Budak et al. [47] investigated the *HH*-type inequalities in 2020 by studying interval-valued fractional operators. In order to show several *HH*-type inequalities, Kara et al. [48] combined novel double-fractional operators with the idea of interval-valued coordinated convexity. In order to extract some novel fractional versions of the *HH*-type inequalities, Bin-Mohsin et al. [49] recently presented the concept of interval-valued coordinated, harmonically convex mappings and double-fractional operators using the modified Mittag-Leffler function introduced by Raina as a kernel. Trigonometric convex functions with exponential weights were investigated by Zhou et al. [50] to create some novel *HH*-like inequalities. A fuzzy order relation was used to execute their discussion of trigonometric convexity and related integral containment in [51]. *IV* convexity and (p, q) calculus were employed by Kalsoom et al. in [52] to establish some fresh refinements of previous findings. To create certain *HH*-like inequalities, the authors of [53] developed the idea of the fuzzy-interval-valued bi-convex function. For interval-coordinated convex functions and products of Hermite–Hadamard-type inequalities, the authors of [54,55], respectively, obtained fractional forms of these inequalities. They wrapped up this work in [56] with some fresh Hermite–Hadamard inequalities involving interval-valued convexity and generalized quantum calculus. See [33,57–66] for additional information and current developments.

The goal of the current study is to use AB-fractional notions to develop new generic inclusion relations of the *HH* type. First, we create a novel class of convexity based on the interval analysis bi-function and monotonically continuous function g. The uniqueness of this study is in the derivation of numerous new and existing fractional counterparts using various values. Additionally, we use numerical simulations to confirm the results of our theoretical work. To the best of our knowledge, these results are more helpful for obtaining variations of *IV HH*-type inequalities for some classes of convexity; see [55,59,67–74].

The work is divided into two sections. In the first half, we review some information about convexity and fractional calculus and discuss the problem's history. The newly proposed class of convexity is introduced in the second section, along with its implications and uses in integral inequalities. Concluding observations are included later.

2. Preliminaries

We will go through the fundamental terminologies and findings in this section, which aid in comprehending the ideas behind our fresh findings.

Definition 1 ([63,64]). *Let $\mathbb{F}_0$ be a fuzzy number space. Given $\widetilde{Ð} \in \mathbb{F}_0$, the level sets or cut sets are given by $\left[\widetilde{Ð}\right]^{\gamma} = \left\{x \in \mathfrak{R} \middle| \widetilde{Ð}(x) > \gamma\right\} \forall \gamma \in [0,\ 1]$ and by*

$$\left[\widetilde{Ð}\right]^{0} = \left\{x \in \mathfrak{R} \middle| \widetilde{Ð}(x) > 0\right\}. \tag{2}$$

These sets are known as γ-level sets or γ-cut sets of $\widetilde{Ð}$.

Proposition 1 ([33]). *Let $\widetilde{Ð}, \widetilde{ſ\!ſ} \in \mathbb{F}_0$. Then, the relation " $\leq_{\mathbb{F}}$ " is given on $\mathbb{F}_0$ by $\widetilde{Ð} \leq_{\mathbb{F}} \widetilde{ſ\!ſ}$ when and only when $\left[\widetilde{Ð}\right]^{\gamma} \leq_I \left[\widetilde{ſ\!ſ}\right]^{\gamma}$ for every $\gamma \in [0,\ 1]$, which are left- and right-order relations or just order relations.*

Proposition 2 ([62])**.** *Let $\widetilde{Đ}, \widetilde{ሰ} \in \mathbb{F}_0$. Then, the relation " $\supseteq_{\mathbb{F}}$ " is given on $\mathbb{F}_0$ by $\widetilde{Đ} \supseteq_{\mathbb{F}} \widetilde{ሰ}$ when and only when $\left[\widetilde{Đ}\right]^{\gamma} \supseteq_I \left[\widetilde{ሰ}\right]^{\gamma}$ for every $\gamma \in [0,\ 1]$, which is the UD−order relation on $\mathbb{F}_0$.*

Remember the approaching notions, which are offered in the literature. If $\widetilde{Đ}, \widetilde{ሰ} \in \mathbb{F}_0$ and $t \in \mathfrak{R}$, then, for every $\gamma \in [0,\ 1]$, the arithmetic operations addition, "$\oplus$", multiplication, "$\otimes$", and scaler multiplication, "$\odot$", are defined by

$$\left[\widetilde{Đ} \oplus \widetilde{ሰ}\right]^{\gamma} = \left[\widetilde{Đ}\right]^{\gamma} + \left[\widetilde{ሰ}\right]^{\gamma}, \tag{3}$$

$$\left[\widetilde{Đ} \otimes \widetilde{ሰ}\right]^{\gamma} = \left[\widetilde{Đ}\right]^{\gamma} \times \left[\widetilde{ሰ}\right]^{\gamma}, \tag{4}$$

$$\left[t \odot \widetilde{Đ}\right]^{\gamma} = t\left[\widetilde{Đ}\right]^{\gamma}, \tag{5}$$

Equations (4) through to (6) have immediate consequences for these outcomes.

Theorem 2 ([33])**.** *The space $\mathbb{F}_0$ dealing with a supremum metric, i.e., for $\widetilde{Đ},\ \widetilde{ሰ} \in \mathbb{F}_0$,*

$$d_\infty\left(\widetilde{Đ}, \widetilde{ሰ}\right) = \sup_{0 \le \gamma \le 1} d_H\left(\left[\widetilde{Đ}\right]^{\gamma}, \left[\widetilde{ሰ}\right]^{\gamma}\right), \tag{6}$$

is a complete metric space, where H indicates the well-known Hausdorff metric on the space of intervals.

Theorem 3 ([33])**.** *Let $\mathbb{R}_I$ be a set of inetervals and $\widetilde{G} : [\mathfrak{u}, \mathfrak{v}] \subset \mathfrak{R} \to \mathbb{F}_0$ be an FNVM; its IVMs are classified according to their γ-levels, $G_\gamma : [\mathfrak{u}, \mathfrak{v}] \subset \mathfrak{R} \to \mathbb{R}_I$ are given by $G_\gamma(x) = [G_*(x,\gamma), G^*(x,\gamma)]\forall x \in [\mathfrak{u}, \mathfrak{v}]$, and $\forall \gamma \in (0,\ 1]$. Then, $\widetilde{G}$* is *FA-integrable over $[\mathfrak{u}, \mathfrak{v}]$ if and only if $G_*(x,\gamma)$ and $G^*(x,\gamma)$ are both A-integrable over $[\mathfrak{u}, \mathfrak{v}]$. Moreover, if $\widetilde{G}$* is *FA-integrable over $[\mathfrak{u}, \mathfrak{v}]$, then*

$$\begin{aligned}\left[(FA)\int_{\mathfrak{u}}^{\mathfrak{v}} \widetilde{G}(x)dx\right]^{\gamma} = &\left[(A)\int_{\mathfrak{u}}^{\mathfrak{v}} G_*(x,\gamma)dx,\ (A)\int_{\mathfrak{u}}^{\mathfrak{v}} G^*(x,\gamma)dx\right] \\ &= (IA)\int_{\mathfrak{u}}^{\mathfrak{v}} G_\gamma(x)dx,\end{aligned} \tag{7}$$

$\forall\ \gamma \in (0,\ 1]$. $\forall\ \gamma \in (0,\ 1]$, $\mathcal{FA}_{([\mathfrak{u},\mathfrak{v}],\ \gamma)}$ denotes the collection of all FA-integrable FNVMs over $[\mathfrak{u}, \mathfrak{v}]$.

Definition 2. ([67])**.** Let *$\mathbb{R}_I^+$ be a set of positive intervals and $G : [\mathfrak{e}, \mathfrak{g}] \to \mathbb{R}_I^+$ be an IVM, where $G \in \mathcal{IR}_{[\mathfrak{e},\mathfrak{g}]}$. Then, interval Riemann–Liouville-type integrals of G are defined as*

$$\mathcal{I}_{\mathfrak{e}^+}^{\alpha} G(y) = \frac{1}{\Gamma(\alpha)} \int_{\mathfrak{e}}^{y} (y - \mathfrak{t})^{\alpha-1} G(\mathfrak{t}) d\mathfrak{t} (y > \mathfrak{e}), \tag{8}$$

$$\mathcal{I}_{\mathfrak{g}^-}^{\alpha} G(y) = \frac{1}{\Gamma(\alpha)} \int_{y}^{\mathfrak{g}} (\mathfrak{t} - y)^{\alpha-1} G(\mathfrak{t}) d\mathfrak{t} (y < \mathfrak{g}), \tag{9}$$

where $\alpha > 0$ and Γ is the gamma function.

Recently, Allahviranloo et al. [68] introduced the fuzzy version of this and defined fractional integrals, resulting in the following:

Definition 3. *Let $\alpha > 0$ and $L([\mathfrak{e},\mathfrak{g}],\mathbb{F}_0)$ be the collection of all Lebesgue measurable FNVMs on $[\mathfrak{e},\mathfrak{g}]$. Then, the fuzzy left and right Riemann–Liouville fractional integrals of $\widetilde{G} \in L([\mathfrak{e},\mathfrak{g}],\mathbb{F}_0)$ with order $\alpha > 0$ are defined by*

$$\mathcal{I}^{\alpha}_{\mathfrak{e}^+}\,\widetilde{G}(y) = \frac{1}{\Gamma(\alpha)}\int_{\mathfrak{e}}^{y}(y-\mathtt{t})^{\alpha-1}\widetilde{G}(\mathtt{t})d\mathtt{t},(y>\mathfrak{e}), \tag{10}$$

and

$$\mathcal{I}^{\alpha}_{\mathfrak{g}^-}\,\widetilde{G}(y) = \frac{1}{\Gamma(\alpha)}\int_{y}^{\mathfrak{g}}(\mathtt{t}-y)^{\alpha-1}\widetilde{G}(\mathtt{t})d\mathtt{t},(y<\mathfrak{g}), \tag{11}$$

respectively, where $\Gamma(y) = \int_0^\infty \mathtt{t}^{y-1}e^{-\mathtt{t}}d\mathtt{t}$ is the Euler gamma function. The fuzzy left and right Riemann–Liouville fractional integral, y, based on the left and right endpoint functions, can be defined, that is

$$\begin{aligned}\left[\mathcal{I}^{\alpha}_{\mathfrak{e}^+}\widetilde{G}(y)\right]^{\gamma} &= \tfrac{1}{\Gamma(\alpha)}\textstyle\int_{\mathfrak{e}}^{y}(y-\mathtt{t})^{\alpha-1}G_{\gamma}(\mathtt{t})d\mathtt{t}\\ &= \tfrac{1}{\Gamma(\alpha)}\textstyle\int_{\mathfrak{e}}^{y}(y-\mathtt{t})^{\alpha-1}[G_*(\mathtt{t},\gamma),G^*(\mathtt{t},\gamma)]d\mathtt{t},(y>\mathfrak{e}),\end{aligned} \tag{12}$$

where

$$\mathcal{I}^{\alpha}_{\mathfrak{e}^+}\,G_*(y,\ \gamma) = \frac{1}{\Gamma(\alpha)}\int_{\mathfrak{e}}^{y}(y-\mathtt{t})^{\alpha-1}G_*(\mathtt{t},\gamma)d\mathtt{t},(y>\mathfrak{e}), \tag{13}$$

and

$$\mathcal{I}^{\alpha}_{\mathfrak{e}^+}\,G^*(y,\ \gamma) = \frac{1}{\Gamma(\alpha)}\int_{\mathfrak{e}}^{y}(y-\mathtt{t})^{\alpha-1}G^*(\mathtt{t},\gamma)d\mathtt{t},(y>\mathfrak{e}), \tag{14}$$

The right Riemann–Liouville fractional integral, denoted by $\left[\mathcal{I}^{\alpha}_{\mathfrak{g}^-}\,\widetilde{G}(y)\right]^{\gamma}$, can also be defined using the left and right endpoint functions.

Theorem 4. ([69]). *Let $\mathbb{F}_0^+$ be a set of positive fuzzy numbers, $\mathcal{J} : [0,\ 1] \to \mathbb{R}^+$, and $\widetilde{G} : [\mathfrak{u},\mathfrak{v}] \to \mathbb{F}_0^+$ be a UD-convex FNVM on $[\mathfrak{u},\mathfrak{v}]$, whose γ-cuts set up the sequence of IVMs $G_\gamma : [\mathfrak{u},\mathfrak{v}] \subset \mathbb{R} \to \mathbb{R}_C{}^+$, which is given by $G_\gamma(y) = [G_*(y,\gamma),\ G^*(y,\gamma)]$ for all $y \in [\mathfrak{u},\mathfrak{v}]$ and for all $\gamma \in [0,\ 1]$. If $\widetilde{G} \in L([\mathfrak{u},\mathfrak{v}],\mathbb{F}_0)$; then,*

$$\frac{1}{\alpha\mathcal{J}\left(\frac{1}{2}\right)}\widetilde{G}\left(\frac{\mathfrak{u}+\mathfrak{v}}{2}\right) \supseteq_{\mathbb{F}} \frac{\Gamma(\alpha)}{(\mathfrak{v}-\mathfrak{u})^{\alpha}}\left[\mathcal{I}^{\alpha}_{\mathfrak{u}^+}\widetilde{G}(\mathfrak{v}) \oplus \mathcal{I}^{\alpha}_{\mathfrak{v}^-}\widetilde{G}(\mathfrak{u})\right] \supseteq_{\mathbb{F}} \left[\widetilde{G}(\mathfrak{u}) \oplus \widetilde{G}(\mathfrak{v})\right]\int_0^1 \tau^{\beta-1}[\mathcal{J}(\tau)+\mathcal{J}(1-\tau)]d\tau. \tag{15}$$

Interval and fuzzy Aumann's type integrals are defined as follows for the coordinated IVM $G(x,y)$ and the coordinated FNVM $\widetilde{G}(x,y)$:

Theorem 5. ([59]). *Let $\widetilde{G} : \Delta[\mathfrak{e},\mathfrak{g}] \times [\mathfrak{u},\mathfrak{v}] \subset \mathbb{R}^2 \to \mathbb{F}_0^+$ be an FNVM on coordinates, whose γ-cuts set up the sequence of IVMs $G_\gamma : \Delta \subset \mathbb{R}^2 \to \mathbb{R}_I$, which is given by $G_\gamma(x,y) = [G_*((x,y),\gamma),\ G^*((x,y),\gamma)]$ for all $(x,y) \in \Delta = [\mathfrak{e},\mathfrak{g}] \times [\mathfrak{u},\mathfrak{v}]$ and for all $\gamma \in [0,\ 1]$. Then, $\widetilde{G}$ is fuzzy double integrable (FD-integrable) over Δ if and only if $G_*(x,\gamma)$ and $G^*(x,\gamma)$ both are D -integrable over Δ. Moreover, if $\widetilde{G}$ is FD-integrable over Δ, then*

$$\begin{aligned}\left[(FD)\textstyle\int_{\mathfrak{e}}^{\mathfrak{g}}\int_{\mathfrak{u}}^{\mathfrak{v}}\widetilde{G}(x,y)dydx\right]^{\gamma} &= \left[(D)\textstyle\int_{\mathfrak{e}}^{\mathfrak{g}}\int_{\mathfrak{u}}^{\mathfrak{v}}G_*((x,y),\gamma)dydx,\ (D)\int_{\mathfrak{e}}^{\mathfrak{g}}\int_{\mathfrak{u}}^{\mathfrak{v}}G^*((x,y),\gamma)dydx\right]\\ &= (ID)\textstyle\int_{\mathfrak{e}}^{\mathfrak{g}}\int_{\mathfrak{u}}^{\mathfrak{v}}G_\gamma(x,y)dydx,\end{aligned} \tag{16}$$

for all $\gamma \in [0,\ 1]$.

The families of all *FD*-integrable *FNVM*s over coordinates and *D*-integrable functions over coordinates are denoted by $\mathcal{FD}_\Delta$ and $\mathfrak{D}_{(\Delta,\ \gamma)}$ for all $\gamma \in [0,\ 1]$.

Here is the main definition of a fuzzy Riemann–Liouville fractional integral on the coordinates of the function $\widetilde{G}(x, y)$ by:

Definition 4 ([70]). *Let* $\widetilde{G} : \Delta \to \mathbb{F}_0$ *and* $\widetilde{G} \in \mathcal{FD}_\Delta$. *The double fuzzy interval Riemann–Liouville-type integrals* $\mathcal{I}^{\alpha,\,\beta}_{\mathfrak{e}^+,\mathfrak{u}^+}$, $\mathcal{I}^{\alpha,\,\beta}_{\mathfrak{e}^+,\mathfrak{v}^-}$, $\mathcal{I}^{\alpha,\,\beta}_{\mathfrak{g}^-,\mathfrak{u}^+}$, $\mathcal{I}^{\alpha,\,\beta}_{\mathfrak{g}^-,\mathfrak{v}^-}$ *of G of the order* $\alpha,\ \beta > 0$ *are defined by:*

$$\mathcal{I}^{\alpha,\,\beta}_{\mathfrak{e}^+,\mathfrak{u}^+}\widetilde{G}(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{\mathfrak{e}}^{x}\int_{\mathfrak{u}}^{y}(x-\mathfrak{t})^{\alpha-1}(y-\mathfrak{s})^{\beta-1}\widetilde{G}(\mathfrak{t},\mathfrak{s})d\mathfrak{s}d\mathfrak{t},\ \ (x > \mathfrak{e}, y > \mathfrak{u}), \tag{17}$$

$$\mathcal{I}^{\alpha,\,\beta}_{\mathfrak{e}^+,\mathfrak{v}^-}\widetilde{G}(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{\mathfrak{e}}^{x}\int_{y}^{\mathfrak{v}}(x-\mathfrak{t})^{\alpha-1}(\mathfrak{s}-y)^{\beta-1}\widetilde{G}(\mathfrak{t},\mathfrak{s})d\mathfrak{s}d\mathfrak{t},\ \ (x > \mathfrak{e}, y < \mathfrak{v}), \tag{18}$$

$$\mathcal{I}^{\alpha,\,\beta}_{\mathfrak{g}^-,\mathfrak{u}^+}\widetilde{G}(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{x}^{\mathfrak{g}}\int_{\mathfrak{u}}^{y}(\mathfrak{t}-x)^{\alpha-1}(y-\mathfrak{s})^{\beta-1}\widetilde{G}(\mathfrak{t},\mathfrak{s})d\mathfrak{s}d\mathfrak{t},\ \ (x < \mathfrak{g}, y > \mathfrak{u}), \tag{19}$$

$$\mathcal{I}^{\alpha,\,\beta}_{\mathfrak{g}^-,\mathfrak{v}^-}\widetilde{G}(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{x}^{\mathfrak{g}}\int_{y}^{\mathfrak{v}}(\mathfrak{t}-x)^{\alpha-1}(\mathfrak{s}-y)^{\beta-1}\widetilde{G}(\mathfrak{t},\mathfrak{s})d\mathfrak{s}d\mathfrak{t},\ \ (x < \mathfrak{g}, y < \mathfrak{v}). \tag{20}$$

Here is the newly defined concept of coordinated *UD*-$\mathcal{J}$-convexity over fuzzy number space in the codomain via the *UD*-relation given by the following:

Definition 5. *The FNVM* $\widetilde{G} : \Delta \to \mathbb{F}_0$ *is referred to as a coordinated UD-*$\mathcal{J}$*-convex FNVM on* Δ *if*

$$\begin{aligned}&\widetilde{G}(\tau\mathfrak{e} + (1-\tau)\mathfrak{g}, \kappa\mathfrak{u} + (1-\kappa)\mathfrak{v})\\ &\supseteq_{\mathbb{F}} \mathcal{J}(\tau)\mathcal{J}(\kappa)\widetilde{G}(\mathfrak{e},\mathfrak{u}) \oplus \mathcal{J}(\tau)\mathcal{J}(1-\kappa)\widetilde{G}(\mathfrak{e},\mathfrak{v}) \oplus \mathcal{J}(1-\tau)\mathcal{J}(\kappa)\widetilde{G}(\mathfrak{g},\mathfrak{u}) \oplus \mathcal{J}(1-\tau)\mathcal{J}(1-\kappa)\widetilde{G}(\mathfrak{g},\mathfrak{v}),\end{aligned} \tag{21}$$

for all $(\mathfrak{e},\mathfrak{g}),\ (\mathfrak{u},\mathfrak{v}) \in \Delta$ *and* $\tau,\kappa \in [0,\ 1]$, *where* $\widetilde{G}(x) \geq_{\mathbb{F}} \widetilde{0}$. *If inequality (21) is reversed, then* $\widetilde{G}$ *is referred to as a coordinate UD-*$\mathcal{J}$*-concave FNVM on* Δ.

Lemma 1. *Let* $\widetilde{G} : \Delta \to \mathbb{F}_0$ *be a coordinated FNVM on* Δ. *Then,* $\widetilde{G}$ *is a coordinated UD-*$\mathcal{J}$*-convex FNVM on* Δ *if and only if there exist two coordinated UD-*$\mathcal{J}$*-convex FNVMs,* $\widetilde{G}_x : [\mathfrak{u},\mathfrak{v}] \to \mathbb{F}_0$, $\widetilde{G}_x(w) = \widetilde{G}(x,w)$ *and* $\widetilde{G}_y : [\mathfrak{e},\mathfrak{g}] \to \mathbb{F}_0$, $\widetilde{G}_y(z) = \widetilde{G}(z,y)$.

Theorem 6. *Let* $\widetilde{G} : \Delta \to \mathbb{F}_0^+$ *be an FNVM on* Δ. *Then, from* γ*-levels, we obtain the collection of IVMs* $G_\gamma : \Delta \to \mathbb{R}_I^+ \subset \mathbb{R}_I$, *which is given by*

$$G_\gamma(x,y) = [G_*((x,y),\gamma),\ G^*((x,y),\gamma)], \tag{22}$$

for all $(x,y) \in \Delta$ *and for all* $\gamma \in [0,\ 1]$. *Then,* $\widetilde{G}$ *is a coordinated UD-*$\mathcal{J}$*-convex FNVM on* Δ *if and only if for all* $\gamma \in [0,\ 1]$, $G_*((x,y),\ \gamma)$ *and* $G^*((x,y),\ \gamma)$ *are coordinated* $\mathcal{J}$*-convex and* $\mathcal{J}$*-concave functions, respectively.*

Proof. Assume that for each $\gamma \in [0,\ 1]$, $G_*(x,\gamma)$ and $G^*(x,\gamma)$ are coordinated $\mathcal{J}$-convex and $\mathcal{J}$-concave on Δ, respectively. Then, from Equation (21), for all $(\mathfrak{e},\mathfrak{g}),\ (\mathfrak{u},\mathfrak{v}) \in \Delta$, τ and $\kappa \in [0,\ 1]$, we have

$$\begin{aligned}&G_*((\tau\mathfrak{e} + (1-\tau)\mathfrak{g}, \kappa\mathfrak{u} + (1-\kappa)\mathfrak{v}),\ \gamma)\\ &\leq \mathcal{J}(\tau)\mathcal{J}(\kappa)G_*((\mathfrak{e},\mathfrak{u}),\ \gamma) + \mathcal{J}(\tau)\mathcal{J}(1-\kappa)G_*((\mathfrak{e},\mathfrak{v}),\ \gamma) + \mathcal{J}(\kappa)\mathcal{J}(1-\tau)G_*((\mathfrak{e},\mathfrak{u}),\ \gamma) +\\ &\mathcal{J}(1-\tau)\mathcal{J}(1-\kappa)G_*((\mathfrak{e},\mathfrak{v}),\ \gamma),\end{aligned}$$

and

$$G^*((\tau\mathfrak{e}+(1-\tau)\mathfrak{g},\kappa\mathfrak{u}+(1-\kappa)\mathfrak{v}),\ \gamma)$$
$$\geq \mathcal{J}(\tau)\mathcal{J}(\kappa)G_*((\mathfrak{e},\mathfrak{u}),\ \gamma)+\mathcal{J}(\tau)\mathcal{J}(1-\kappa)G^*((\mathfrak{e},\mathfrak{v}),\ \gamma)+\mathcal{J}(\kappa)\mathcal{J}(1-\tau)G^*((\mathfrak{e},\mathfrak{u}),\ \gamma)+$$
$$\mathcal{J}(1-\tau)\mathcal{J}(1-\kappa)G^*((\mathfrak{e},\mathfrak{v}),\ \gamma),$$

Then, from Equations (3), (5), and (22), we obtain

$$G_\gamma((\tau\mathfrak{e}+(1-\tau)\mathfrak{g},\kappa\mathfrak{u}+(1-\kappa)\mathfrak{v}))$$
$$=[G_*((\tau\mathfrak{e}+(1-\tau)\mathfrak{g},\kappa\mathfrak{u}+(1-\kappa)\mathfrak{v}),\ \gamma),\ G^*((\tau\mathfrak{e}+(1-\tau)\mathfrak{g},\kappa\mathfrak{u}+(1-\kappa)\mathfrak{v}),\ \gamma)]$$
$$\supseteq_I \mathcal{J}(\tau)\mathcal{J}(\kappa)[G_*((\mathfrak{e},\mathfrak{u}),\ \gamma),\ G^*((\mathfrak{e},\mathfrak{u}),\ \gamma)]+\mathcal{J}(\tau)\mathcal{J}(1-\kappa)[G_*(\,(\mathfrak{e},\mathfrak{v}),\ \gamma),\ G^*((\mathfrak{e},\mathfrak{v}),\ \gamma)]+$$
$$\mathcal{J}(\kappa)\mathcal{J}(1-\tau)[G_*((\mathfrak{e},\mathfrak{u}),\ \gamma),\ G^*((\mathfrak{e},\mathfrak{u}),\ \gamma)]+\mathcal{J}(1-\tau)\mathcal{J}(1-\kappa)[G_*((\mathfrak{e},\mathfrak{v}),\ \gamma),\ G^*((\mathfrak{e},\mathfrak{v}),\ \gamma)]$$

That is

$$\widetilde{G}(\tau\mathfrak{e}+(1-\tau)\mathfrak{g},\kappa\mathfrak{u}+(1-\kappa)\mathfrak{v})$$
$$\supseteq_{\mathbb{F}} \mathcal{J}(\tau)\mathcal{J}(\kappa)\widetilde{G}(\mathfrak{e},\mathfrak{u})\oplus\mathcal{J}(\tau)\mathcal{J}(1-\kappa)\widetilde{G}(\mathfrak{e},\mathfrak{v})\oplus\mathcal{J}(1-\tau)\mathcal{J}(1-\kappa)\widetilde{G}(\mathfrak{g},\mathfrak{u})\oplus$$
$$\mathcal{J}(1-\tau)\mathcal{J}(1-\kappa)\widetilde{G}(\mathfrak{g},\mathfrak{v}),$$

and hence, $\widetilde{G}$ is a coordinated *UD*-$\mathcal{J}$-convex *FNVM* on Δ.

Conversely, let $\widetilde{G}$ be a coordinated *UD*-$\mathcal{J}$-convex *FNVM* on Δ. Then, for all $(\mathfrak{e},\mathfrak{g}),\ (\mathfrak{u},\mathfrak{v})\in\Delta$, τ and $\kappa\in[0,\ 1]$, we have

$$\widetilde{G}(\tau\mathfrak{e}+(1-\tau)\mathfrak{g},\kappa\mathfrak{u}+(1-\kappa)\mathfrak{v})$$
$$\supseteq_{\mathbb{F}} \mathcal{J}(\tau)\mathcal{J}(\kappa)\widetilde{G}(\mathfrak{e},\mathfrak{u})\oplus\mathcal{J}(\tau)\mathcal{J}(1-\kappa)\widetilde{G}(\mathfrak{e},\mathfrak{v})\oplus\mathcal{J}(1-\tau)\mathcal{J}(\kappa)\widetilde{G}(\mathfrak{g},\mathfrak{u})\oplus\mathcal{J}(1-\tau)\mathcal{J}(1-\kappa)\widetilde{G}(\mathfrak{g},\mathfrak{v}).$$

Therefore, again from Equation (22), for each $\gamma\in[0,\ 1]$, we have

$$G_\gamma((\tau\mathfrak{e}+(1-\tau)\mathfrak{g},\kappa\mathfrak{u}+(1-\kappa)\mathfrak{v}))$$
$$=[G_*((\tau\mathfrak{e}+(1-\tau)\mathfrak{g},\kappa\mathfrak{u}+(1-\kappa)\mathfrak{v}),\ \gamma),\ G^*((\tau\mathfrak{e}+(1-\tau)\mathfrak{g},\kappa\mathfrak{u}+(1-\kappa)\mathfrak{v}),\ \gamma)].$$

Again, from Equations (3) and (5), we obtain

$$\mathcal{J}(\tau)\mathcal{J}(\kappa)G_\gamma(\mathfrak{e},\mathfrak{u})+\mathcal{J}(\tau)\mathcal{J}(1-\kappa)G_\gamma(\mathfrak{e},\mathfrak{v})+\mathcal{J}(1-\tau)\mathcal{J}(\kappa)G_\gamma(\mathfrak{g},\mathfrak{u})+\mathcal{J}(1-\tau)\mathcal{J}(1-\kappa)G_\gamma(\mathfrak{g},\mathfrak{v})$$
$$=\mathcal{J}(\tau)\mathcal{J}(\kappa)[G_*((\mathfrak{e},\mathfrak{u}),\ \gamma),\ G^*((\mathfrak{e},\mathfrak{u}),\ \gamma)]+\mathcal{J}(\tau)\mathcal{J}(1-\kappa)[G_*((\mathfrak{e},\mathfrak{v}),\ \gamma),\ G^*((\mathfrak{e},\mathfrak{v}),\ \gamma)]$$
$$+\mathcal{J}(\kappa)\mathcal{J}(1-\tau)[G_*((\mathfrak{e},\mathfrak{u}),\ \gamma),\ G^*((\mathfrak{e},\mathfrak{u}),\ \gamma)]+\mathcal{J}(1-\tau)\mathcal{J}(1-\kappa)[G_*((\mathfrak{e},\mathfrak{v}),\ \gamma),\ G^*((\mathfrak{e},\mathfrak{v}),\ \gamma)],$$

for all $x,\omega\in\Delta$ and $\tau\in[0,\ 1]$. Then, through the coordinated *UD*-$\mathcal{J}$-convexity of $\widetilde{G}$, we have, for all $x,\omega\in\Delta$ and $\tau\in[0,\ 1]$, that

$$G_*((\tau\mathfrak{e}+(1-\tau)\mathfrak{g},\kappa\mathfrak{u}+(1-\kappa)\mathfrak{v}),\ \gamma)$$
$$\leq\mathcal{J}(\tau)\mathcal{J}(\kappa)G_*(\mathfrak{e},\mathfrak{u})+\mathcal{J}(\tau)\mathcal{J}(1-\kappa)G_*(\mathfrak{e},\mathfrak{v})+\mathcal{J}(1-\tau)\mathcal{J}(\kappa)G_*(\mathfrak{g},\mathfrak{u})+\mathcal{J}(1-\tau)\mathcal{J}(1-\kappa)G_*(\mathfrak{g},\mathfrak{v}),$$

and

$$G^*((\tau\mathfrak{e}+(1-\tau)\mathfrak{g},\kappa\mathfrak{u}+(1-\kappa)\mathfrak{v}),\ \gamma)$$
$$\geq\mathcal{J}(\tau)\mathcal{J}(\kappa)G^*(\mathfrak{e},\mathfrak{u})+\mathcal{J}(\tau)\mathcal{J}(1-\kappa)G^*(\mathfrak{e},\mathfrak{v})+\mathcal{J}(1-\tau)\mathcal{J}(\kappa)G^*(\mathfrak{g},\mathfrak{u})+\mathcal{J}(1-\tau)\mathcal{J}(1-\kappa)G^*(\mathfrak{g},\mathfrak{v}),$$

for each $\gamma\in[0,\ 1]$. Hence, the result follows. □

Example 1. *We consider the FNVM* $\widetilde{G} : [0,\ 1] \times [0,\ 1] \to \mathbb{F}_0$ *defined by*

$$G(x)(\sigma) = \begin{cases} \dfrac{\sigma - xy}{5 - xy},\ \sigma \in [xy,\ 5] \\ \dfrac{(6+e^x)(6+e^y) - \sigma}{(6+e^x)(6+e^y) - 5},\ \sigma \in (5,\ (6+e^x)(6+e^y)] \\ 0\ ,\ otherwise, \end{cases} \tag{23}$$

Then, for each $\gamma \in [0,\ 1]$, *we have* $G_\gamma(x) = [(1-\gamma)xy + 5\gamma, (1-\gamma)(6+e^x)(6+e^y) + 5\gamma]$. *Since the endpoint functions* $G_*((x,y),\gamma)$ *and* $G^*((x,y),\gamma)$ *are coordinate* Л*-concave functions for each* $\gamma \in [0,\ 1]$, $\widetilde{G}(x,y)$ *is a coordinate UD-*Л*-convex FNVM.*

From Lemma 1 and Example 1, we can easily note that each UD-Л*-convex FNVM is a coordinated UD-*Л*-convex FNVM. But the converse is not true.*

Remark 1. *If one assumes that* $\text{Л}(\tau) = \tau$, $\text{Л}(\kappa) = \kappa$ *and* $G_*((x,y),\gamma) = G^*((x,y),\gamma)$ *with* $\gamma = 1$, *then G is referred to as a coordinated convex function if G meets the stated inequality here:*

$$\begin{aligned} &G(\tau\mathfrak{e} + (1-\tau)\mathfrak{g}, \kappa\mathfrak{u} + (1-\kappa)\mathfrak{v}) \\ &\leq \tau\kappa G(\mathfrak{e},\mathfrak{u}) + \tau(1-\kappa)G(\mathfrak{e},\mathfrak{v}) + (1-\tau)\kappa G(\mathfrak{g},\mathfrak{u}) + (1-\tau)(1-\kappa)G(\mathfrak{g},\mathfrak{v}). \end{aligned} \tag{24}$$

Let one assume that $\text{Л}(\tau) = \tau$, $\text{Л}(\kappa) = \kappa$ *and* $G_*((x,y),\gamma) \neq G^*((x,y),\gamma)$ *with* $\gamma = 1$, $G_*((x,y),\gamma)$ *is an affine function, and* $G^*((x,y),\gamma)$ *is a concave function. Then, the stated inequality here, (see [68])*

$$\begin{aligned} &G(\tau\mathfrak{e} + (1-\tau)\mathfrak{g}, \kappa\mathfrak{u} + (1-\kappa)\mathfrak{v}) \\ &\supseteq \tau\kappa G(\mathfrak{e},\mathfrak{u}) + \tau(1-\kappa)G(\mathfrak{e},\mathfrak{v}) + (1-\tau)\kappa G(\mathfrak{g},\mathfrak{u}) + (1-\tau)(1-\kappa)G(\mathfrak{g},\mathfrak{v}), \end{aligned} \tag{25}$$

is true.

Definition 6. *Let* $\widetilde{G} : \Delta \to \mathbb{F}_0$ *be an FNVM on* Δ. *Then, from* γ*-levels, we obtain that the collection of IVMs* $G_\gamma : \Delta \to \mathbb{R}_I^+ \subset \mathbb{R}_I$ *is given by*

$$G_\gamma(x,y) = [G_*((x,y),\gamma),\ G^*((x,y),\gamma)], \tag{26}$$

for all $(x,y) \in \Delta$ *and for all* $\gamma \in [0,\ 1]$. *Then,* $\widetilde{G}$ *is a coordinated left-UD-*Л*-convex (concave) FNVM on* Δ *if and only if for all* $\gamma \in [0,\ 1]$, $G_*((x,y),\ \gamma)$ *and* $G^*((x,y),\ \gamma)$ *are coordinated* Л*-convex (concave) and affine functions on* Δ*, respectively.*

Definition 7. *Let* $\widetilde{G} : \Delta \to \mathbb{F}_0$ *be an FNVM on* Δ. *Then, from* γ*-levels, we obtain that the collection of IVMs* $G_\gamma : \Delta \to \mathbb{R}_I^+ \subset \mathbb{R}_I$ *is given by*

$$G_\gamma(x,y) = [G_*((x,y),\gamma),\ G^*((x,y),\gamma)],$$

for all $(x,y) \in \Delta$ *and for all* $\gamma \in [0,\ 1]$. *Then,* $\widetilde{G}$ *is a coordinated right-UD-*Л*-convex (concave) FNVM on* Δ *if and only if for all* $\gamma \in [0,\ 1]$, $G_*((x,y),\ \gamma)$ *and* $G^*((x,y),\ \gamma)$ *are coordinated* Л*-affine and* Л*-convex (concave) functions on* Δ*, respectively.*

Theorem 7. *Let* Δ *be a coordinated convex set, and let* $\widetilde{G} : \Delta \to \mathbb{F}_0^+$ *be an FNVM. Then, from* γ*-levels, we obtain that the collection of IVMs* $G_\gamma : \Delta \to \mathbb{R}_I^+ \subset \mathbb{R}_I$ *is given by*

$$G_\gamma(x,y) = [G_*((x,y),\gamma),\ G^*((x,y),\gamma)],$$

for all $(x, y) \in \Delta$ *and for all* $\gamma \in [0,\ 1]$. *Then,* $\widetilde{G}$ *is a coordinated* UD-Ʝ*-concave* FNVM *on* Δ *if and only if for all* $\gamma \in [0,\ 1]$, $G_*((x, y),\ \gamma)$ *and* $G^*((x, y),\ \gamma)$ *are coordinated* Ʝ*-concave and* Ʝ*-convex functions, respectively.*

Proof. The demonstration of the proof of Theorem 7 is similar to the demonstration of the proof of Theorem 6. □

Example 2. *We consider the* FNVMs $\widetilde{G} : [0,\ 1] \times [0,\ 1] \to \mathbb{F}_0^+$ *defined by*

$$\widetilde{G}(x, y)(\sigma) = \begin{cases} \dfrac{\sigma - (6 - e^x)(6 - e^y)}{(6 - e^x)(6 - e^y) - 25},\ \sigma \in [(6 - e^x)(6 - e^y),\ 25] \\ \dfrac{35xy - \sigma}{35xy - 25},\ \sigma \in (25,\ 35xy] \\ 0,\ otherwise. \end{cases} \quad (27)$$

Then, for each $\gamma \in [0,\ 1]$, *we have* $G_\gamma(x, y) = [(1 - \gamma)(6 - e^x)(6 - e^y) + 25\gamma, 35(1 - \gamma)xy + 25\gamma]$. *Since the endpoint functions* $G_*((x, y), \gamma)$ *and* $G^*((x, y), \gamma)$ *are coordinated* Ʝ*-concave and* Ʝ*-convex functions for each* $\gamma \in [0,\ 1]$, $\widetilde{G}(x, y)$ *is a coordinated* UD-Ʝ*-concave* FNVM.

3. Main Results

Here is the first result of the coordinated integral inequalities of the Hermite–Hadamard type using fuzzy fractional operators via coordinated *UD*-Ʝ-concave *FNVM*s.

Theorem 8. *Let* $\widetilde{G} : \Delta \to \mathbb{F}_0^+$ *be a coordinated* UD-Ʝ*-convex* FNVM *on* Δ, *and let* $Ʝ : [0,\ 1] \to \mathbb{R}^+$. *Then, from* γ*-cuts, we set up the sequence of* IVMs $G_\gamma : \Delta \to \mathbb{R}_I^+$, *which is given by* $G_\gamma(x, y) = [G_*((x, y), \gamma),\ G^*((x, y), \gamma)]$ *for all* $(x, y) \in \Delta$ *and for all* $\gamma \in [0,\ 1]$. *If* $\widetilde{G} \in \mathcal{F}\mathfrak{O}_\Delta$, *then the following inequalities hold:*

$$\begin{aligned} &\frac{1}{Ʝ^2\left(\frac{1}{2}\right)}\widetilde{G}\left(\frac{\mathfrak{e}+\mathfrak{g}}{2},\ \frac{\mathfrak{u}+\mathfrak{v}}{2}\right) \\ &\supseteq_{\mathbb{F}} \frac{\Gamma(\alpha+1)}{2Ʝ\left(\frac{1}{2}\right)(\mathfrak{g}-\mathfrak{e})^\alpha}\left[\mathcal{I}^\alpha_{\mathfrak{e}^+}\widetilde{G}\left(\mathfrak{g},\ \frac{\mathfrak{u}+\mathfrak{v}}{2}\right) \oplus \mathcal{I}^\alpha_{\mathfrak{g}^-}\widetilde{G}\left(\mathfrak{e},\ \frac{\mathfrak{u}+\mathfrak{v}}{2}\right)\right] \oplus \frac{\Gamma(\beta+1)}{2Ʝ\left(\frac{1}{2}\right)(\mathfrak{v}-\mathfrak{u})^\beta}\left[\mathcal{I}^\beta_{\mathfrak{u}^+}\widetilde{G}\left(\frac{\mathfrak{e}+\mathfrak{g}}{2}, \mathfrak{v}\right) \oplus \mathcal{I}^\beta_{\mathfrak{v}^-}\widetilde{G}\left(\frac{\mathfrak{e}+\mathfrak{g}}{2}, \mathfrak{u}\right)\right] \\ &\supseteq_{\mathbb{F}} \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{(\mathfrak{g}-\mathfrak{e})^\alpha(\mathfrak{v}-\mathfrak{u})^\beta}\left[\mathcal{I}^{\alpha,\ \beta}_{\mathfrak{e}^+,\mathfrak{u}^+}\widetilde{G}(\mathfrak{g}, \mathfrak{v}) \oplus \mathcal{I}^{\alpha,\ \beta}_{\mathfrak{e}^+,\mathfrak{v}^-}\widetilde{G}(\mathfrak{g}, \mathfrak{u}) \oplus \mathcal{I}^{\alpha,\ \beta}_{\mathfrak{g}^-,\mathfrak{u}^+}\widetilde{G}(\mathfrak{e}, \mathfrak{v}) \oplus \mathcal{I}^{\alpha,\ \beta}_{\mathfrak{g}^-,\mathfrak{v}^-}\widetilde{G}(\mathfrak{e}, \mathfrak{u})\right] \\ &\supseteq_{\mathbb{F}} \frac{\beta\Gamma(\alpha+1)}{(\mathfrak{g}-\mathfrak{e})^\alpha}\left[\mathcal{I}^\alpha_{\mathfrak{e}^+}\widetilde{G}(\mathfrak{g}, \mathfrak{u}) \oplus \mathcal{I}^\alpha_{\mathfrak{e}^+}\widetilde{G}(\mathfrak{g}, \mathfrak{v}) \oplus \mathcal{I}^\alpha_{\mathfrak{g}^-}\widetilde{G}(\mathfrak{e}, \mathfrak{u}) \oplus \mathcal{I}^\alpha_{\mathfrak{g}^-}\widetilde{G}(\mathfrak{e}, \mathfrak{v})\right] \times \int_0^1 \kappa^{\beta-1}[Ʝ(\kappa) + Ʝ(1-\kappa)]d\kappa \\ &\oplus \frac{\alpha\Gamma(\beta+1)}{(\mathfrak{v}-\mathfrak{u})^\beta}\left[\mathcal{I}^\beta_{\mathfrak{u}^+}\widetilde{G}(\mathfrak{e}, \mathfrak{v}) \oplus \mathcal{I}^\beta_{\mathfrak{v}^-}\widetilde{G}(\mathfrak{g}, \mathfrak{u}) \oplus \mathcal{I}^\beta_{\mathfrak{u}^+}\widetilde{G}(\mathfrak{g}, \mathfrak{v}) \oplus \mathcal{I}^\beta_{\mathfrak{v}^-}\widetilde{G}(\mathfrak{g}, \mathfrak{u})\right] \times \int_0^1 \tau^{\alpha-1}Ʝ(\tau) + Ʝ(1-\tau)d\tau \\ &\supseteq_{\mathbb{F}} \alpha\beta\left[\widetilde{G}(\mathfrak{e}, \mathfrak{u}) \oplus \widetilde{G}(\mathfrak{g}, \mathfrak{u}) \oplus \widetilde{G}(\mathfrak{e}, \mathfrak{v}) \oplus \widetilde{G}(\mathfrak{g}, \mathfrak{v})\right] \times \int_0^1 \kappa^{\beta-1}[Ʝ(\kappa) + Ʝ(1-\kappa)]d\kappa \int_0^1 \tau^{\alpha-1}[Ʝ(\tau) + Ʝ(1-\tau)]d\tau. \end{aligned} \quad (28)$$

If $\widetilde{G}(x, y)$ is a coordinated *UD*-Ʝ-concave *FNVM*, then

$$
\begin{aligned}
&\frac{1}{J^2\left(\frac{1}{2}\right)}\widetilde{G}\left(\frac{\mathfrak{e}+\mathfrak{g}}{2},\ \frac{\mathfrak{u}+\mathfrak{v}}{2}\right)\\
&\supseteq_{\mathbb{F}} \frac{\Gamma(\alpha+1)}{2J\left(\frac{1}{2}\right)(\mathfrak{g}-\mathfrak{e})^{\alpha}}\left[\mathcal{I}^{\alpha}_{\mathfrak{e}^+}\widetilde{G}\left(\mathfrak{g},\frac{\mathfrak{u}+\mathfrak{v}}{2}\right)\oplus \mathcal{I}^{\alpha}_{\mathfrak{g}^-}\widetilde{G}\left(\mathfrak{e},\frac{\mathfrak{u}+\mathfrak{v}}{2}\right)\right]\oplus \frac{\Gamma(\beta+1)}{2J\left(\frac{1}{2}\right)(\mathfrak{v}-\mathfrak{u})^{\beta}}\left[\mathcal{I}^{\beta}_{\mathfrak{u}^+}\widetilde{G}\left(\frac{\mathfrak{e}+\mathfrak{g}}{2},\mathfrak{v}\right)\oplus \mathcal{I}^{\beta}_{\mathfrak{v}^-}\widetilde{G}\left(\frac{\mathfrak{e}+\mathfrak{g}}{2},\mathfrak{u}\right)\right]\\
&\supseteq_{\mathbb{F}} \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{(\mathfrak{g}-\mathfrak{e})^{\alpha}(\mathfrak{v}-\mathfrak{u})^{\beta}}\left[\mathcal{I}^{\alpha,\,\beta}_{\mathfrak{e}^+,\mathfrak{u}^+}\widetilde{G}(\mathfrak{g},\mathfrak{v})\oplus \mathcal{I}^{\alpha,\,\beta}_{\mathfrak{e}^+,\mathfrak{v}^-}\widetilde{G}(\mathfrak{g},\mathfrak{u})\oplus \mathcal{I}^{\alpha,\,\beta}_{\mathfrak{g}^-,\mathfrak{u}^+}\widetilde{G}(\mathfrak{e},\mathfrak{v})\oplus \mathcal{I}^{\alpha,\,\beta}_{\mathfrak{g}^-,\mathfrak{v}^-}\widetilde{G}(\mathfrak{e},\mathfrak{u})\right]\\
&\supseteq_{\mathbb{F}} \frac{\beta\Gamma(\alpha+1)}{(\mathfrak{g}-\mathfrak{e})^{\alpha}}\left[\mathcal{I}^{\alpha}_{\mathfrak{e}^+}\widetilde{G}(\mathfrak{g},\mathfrak{u})\oplus \mathcal{I}^{\alpha}_{\mathfrak{e}^+}\widetilde{G}(\mathfrak{g},\mathfrak{v})\oplus \mathcal{I}^{\alpha}_{\mathfrak{g}^-}\widetilde{G}(\mathfrak{e},\mathfrak{u})\oplus \mathcal{I}^{\alpha}_{\mathfrak{g}^-}\widetilde{G}(\mathfrak{e},\mathfrak{v})\right]\times\int_0^1 \kappa^{\beta-1}[J(\kappa)+J(1-\kappa)]d\kappa\\
&\oplus \frac{\alpha\Gamma(\beta+1)}{(\mathfrak{v}-\mathfrak{u})^{\beta}}\left[\mathcal{I}^{\beta}_{\mathfrak{u}^+}\widetilde{G}(\mathfrak{e},\mathfrak{v})\oplus \mathcal{I}^{\beta}_{\mathfrak{v}^-}\widetilde{G}(\mathfrak{g},\mathfrak{u})\oplus \mathcal{I}^{\beta}_{\mathfrak{u}^+}\widetilde{G}(\mathfrak{g},\mathfrak{v})\oplus \mathcal{I}^{\beta}_{\mathfrak{v}^-}\widetilde{G}(\mathfrak{g},\mathfrak{u})\right]\times\int_0^1 \tau^{\alpha-1}[J(\tau)+J(1-\tau)]d\tau\\
&\supseteq_{\mathbb{F}} \alpha\beta\left[\widetilde{G}(\mathfrak{e},\mathfrak{u})\oplus \widetilde{G}(\mathfrak{g},\mathfrak{u})\oplus \widetilde{G}(\mathfrak{e},\mathfrak{v})\oplus \widetilde{G}(\mathfrak{g},\mathfrak{v})\right]\times\int_0^1 \kappa^{\beta-1}[J(\kappa)+J(1-\kappa)]d\kappa\int_0^1 \tau^{\alpha-1}[J(\tau)+J(1-\tau)]d\tau.
\end{aligned}
\tag{29}
$$

Proof. Let $\widetilde{G}:[\mathfrak{e},\mathfrak{g}]\to\mathbb{F}_0$ be a coordinated *UD-J*-convex *FNVM*. Then, from our hypothesis, we have

$$
\frac{1}{J^2\left(\frac{1}{2}\right)}\widetilde{G}\left(\frac{\mathfrak{e}+\mathfrak{g}}{2},\frac{\mathfrak{u}+\mathfrak{v}}{2}\right)\supseteq_{\mathbb{F}}\widetilde{G}(\tau\mathfrak{e}+(1-\tau)\mathfrak{g},\tau\mathfrak{u}+(1-\tau)\mathfrak{v})\oplus\widetilde{G}((1-\tau)\mathfrak{e}+\tau\mathfrak{g},(1-\tau)\mathfrak{u}+\tau\mathfrak{v}).
$$

By using Theorem 6, for every $\gamma\in[0,\ 1]$, we have

$$
\begin{aligned}
&\frac{1}{J^2\left(\frac{1}{2}\right)}G_*\left(\left(\frac{\mathfrak{e}+\mathfrak{g}}{2},\frac{\mathfrak{u}+\mathfrak{v}}{2}\right),\ \gamma\right)\\
&\le G_*((\tau\mathfrak{e}+(1-\tau)\mathfrak{g},\tau\mathfrak{u}+(1-\tau)\mathfrak{v}),\ \gamma)+G_*(((1-\tau)\mathfrak{e}+\tau\mathfrak{g},(1-\tau)\mathfrak{u}+\tau\mathfrak{v}),\ \gamma),\\
&\frac{1}{J^2\left(\frac{1}{2}\right)}G^*\left(\left(\frac{\mathfrak{e}+\mathfrak{g}}{2},\frac{\mathfrak{u}+\mathfrak{v}}{2}\right),\ \gamma\right)\\
&\ge G^*((\tau\mathfrak{e}+(1-\tau)\mathfrak{g},\tau\mathfrak{u}+(1-\tau)\mathfrak{v}),\ \gamma)+G^*(((1-\tau)\mathfrak{e}+\tau\mathfrak{g},(1-\tau)\mathfrak{u}+\tau\mathfrak{v}),\gamma).
\end{aligned}
$$

By using Lemma 1, we have

$$
\begin{aligned}
&\frac{1}{J\left(\frac{1}{2}\right)}G_*\left(\left(x,\frac{\mathfrak{u}+\mathfrak{v}}{2}\right),\ \gamma\right)\le G_*((x,\ \tau\mathfrak{u}+(1-\tau)\mathfrak{v}),\ \gamma)+G_*((x,\ (1-\tau)\mathfrak{u}+\tau\mathfrak{v}),\ \gamma),\\
&\frac{1}{J\left(\frac{1}{2}\right)}G^*\left(\left(x,\frac{\mathfrak{u}+\mathfrak{v}}{2}\right),\ \gamma\right)\ge G^*((x,\ \tau\mathfrak{u}+(1-\tau)\mathfrak{v}),\ \gamma)+G^*((x,\ (1-\tau)\mathfrak{u}+\tau\mathfrak{v}),\gamma),
\end{aligned}
\tag{30}
$$

$$
\begin{aligned}
&\frac{1}{J\left(\frac{1}{2}\right)}G_*\left(\left(\frac{\mathfrak{e}+\mathfrak{g}}{2},y\right),\ \gamma\right)\le G_*((\tau\mathfrak{e}+(1-\tau)\mathfrak{g},y),\ \gamma)+G_*(((1-\tau)\mathfrak{e}+\tau\mathfrak{g},y),\ \gamma),\\
&\frac{1}{J\left(\frac{1}{2}\right)}G^*\left(\left(\frac{\mathfrak{e}+\mathfrak{g}}{2},y\right),\ \gamma\right)\ge G^*((\tau\mathfrak{e}+(1-\tau)\mathfrak{g},y),\ \gamma)+G^*(((1-\tau)\mathfrak{e}+\tau\mathfrak{g},y),\gamma).
\end{aligned}
\tag{31}
$$

From (30) and (31), we have

$$
\begin{aligned}
&\frac{1}{J\left(\frac{1}{2}\right)}\left[G_*\left(\left(x,\frac{\mathfrak{u}+\mathfrak{v}}{2}\right),\ \gamma\right),G^*\left(\left(x,\frac{\mathfrak{u}+\mathfrak{v}}{2}\right),\ \gamma\right)\right]\\
&\supseteq_I [G_*((x,\ \tau\mathfrak{u}+(1-\tau)\mathfrak{v}),\ \gamma),G^*((x,\ \tau\mathfrak{u}+(1-\tau)\mathfrak{v}),\ \gamma)]\\
&+[G_*((x,\ (1-\tau)\mathfrak{u}+\tau\mathfrak{v}),\ \gamma),G^*((x,\ (1-\tau)\mathfrak{u}+\tau\mathfrak{v}),\gamma)],
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{1}{J\left(\frac{1}{2}\right)}\left[G_*\left(\left(\frac{\mathfrak{e}+\mathfrak{g}}{2},y\right),\ \gamma\right),G^*\left(\left(\frac{\mathfrak{e}+\mathfrak{g}}{2},y\right),\ \gamma\right)\right]\\
&\supseteq_I [G_*((\tau\mathfrak{e}+(1-\tau)\mathfrak{g},y),\ \gamma),G^*((\tau\mathfrak{e}+(1-\tau)\mathfrak{g},y),\ \gamma)]\\
&+[G_*((\tau\mathfrak{e}+(1-\tau)\mathfrak{g},y),\ \gamma),G^*((\tau\mathfrak{e}+(1-\tau)\mathfrak{g},y),\gamma)],
\end{aligned}
$$

It follows that

$$\frac{1}{Ɉ\left(\frac{1}{2}\right)} G_\gamma\left(x, \frac{\mathfrak{u}+\mathfrak{v}}{2}\right) \supseteq_I G_\gamma(x,\ \tau\mathfrak{u}+(1-\tau)\mathfrak{v}) + G_\gamma(x,\ (1-\tau)\mathfrak{u}+\tau\mathfrak{v}), \tag{32}$$

and

$$\frac{1}{Ɉ\left(\frac{1}{2}\right)} G_\gamma\left(\frac{\mathfrak{e}+\mathfrak{g}}{2}, y\right) \supseteq_I G_\gamma(\tau\mathfrak{e}+(1-\tau)\mathfrak{g}, y) + G_\gamma(\tau\mathfrak{e}+(1-\tau)\mathfrak{g}, y). \tag{33}$$

Since $G_\gamma(x,.)$ and $G_\gamma(.,y)$ are both coordinated *UD*-Ɉ-convex-*IVM*s, then from inequality (15), for every $\gamma \in [0,\ 1]$, and from inequalities (32) and (43), we have

$$\begin{aligned}\frac{1}{\beta Ɉ\left(\frac{1}{2}\right)} G_{\gamma_x}\left(\tfrac{\mathfrak{u}+\mathfrak{v}}{2}\right) &\supseteq_I \frac{\Gamma(\beta)}{(\mathfrak{v}-\mathfrak{u})^\beta}\left[\mathcal{I}^\beta_{\mathfrak{u}^+} G_{\gamma_x}(\mathfrak{v}) + \mathcal{I}^\beta_{\mathfrak{v}^-}\ G_{\gamma_x}(\mathfrak{u})\right] \\ &\supseteq_I \left[G_{\gamma_x}(\mathfrak{u}) + G_{\gamma_x}(\mathfrak{v})\right]\int_0^1 \kappa^{\beta-1}[Ɉ(\kappa)+Ɉ(1-\kappa)]d\kappa\end{aligned} \tag{34}$$

and

$$\begin{aligned}\frac{1}{\alpha Ɉ\left(\frac{1}{2}\right)} G_{\gamma_y}\left(\tfrac{\mathfrak{e}+\mathfrak{g}}{2}\right) &\supseteq_I \frac{\Gamma(\alpha)}{(\mathfrak{g}-\mathfrak{e})^\alpha}\left[\mathcal{I}^\alpha_{\mathfrak{e}^+} G_{\gamma_y}(\mathfrak{g}) + \mathcal{I}^\alpha_{\mathfrak{g}^-} G_{\gamma_y}(\mathfrak{e})\right] \\ &\supseteq_I \left[G_{\gamma_y}(\mathfrak{e}) + G_{\gamma_y}(\mathfrak{g})\right]\int_0^1 \tau^{\alpha-1}Ɉ(\tau)+Ɉ(1-\tau)d\tau\end{aligned} \tag{35}$$

Since $G_{\gamma_x}(w) = G_\gamma(x,w)$, (34) can be written as

$$\begin{aligned}\frac{1}{\beta Ɉ\left(\frac{1}{2}\right)} G_\gamma\left(x, \tfrac{\mathfrak{u}+\mathfrak{v}}{2}\right) &\supseteq_I \frac{\Gamma(\beta)}{(\mathfrak{v}-\mathfrak{u})^\beta}\left[\mathcal{I}^\alpha_{\mathfrak{u}^+} G_\gamma(x,\mathfrak{v}) + \mathcal{I}^\alpha_{\mathfrak{v}^-} G_\gamma(x,\mathfrak{u})\right] \\ &\supseteq_I [G_\gamma(x,\mathfrak{u}) + G_\gamma(x,\mathfrak{v})]\int_0^1 \kappa^{\beta-1}[Ɉ(\kappa)+Ɉ(1-\kappa)]d\kappa.\end{aligned} \tag{36}$$

That is

$$\begin{aligned}\frac{1}{\beta Ɉ\left(\frac{1}{2}\right)} G_\gamma\left(x, \tfrac{\mathfrak{u}+\mathfrak{v}}{2}\right) &\supseteq_I \frac{1}{(\mathfrak{v}-\mathfrak{u})^\beta}\left[\int_\mathfrak{u}^\mathfrak{v} (\mathfrak{v}-\kappa)^{\beta-1} G_\gamma(x,\kappa)d\kappa + \int_\mathfrak{u}^\mathfrak{v} (\kappa-\mathfrak{u})^{\beta-1} G_\gamma(x,\kappa)d\kappa\right] \\ &\supseteq_I [G_\gamma(x,\mathfrak{u}) + G_\gamma(x,\mathfrak{v})]\int_0^1 \kappa^{\beta-1}[Ɉ(\kappa)+Ɉ(1-\kappa)]d\kappa.\end{aligned}$$

Multiplying the double inequality of (36) by $\frac{(\mathfrak{g}-x)^{\alpha-1}}{(\mathfrak{g}-\mathfrak{e})^\alpha}$ and integrating with respect to x over $[\mathfrak{e},\mathfrak{g}]$, we have

$$\begin{aligned}&\frac{1}{\beta(\mathfrak{g}-\mathfrak{e})^\alpha Ɉ\left(\frac{1}{2}\right)}\int_\mathfrak{e}^\mathfrak{g} G_\gamma\left(x, \tfrac{\mathfrak{u}+\mathfrak{v}}{2}\right)(\mathfrak{g}-x)^{\alpha-1}dx \\ &\supseteq_I \frac{1}{(\mathfrak{g}-\mathfrak{e})^\alpha(\mathfrak{v}-\mathfrak{u})^\beta}\int_\mathfrak{e}^\mathfrak{g}\int_\mathfrak{u}^\mathfrak{v} (\mathfrak{g}-x)^{\alpha-1}(\mathfrak{v}-\kappa)^{\beta-1}G_\gamma(x,\kappa)d\kappa dx + \int_\mathfrak{e}^\mathfrak{g}\int_\mathfrak{u}^\mathfrak{v} (\mathfrak{g}-x)^{\alpha-1}(\kappa-\mathfrak{u})^{\beta-1}G_\gamma(x,\kappa)d\kappa dx \\ &\supseteq_I \frac{1}{(\mathfrak{g}-\mathfrak{e})^\alpha}\left[\int_\mathfrak{e}^\mathfrak{g} (\mathfrak{g}-x)^{\alpha-1}G_\gamma(x,\mathfrak{u})dx + \int_\mathfrak{e}^\mathfrak{g} (\mathfrak{g}-x)^{\alpha-1}G_\gamma(x,\mathfrak{v})dx\right]\int_0^1 \kappa^{\beta-1}[Ɉ(\kappa)+Ɉ(1-\kappa)]d\kappa.\end{aligned} \tag{37}$$

Again, multiplying the double inequality of (36) by $\frac{(x-\mathfrak{e})^{\alpha-1}}{(\mathfrak{g}-\mathfrak{e})^\alpha}$ and integrating with respect to x over $[\mathfrak{e},\mathfrak{g}]$, we have

$$\begin{aligned}&\frac{1}{\beta(\mathfrak{g}-\mathfrak{e})^\alpha Ɉ\left(\frac{1}{2}\right)}\int_\mathfrak{e}^\mathfrak{g} G_\gamma\left(x, \tfrac{\mathfrak{u}+\mathfrak{v}}{2}\right)(x-\mathfrak{e})^{\alpha-1}dx \\ &\supseteq_I \frac{1}{(\mathfrak{g}-\mathfrak{e})^\alpha(\mathfrak{v}-\mathfrak{u})^\beta}\int_\mathfrak{e}^\mathfrak{g}\int_\mathfrak{u}^\mathfrak{v} (x-\mathfrak{e})^{\alpha-1}(\mathfrak{v}-\kappa)^{\beta-1}G_\gamma(x,\kappa)d\kappa dx \\ &+\frac{1}{(\mathfrak{g}-\mathfrak{e})^\alpha(\mathfrak{v}-\mathfrak{u})^\beta}\int_\mathfrak{e}^\mathfrak{g}\int_\mathfrak{u}^\mathfrak{v} (x-\mathfrak{e})^{\alpha-1}(\kappa-\mathfrak{u})^{\beta-1}G_\gamma(x,\kappa)d\kappa dx \\ &\supseteq_I \frac{1}{(\mathfrak{g}-\mathfrak{e})^\alpha}\left[\int_\mathfrak{e}^\mathfrak{g} (x-\mathfrak{e})^{\alpha-1}G_\gamma(x,\mathfrak{u})dx + \int_\mathfrak{e}^\mathfrak{g} (x-\mathfrak{e})^{\alpha-1}G_\gamma(x,\mathfrak{v})dx\right]\int_0^1 \kappa^{\beta-1}[Ɉ(\kappa)+Ɉ(1-\kappa)]d\kappa.\end{aligned} \tag{38}$$

From (37), we have

$$\begin{aligned}
&\frac{\Gamma(\alpha+1)}{2\mathfrak{J}\left(\frac{1}{2}\right)(\mathfrak{g}-\mathfrak{e})^{\alpha}}\left[\mathcal{I}^{\alpha}_{\mathfrak{e}^{+}}G_{\gamma}\left(\mathfrak{g},\frac{\mathfrak{u}+\mathfrak{v}}{2}\right)\right]\\
&\supseteq_{I}\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{(\mathfrak{g}-\mathfrak{e})^{\alpha}(\mathfrak{v}-\mathfrak{u})^{\beta}}\left[\mathcal{I}^{\alpha,\,\beta}_{\mathfrak{e}^{+},\mathfrak{u}^{+}}G_{\gamma}(\mathfrak{g},\mathfrak{v})+\mathcal{I}^{\alpha,\,\beta}_{\mathfrak{g}^{-},\mathfrak{u}^{+}}G_{\gamma}(\mathfrak{g},\mathfrak{u})\right]\\
&\supseteq_{I}\frac{\beta\Gamma(\alpha+1)}{(\mathfrak{g}-\mathfrak{e})^{\alpha}}\left[\mathcal{I}^{\alpha}_{\mathfrak{e}^{+}}G_{\gamma}(\mathfrak{g},\mathfrak{u})+\mathcal{I}^{\alpha}_{\mathfrak{e}^{+}}G_{\gamma}(\mathfrak{g},\mathfrak{v})\right]\int_0^1\kappa^{\beta-1}[\mathfrak{J}(\kappa)+\mathfrak{J}(1-\kappa)]d\kappa.
\end{aligned}\tag{39}$$

From (38), we have

$$\begin{aligned}
&\frac{\Gamma(\alpha+1)}{2\mathfrak{J}\left(\frac{1}{2}\right)(\mathfrak{g}-\mathfrak{e})^{\alpha}}\left[\mathcal{I}^{\alpha}_{\mathfrak{g}^{-}}G_{\gamma}\left(\mathfrak{e},\frac{\mathfrak{u}+\mathfrak{v}}{2}\right)\right]\\
&\supseteq_{I}\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{(\mathfrak{g}-\mathfrak{e})^{\alpha}(\mathfrak{v}-\mathfrak{u})^{\beta}}\left[\mathcal{I}^{\alpha,\,\beta}_{\mathfrak{g}^{-},\mathfrak{u}^{+}}G_{\gamma}(\mathfrak{e},\mathfrak{v})+\mathcal{I}^{\alpha,\,\beta}_{\mathfrak{g}^{-},\mathfrak{v}^{-}}G_{\gamma}(\mathfrak{e},\mathfrak{u})\right]\\
&\supseteq_{I}\frac{\beta\Gamma(\alpha+1)}{(\mathfrak{g}-\mathfrak{e})^{\alpha}}\left[\mathcal{I}^{\alpha}_{\mathfrak{g}^{-}}G_{\gamma}(\mathfrak{e},\mathfrak{u})+\mathcal{I}^{\alpha}_{\mathfrak{g}^{-}}G_{\gamma}(\mathfrak{e},\mathfrak{v})\right]\int_0^1\kappa^{\beta-1}[\mathfrak{J}(\kappa)+\mathfrak{J}(1-\kappa)]d\kappa.
\end{aligned}\tag{40}$$

Since, from γ-cuts, we obtain the collection of *IVMs* $G_{\gamma}:\Delta\to\mathbb{R}^{+}_{I}$, we have

$$\begin{aligned}
&\frac{\Gamma(\alpha+1)}{2\mathfrak{J}\left(\frac{1}{2}\right)(\mathfrak{g}-\mathfrak{e})^{\alpha}}\left[\mathcal{I}^{\alpha}_{\mathfrak{e}^{+}}\widetilde{G}\left(\mathfrak{g},\frac{\mathfrak{u}+\mathfrak{v}}{2}\right)\right]\\
&\supseteq_{\mathbb{F}}\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{(\mathfrak{g}-\mathfrak{e})^{\alpha}(\mathfrak{v}-\mathfrak{u})^{\beta}}\left[\mathcal{I}^{\alpha,\,\beta}_{\mathfrak{e}^{+},\mathfrak{u}^{+}}\widetilde{G}(\mathfrak{g},\mathfrak{v})\oplus\mathcal{I}^{\alpha,\,\beta}_{\mathfrak{g}^{-},\mathfrak{u}^{+}}\widetilde{G}(\mathfrak{g},\mathfrak{u})\right]\\
&\supseteq_{\mathbb{F}}\frac{\beta\Gamma(\alpha+1)}{(\mathfrak{g}-\mathfrak{e})^{\alpha}}\left[\mathcal{I}^{\alpha}_{\mathfrak{e}^{+}}\widetilde{G}(\mathfrak{g},\mathfrak{u})\oplus\mathcal{I}^{\alpha}_{\mathfrak{e}^{+}}\widetilde{G}(\mathfrak{g},\mathfrak{v})\right]\int_0^1\kappa^{\beta-1}[\mathfrak{J}(\kappa)+\mathfrak{J}(1-\kappa)]d\kappa.
\end{aligned}\tag{41}$$

And

$$\begin{aligned}
&\frac{\Gamma(\alpha+1)}{2\mathfrak{J}\left(\frac{1}{2}\right)(\mathfrak{g}-\mathfrak{e})^{\alpha}}\left[\mathcal{I}^{\alpha}_{\mathfrak{g}^{-}}\widetilde{G}\left(\mathfrak{e},\frac{\mathfrak{u}+\mathfrak{v}}{2}\right)\right]\\
&\supseteq_{\mathbb{F}}\frac{\beta\Gamma(\alpha+1)\Gamma(\beta+1)}{(\mathfrak{g}-\mathfrak{e})^{\alpha}(\mathfrak{v}-\mathfrak{u})^{\beta}}\left[\mathcal{I}^{\alpha,\,\beta}_{\mathfrak{g}^{-},\mathfrak{u}^{+}}\widetilde{G}(\mathfrak{e},\mathfrak{v})\oplus\mathcal{I}^{\alpha,\,\beta}_{\mathfrak{g}^{-},\mathfrak{v}^{-}}\widetilde{G}(\mathfrak{e},\mathfrak{u})\right]\\
&\supseteq_{\mathbb{F}}\frac{\beta\Gamma(\alpha+1)}{(\mathfrak{g}-\mathfrak{e})^{\alpha}}\left[\mathcal{I}^{\alpha}_{\mathfrak{g}^{-}}\widetilde{G}(\mathfrak{e},\mathfrak{u})\oplus\mathcal{I}^{\alpha}_{\mathfrak{g}^{-}}\widetilde{G}(\mathfrak{e},\mathfrak{v})\right]\int_0^1\kappa^{\beta-1}[\mathfrak{J}(\kappa)+\mathfrak{J}(1-\kappa)]d\kappa.
\end{aligned}\tag{42}$$

Similarly, since $\widetilde{G}_{y}(z)=\widetilde{G}(z,y)$, from (35), (41), and (42), we have

$$\begin{aligned}
&\frac{\Gamma(\beta+1)}{2\mathfrak{J}\left(\frac{1}{2}\right)(\mathfrak{v}-\mathfrak{u})^{\beta}}\left[\mathcal{I}^{\beta}_{\mathfrak{u}^{+}}\widetilde{G}\left(\frac{\mathfrak{e}+\mathfrak{g}}{2},\mathfrak{v}\right)\right]\\
&\supseteq_{\mathbb{F}}\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{(\mathfrak{g}-\mathfrak{e})^{\alpha}(\mathfrak{v}-\mathfrak{u})^{\beta}}\left[\mathcal{I}^{\alpha,\,\beta}_{\mathfrak{e}^{+},\mathfrak{u}^{+}}\widetilde{G}(\mathfrak{g},\mathfrak{v})\oplus\mathcal{I}^{\alpha,\,\beta}_{\mathfrak{g}^{-},\mathfrak{u}^{+}}\widetilde{G}(\mathfrak{e},\mathfrak{v})\right]\\
&\supseteq_{\mathbb{F}}\frac{\alpha\Gamma(\beta+1)}{(\mathfrak{v}-\mathfrak{u})^{\beta}}\left[\mathcal{I}^{\beta}_{\mathfrak{u}^{+}}\widetilde{G}(\mathfrak{e},\mathfrak{v})\oplus\mathcal{I}^{\beta}_{\mathfrak{u}^{+}}\widetilde{G}(\mathfrak{g},\mathfrak{v})\right].
\end{aligned}\tag{43}$$

And

$$\begin{aligned}
&\frac{\Gamma(\beta+1)}{2\mathfrak{J}\left(\frac{1}{2}\right)(\mathfrak{v}-\mathfrak{u})^{\alpha}}\left[\mathcal{I}^{\beta}_{\mathfrak{v}^{-}}\widetilde{G}\left(\frac{\mathfrak{e}+\mathfrak{g}}{2},\mathfrak{u}\right)\right]\\
&\supseteq_{\mathbb{F}}\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{(\mathfrak{g}-\mathfrak{e})^{\alpha}(\mathfrak{v}-\mathfrak{u})^{\beta}}\left[\mathcal{I}^{\alpha,\,\beta}_{\mathfrak{e}^{+},\mathfrak{v}^{-}}\widetilde{G}(\mathfrak{g},\mathfrak{u})\oplus\mathcal{I}^{\alpha,\,\beta}_{\mathfrak{g}^{-},\mathfrak{v}^{-}}\widetilde{G}(\mathfrak{e},\mathfrak{u})\right]\\
&\supseteq_{\mathbb{F}}\frac{\alpha\Gamma(\beta+1)}{(\mathfrak{v}-\mathfrak{u})^{\beta}}\left[\mathcal{I}^{\beta}_{\mathfrak{v}^{-}}\widetilde{G}(\mathfrak{e},\mathfrak{u})\oplus\mathcal{I}^{\beta}_{\mathfrak{v}^{-}}\widetilde{G}(\mathfrak{g},\mathfrak{u})\right].
\end{aligned}\tag{44}$$

The second, third, and fourth inequalities of (28) will be the consequence of adding the inequalities (41)–(44).

Now, for any $\gamma \in [0,\ 1]$, we have inequality (15)'s left side:

$$\frac{1}{\text{Ꭻ}^2\left(\frac{1}{2}\right)} G_\gamma\left(\frac{\mathfrak{e}+\mathfrak{g}}{2}, \frac{\mathfrak{u}+\mathfrak{v}}{2}\right) \supseteq_I \frac{\Gamma(\beta+1)}{\text{Ꭻ}\left(\frac{1}{2}\right)(\mathfrak{v}-\mathfrak{u})^\beta}\left[\mathcal{I}^\beta_{\mathfrak{u}^+} G_\gamma\left(\frac{\mathfrak{e}+\mathfrak{g}}{2}, \mathfrak{v}\right) + \mathcal{I}^\beta_{\mathfrak{v}^-} G_\gamma\left(\frac{\mathfrak{e}+\mathfrak{g}}{2}, \mathfrak{u}\right)\right] \tag{45}$$

And

$$\frac{1}{\text{Ꭻ}^2\left(\frac{1}{2}\right)} G_\gamma\left(\frac{\mathfrak{e}+\mathfrak{g}}{2}, \frac{\mathfrak{u}+\mathfrak{v}}{2}\right) \supseteq_I \frac{\Gamma(\alpha+1)}{\text{Ꭻ}\left(\frac{1}{2}\right)(\mathfrak{g}-\mathfrak{e})^\alpha}\left[\mathcal{I}^\alpha_{\mathfrak{e}^+} G_\gamma\left(\mathfrak{g}, \frac{\mathfrak{u}+\mathfrak{v}}{2}\right) + \mathcal{I}^\alpha_{\mathfrak{g}^-} G_\gamma\left(\mathfrak{e}, \frac{\mathfrak{u}+\mathfrak{v}}{2}\right)\right] \tag{46}$$

The following inequality is created by adding the two inequalities (45) and (46):

$$\begin{aligned}&\tfrac{1}{\text{Ꭻ}^2\left(\frac{1}{2}\right)} G_\gamma\left(\tfrac{\mathfrak{e}+\mathfrak{g}}{2}, \tfrac{\mathfrak{u}+\mathfrak{v}}{2}\right) \supseteq_I \tfrac{\Gamma(\alpha+1)}{\text{Ꭻ}\left(\frac{1}{2}\right)(\mathfrak{g}-\mathfrak{e})^\alpha}\left[\mathcal{I}^\alpha_{\mathfrak{e}^+} G_\gamma\left(\mathfrak{g}, \tfrac{\mathfrak{u}+\mathfrak{v}}{2}\right) + \mathcal{I}^\alpha_{\mathfrak{g}^-} G_\gamma\left(\mathfrak{e}, \tfrac{\mathfrak{u}+\mathfrak{v}}{2}\right)\right]\\ &+\tfrac{\Gamma(\beta+1)}{\text{Ꭻ}\left(\frac{1}{2}\right)(\mathfrak{v}-\mathfrak{u})^\beta}\left[\mathcal{I}^\beta_{\mathfrak{u}^+} G_\gamma\left(\tfrac{\mathfrak{e}+\mathfrak{g}}{2}, \mathfrak{v}\right) + \mathcal{I}^\beta_{\mathfrak{v}^-} G_\gamma\left(\tfrac{\mathfrak{e}+\mathfrak{g}}{2}, \mathfrak{u}\right)\right].\end{aligned}$$

Similarly, since we obtain the set of *IVM*s $G_\gamma : \Delta \to \mathbb{R}^+_I$ for $\gamma \in [0,\ 1]$, the inequality can be expressed as follows:

$$\begin{aligned}&\tfrac{1}{\text{Ꭻ}^2\left(\frac{1}{2}\right)} \widetilde{G}\left(\tfrac{\mathfrak{e}+\mathfrak{g}}{2}, \tfrac{\mathfrak{u}+\mathfrak{v}}{2}\right)\\ &\supseteq_{\mathbb{F}} \tfrac{\Gamma(\alpha+1)}{\text{Ꭻ}\left(\frac{1}{2}\right)(\mathfrak{g}-\mathfrak{e})^\alpha}\left[\mathcal{I}^\alpha_{\mathfrak{e}^+} \widetilde{G}\left(\mathfrak{g}, \tfrac{\mathfrak{u}+\mathfrak{v}}{2}\right) \oplus \mathcal{I}^\alpha_{\mathfrak{g}^-} \widetilde{G}\left(\mathfrak{e}, \tfrac{\mathfrak{u}+\mathfrak{v}}{2}\right)\right] \oplus \tfrac{\Gamma(\beta+1)}{\text{Ꭻ}\left(\frac{1}{2}\right)(\mathfrak{v}-\mathfrak{u})^\beta}\left[\mathcal{I}^\beta_{\mathfrak{u}^+} \widetilde{G}\left(\tfrac{\mathfrak{e}+\mathfrak{g}}{2}, \mathfrak{v}\right) \oplus \mathcal{I}^\beta_{\mathfrak{v}^-} \widetilde{G}\left(\tfrac{\mathfrak{e}+\mathfrak{g}}{2}, \mathfrak{u}\right)\right].\end{aligned} \tag{47}$$

The first inequality of (28) is this one.

Now, for any $\gamma \in [0,\ 1]$, we have inequality (15)'s right side:

$$\frac{\Gamma(\beta)}{(\mathfrak{v}-\mathfrak{u})^\beta}\left[\mathcal{I}^\beta_{\mathfrak{u}^+} G_\gamma(\mathfrak{e},\mathfrak{v}) + \mathcal{I}^\beta_{\mathfrak{v}^-}\ G_\gamma(\mathfrak{e},\mathfrak{u})\right] \supseteq_I [G_\gamma(\mathfrak{e},\mathfrak{u}) + G_\gamma(\mathfrak{e},\mathfrak{v})] \times \int_0^1 \kappa^{\beta-1}[\text{Ꭻ}(\kappa) + \text{Ꭻ}(1-\kappa)]d\kappa \tag{48}$$

$$\frac{\Gamma(\beta)}{(\mathfrak{v}-\mathfrak{u})^\beta}\left[\mathcal{I}^\beta_{\mathfrak{u}^+} G_\gamma(\mathfrak{g},\mathfrak{v}) + \mathcal{I}^\beta_{\mathfrak{v}^-}\ G_\gamma(\mathfrak{g},\mathfrak{u})\right] \supseteq_I [G_\gamma(\mathfrak{g},\mathfrak{u}) + G_\gamma(\mathfrak{g},\mathfrak{v})] \times \int_0^1 \kappa^{\beta-1}[\text{Ꭻ}(\kappa) + \text{Ꭻ}(1-\kappa)]d\kappa \tag{49}$$

$$\frac{\Gamma(\alpha)}{(\mathfrak{g}-\mathfrak{e})^\alpha}\left[\mathcal{I}^\alpha_{\mathfrak{e}^+} G_\gamma(\mathfrak{g},\mathfrak{u}) + \mathcal{I}^\alpha_{\mathfrak{g}^-}\ G_\gamma(\mathfrak{e},\mathfrak{u})\right] \supseteq_I [G_\gamma(\mathfrak{e},\mathfrak{u}) + G_\gamma(\mathfrak{g},\mathfrak{u})] \times \int_0^1 \tau^{\alpha-1}\text{Ꭻ}(\tau) + \text{Ꭻ}(1-\tau)d\tau \tag{50}$$

$$\frac{\Gamma(\alpha)}{(\mathfrak{g}-\mathfrak{e})^\alpha}\left[\mathcal{I}^\alpha_{\mathfrak{e}^+} G_\gamma(\mathfrak{g},\mathfrak{v}) + \mathcal{I}^\alpha_{\mathfrak{g}^-}\ G_\gamma(\mathfrak{e},\mathfrak{v})\right] \supseteq_I [G_\gamma(\mathfrak{e},\mathfrak{v}) + G_\gamma(\mathfrak{g},\mathfrak{v})] \times \int_0^1 \tau^{\alpha-1}\text{Ꭻ}(\tau) + \text{Ꭻ}(1-\tau)d\tau \tag{51}$$

Summing inequalities (48)–(51) and then taking the multiplication of the result with $\alpha\beta$, we have

$$\begin{aligned}&\tfrac{\beta\Gamma(\alpha+1)}{(\mathfrak{g}-\mathfrak{e})^\alpha}\left[\mathcal{I}^\alpha_{\mathfrak{e}^+} G_\gamma(\mathfrak{g},\mathfrak{u}) + \mathcal{I}^\alpha_{\mathfrak{g}^-} G_\gamma(\mathfrak{e},\mathfrak{u}) + \mathcal{I}^\alpha_{\mathfrak{e}^+} G_\gamma(\mathfrak{g},\mathfrak{v}) + \mathcal{I}^\alpha_{\mathfrak{g}^-} G_\gamma(\mathfrak{e},\mathfrak{v})\right]\\ &+\tfrac{\alpha\Gamma(\beta+1)}{(\mathfrak{v}-\mathfrak{u})^\beta}\left[\mathcal{I}^\beta_{\mathfrak{u}^+} G_\gamma(\mathfrak{e},\mathfrak{v}) + \mathcal{I}^\beta_{\mathfrak{v}^-} G_\gamma(\mathfrak{e},\mathfrak{u}) + \mathcal{I}^\beta_{\mathfrak{u}^+} G_\gamma(\mathfrak{g},\mathfrak{v}) + \mathcal{I}^\beta_{\mathfrak{v}^-} G_\gamma(\mathfrak{g},\mathfrak{u})\right]\\ &\supseteq_I [G_\gamma(\mathfrak{e},\mathfrak{u}) + G_\gamma(\mathfrak{e},\mathfrak{v}) + G_\gamma(\mathfrak{g},\mathfrak{u}) + G_\gamma(\mathfrak{g},\mathfrak{v})] \times \textstyle\int_0^1 \kappa^{\beta-1}[\text{Ꭻ}(\kappa) + \text{Ꭻ}(1-\kappa)]\\ &d\kappa \textstyle\int_0^1 \tau^{\alpha-1}\text{Ꭻ}(\tau) + \text{Ꭻ}(1-\tau)d\tau.\end{aligned}$$

Since we receive the collection of *IVM*s $G_\gamma : \Delta \to \mathbb{R}^+_I$ from γ-cuts, we have

$$\begin{aligned}
&\tfrac{\beta\Gamma(\alpha+1)}{(\mathfrak{g}-\mathfrak{e})^{\alpha}}\left[\mathcal{I}^{\alpha}_{\mathfrak{e}^+}\widetilde{G}(\mathfrak{g},\mathfrak{u})\oplus\mathcal{I}^{\alpha}_{\mathfrak{g}^-}\widetilde{G}(\mathfrak{e},\mathfrak{u})\oplus\mathcal{I}^{\alpha}_{\mathfrak{e}^+}\widetilde{G}(\mathfrak{g},\mathfrak{v})\oplus\mathcal{I}^{\alpha}_{\mathfrak{g}^-}\widetilde{G}(\mathfrak{e},\mathfrak{v})\right]\\
&\oplus\tfrac{\alpha\Gamma(\beta+1)}{(\mathfrak{v}-\mathfrak{u})^{\beta}}\left[\mathcal{I}^{\beta}_{\mathfrak{u}^+}\widetilde{G}(\mathfrak{e},\mathfrak{v})\oplus\mathcal{I}^{\beta}_{\mathfrak{v}^-}\widetilde{G}(\mathfrak{e},\mathfrak{u})\oplus\mathcal{I}^{\beta}_{\mathfrak{u}^+}\widetilde{G}(\mathfrak{g},\mathfrak{v})\oplus\mathcal{I}^{\beta}_{\mathfrak{v}^-}\widetilde{G}(\mathfrak{g},\mathfrak{u})\right]\\
&\supseteq_{\mathbb{F}}\left[\widetilde{G}(\mathfrak{e},\mathfrak{u})\oplus\widetilde{G}(\mathfrak{e},\mathfrak{v})\oplus\widetilde{G}(\mathfrak{g},\mathfrak{u})\oplus\widetilde{G}(\mathfrak{g},\mathfrak{v})\right]\times\int_0^1\kappa^{\beta-1}[\mathcal{J}(\kappa)+\mathcal{J}(1-\kappa)]\\
&d\kappa\int_0^1\tau^{\alpha-1}\mathcal{J}(\tau)+\mathcal{J}(1-\tau)d\tau.
\end{aligned}\tag{52}$$

This is the final inequality of (28), and a conclusion has been established. □

Example 3. *We assume the FNVMs* $\widetilde{G}:[0,\ 2]\times[0,\ 2]\to\mathbb{F}_0$ *are defined by*

$$G(x,y)(\sigma)=\begin{cases}\frac{\sigma-(2-\sqrt{x})(2-\sqrt{y})}{4-(2-\sqrt{x})(2-\sqrt{y})}, & \sigma\in\left[(2-\sqrt{x})(2-\sqrt{y}),\ 4\right]\\ \frac{(2+\sqrt{x})(2+\sqrt{y})-\sigma}{(2+\sqrt{x})(2+\sqrt{y})-4}, & \sigma\in\left(4,\ (2+\sqrt{x})(2+\sqrt{y})\right]\\ 0, & otherwise,\end{cases}\tag{53}$$

and then, for each $\gamma\in[0,\ 1]$, *we have* $G_{\gamma}(x,y)=\left[(1-\gamma)(2-\sqrt{x})(2-\sqrt{y})+4\gamma,(1-\gamma)(2+\sqrt{x})(2+\sqrt{y})+4\gamma\right]$. *Since the endpoint functions* $G_*((x,y),\gamma)$ *and* $G^*((x,y),\gamma)$ *are coordinate* $\mathcal{J}$*-convex and* $\mathcal{J}$*-concave functions for each* $\gamma\in[0,\ 1]$, $\widetilde{G}(x,y)$ *is a UD-*$\mathcal{J}$*-coordinate convex FNVM.*

$$\begin{aligned}
&G_{\gamma}\left(\tfrac{\mathfrak{e}+\mathfrak{g}}{2},\ \tfrac{\mathfrak{u}+\mathfrak{v}}{2}\right)=[(1-\gamma)+4\gamma,9(1-\gamma)+4\gamma],\\
&\tfrac{\Gamma(\alpha+1)}{4(\mathfrak{g}-\mathfrak{e})^{\alpha}}\left[\mathcal{I}^{\alpha}_{\mathfrak{e}^+}\widetilde{G}\left(\mathfrak{g},\tfrac{\mathfrak{u}+\mathfrak{v}}{2}\right)\oplus\mathcal{I}^{\alpha}_{\mathfrak{g}^-}\widetilde{G}\left(\mathfrak{e},\tfrac{\mathfrak{u}+\mathfrak{v}}{2}\right)\right]\oplus\tfrac{\Gamma(\beta+1)}{4(\mathfrak{v}-\mathfrak{u})^{\beta}}\left[\mathcal{I}^{\beta}_{\mathfrak{u}^+}\widetilde{G}\left(\tfrac{\mathfrak{e}+\mathfrak{g}}{2},\mathfrak{v}\right)\oplus\mathcal{I}^{\beta}_{\mathfrak{v}^-}\widetilde{G}\left(\tfrac{\mathfrak{e}+\mathfrak{g}}{2},\mathfrak{u}\right)\right]\\
&=\left[(1-\gamma)\left(2-\tfrac{\sqrt{2}}{4}-\tfrac{\sqrt{2}}{8}\pi\right)+4\gamma,(1-\gamma)\left(2+\tfrac{\sqrt{2}}{4}+\tfrac{\sqrt{2}}{8}\pi\right)+4\gamma\right]\\
&\tfrac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\mathfrak{g}-\mathfrak{e})^{\alpha}(\mathfrak{v}-\mathfrak{u})^{\beta}}\left[\mathcal{I}^{\alpha,\ \beta}_{\mathfrak{e}^+,\mathfrak{u}^+}G_{\gamma}(\mathfrak{g},\mathfrak{v})\oplus\mathcal{I}^{\alpha,\ \beta}_{\mathfrak{e}^+,\mathfrak{v}^-}G_{\gamma}(\mathfrak{g},\mathfrak{u})\oplus\mathcal{I}^{\alpha,\ \beta}_{\mathfrak{g}^-,\mathfrak{u}^+}G_{\gamma}(\mathfrak{e},\mathfrak{v})\oplus\mathcal{I}^{\alpha,\ \beta}_{\mathfrak{g}^-,\mathfrak{v}^-}G_{\gamma}(\mathfrak{e},\mathfrak{u})\right]\\
&=\left[(1-\gamma)\left(\tfrac{33}{8}-\sqrt{2}-\tfrac{\sqrt{2}}{2}\pi+\tfrac{\pi}{8}+\tfrac{\pi^2}{32}\right)+4\gamma,(1-\gamma)\left(\tfrac{33}{8}+\sqrt{2}+\tfrac{\sqrt{2}}{2}\pi+\tfrac{\pi}{8}+\tfrac{\pi^2}{32}\right)+4\gamma\right]\\
&\tfrac{\Gamma(\alpha+1)}{8(\mathfrak{g}-\mathfrak{e})^{\alpha}}\left[\mathcal{I}^{\alpha}_{\mathfrak{e}^+}\widetilde{G}(\mathfrak{g},\mathfrak{u})\oplus\mathcal{I}^{\alpha}_{\mathfrak{e}^+}\widetilde{G}(\mathfrak{g},\mathfrak{v})\oplus\mathcal{I}^{\alpha}_{\mathfrak{g}^-}\widetilde{G}(\mathfrak{e},\mathfrak{u})\oplus\mathcal{I}^{\alpha}_{\mathfrak{g}^-}\widetilde{G}(\mathfrak{e},\mathfrak{v})\right]\\
&\oplus\tfrac{\Gamma(\beta+1)}{8(\mathfrak{v}-\mathfrak{u})^{\beta}}\left[\mathcal{I}^{\beta}_{\mathfrak{u}^+}\widetilde{G}(\mathfrak{e},\mathfrak{v})\oplus\mathcal{I}^{\beta}_{\mathfrak{u}^+}\widetilde{G}(\mathfrak{g},\mathfrak{v})\oplus\mathcal{I}^{\beta}_{\mathfrak{v}^-}\widetilde{G}(\mathfrak{e},\mathfrak{u})\oplus\mathcal{I}^{\beta}_{\mathfrak{v}^-}\widetilde{G}(\mathfrak{g},\mathfrak{u})\right]\\
&=\left[\tfrac{34\sqrt{2}+(\sqrt{2}-4)\pi-24}{8\sqrt{2}}(1-\gamma)+4\gamma,\tfrac{34\sqrt{2}+(\sqrt{2}+4)\pi+24}{8\sqrt{2}}(1-\gamma)+4\gamma\right]\\
&\tfrac{G_{\gamma}(\mathfrak{u},\mathfrak{g})+G_{\gamma}(\sigma,\mathfrak{g})+G_{\gamma}(\mathfrak{u},\mathfrak{v})+G_{\gamma}(\sigma,\mathfrak{v})}{4}=\left[(1-\gamma)\left(\tfrac{9}{2}-2\sqrt{2}\right)+4\gamma,(1-\gamma)\left(\tfrac{9}{2}+2\sqrt{2}\right)+4\gamma\right].
\end{aligned}$$

That is,

$$\begin{aligned}
&[(1-\gamma)+4\gamma,9(1-\gamma)+4\gamma]\supseteq_I\left[(1-\gamma)\left(2-\tfrac{\sqrt{2}}{4}-\tfrac{\sqrt{2}}{8}\pi\right)+4\gamma,(1-\gamma)\left(2+\tfrac{\sqrt{2}}{4}+\tfrac{\sqrt{2}}{8}\pi\right)+4\gamma\right]\\
&\supseteq_I\left[(1-\gamma)\left(\tfrac{33}{8}-\sqrt{2}-\tfrac{\sqrt{2}}{2}\pi+\tfrac{\pi}{8}+\tfrac{\pi^2}{32}\right)+4\gamma,(1-\gamma)\left(\tfrac{33}{8}+\sqrt{2}+\tfrac{\sqrt{2}}{2}\pi+\tfrac{\pi}{8}+\tfrac{\pi^2}{32}\right)+4\gamma\right]\\
&\supseteq_I\left[\tfrac{34\sqrt{2}+(\sqrt{2}-4)\pi-24}{8\sqrt{2}}(1-\gamma)+4\gamma,\tfrac{34\sqrt{2}+(\sqrt{2}+4)\pi+24}{8\sqrt{2}}(1-\gamma)+4\gamma\right]\\
&\supseteq_I\ \tfrac{34\sqrt{2}+(\sqrt{2}-4)\pi-24}{8\sqrt{2}}(1-\gamma)+4\gamma.
\end{aligned}$$

Hence, Theorem 8 has been verified.

Remark 2. *If one assumes that $\alpha = 1$, $\beta = 1$, and $\daleth(\tau) = \tau$, $\daleth(\kappa) = \kappa$, then, from (28), as a result, there will be an inequality (see [70]):*

$$\begin{aligned}
&\widetilde{G}\left(\tfrac{\mathfrak{e}+\mathfrak{g}}{2}, \tfrac{\mathfrak{u}+\mathfrak{v}}{2}\right) \\
&\supseteq_{\mathbb{F}} \tfrac{1}{2}\left[\tfrac{1}{\mathfrak{g}-\mathfrak{e}}\int_{\mathfrak{e}}^{\mathfrak{g}} \widetilde{G}\left(x, \tfrac{\mathfrak{u}+\mathfrak{v}}{2}\right)dx \oplus \tfrac{1}{\mathfrak{v}-\mathfrak{u}}\int_{\mathfrak{u}}^{\mathfrak{v}} \widetilde{G}\left(\tfrac{\mathfrak{e}+\mathfrak{g}}{2}, y\right)dy\right] \supseteq_{\mathbb{F}} \tfrac{1}{(\mathfrak{g}-\mathfrak{e})(\mathfrak{v}-\mathfrak{u})}\int_{\mathfrak{e}}^{\mathfrak{g}}\int_{\mathfrak{u}}^{\mathfrak{v}} \widetilde{G}(x,y)dydx \\
&\supseteq_{\mathbb{F}} \tfrac{1}{4(\mathfrak{g}-\mathfrak{e})}\left[\int_{\mathfrak{e}}^{\mathfrak{g}} \widetilde{G}(x,\mathfrak{u})dx \oplus \int_{\mathfrak{e}}^{\mathfrak{g}} \widetilde{G}(x,\mathfrak{v})dx\right] \oplus \tfrac{1}{4(\mathfrak{v}-\mathfrak{u})}\left[\int_{\mathfrak{u}}^{\mathfrak{v}} \widetilde{G}(\mathfrak{e},y)dy \oplus \int_{\mathfrak{u}}^{\mathfrak{v}} \widetilde{G}(\mathfrak{g},y)dy\right] \\
&\supseteq_{\mathbb{F}} \tfrac{\widetilde{G}(\mathfrak{e},\mathfrak{u})\oplus\widetilde{G}(\mathfrak{g},\mathfrak{u})\oplus\widetilde{G}(\mathfrak{e},\mathfrak{v})\oplus\widetilde{G}(\mathfrak{g},\mathfrak{v})}{4}.
\end{aligned} \tag{54}$$

If one assumes that $\alpha = 1$, $\beta = 1$, $\daleth(\tau) = \tau$, $\daleth(\kappa) = \kappa$, and $\widetilde{G}$ is a coordinated left-UD-$\daleth$-convex, then, from (28), as a result, there will be an inequality (see [59]):

$$\begin{aligned}
&\widetilde{G}\left(\tfrac{\mathfrak{e}+\mathfrak{g}}{2}, \tfrac{\mathfrak{u}+\mathfrak{v}}{2}\right) \\
&\leq_{\mathbb{F}} \tfrac{1}{2}\left[\tfrac{1}{\mathfrak{g}-\mathfrak{e}}\int_{\mathfrak{e}}^{\mathfrak{g}} \widetilde{G}\left(x, \tfrac{\mathfrak{u}+\mathfrak{v}}{2}\right)dx \oplus \tfrac{1}{\mathfrak{v}-\mathfrak{u}}\int_{\mathfrak{u}}^{\mathfrak{v}} \widetilde{G}\left(\tfrac{\mathfrak{e}+\mathfrak{g}}{2}, y\right)dy\right] \leq_{\mathbb{F}} \tfrac{1}{(\mathfrak{g}-\mathfrak{e})(\mathfrak{v}-\mathfrak{u})}\int_{\mathfrak{e}}^{\mathfrak{g}}\int_{\mathfrak{u}}^{\mathfrak{v}} \widetilde{G}(x,y)dydx \\
&\leq_{\mathbb{F}} \tfrac{1}{4(\mathfrak{g}-\mathfrak{e})}\left[\int_{\mathfrak{e}}^{\mathfrak{g}} \widetilde{G}(x,\mathfrak{u})dx \oplus \int_{\mathfrak{e}}^{\mathfrak{g}} \widetilde{G}(x,\mathfrak{v})dx\right] \oplus \tfrac{1}{4(\mathfrak{v}-\mathfrak{u})}\left[\int_{\mathfrak{u}}^{\mathfrak{v}} \widetilde{G}(\mathfrak{e},y)dy \oplus \int_{\mathfrak{u}}^{\mathfrak{v}} \widetilde{G}(\mathfrak{g},y)dy\right] \\
&\leq_{\mathbb{F}} \tfrac{\widetilde{G}(\mathfrak{e},\mathfrak{u})\oplus\widetilde{G}(\mathfrak{g},\mathfrak{u})\oplus\widetilde{G}(\mathfrak{e},\mathfrak{v})\oplus\widetilde{G}(\mathfrak{g},\mathfrak{v})}{4}.
\end{aligned} \tag{55}$$

If $\daleth(\tau) = \tau$, $\daleth(\kappa) = \kappa$, and $G_*((x,y), \gamma) \neq G^*((x,y), \gamma)$ with $\gamma = 1$, then, from (28), we succeed in bringing about the upcoming inequality (see [55]):

$$\begin{aligned}
&G\left(\tfrac{\mathfrak{e}+\mathfrak{g}}{2}, \tfrac{\mathfrak{u}+\mathfrak{v}}{2}\right) \\
&\supseteq \tfrac{\Gamma(\alpha+1)}{4(\mathfrak{g}-\mathfrak{e})^{\alpha}}\left[\mathcal{I}_{\mathfrak{e}^+}^{\alpha} G\left(\mathfrak{g}, \tfrac{\mathfrak{u}+\mathfrak{v}}{2}\right) + \mathcal{I}_{\mathfrak{g}^-}^{\alpha} G\left(\mathfrak{e}, \tfrac{\mathfrak{u}+\mathfrak{v}}{2}\right)\right] + \tfrac{\Gamma(\beta+1)}{4(\mathfrak{v}-\mathfrak{u})^{\beta}}\left[\mathcal{I}_{\mathfrak{u}^+}^{\beta} G\left(\tfrac{\mathfrak{e}+\mathfrak{g}}{2}, \mathfrak{v}\right) + \mathcal{I}_{\mathfrak{v}^-}^{\beta} G\left(\tfrac{\mathfrak{e}+\mathfrak{g}}{2}, \mathfrak{u}\right)\right] \\
&\supseteq \tfrac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\mathfrak{g}-\mathfrak{e})^{\alpha}(\mathfrak{v}-\mathfrak{u})^{\beta}}\left[\mathcal{I}_{\mathfrak{e}^+,\mathfrak{u}^+}^{\alpha,\,\beta} G(\mathfrak{g},\mathfrak{v}) + \mathcal{I}_{\mathfrak{e}^+,\mathfrak{v}^-}^{\alpha,\,\beta} G(\mathfrak{g},\mathfrak{u}) + \mathcal{I}_{\mathfrak{g}^-,\mathfrak{u}^+}^{\alpha,\,\beta} G(\mathfrak{e},\mathfrak{v}) + \mathcal{I}_{\mathfrak{g}^-,\mathfrak{v}^-}^{\alpha,\,\beta} G(\mathfrak{e},\mathfrak{u})\right] \\
&\supseteq \tfrac{\Gamma(\alpha+1)}{8(\mathfrak{g}-\mathfrak{e})^{\alpha}}\left[\mathcal{I}_{\mathfrak{e}^+}^{\alpha} G(\mathfrak{g},\mathfrak{u}) + \mathcal{I}_{\mathfrak{e}^+}^{\alpha} G(\mathfrak{g},\mathfrak{v}) + \mathcal{I}_{\mathfrak{g}^-}^{\alpha} G(\mathfrak{e},\mathfrak{u}) + \mathcal{I}_{\mathfrak{g}^-}^{\alpha} G(\mathfrak{e},\mathfrak{v})\right] \\
&+ \tfrac{\Gamma(\beta+1)}{8(\mathfrak{v}-\mathfrak{u})^{\beta}}\left[\mathcal{I}_{\mathfrak{u}^+}^{\beta} G(\mathfrak{e},\mathfrak{v}) + \mathcal{I}_{\mathfrak{v}^-}^{\beta} G(\mathfrak{e},\mathfrak{u}) + \mathcal{I}_{\mathfrak{u}^+}^{\beta} G(\mathfrak{g},\mathfrak{v}) + \mathcal{I}_{\mathfrak{v}^-}^{\beta} G(\mathfrak{g},\mathfrak{u})\right] \\
&\supseteq \tfrac{G(\mathfrak{e},\mathfrak{u})+G(\mathfrak{g},\mathfrak{u})+G(\mathfrak{e},\mathfrak{v})+G(\mathfrak{g},\mathfrak{v})}{4}.
\end{aligned} \tag{56}$$

If $\daleth(\tau) = \tau$, $\daleth(\kappa) = \kappa$ and $G_*((x,y), \gamma) \neq G^*((x,y), \gamma)$ with $\gamma = 1$, then, from (28), we succeed in bringing about the upcoming inequality (see [68]):

$$\begin{aligned}
&G\left(\tfrac{\mathfrak{e}+\mathfrak{g}}{2}, \tfrac{\mathfrak{u}+\mathfrak{v}}{2}\right) \\
&\supseteq \tfrac{1}{2}\left[\tfrac{1}{\mathfrak{g}-\mathfrak{e}}\int_{\mathfrak{e}}^{\mathfrak{g}} G\left(x, \tfrac{\mathfrak{u}+\mathfrak{v}}{2}\right)dx + \tfrac{1}{\mathfrak{v}-\mathfrak{u}}\int_{\mathfrak{u}}^{\mathfrak{v}} G\left(\tfrac{\mathfrak{e}+\mathfrak{g}}{2}, y\right)dy\right] \subseteq \tfrac{1}{(\mathfrak{g}-\mathfrak{e})(\mathfrak{v}-\mathfrak{u})}\int_{\mathfrak{e}}^{\mathfrak{g}}\int_{\mathfrak{u}}^{\mathfrak{v}} G(x,y)dydx \\
&\supseteq \tfrac{1}{4(\mathfrak{g}-\mathfrak{e})}\left[\int_{\mathfrak{e}}^{\mathfrak{g}} G(x,\mathfrak{u})dx + \int_{\mathfrak{e}}^{\mathfrak{g}} G(x,\mathfrak{v})dx\right] + \tfrac{1}{4(\mathfrak{v}-\mathfrak{u})}\left[\int_{\mathfrak{u}}^{\mathfrak{v}} G(\mathfrak{e},y)dy + \int_{\mathfrak{u}}^{\mathfrak{v}} G(\mathfrak{g},y)dy\right] \\
&\supseteq \tfrac{G(\mathfrak{e},\mathfrak{u})+G(\mathfrak{g},\mathfrak{u})+G(\mathfrak{e},\mathfrak{v})+G(\mathfrak{g},\mathfrak{v})}{4}.
\end{aligned} \tag{57}$$

If $\widetilde{G}$ is a coordinated right-UD-$\daleth$-convex function with $\daleth(\tau) = \tau$, $\daleth(\kappa) = \kappa$ and $G_*((x,y), \gamma)$ $= G^*((x,y), \gamma)$ with $\gamma = 1$, then, from (28), we succeed in bringing about the upcoming inequality (see [71]):

$$
\begin{aligned}
&G\left(\tfrac{\mathfrak{e}+\mathfrak{g}}{2},\ \tfrac{\mathfrak{u}+\mathfrak{v}}{2}\right)\\
&\leq \tfrac{\Gamma(\alpha+1)}{4(\mathfrak{g}-\mathfrak{e})^{\alpha}}\left[\mathcal{I}^{\alpha}_{\mathfrak{e}^{+}}\ G\left(\mathfrak{g},\tfrac{\mathfrak{u}+\mathfrak{v}}{2}\right)+\mathcal{I}^{\alpha}_{\mathfrak{g}^{-}}G\left(\mathfrak{e},\tfrac{\mathfrak{u}+\mathfrak{v}}{2}\right)\right]+\tfrac{\Gamma(\beta+1)}{4(\mathfrak{v}-\mathfrak{u})^{\beta}}\left[\mathcal{I}^{\beta}_{\mathfrak{u}^{+}}\ G\left(\tfrac{\mathfrak{e}+\mathfrak{g}}{2},\mathfrak{v}\right)+\mathcal{I}^{\beta}_{\mathfrak{v}^{-}}\ G\left(\tfrac{\mathfrak{e}+\mathfrak{g}}{2},\mathfrak{u}\right)\right]\\
&\leq \tfrac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\mathfrak{g}-\mathfrak{e})^{\alpha}(\mathfrak{v}-\mathfrak{u})^{\beta}}\left[\mathcal{I}^{\alpha,\ \beta}_{\mathfrak{e}^{+},\mathfrak{u}^{+}}\ G(\mathfrak{g},\mathfrak{v})+\mathcal{I}^{\alpha,\ \beta}_{\mathfrak{e}^{+},\mathfrak{v}^{-}}\ G(\mathfrak{g},\mathfrak{u})+\mathcal{I}^{\alpha,\ \beta}_{\mathfrak{g}^{-},\mathfrak{u}^{+}}\ G(\mathfrak{e},\mathfrak{v})+\mathcal{I}^{\alpha,\ \beta}_{\mathfrak{g}^{-},\mathfrak{v}^{-}}\ G(\mathfrak{e},\mathfrak{u})\right]\\
&\leq \tfrac{\Gamma(\alpha+1)}{8(\mathfrak{g}-\mathfrak{e})^{\alpha}}\left[\mathcal{I}^{\alpha}_{\mathfrak{e}^{+}}\ G(\mathfrak{g},\mathfrak{u})G\mathcal{I}^{\alpha}_{\mathfrak{e}^{+}}\ G(\mathfrak{g},\mathfrak{v})+\mathcal{I}^{\alpha}_{\mathfrak{g}^{-}}\ G(\mathfrak{e},\mathfrak{u})+\mathcal{I}^{\alpha}_{\mathfrak{g}^{-}}\ G(\mathfrak{e},\mathfrak{v})\right].\\
&+\tfrac{\Gamma(\beta+1)}{8(\mathfrak{v}-\mathfrak{u})^{\beta}}\left[\mathcal{I}^{\ \beta}_{\mathfrak{u}^{+}}\ G(\mathfrak{e},\mathfrak{v})\ \widetilde{+}\ \mathcal{I}^{\beta}_{\mathfrak{v}^{-}}\ G(\mathfrak{e},\mathfrak{u})+\mathcal{I}^{\beta}_{\mathfrak{u}^{+}}\ G(\mathfrak{g},\mathfrak{v})+\mathcal{I}^{\beta}_{\mathfrak{v}^{-}}\ G(\mathfrak{g},\mathfrak{u})\right]\\
&\leq \tfrac{G(\mathfrak{e},\mathfrak{u})+G(\mathfrak{g},\mathfrak{u})+G(\mathfrak{e},\mathfrak{v})+G(\mathfrak{g},\mathfrak{v})}{4}.
\end{aligned}
\tag{58}
$$

In the next section, we are going to find very interesting outcomes that will be obtained over a product of two coordinate UD-Л-convex FNVMs. These inequalities are known as Pachpatte inequalities.

Theorem 9. *Let $\widetilde{G},\ \widetilde{\mathcal{J}}:\Delta\to\mathbb{F}_0^{+}$ be two coordinated UD-Л-convex FNVMs on Δ, and let $Л_1, Л_2:[0,\ 1]\to\mathbb{R}^{+}$. Then, from γ-cuts, we set up the sequence of IVMs $G_\gamma,\mathcal{J}_\gamma:\Delta\to\mathbb{R}_I^{+}$, which is given by $G_\gamma(x,y)=[G_*((x,y),\gamma),\ G^*((x,y),\gamma)]$ and $\mathcal{J}_\gamma(x,y)=[\mathcal{J}_*((x,y),\gamma),\ \mathcal{J}^*((x,y),\gamma)]$ for all $(x,y)\in\Delta$ and for all $\gamma\in[0,\ 1]$. If $\widetilde{G}\otimes\widetilde{\mathcal{J}}\in\mathcal{FD}_\Delta$, then the following inequalities hold:*

$$
\begin{aligned}
&\tfrac{\Gamma(\alpha)\Gamma(\beta)}{(\mathfrak{g}-\mathfrak{e})^{\alpha}(\mathfrak{v}-\mathfrak{u})^{\beta}}\left[\mathcal{I}^{\alpha,\ \beta}_{\mathfrak{e}^{+},\mathfrak{u}^{+}}\widetilde{G}(\mathfrak{g},\mathfrak{v})\otimes\widetilde{\mathcal{J}}(\mathfrak{g},\mathfrak{v})\oplus\mathcal{I}^{\alpha,\ \beta}_{\mathfrak{e}^{+},\mathfrak{v}^{-}}\widetilde{G}(\mathfrak{g},\mathfrak{u})\otimes\widetilde{\mathcal{J}}(\mathfrak{g},\mathfrak{u})\right]\\
&\oplus\tfrac{\Gamma(\alpha)\Gamma(\beta)}{(\mathfrak{g}-\mathfrak{e})^{\alpha}(\mathfrak{v}-\mathfrak{u})^{\beta}}\left[\mathcal{I}^{\alpha,\ \beta}_{\mathfrak{g}^{-},\mathfrak{u}^{+}}\widetilde{G}(\mathfrak{e},\mathfrak{v})\otimes\widetilde{\mathcal{J}}(\mathfrak{e},\mathfrak{v})\oplus\mathcal{I}^{\alpha,\ \beta}_{\mathfrak{g}^{-},\mathfrak{v}^{-}}\widetilde{G}(\mathfrak{e},\mathfrak{u})\otimes\widetilde{\mathcal{J}}(\mathfrak{e},\mathfrak{u})\right]\\
&\supseteq_{\mathbb{F}}\widetilde{\mathcal{M}}(\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v})\int_0^1\tau^{\alpha-1}\kappa^{\beta-1}[Л_1(1-\tau)Л_2(1-\tau)Л_1(1-\kappa)Л_2(1-\kappa)+Л_1(1-\tau)\\
&Л_2(1-\tau)Л_1(\kappa)Л_2(\kappa)+Л_1(\tau)Л_2(\tau)Л_1(1-\kappa)Л_2(1-\kappa)+\\
&Л_1(\tau)Л_2(\tau)Л_1(\kappa)Л_2(\kappa)]d\tau d\kappa\\
&\oplus\widetilde{P}(\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v})\int_0^1\tau^{\alpha-1}\kappa^{\beta-1}[Л_1(\tau)Л_2(1-\tau)Л_1(1-\kappa)Л_2(1-\kappa)+Л_1(1-\tau)\\
&Л_2(\tau)Л_1(1-\kappa)Л_2(1-\kappa)+Л_1(\tau)Л_2(1-\tau)Л_1(\kappa)Л_2(\kappa)+Л_1(1-\tau)\\
&Л_2(\tau)Л_1(\kappa)Л_2(\kappa)]d\tau d\kappa\\
&\oplus\widetilde{\mathcal{N}}(\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v})\int_0^1\tau^{\alpha-1}\kappa^{\beta-1}[Л_1(1-\tau)Л_2(1-\tau)Л_1(\kappa)Л_2(1-\kappa)+Л_1(1-\tau)Л_2(1-\tau)\\
&Л_1(1-\kappa)Л_2(\kappa)+Л_1(\tau)Л_2(\tau)Л_1(1-\kappa)Л_2(\kappa)+Л_1(\tau)Л_2(\tau)Л_1(\kappa)Л_2(1-\kappa)]d\tau d\kappa\\
&\oplus\widetilde{Q}(\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v})\int_0^1\tau^{\alpha-1\beta-1}[Л_1(\tau)Л_2(1-\tau)Л_1(\kappa)Л_2(1-\kappa)+Л_1(\tau)Л_2(1-\tau)Л_1(1-\kappa)\\
&Л_2(\kappa)+Л_1(1-\tau)Л_2(\tau)Л_1(\kappa)Л_2(1-\kappa)+Л_1(\tau)Л_2(1-\tau)Л_1(\kappa)Л_2(1-\kappa)]d\tau d\kappa.
\end{aligned}
\tag{59}
$$

If $\widetilde{G}$ and $\widetilde{\mathcal{J}}$ are both coordinated UD-Л-concave FNVMs on Δ, then the inequality above can be expressed as follows:

$$\begin{aligned}
&\frac{\Gamma(\alpha)\Gamma(\beta)}{(\mathfrak{g}-\mathfrak{e})^{\alpha}(\mathfrak{v}-\mathfrak{u})^{\beta}}\left[\mathcal{I}^{\alpha,\,\beta}_{\mathfrak{e}^+,\mathfrak{u}^+}\widetilde{G}(\mathfrak{g},\mathfrak{v})\otimes\widetilde{\mathcal{J}}(\mathfrak{g},\mathfrak{v})\oplus\mathcal{I}^{\alpha,\,\beta}_{\mathfrak{e}^+,\mathfrak{v}^-}\widetilde{G}(\mathfrak{g},\mathfrak{u})\otimes\widetilde{\mathcal{J}}(\mathfrak{g},\mathfrak{u})\right]\\
&\oplus\frac{\Gamma(\alpha)\Gamma(\beta)}{(\mathfrak{g}-\mathfrak{e})^{\alpha}(\mathfrak{v}-\mathfrak{u})^{\beta}}\left[\mathcal{I}^{\alpha,\,\beta}_{\mathfrak{g}^-,\mathfrak{u}^+}\widetilde{G}(\mathfrak{e},\mathfrak{v})\otimes\widetilde{\mathcal{J}}(\mathfrak{e},\mathfrak{v})\oplus\mathcal{I}^{\alpha,\,\beta}_{\mathfrak{g}^-,\mathfrak{v}^-}\widetilde{G}(\mathfrak{e},\mathfrak{u})\otimes\widetilde{\mathcal{J}}(\mathfrak{e},\mathfrak{u})\right]\\
&\subseteq_{\mathbb{F}}\widetilde{\mathcal{M}}(\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v})\int_0^1\tau^{\alpha-1}\kappa^{\beta-1}[Л_1(1-\tau)Л_2(1-\tau)Л_1(1-\kappa)Л_2(1-\kappa)+Л_1(1-\tau)\\
&Л_2(1-\tau)Л_1(\kappa)Л_2(\kappa)+Л_1(\tau)Л_2(\tau)Л_1(1-\kappa)Л_2(1-\kappa)+\\
&Л_1(\tau)Л_2(\tau)Л_1(\kappa)Л_2(\kappa)]d\tau d\kappa\\
&\oplus\widetilde{P}(\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v})\int_0^1\tau^{\alpha-1}\kappa^{\beta-1}[Л_1(\tau)Л_2(1-\tau)Л_1(1-\kappa)Л_2(1-\kappa)+\\
&Л_1(1-\tau)Л_2(\tau)Л_1(1-\kappa)Л_2(1-\kappa)+Л_1(\tau)Л_2(1-\tau)Л_1(\kappa)Л_2(\kappa)+Л_1(1-\tau)\\
&Л_2(\tau)Л_1(\kappa)Л_2(\kappa)]d\tau d\kappa\\
&\oplus\widetilde{\mathcal{N}}(\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v})\int_0^1\tau^{\alpha-1}\kappa^{\beta-1}[Л_1(1-\tau)Л_2(1-\tau)Л_1(\kappa)Л_2(1-\kappa)+Л_1(1-\tau)Л_2(1-\tau)\\
&Л_1(1-\kappa)Л_2(\kappa)+Л_1(\tau)Л_2(\tau)Л_1(1-\kappa)Л_2(\kappa)+Л_1(\tau)Л_2(\tau)Л_1(\kappa)Л_2(1-\kappa)]d\tau d\kappa\\
&\oplus\widetilde{Q}(\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v})\int_0^1\tau^{\alpha-1}\kappa^{\beta-1}[Л_1(\tau)Л_2(1-\tau)Л_1(\kappa)Л_2(1-\kappa)+Л_1(\tau)Л_2(1-\tau)Л_1(1-\kappa)\\
&Л_2(\kappa)+Л_1(1-\tau)Л_2(\tau)Л_1(\kappa)Л_2(1-\kappa)+Л_1(\tau)Л_2(1-\tau)Л_1(\kappa)Л_2(1-\kappa)]d\tau d\kappa
\end{aligned}\tag{60}$$

where

$$\widetilde{\mathcal{M}}(\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v})=\widetilde{G}(\mathfrak{e},\mathfrak{u})\otimes\widetilde{\mathcal{J}}(\mathfrak{e},\mathfrak{u})\oplus\widetilde{G}(\mathfrak{g},\mathfrak{u})\otimes\widetilde{\mathcal{J}}(\mathfrak{g},\mathfrak{u})\oplus\widetilde{G}(\mathfrak{e},\mathfrak{v})\otimes\widetilde{\mathcal{J}}(\mathfrak{e},\mathfrak{v})\oplus\widetilde{G}(\mathfrak{g},\mathfrak{v})\otimes\widetilde{\mathcal{J}}(\mathfrak{g},\mathfrak{v}),$$

$$\widetilde{P}(\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v})=\widetilde{G}(\mathfrak{e},\mathfrak{u})\otimes\widetilde{\mathcal{J}}(\mathfrak{g},\mathfrak{u})\oplus\widetilde{G}(\mathfrak{g},\mathfrak{u})\otimes\widetilde{\mathcal{J}}(\mathfrak{e},\mathfrak{u})\oplus\widetilde{G}(\mathfrak{e},\mathfrak{v})\otimes\widetilde{\mathcal{J}}(\mathfrak{g},\mathfrak{v})\oplus\widetilde{G}(\mathfrak{g},\mathfrak{v})\otimes\widetilde{\mathcal{J}}(\mathfrak{e},\mathfrak{v}),$$

$$\widetilde{\mathcal{N}}(\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v})=\widetilde{G}(\mathfrak{e},\mathfrak{u})\otimes\widetilde{\mathcal{J}}(\mathfrak{e},\mathfrak{v})\oplus\widetilde{G}(\mathfrak{g},\mathfrak{u})\otimes\widetilde{\mathcal{J}}(\mathfrak{g},\mathfrak{v})\oplus\widetilde{G}(\mathfrak{e},\mathfrak{v})\otimes\widetilde{\mathcal{J}}(\mathfrak{e},\mathfrak{u})\oplus\widetilde{G}(\mathfrak{g},\mathfrak{v})\otimes\widetilde{\mathcal{J}}(\mathfrak{g},\mathfrak{u}),$$

$$\widetilde{Q}(\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v})=\widetilde{G}(\mathfrak{e},\mathfrak{u})\otimes\widetilde{\mathcal{J}}(\mathfrak{g},\mathfrak{v})\oplus\widetilde{G}(\mathfrak{g},\mathfrak{u})\otimes\widetilde{\mathcal{J}}(\mathfrak{e},\mathfrak{v})\oplus\widetilde{G}(\mathfrak{e},\mathfrak{v})\otimes\widetilde{\mathcal{J}}(\mathfrak{g},\mathfrak{u})\oplus\widetilde{G}(\mathfrak{g},\mathfrak{v})\otimes\widetilde{\mathcal{J}}(\mathfrak{e},\mathfrak{u}),$$

and for each $\gamma\in[0,\ 1]$, $\widetilde{\mathcal{M}}(\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v})$, $\widetilde{P}(\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v})$, $\widetilde{\mathcal{N}}(\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v})$, *and* $\widetilde{Q}(\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v})$ *are defined as follows:*

$$\begin{aligned}
\mathcal{M}_\gamma(\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v})&=[\mathcal{M}_*((\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v}),\ \gamma),\ \mathcal{M}^*((\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v}),\ \gamma)],\\
P_\gamma(\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v})&=[P_*((\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v}),\ \gamma),\ P^*((\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v}),\ \gamma)],\\
\mathcal{N}_\gamma(\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v})&=[\mathcal{N}_*((\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v}),\ \gamma),\ \mathcal{N}^*((\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v}),\ \gamma)],\\
Q_\gamma(\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v})&=[Q_*((\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v}),\ \gamma),\ Q^*((\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v}),\ \gamma)].
\end{aligned}$$

Proof. Let $\widetilde{G}$ and $\widetilde{\mathcal{J}}$ be two coordinated *UD*-$Л_1$ and $Л_2$-convex *FNVM*s on $[\mathfrak{e},\mathfrak{g}]\times[\mathfrak{u},\mathfrak{v}]$, respectively. Then,

$$\begin{aligned}
&\widetilde{G}(\tau\mathfrak{e}+(1-\tau)\mathfrak{g},\kappa\mathfrak{u}+(1-\kappa)\mathfrak{v})\\
&\supseteq_{\mathbb{F}}Л_1(\tau)Л_1(\kappa)\widetilde{G}(\mathfrak{e},\mathfrak{u})\oplus Л_1(\tau)Л_1(1-\kappa)\widetilde{G}(\mathfrak{e},\mathfrak{v})\oplus Л_1(1-\tau)Л_1(\kappa)\widetilde{G}(\mathfrak{g},\mathfrak{u})\oplus Л_1(1-\tau)\\
&Л_1(1-\kappa)\widetilde{G}(\mathfrak{g},\mathfrak{v}),\\
&\widetilde{G}(\tau\mathfrak{e}+(1-\tau)\mathfrak{g},(1-\kappa)\mathfrak{u}+\kappa\mathfrak{v})\\
&\supseteq_{\mathbb{F}}Л_1(\tau)Л_1(1-\kappa)\widetilde{G}(\mathfrak{e},\mathfrak{u})\oplus Л_1(\tau)Л_1(\kappa)\widetilde{G}(\mathfrak{e},\mathfrak{v})\oplus Л_1(1-\tau)Л_1(1-\kappa)\widetilde{G}(\mathfrak{g},\mathfrak{u})\oplus\\
&Л_1(1-\tau)Л_1(\kappa)\widetilde{G}(\mathfrak{g},\mathfrak{v}),
\end{aligned}$$

$$\widetilde{G}((1-\tau)\mathfrak{e}+\tau\mathfrak{g},\kappa\mathfrak{u}+(1-\kappa)\mathfrak{v})$$
$$\supseteq_{\mathbb{F}} Л_1(1-\tau)Л_1(\kappa)\widetilde{G}(\mathfrak{e},\mathfrak{u}) \oplus Л_1(1-\tau)Л_1(1-\kappa)\widetilde{G}(\mathfrak{e},\mathfrak{v}) \oplus Л_1(\tau)Л_1(\kappa)\widetilde{G}(\mathfrak{g},\mathfrak{u}) \oplus Л_1(\tau)Л_1(1-\kappa)\widetilde{G}(\mathfrak{g},\mathfrak{v}),$$
$$\widetilde{G}((1-\tau)\mathfrak{e}+\tau\mathfrak{g},(1-\kappa)\mathfrak{u}+\kappa\mathfrak{v})$$
$$\supseteq_{\mathbb{F}} Л_1(1-\tau)Л_1(1-\kappa)\widetilde{G}(\mathfrak{e},\mathfrak{u}) \oplus Л_1(1-\tau)Л_1(\kappa)\widetilde{G}(\mathfrak{e},\mathfrak{v}) \oplus Л_1(\tau)Л_1(1-\kappa)\widetilde{G}(\mathfrak{g},\mathfrak{u}) \oplus Л_1(\tau)Л_1(\kappa)\widetilde{G}(\mathfrak{g},\mathfrak{v}),$$

and

$$\widetilde{\mathcal{J}}(\tau\mathfrak{e}+(1-\tau)\mathfrak{g},\kappa\mathfrak{u}+(1-\kappa)\mathfrak{v})$$
$$\supseteq_{\mathbb{F}} Л_2(\tau)Л_2(\kappa)\widetilde{\mathcal{J}}(\mathfrak{e},\mathfrak{u}) \oplus Л_2(\tau)Л_2(1-\kappa)\widetilde{\mathcal{J}}(\mathfrak{e},\mathfrak{v}) \oplus Л_2(1-\tau)Л_2(\kappa)\widetilde{\mathcal{J}}(\mathfrak{g},\mathfrak{u}) \oplus Л_2(1-\tau) Л_2(1-\kappa)\widetilde{\mathcal{J}}(\mathfrak{g},\mathfrak{v}),$$
$$\widetilde{\mathcal{J}}(\tau\mathfrak{e}+(1-\tau)\mathfrak{g},(1-\kappa)\mathfrak{u}+\kappa\mathfrak{v})$$
$$\supseteq_{\mathbb{F}} Л_2(\tau)Л_2(1-\kappa)\widetilde{\mathcal{J}}(\mathfrak{e},\mathfrak{u}) \oplus Л_2(\tau)Л_2(\kappa)\widetilde{\mathcal{J}}(\mathfrak{e},\mathfrak{v}) \oplus Л_2(1-\tau)Л_2(1-\kappa)\widetilde{\mathcal{J}}(\mathfrak{g},\mathfrak{u}) \oplus Л_2(1-\tau)Л_2(\kappa)\widetilde{\mathcal{J}}(\mathfrak{g},\mathfrak{v}),$$

$$\widetilde{\mathcal{J}}((1-\tau)\mathfrak{e}+\tau\mathfrak{g},\kappa\mathfrak{u}+(1-\kappa)\mathfrak{v})$$
$$\supseteq_{\mathbb{F}} Л_2(1-\tau)Л_2(\kappa)\widetilde{\mathcal{J}}(\mathfrak{e},\mathfrak{u}) \oplus Л_2(1-\tau)Л_2(1-\kappa)\widetilde{\mathcal{J}}(\mathfrak{e},\mathfrak{v}) \oplus Л_2(\tau)Л_2(\kappa)\widetilde{\mathcal{J}}(\mathfrak{g},\mathfrak{u}) \oplus Л_2(\tau)Л_2(1-\kappa)\widetilde{\mathcal{J}}(\mathfrak{g},\mathfrak{v}),$$
$$\widetilde{\mathcal{J}}((1-\tau)\mathfrak{e}+\tau\mathfrak{g},(1-\kappa)\mathfrak{u}+\kappa\mathfrak{v})$$
$$\supseteq_{\mathbb{F}} Л_2(1-\tau)Л_2(1-\kappa)\widetilde{\mathcal{J}}(\mathfrak{e},\mathfrak{u}) \oplus Л_2(1-\tau)Л_2(\kappa)\widetilde{\mathcal{J}}(\mathfrak{e},\mathfrak{v}) \oplus Л_2(\tau)Л_2(1-\kappa)\widetilde{\mathcal{J}}(\mathfrak{g},\mathfrak{u}) \oplus Л_2(\tau)Л_2(\kappa)\widetilde{\mathcal{J}}(\mathfrak{g},\mathfrak{v}),$$

Since $\widetilde{G}$ and $\widetilde{\mathcal{J}}$ are both coordinated *UD*-$Л_1$- and $Л_2$-convex *FNVM*s on $[\mathfrak{e},\mathfrak{g}]\times[\mathfrak{u},\mathfrak{v}]$, respectively, then, for any $\gamma\in[0,\ 1]$, we have

$$G_\gamma(\tau\mathfrak{e}+(1-\tau)\mathfrak{g},\kappa\mathfrak{u}+(1-\kappa)\mathfrak{v})\times\mathcal{J}_\gamma(\tau\mathfrak{e}+(1-\tau)\mathfrak{g},\kappa\mathfrak{u}+(1-\kappa)\mathfrak{v})$$
$$+G_\gamma(\tau\mathfrak{e}+(1-\tau)\mathfrak{g},(1-\kappa)\mathfrak{u}+\kappa\mathfrak{v})\times\mathcal{J}_\gamma(\tau\mathfrak{e}+(1-\tau)\mathfrak{g},(1-\kappa)\mathfrak{u}+\kappa\mathfrak{v})$$
$$+G_\gamma((1-\tau)\mathfrak{e}+\tau\mathfrak{g},\kappa\mathfrak{u}+(1-\kappa)\mathfrak{v})\times\mathcal{J}_\gamma((1-\tau)\mathfrak{e}+\tau\mathfrak{g},\kappa\mathfrak{u}+(1-\kappa)\mathfrak{v})$$
$$+G_\gamma((1-\tau)\mathfrak{e}+\tau\mathfrak{g},(1-\kappa)\mathfrak{u}+\kappa\mathfrak{v})\times\mathcal{J}_\gamma((1-\tau)\mathfrak{e}+\tau\mathfrak{g},(1-\kappa)\mathfrak{u}+\kappa\mathfrak{v})$$
$$\supseteq_I \mathcal{M}_\gamma(\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v})[Л_1(1-\tau)Л_2(1-\tau)Л_1(1-\kappa)Л_2(1-\kappa)+Л_1(1-\tau)Л_2(1-\tau) Л_1(\kappa)Л_2(\kappa)+Л_1(\tau)Л_2(\tau)Л_1(1-\kappa)Л_2(1-\kappa)+Л_1(\tau)Л_2(\tau)Л_1(\kappa)Л_2(\kappa)]$$
$$+P_\gamma(\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v})[Л_1(\tau)Л_2(1-\tau)Л_1(1-\kappa)Л_2(1-\kappa)+Л_1(1-\tau)Л_2(\tau)Л_1(1-\kappa)Л_2(1-\kappa) +Л_1(\tau)Л_2(1-\tau)Л_1(\kappa)Л_2(\kappa)+Л_1(1-\tau)Л_2(\tau)Л_1(\kappa)Л_2(\kappa)]$$
$$+\mathcal{N}_\gamma(\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v})[Л_1(1-\tau)Л_2(1-\tau)Л_1(\kappa)Л_2(1-\kappa)+Л_1(1-\tau)Л_2(1-\tau)Л_1(1-\kappa) Л_2(\kappa)+Л_1(\tau)Л_2(\tau)Л_1(1-\kappa)Л_2(\kappa)+Л_1(\tau)Л_2(\tau)Л_1(\kappa)Л_2(1-\kappa)]$$
$$+Q_\gamma(\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v})[Л_1(\tau)Л_2(1-\tau)Л_1(\kappa)Л_2(1-\kappa)+Л_1(\tau)Л_2(1-\tau)Л_1(1-\kappa)Л_2(\kappa)+ Л_1(1-\tau)Л_2(\tau)Л_1(\kappa)Л_2(1-\kappa)+Л_1(\tau)Л_2(1-\tau)Л_1(\kappa)Л_2(1-\kappa)].$$

Taking the multiplication of the above fuzzy inclusion with $\tau^{\alpha-1}\kappa^{\beta-1}$ and then taking the double integration of the result over $[0,\ 1]\times[0,\ 1]$ with respect to $(\tau,\ \kappa)$ gives

$$\begin{aligned}
&\int_0^1\int_0^1 \tau^{\alpha-1}\kappa^{\beta-1} G_\gamma(\tau\mathfrak{e}+(1-\tau)\mathfrak{g},\kappa\mathfrak{u}+(1-\kappa)\mathfrak{v})\times\mathcal{J}_\gamma(\tau\mathfrak{e}+(1-\tau)\mathfrak{g},\kappa\mathfrak{u}+(1-\kappa)\mathfrak{v})d\tau d\kappa\\
&+\int_0^1\int_0^1 \tau^{\alpha-1}\kappa^{\beta-1} G_\gamma(\tau\mathfrak{e}+(1-\tau)\mathfrak{g},(1-\kappa)\mathfrak{u}+\kappa\mathfrak{v})\times\mathcal{J}_\gamma(\tau\mathfrak{e}+(1-\tau)\mathfrak{g},(1-\kappa)\mathfrak{u}+\kappa\mathfrak{v})d\tau d\kappa\\
&+\int_0^1\int_0^1 \tau^{\alpha-1}\kappa^{\beta-1} G_\gamma((1-\tau)\mathfrak{e}+\tau\mathfrak{g},\kappa\mathfrak{u}+(1-\kappa)\mathfrak{v})\times\mathcal{J}_\gamma((1-\tau)\mathfrak{e}+\tau\mathfrak{g},\kappa\mathfrak{u}+(1-\kappa)\mathfrak{v})d\tau d\kappa\\
&+\int_0^1\int_0^1 \tau^{\alpha-1}\kappa^{\beta-1} G_\gamma((1-\tau)\mathfrak{e}+\tau\mathfrak{g},(1-\kappa)\mathfrak{u}+\kappa\mathfrak{v})\times\mathcal{J}_\gamma((1-\tau)\mathfrak{e}+\tau\mathfrak{g},(1-\kappa)\mathfrak{u}+\kappa\mathfrak{v})d\tau d\kappa\\
&\supseteq_I \mathcal{M}_\gamma(\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v}) \quad \int_0^1\int_0^1 \tau^{\alpha-1}\kappa^{\beta-1}[Ɉ_1(1-\tau)Ɉ_2(1-\tau)Ɉ_1(1-\kappa)Ɉ_2(1-\kappa)\\
&\qquad +Ɉ_1(1-\tau)Ɉ_2(1-\tau)Ɉ_1(\kappa)Ɉ_2(\kappa)+Ɉ_1(\tau)Ɉ_2(\tau)Ɉ_1(1-\kappa)Ɉ_2(1-\kappa)\\
&\qquad +Ɉ_1(\tau)Ɉ_2(\tau)Ɉ_1(\kappa)Ɉ_2(\kappa)]d\tau d\kappa\\
&+P_\gamma(\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v}) \quad \int_0^1\int_0^1 \tau^{\alpha-1}\kappa^{\beta-1}[Ɉ_1(\tau)Ɉ_2(1-\tau)Ɉ_1(1-\kappa)Ɉ_2(1-\kappa)\\
&\qquad +Ɉ_1(1-\tau)Ɉ_2(\tau)Ɉ_1(1-\kappa)Ɉ_2(1-\kappa)+Ɉ_1(\tau)Ɉ_2(1-\tau)Ɉ_1(\kappa)Ɉ_2(\kappa)\\
&\qquad +Ɉ_1(1-\tau)Ɉ_2(\tau)Ɉ_1(\kappa)Ɉ_2(\kappa)]d\tau d\kappa\\
&+\mathcal{N}_\gamma(\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v}) \quad \int_0^1\int_0^1 \tau^{\alpha-1}\kappa^{\beta-1}[Ɉ_1(1-\tau)Ɉ_2(1-\tau)Ɉ_1(\kappa)Ɉ_2(1-\kappa)\\
&\qquad +Ɉ_1(1-\tau)Ɉ_2(1-\tau)Ɉ_1(1-\kappa)Ɉ_2(\kappa)+Ɉ_1(\tau)Ɉ_2(\tau)Ɉ_1(1-\kappa)Ɉ_2(\kappa)\\
&\qquad +Ɉ_1(\tau)Ɉ_2(\tau)Ɉ_1(\kappa)Ɉ_2(1-\kappa)]d\tau d\kappa\\
&+Q_\gamma(\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v}) \quad \int_0^1\int_0^1 \tau^{\alpha-1}\kappa^{\beta-1}[Ɉ_1(\tau)Ɉ_2(1-\tau)Ɉ_1(\kappa)Ɉ_2(1-\kappa)\\
&\qquad +Ɉ_1(\tau)Ɉ_2(1-\tau)Ɉ_1(1-\kappa)Ɉ_2(\kappa)+Ɉ_1(1-\tau)Ɉ_2(\tau)Ɉ_1(\kappa)Ɉ_2(1-\kappa)\\
&\qquad +Ɉ_1(\tau)Ɉ_2(1-\tau)Ɉ_1(\kappa)Ɉ_2(1-\kappa)]d\tau d\kappa
\end{aligned} \tag{61}$$

From the right-hand side of (61), we have

$$\begin{aligned}
&\int_0^1\int_0^1 \tau^{\alpha-1}\kappa^{\beta-1} G_\gamma(\tau\mathfrak{e}+(1-\tau)\mathfrak{g},\kappa\mathfrak{u}+(1-\kappa)\mathfrak{v})\times\mathcal{J}_\gamma(\tau\mathfrak{e}+(1-\tau)\mathfrak{g},\kappa\mathfrak{u}+(1-\kappa)\mathfrak{v})d\tau d\kappa\\
&+\int_0^1\int_0^1 \tau^{\alpha-1}\kappa^{\beta-1} G_\gamma(\tau\mathfrak{e}+(1-\tau)\mathfrak{g},(1-\kappa)\mathfrak{u}+\kappa\mathfrak{v})\times\mathcal{J}_\gamma(\tau\mathfrak{e}+(1-\tau)\mathfrak{g},(1-\kappa)\mathfrak{u}+\kappa\mathfrak{v})d\tau d\kappa\\
&+\int_0^1\int_0^1 \tau^{\alpha-1}\kappa^{\beta-1} G_\gamma((1-\tau)\mathfrak{e}+\tau\mathfrak{g},\kappa\mathfrak{u}+(1-\kappa)\mathfrak{v})\times\mathcal{J}_\gamma((1-\tau)\mathfrak{e}+\tau\mathfrak{g},\kappa\mathfrak{u}+(1-\kappa)\mathfrak{v})d\tau d\kappa\\
&+\int_0^1\int_0^1 \tau^{\alpha-1}\kappa^{\beta-1} G_\gamma((1-\tau)\mathfrak{e}+\tau\mathfrak{g},(1-\kappa)\mathfrak{u}+\kappa\mathfrak{v})\times\mathcal{J}_\gamma((1-\tau)\mathfrak{e}+\tau\mathfrak{g},(1-\kappa)\mathfrak{u}+\kappa\mathfrak{v})d\tau d\kappa\\
&=\frac{\Gamma(\alpha)\Gamma(\beta)}{(\mathfrak{g}-\mathfrak{e})^\alpha(\mathfrak{v}-\mathfrak{u})^\beta}\left[\mathcal{I}^{\alpha,\ \beta}_{\mathfrak{e}^+,\mathfrak{u}^+}G_\gamma(\mathfrak{g},\mathfrak{v})\times\mathcal{J}_\gamma(\mathfrak{g},\mathfrak{v})+\mathcal{I}^{\alpha,\ \beta}_{\mathfrak{e}^+,\mathfrak{v}^-}G_\gamma(\mathfrak{g},\mathfrak{u})\times\mathcal{J}_\gamma(\mathfrak{g},\mathfrak{u})\right]
\end{aligned} \tag{62}$$

Combining (61) and (62), for each $\gamma\in[0,\ 1]$, we have

$$\frac{\Gamma(\alpha)\Gamma(\beta)}{(\mathfrak{g}-\mathfrak{e})^{\alpha}(\mathfrak{v}-\mathfrak{u})^{\beta}}\left[\mathcal{I}^{\alpha,\,\beta}_{\mathfrak{e}^+,\mathfrak{u}^+}G_{\gamma}(\mathfrak{g},\mathfrak{v})\times\mathcal{J}_{\gamma}(\mathfrak{g},\mathfrak{v})+\mathcal{I}^{\alpha,\,\beta}_{\mathfrak{e}^+,\mathfrak{v}^-}G_{\gamma}(\mathfrak{g},\mathfrak{u})\times\mathcal{J}_{\gamma}(\mathfrak{g},\mathfrak{u})\right]$$

$$\begin{aligned}\supseteq_I \mathcal{M}_{\gamma}(\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v}) \quad & \int_0^1\int_0^1 \tau^{\alpha-1}\kappa^{\beta-1}[Л_1(1-\tau)Л_2(1-\tau)Л_1(1-\kappa)Л_2(1-\kappa)\\ &+Л_1(1-\tau)Л_2(1-\tau)Л_1(\kappa)Л_2(\kappa)+Л_1(\tau)Л_2(\tau)Л_1(1-\kappa)Л_2(1-\kappa)\\ &+Л_1(\tau)Л_2(\tau)Л_1(\kappa)Л_2(\kappa)]d\tau d\kappa\\ +P_{\gamma}(\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v}) \quad & \int_0^1\int_0^1 \tau^{\alpha-1}\kappa^{\beta-1}[Л_1(\tau)Л_2(1-\tau)Л_1(1-\kappa)Л_2(1-\kappa)\\ &+Л_1(1-\tau)Л_2(\tau)Л_1(1-\kappa)Л_2(1-\kappa)+Л_1(\tau)Л_2(1-\tau)Л_1(\kappa)Л_2(\kappa)\\ &+Л_1(1-\tau)Л_2(\tau)Л_1(\kappa)Л_2(\kappa)]d\tau d\kappa\\ +\mathcal{N}_{\gamma}(\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v}) \quad & \int_0^1\int_0^1 \tau^{\alpha-1}\kappa^{\beta-1}[Л_1(1-\tau)Л_2(1-\tau)Л_1(\kappa)Л_2(1-\kappa)\\ &+Л_1(1-\tau)Л_2(1-\tau)Л_1(1-\kappa)Л_2(\kappa)+Л_1(\tau)Л_2(\tau)Л_1(1-\kappa)Л_2(\kappa)\\ &+Л_1(\tau)Л_2(\tau)Л_1(\kappa)Л_2(1-\kappa)]d\tau d\kappa\\ +Q_{\gamma}(\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v}) \quad & \int_0^1\int_0^1 \tau^{\alpha-1}\kappa^{\beta-1}[Л_1(\tau)Л_2(1-\tau)Л_1(\kappa)Л_2(1-\kappa)\\ &+Л_1(\tau)Л_2(1-\tau)Л_1(1-\kappa)Л_2(\kappa)+Л_1(1-\tau)Л_2(\tau)Л_1(\kappa)Л_2(1-\kappa)\\ &+Л_1(\tau)Л_2(1-\tau)Л_1(\kappa)Л_2(1-\kappa)]d\tau d\kappa.\end{aligned}$$

Moreover, we have

$$\begin{aligned}&\frac{\Gamma(\alpha)\Gamma(\beta)}{(\mathfrak{g}-\mathfrak{e})^{\alpha}(\mathfrak{v}-\mathfrak{u})^{\beta}}\left[\mathcal{I}^{\alpha,\,\beta}_{\mathfrak{e}^+,\mathfrak{u}^+}\widetilde{G}(\mathfrak{g},\mathfrak{v})\otimes\widetilde{\mathcal{J}}(\mathfrak{g},\mathfrak{v})\oplus\mathcal{I}^{\alpha,\,\beta}_{\mathfrak{e}^+,\mathfrak{v}^-}\widetilde{G}(\mathfrak{g},\mathfrak{u})\otimes\widetilde{\mathcal{J}}(\mathfrak{g},\mathfrak{u})\right]\\ &\oplus\frac{\Gamma(\alpha)\Gamma(\beta)}{(\mathfrak{g}-\mathfrak{e})^{\alpha}(\mathfrak{v}-\mathfrak{u})^{\beta}}\left[\mathcal{I}^{\alpha,\,\beta}_{\mathfrak{g}^-,\mathfrak{u}^+}\widetilde{G}(\mathfrak{e},\mathfrak{v})\otimes\widetilde{\mathcal{J}}(\mathfrak{e},\mathfrak{v})\oplus\mathcal{I}^{\alpha,\,\beta}_{\mathfrak{g}^-,\mathfrak{v}^-}\widetilde{G}(\mathfrak{e},\mathfrak{u})\otimes\widetilde{\mathcal{J}}(\mathfrak{e},\mathfrak{u})\right]\\ &\supseteq_{\mathbb{F}}\widetilde{\mathcal{M}}(\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v})\int_0^1\tau^{\alpha-1}\kappa^{\beta-1}[Л_1(1-\tau)Л_2(1-\tau)Л_1(1-\kappa)Л_2(1-\kappa)+Л_1(1-\tau)\\ &Л_2(1-\tau)Л_1(\kappa)Л_2(\kappa)+Л_1(\tau)Л_2(\tau)Л_1(1-\kappa)Л_2(1-\kappa)+\\ &Л_1(\tau)Л_2(\tau)Л_1(\kappa)Л_2(\kappa)]d\tau d\kappa\\ &\oplus\widetilde{P}(\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v})\int_0^1\tau^{\alpha-1}\kappa^{\beta-1}[Л_1(\tau)Л_2(1-\tau)Л_1(1-\kappa)Л_2(1-\kappa)+Л_1(1-\tau)\\ &Л_2(\tau)Л_1(1-\kappa)Л_2(1-\kappa)+Л_1(\tau)Л_2(1-\tau)Л_1(\kappa)Л_2(\kappa)+Л_1(1-\tau)\\ &Л_2(\tau)Л_1(\kappa)Л_2(\kappa)]d\tau d\kappa\\ &\oplus\widetilde{\mathcal{N}}(\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v})\int_0^1\tau^{\alpha-1}\kappa^{\beta-1}[Л_1(1-\tau)Л_2(1-\tau)Л_1(\kappa)Л_2(1-\kappa)+Л_1(1-\tau)Л_2(1-\tau)\\ &Л_1(1-\kappa)Л_2(\kappa)+Л_1(\tau)Л_2(\tau)Л_1(1-\kappa)Л_2(\kappa)+Л_1(\tau)Л_2(\tau)Л_1(\kappa)Л_2(1-\kappa)]d\tau d\kappa\\ &\oplus\widetilde{Q}(\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v})\int_0^1\tau^{\alpha-1}\kappa^{\beta-1}[Л_1(\tau)Л_2(1-\tau)Л_1(\kappa)Л_2(1-\kappa)+Л_1(\tau)Л_2(1-\tau)Л_1(1-\kappa)\\ &Л_2(\kappa)+Л_1(1-\tau)Л_2(\tau)Л_1(\kappa)Л_2(1-\kappa)+Л_1(\tau)Л_2(1-\tau)Л_1(\kappa)Л_2(1-\kappa)]d\tau d\kappa.\end{aligned}$$

Hence, we obtain the required result. □

Remark 3. *If one assumes that* $Л(\tau)=\tau$, $Л(\kappa)=\kappa$, $\alpha=1$, *and* $\beta=1$, *then, from (59), as a result, there will be an inequality (see [70]):*

$$\begin{aligned}&\frac{1}{(\mathfrak{g}-\mathfrak{e})(\mathfrak{v}-\mathfrak{u})}\int_{\mathfrak{e}}^{\mathfrak{g}}\int_{\mathfrak{u}}^{\mathfrak{v}}\widetilde{G}(x,y)\otimes\widetilde{\mathcal{J}}(x,y)dydx\\ &\supseteq_{\mathbb{F}}\frac{1}{9}\widetilde{\mathcal{M}}(\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v})\oplus\frac{1}{18}\left[\widetilde{P}(\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v})\oplus\widetilde{\mathcal{N}}(\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v})\right]\oplus\frac{1}{36}\widetilde{Q}(\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v}).\end{aligned} \tag{63}$$

If $\widetilde{G}$ is a coordinated left-UD-Ɉ-convex function with $Ɉ(\tau) = \tau,\ Ɉ(\kappa) = \kappa$ and one assumes that $\alpha = 1$ and $\beta = 1$, then, from (59), as a result, there will be an inequality (see [59]):

$$\begin{aligned}&\frac{1}{(\mathfrak{g}-\mathfrak{e})(\mathfrak{v}-\mathfrak{u})}\int_{\mathfrak{e}}^{\mathfrak{g}}\int_{\mathfrak{u}}^{\mathfrak{v}}\widetilde{G}(x,y)\otimes\widetilde{\mathcal{J}}(x,y)dydx\\ &\leq_{\mathbb{F}}\tfrac{1}{9}\widetilde{\mathcal{M}}(\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v})\oplus\tfrac{1}{18}\Big[\widetilde{P}(\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v})\oplus\widetilde{\mathcal{N}}(\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v})\Big]\oplus\tfrac{1}{36}\widetilde{Q}(\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v}).\end{aligned}\tag{64}$$

If $G_*((x,y),\ \gamma) \neq G^*((x,y),\ \gamma)$ with $\gamma = 1$ and $Ɉ(\tau) = \tau,\ Ɉ(\kappa) = \kappa$, then, from (59), we succeed in bringing about the upcoming inequality (see [55]):

$$\begin{aligned}&\tfrac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\mathfrak{g}-\mathfrak{e})^{\alpha}(\mathfrak{v}-\mathfrak{u})^{\beta}}\Big[\mathcal{I}^{\alpha,\,\beta}_{\mathfrak{e}^+,\mathfrak{u}^+}G(\mathfrak{g},\mathfrak{v})\times\mathcal{J}(\mathfrak{g},\mathfrak{v})+\mathcal{I}^{\alpha,\,\beta}_{\mathfrak{e}^+,\mathfrak{v}^-}G(\mathfrak{g},\mathfrak{u})\times\mathcal{J}(\mathfrak{g},\mathfrak{u})\Big]\\ &+\tfrac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\mathfrak{g}-\mathfrak{e})^{\alpha}(\mathfrak{v}-\mathfrak{u})^{\beta}}\Big[\mathcal{I}^{\alpha,\,\beta}_{\mathfrak{g}^-,\mathfrak{u}^+}G(\mathfrak{e},\mathfrak{v})\times\mathcal{J}(\mathfrak{e},\mathfrak{v})+\mathcal{I}^{\alpha,\,\beta}_{\mathfrak{g}^-,\mathfrak{v}^-}G(\mathfrak{e},\mathfrak{u})\times\mathcal{J}(\mathfrak{e},\mathfrak{u})\Big]\\ &\supseteq\Big(\tfrac{1}{2}-\tfrac{\alpha}{(\alpha+1)(\alpha+2)}\Big)\Big(\tfrac{1}{2}-\tfrac{\beta}{(\beta+1)(\beta+2)}\Big)\mathcal{M}(\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v})+\tfrac{\alpha}{(\alpha+1)(\alpha+2)}\Big(\tfrac{1}{2}-\tfrac{\beta}{(\beta+1)(\beta+2)}\Big)P(\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v})\\ &+\Big(\tfrac{1}{2}-\tfrac{\alpha}{(\alpha+1)(\alpha+2)}\Big)\tfrac{\beta}{(\beta+1)(\beta+2)}\mathcal{N}(\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v})+\tfrac{\beta}{(\beta+1)(\beta+2)}\tfrac{\alpha}{(\alpha+1)(\alpha+2)}Q(\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v}).\end{aligned}\tag{65}$$

If $Ɉ(\tau) = \tau,\ Ɉ(\kappa) = \kappa$, and $G_*((x,y),\ \gamma) \neq G^*((x,y),\ \gamma)$ with $\gamma = 1$, then, from (59), we succeed in bringing about the upcoming inequality (see [68]):

$$\begin{aligned}&\frac{1}{(\mathfrak{g}-\mathfrak{e})(\mathfrak{v}-\mathfrak{u})}\int_{\mathfrak{e}}^{\mathfrak{g}}\int_{\mathfrak{u}}^{\mathfrak{v}}G(x,y)\times\mathcal{J}(x,y)dydx\\ &\supseteq\frac{1}{9}\mathcal{M}(\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v})+\frac{1}{18}[P(\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v})+\mathcal{N}(\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v})]+\frac{1}{36}Q(\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v}).\end{aligned}\tag{66}$$

If $G_*((x,y),\ \gamma) = G^*((x,y),\ \gamma)$ and $\mathcal{J}_*((x,y),\ \gamma) = \mathcal{J}^*((x,y),\ \gamma)$ with $\gamma = 1$ and $Ɉ(\tau) = \tau,\ Ɉ(\kappa) = \kappa$, then, from (59), we succeed in bringing about the upcoming inequality (see [69]):

$$\begin{aligned}&\tfrac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\mathfrak{g}-\mathfrak{e})^{\alpha}(\mathfrak{v}-\mathfrak{u})^{\beta}}\Big[\mathcal{I}^{\alpha,\,\beta}_{\mathfrak{e}^+,\mathfrak{u}^+}G(\mathfrak{g},\mathfrak{v})\times\mathcal{J}(\mathfrak{g},\mathfrak{v})+\mathcal{I}^{\alpha,\,\beta}_{\mathfrak{e}^+,\mathfrak{v}^-}G(\mathfrak{g},\mathfrak{u})\times\mathcal{J}(\mathfrak{g},\mathfrak{u})\Big]\\ &+\tfrac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\mathfrak{g}-\mathfrak{e})^{\alpha}(\mathfrak{v}-\mathfrak{u})^{\beta}}\Big[+\mathcal{I}^{\alpha,\,\beta}_{\mathfrak{g}^-,\mathfrak{u}^+}G(\mathfrak{e},\mathfrak{v})\times\mathcal{J}(\mathfrak{e},\mathfrak{v})+\mathcal{I}^{\alpha,\,\beta}_{\mathfrak{g}^-,\mathfrak{v}^-}G(\mathfrak{e},\mathfrak{u})\times\mathcal{J}(\mathfrak{e},\mathfrak{u})\Big]\\ &\leq\Big(\tfrac{1}{2}-\tfrac{\alpha}{(\alpha+1)(\alpha+2)}\Big)\Big(\tfrac{1}{2}-\tfrac{\beta}{(\beta+1)(\beta+2)}\Big)\mathcal{M}(\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v})+\tfrac{\alpha}{(\alpha+1)(\alpha+2)}\Big(\tfrac{1}{2}-\tfrac{\beta}{(\beta+1)(\beta+2)}\Big)P(\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v})\\ &+\Big(\tfrac{1}{2}-\tfrac{\alpha}{(\alpha+1)(\alpha+2)}\Big)\tfrac{\beta}{(\beta+1)(\beta+2)}\mathcal{N}(\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v})+\tfrac{\beta}{(\beta+1)(\beta+2)}\tfrac{\alpha}{(\alpha+1)(\alpha+2)}Q(\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v}).\end{aligned}\tag{67}$$

Theorem 10. *Let $\widetilde{G},\ \widetilde{\mathcal{J}} : \Delta \to \mathbb{F}_0^+$ be a coordinated UD-Ɉ-convex FNVM on Δ, and let $Ɉ\ : [0,\ 1] \to \mathbb{R}^+$. Then, from γ-cuts, we set up the sequence of IVMs $G_\gamma, \mathcal{J}_\gamma : \Delta \to \mathbb{R}_I^+$, which is given by $G_\gamma(x,y) = [G_*((x,y),\gamma),\ G^*((x,y),\gamma)]$ and $\mathcal{J}_\gamma(x,y) = [\mathcal{J}_*((x,y),\gamma),\ \mathcal{J}^*((x,y),\gamma)]$ for all $(x,y) \in \Delta$ and for all $\gamma \in [0,\ 1]$. If $\widetilde{G} \otimes \widetilde{\mathcal{J}} \in \mathcal{FD}_\Delta$, then the following inequalities holds:*

$$
\begin{aligned}
&\frac{1}{2\alpha\beta J_1{}^2\left(\frac{1}{2}\right)J_2{}^2\left(\frac{1}{2}\right)}\widetilde{G}\left(\tfrac{\mathfrak{e}+\mathfrak{g}}{2},\ \tfrac{\mathfrak{u}+\mathfrak{v}}{2}\right)\otimes\widetilde{\mathcal{J}}\left(\tfrac{\mathfrak{e}+\mathfrak{g}}{2},\ \tfrac{\mathfrak{u}+\mathfrak{v}}{2}\right)\\
&\supseteq_{\mathbb{F}}\ \frac{\Gamma(\alpha)\Gamma(\beta)}{2(\mathfrak{g}-\mathfrak{e})^{\alpha}(\mathfrak{v}-\mathfrak{u})^{\beta}}\left[\mathcal{I}^{\alpha,\ \beta}_{\mathfrak{e}^+,\mathfrak{u}^+}\widetilde{G}(\mathfrak{g},\mathfrak{v})\otimes\widetilde{\mathcal{J}}(\mathfrak{g},\mathfrak{v})\oplus\mathcal{I}^{\alpha,\ \beta}_{\mathfrak{e}^+,\mathfrak{v}^-}\widetilde{G}(\mathfrak{g},\mathfrak{u})\otimes\widetilde{\mathcal{J}}(\mathfrak{g},\mathfrak{u})\right]\\
&\oplus\frac{\Gamma(\alpha)\Gamma(\beta)}{2(\mathfrak{g}-\mathfrak{e})^{\alpha}(\mathfrak{v}-\mathfrak{u})^{\beta}}\left[\mathcal{I}^{\alpha,\ \beta}_{\mathfrak{g}^-,\mathfrak{u}^+}\widetilde{G}(\mathfrak{e},\mathfrak{v})\otimes\widetilde{\mathcal{J}}(\mathfrak{e},\mathfrak{v})\oplus\mathcal{I}^{\alpha,\ \beta}_{\mathfrak{g}^-,\mathfrak{v}^-}\widetilde{G}(\mathfrak{e},\mathfrak{u})\otimes\widetilde{\mathcal{J}}(\mathfrak{e},\mathfrak{u})\right]\\
&\oplus\widetilde{\mathcal{M}}(\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v})\int_0^1\tau^{\alpha-1}\kappa^{\beta-1}[J_1(\tau)J_1(\kappa)[J_2(\tau)J_2(1-\kappa)+J_2(1-\tau)J_2(\kappa)+J_2(1-\tau)\\
&J_2(1-\kappa)]+J_1(\tau)J_1(1-\kappa)[J_2(\tau)J_2(\kappa)+J_2(1-\tau)J_2(1-\kappa)+J_2(1-\tau)J_2(\kappa)]]d\tau d\kappa\\
&\oplus\widetilde{P}(\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v})\int_0^1\tau^{\alpha-1}\kappa^{\beta-1}[J_1(\tau)J_1(\kappa)[J_2(1-\tau)J_2(1-\kappa)+J_2(\tau)J_2(\kappa)+\\
&J_2(\tau)J_2(1-\kappa)]+J_1(\tau)J_1(1-\kappa)[J_2(1-\tau)J_2(\kappa)+J_2(\tau)J_2(1-\kappa)+\\
&J_2(\tau)J_2(\kappa)]d\tau d\kappa\\
&\oplus\widetilde{\mathcal{N}}(\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v})\int_0^1\tau^{\alpha-1}\kappa^{\beta-1}[J_1(\tau)J_1(\kappa)[J_2(\tau)J_2(\kappa)+J_2(1-\tau)J_2(1-\kappa)+J_2(1-\tau)\\
&J_2(\kappa)]+J_1(\tau)J_1(1-\kappa)[J_2(\tau)J_2(1-\kappa)+J_2(1-\tau)J_2(\kappa)+J_2(1-\tau)J_2(1-\kappa)]]d\tau d\kappa\\
&\oplus\widetilde{Q}(\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v})\int_0^1\tau^{\alpha-1}\kappa^{\beta-1}[J_1(\tau)J_1(\kappa)[J_2(1-\tau)J_2(\kappa)+J_2(\tau)J_2(1-\kappa)+J_2(\tau)J_2(\kappa)]\\
&+J_1(\tau)J_1(1-\kappa)[J_2(1-\tau)J_2(1-\kappa)+J_2(\tau)J_2(\kappa)+J_2(\tau)J_2(1-\kappa)]]d\tau d\kappa.
\end{aligned}
\tag{68}
$$

If $\widetilde{G}$ and $\widetilde{\mathcal{J}}$ are both coordinate UD-J-concave FNVMs on Δ, then the inequality above can be expressed as follows:

$$
\begin{aligned}
&\frac{1}{2\alpha\beta J_1{}^2\left(\frac{1}{2}\right)J_2{}^2\left(\frac{1}{2}\right)}\widetilde{G}\left(\tfrac{\mathfrak{e}+\mathfrak{g}}{2},\ \tfrac{\mathfrak{u}+\mathfrak{v}}{2}\right)\otimes\widetilde{\mathcal{J}}\left(\tfrac{\mathfrak{e}+\mathfrak{g}}{2},\ \tfrac{\mathfrak{u}+\mathfrak{v}}{2}\right)\\
&\subseteq_{\mathbb{F}}\ \frac{\Gamma(\alpha)\Gamma(\beta)}{2(\mathfrak{g}-\mathfrak{e})^{\alpha}(\mathfrak{v}-\mathfrak{u})^{\beta}}\left[\mathcal{I}^{\alpha,\ \beta}_{\mathfrak{e}^+,\mathfrak{u}^+}\widetilde{G}(\mathfrak{g},\mathfrak{v})\otimes\widetilde{\mathcal{J}}(\mathfrak{g},\mathfrak{v})\oplus\mathcal{I}^{\alpha,\ \beta}_{\mathfrak{e}^+,\mathfrak{v}^-}\widetilde{G}(\mathfrak{g},\mathfrak{u})\otimes\widetilde{\mathcal{J}}(\mathfrak{g},\mathfrak{u})\right]\\
&\oplus\frac{\Gamma(\alpha)\Gamma(\beta)}{2(\mathfrak{g}-\mathfrak{e})^{\alpha}(\mathfrak{v}-\mathfrak{u})^{\beta}}\left[\mathcal{I}^{\alpha,\ \beta}_{\mathfrak{g}^-,\mathfrak{u}^+}\widetilde{G}(\mathfrak{e},\mathfrak{v})\otimes\widetilde{\mathcal{J}}(\mathfrak{e},\mathfrak{v})\oplus\mathcal{I}^{\alpha,\ \beta}_{\mathfrak{g}^-,\mathfrak{v}^-}\widetilde{G}(\mathfrak{e},\mathfrak{u})\otimes\widetilde{\mathcal{J}}(\mathfrak{e},\mathfrak{u})\right]\\
&\oplus\widetilde{\mathcal{M}}(\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v})\int_0^1\tau^{\alpha-1}\kappa^{\beta-1}[J_1(\tau)J_1(\kappa)[J_2(\tau)J_2(1-\kappa)+J_2(1-\tau)J_2(\kappa)+J_2(1-\tau)\\
&J_2(1-\kappa)]+J_1(\tau)J_1(1-\kappa)[J_2(\tau)J_2(\kappa)+J_2(1-\tau)J_2(1-\kappa)+J_2(1-\tau)J_2(\kappa)]]d\tau d\kappa\\
&\oplus\widetilde{P}(\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v})\int_0^1\tau^{\alpha-1}\kappa^{\beta-1}[J_1(\tau)J_1(\kappa)[J_2(1-\tau)J_2(1-\kappa)+J_2(\tau)J_2(\kappa)+\\
&J_2(\tau)J_2(1-\kappa)]+J_1(\tau)J_1(1-\kappa)[J_2(1-\tau)J_2(\kappa)+J_2(\tau)J_2(1-\kappa)+\\
&J_2(\tau)J_2(\kappa)]]d\tau d\kappa\\
&\oplus\widetilde{\mathcal{N}}(\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v})\int_0^1\tau^{\alpha-1}\kappa^{\beta-1}[J_1(\tau)J_1(\kappa)[J_2(\tau)J_2(\kappa)+J_2(1-\tau)J_2(1-\kappa)+J_2(1-\tau)J_2(\kappa)]\\
&+J_1(\tau)J_1(1-\kappa)[J_2(\tau)J_2(1-\kappa)+J_2(1-\tau)J_2(\kappa)+J_2(1-\tau)J_2(1-\kappa)]]d\tau d\kappa\\
&\oplus\widetilde{Q}(\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v})\int_0^1\tau^{\alpha-1}\kappa^{\beta-1}[J_1(\tau)J_1(\kappa)[J_2(1-\tau)J_2(\kappa)+J_2(\tau)J_2(1-\kappa)+\\
&J_2(\tau)J_2(\kappa)]+J_1(\tau)J_1(1-\kappa)[J_2(1-\tau)J_2(1-\kappa)+J_2(\tau)J_2(\kappa)+J_2(\tau)J_2(1-\kappa)]]d\tau d\kappa.
\end{aligned}
\tag{69}
$$

where $\widetilde{\mathcal{M}}(\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v})$, $\widetilde{P}(\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v})$, $\widetilde{\mathcal{N}}(\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v})$, and $\widetilde{Q}(\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v})$ are given in Theorem 9.

Proof. Since $\widetilde{G},\widetilde{\mathcal{J}}\ :\Delta\to\mathbb{F}_0$ are two *UD*-J-convex *FNVM*s, then, from inequality (17) and for each $\gamma\in[0,\ 1]$, we have

$$
\begin{aligned}
& G_\gamma\left(\tfrac{\mathfrak{e}+\mathfrak{g}}{2}, \tfrac{\mathfrak{u}+\mathfrak{v}}{2}\right) \times \mathcal{J}_\gamma\left(\tfrac{\mathfrak{e}+\mathfrak{g}}{2}, \tfrac{\mathfrak{u}+\mathfrak{v}}{2}\right) \\
&= G_\gamma\left(\tfrac{\tau\mathfrak{e}+(1-\tau)\mathfrak{g}}{2} + \tfrac{(1-\tau)\mathfrak{e}+\tau\mathfrak{g}}{2}, \tfrac{\kappa\mathfrak{u}+(1-\kappa)\mathfrak{v}}{2} + \tfrac{\mathfrak{u}+\mathfrak{v}}{2}\right) \times \mathcal{J}_\gamma\left(\tfrac{\tau\mathfrak{e}+(1-\tau)\mathfrak{g}}{2} + \tfrac{(1-\tau)\mathfrak{e}+\tau\mathfrak{g}}{2}, \tfrac{\kappa\mathfrak{u}+(1-\kappa)\mathfrak{v}}{2} + \tfrac{(1-\kappa)\mathfrak{u}+\kappa\mathfrak{v}}{2}\right) \\
&\supseteq_I \text{Л}_1{}^2\left(\tfrac{1}{2}\right)\text{Л}_2{}^2\left(\tfrac{1}{2}\right) \times \left[\begin{array}{l} G_\gamma(\tau\mathfrak{e} + (1-\tau)\mathfrak{g}, \kappa\mathfrak{u} + (1-\kappa)\mathfrak{v}) + G_\gamma((1-\tau)\mathfrak{e} + \tau\mathfrak{g}, \kappa\mathfrak{u} + (1-\kappa)\mathfrak{v}) \\ +G_\gamma(\tau\mathfrak{e} + (1-\tau)\mathfrak{g}, (1-\kappa)\mathfrak{u} + \kappa\mathfrak{v}) + G_\gamma((1-\tau)\mathfrak{e} + \tau\mathfrak{g}, (1-\kappa)\mathfrak{u} + \kappa\mathfrak{v}) \end{array}\right] \\
&\quad \times \left[\begin{array}{l} \mathcal{J}_\gamma(\tau\mathfrak{e} + (1-\tau)\mathfrak{g}, \kappa\mathfrak{u} + (1-\kappa)\mathfrak{v}) + \mathcal{J}_\gamma((1-\tau)\mathfrak{e} + \tau\mathfrak{g}, \kappa\mathfrak{u} + (1-\kappa)\mathfrak{v}) \\ +\mathcal{J}_\gamma(\tau\mathfrak{e} + (1-\tau)\mathfrak{g}, (1-\kappa)\mathfrak{u} + \kappa\mathfrak{v}) + \mathcal{J}_\gamma((1-\tau)\mathfrak{e} + \tau\mathfrak{g}, (1-\kappa)\mathfrak{u} + \kappa\mathfrak{v}) \end{array}\right] \\
&\supseteq_I \text{Л}_1{}^2\left(\tfrac{1}{2}\right)\text{Л}_2{}^2\left(\tfrac{1}{2}\right) \times \left[\begin{array}{l} G_\gamma(\tau\mathfrak{e} + (1-\tau)\mathfrak{g}, \kappa\mathfrak{u} + (1-\kappa)\mathfrak{v}) \times \mathcal{J}_\gamma(\tau\mathfrak{e} + (1-\tau)\mathfrak{g}, \kappa\mathfrak{u} + (1-\kappa)\mathfrak{v}) \\ +G_\gamma((1-\tau)\mathfrak{e} + \tau\mathfrak{g}, \kappa\mathfrak{u} + (1-\kappa)\mathfrak{v}) \times \mathcal{J}_\gamma((1-\tau)\mathfrak{e} + \tau\mathfrak{g}, \kappa\mathfrak{u} + (1-\kappa)\mathfrak{v}) \\ +G_\gamma(\tau\mathfrak{e} + (1-\tau)\mathfrak{g}, (1-\kappa)\mathfrak{u} + \kappa\mathfrak{v}) \times \mathcal{J}_\gamma(\tau\mathfrak{e} + (1-\tau)\mathfrak{g}, (1-\kappa)\mathfrak{u} + \kappa\mathfrak{v}) \\ +G_\gamma((1-\tau)\mathfrak{e} + \tau\mathfrak{g}, (1-\kappa)\mathfrak{u} + \kappa\mathfrak{v}) \times \mathcal{J}_\gamma((1-\tau)\mathfrak{e} + \tau\mathfrak{g}, (1-\kappa)\mathfrak{u} + \kappa\mathfrak{v}) \end{array}\right] \\
&\quad +\text{Л}_1{}^2\left(\tfrac{1}{2}\right)\text{Л}_2{}^2\left(\tfrac{1}{2}\right) \times \\
&\left[\begin{array}{c} \text{Л}_1(\tau)\text{Л}_1(\kappa)[\text{Л}_2(\tau)\text{Л}_2(1-\kappa) + \text{Л}_2(1-\tau)\text{Л}_2(\kappa) + \text{Л}_2(1-\tau)\text{Л}_2(1-\kappa)] \\ +\text{Л}_1(\tau)\text{Л}_1(1-\kappa)[\text{Л}_2(\tau)\text{Л}_2(\kappa) + \text{Л}_2(1-\tau)\text{Л}_2(1-\kappa) + \text{Л}_2(1-\tau)\text{Л}_2(\kappa)] \\ +\text{Л}_1(1-\tau)\text{Л}_1(\kappa)[\text{Л}_2(1-\tau)\text{Л}_2(1-\kappa) + \text{Л}_2(\tau)\text{Л}_2(\kappa) + \text{Л}_2(\tau)\text{Л}_2(1-\kappa)] \\ +\text{Л}_1(1-\tau)\text{Л}_1(1-\kappa)[\text{Л}_2(\tau)\text{Л}_2(\kappa) + \text{Л}_2(1-\tau)\text{Л}_2(\kappa) + \text{Л}_2(\tau)\text{Л}_2(1-\kappa)] \end{array}\right] \mathcal{M}_\gamma(\mathfrak{e}, \mathfrak{g}, \mathfrak{u}, \mathfrak{v}) \\
&\quad +\text{Л}_1{}^2\left(\tfrac{1}{2}\right)\text{Л}_2{}^2\left(\tfrac{1}{2}\right) \times \\
&\left[\begin{array}{c} \text{Л}_1(\tau)\text{Л}_1(\kappa)[\text{Л}_2(1-\tau)\text{Л}_2(1-\kappa) + \text{Л}_2(\tau)\text{Л}_2(\kappa) + \text{Л}_2(\tau)\text{Л}_2(1-\kappa)] \\ +\text{Л}_1(\tau)\text{Л}_1(1-\kappa)[\text{Л}_2(1-\tau)\text{Л}_2(\kappa) + \text{Л}_2(\tau)\text{Л}_2(1-\kappa) + \text{Л}_2(\tau)\text{Л}_2(\kappa)] \\ +\text{Л}_1(1-\tau)\text{Л}_1(\kappa)[\text{Л}_2(\tau)\text{Л}_2(1-\kappa) + \text{Л}_2(1-\tau)\text{Л}_2(\kappa) + \text{Л}_2(1-\tau)\text{Л}_2(1-\kappa)] \\ +\text{Л}_1(1-\tau)\text{Л}_1(1-\kappa)[\text{Л}_2(\tau)\text{Л}_2(\kappa) + \text{Л}_2(1-\tau)\text{Л}_2(1-\kappa) + \text{Л}_2(1-\tau)\text{Л}_2(\kappa)] \end{array}\right] P_\gamma(\mathfrak{e}, \mathfrak{g}, \mathfrak{u}, \mathfrak{v}) \\
&\quad +\text{Л}_1{}^2\left(\tfrac{1}{2}\right)\text{Л}_2{}^2\left(\tfrac{1}{2}\right) \times \\
&\left[\begin{array}{c} \text{Л}_1(\tau)\text{Л}_1(\kappa)[\text{Л}_2(\tau)\text{Л}_2(\kappa) + \text{Л}_2(1-\tau)\text{Л}_2(1-\kappa) + \text{Л}_2(1-\tau)\text{Л}_2(\kappa)] \\ +\text{Л}_1(\tau)\text{Л}_1(1-\kappa)[\text{Л}_2(\tau)\text{Л}_2(1-\kappa) + \text{Л}_2(1-\tau)\text{Л}_2(\kappa) + \text{Л}_2(1-\tau)\text{Л}_2(1-\kappa)] \\ +\text{Л}_1(1-\tau)\text{Л}_1(\kappa)[\text{Л}_2(1-\tau)\text{Л}_2(\kappa) + \text{Л}_2(\tau)\text{Л}_2(1-\kappa) + \text{Л}_2(\tau)\text{Л}_2(\kappa)] \\ +\text{Л}_1(1-\tau)\text{Л}_1(1-\kappa)[\text{Л}_2(1-\tau)\text{Л}_2(1-\kappa) + \text{Л}_2(\tau)\text{Л}_2(\kappa) + \text{Л}_2(\tau)\text{Л}_2(1-\kappa)] \end{array}\right] \mathcal{N}_\gamma(\mathfrak{e}, \mathfrak{g}, \mathfrak{u}, \mathfrak{v}) \\
&\quad +\text{Л}_1{}^2\left(\tfrac{1}{2}\right)\text{Л}_2{}^2\left(\tfrac{1}{2}\right) \times \\
&\left[\begin{array}{c} \text{Л}_1(\tau)\text{Л}_1(\kappa)[\text{Л}_2(1-\tau)\text{Л}_2(\kappa) + \text{Л}_2(\tau)\text{Л}_2(1-\kappa) + \text{Л}_2(\tau)\text{Л}_2(\kappa)] \\ +\text{Л}_1(\tau)\text{Л}_1(1-\kappa)[\text{Л}_2(1-\tau)\text{Л}_2(1-\kappa) + \text{Л}_2(\tau)\text{Л}_2(\kappa) + \text{Л}_2(\tau)\text{Л}_2(1-\kappa)] \\ +\text{Л}_1(1-\tau)\text{Л}_1(\kappa)[\text{Л}_2(\tau)\text{Л}_2(\kappa) + \text{Л}_2(1-\tau)\text{Л}_2(1-\kappa) + \text{Л}_2(1-\tau)\text{Л}_2(\kappa)] \\ +\text{Л}_1(\tau)\text{Л}_1(1-\kappa)[\text{Л}_2(\tau)\text{Л}_2(1-\kappa) + \text{Л}_2(1-\tau)\text{Л}_2(1-\kappa) + \text{Л}_2(1-\tau)\text{Л}_2(\kappa)] \end{array}\right] Q_\gamma(\mathfrak{e}, \mathfrak{g}, \mathfrak{u}, \mathfrak{v}).
\end{aligned}
$$

Taking the multiplication of the above fuzzy inclusion with $\tau^{\alpha-1}\kappa^{\beta-1}$ and then taking the double integration of the result over $[0,\ 1] \times [0,\ 1]$ with respect to $(\tau,\ \kappa)$, we have

$$\int_0^1\int_0^1 \tau^{\alpha-1}\kappa^{\beta-1} G_\gamma\left(\tfrac{\mathfrak{e}+\mathfrak{g}}{2},\tfrac{\mathfrak{u}+\mathfrak{v}}{2}\right)\times\mathcal{J}_\gamma\left(\tfrac{\mathfrak{e}+\mathfrak{g}}{2},\tfrac{\mathfrak{u}+\mathfrak{v}}{2}\right)d\tau d\kappa$$

$$\supseteq_I \text{ЛJ}_1{}^2\left(\tfrac{1}{2}\right)\text{ЛJ}_2{}^2\left(\tfrac{1}{2}\right)$$

$$\times\int_0^1\int_0^1 \tau^{\alpha-1}\kappa^{\beta-1}\begin{bmatrix} G_\gamma(\tau\mathfrak{e}+(1-\tau)\mathfrak{g},\kappa\mathfrak{u}+(1-\kappa)\mathfrak{v})\times\mathcal{J}_\gamma(\tau\mathfrak{e}+(1-\tau)\mathfrak{g},\kappa\mathfrak{u}+(1-\kappa)\mathfrak{v}) \\ +G_\gamma((1-\tau)\mathfrak{e}+\tau\mathfrak{g},\kappa\mathfrak{u}+(1-\kappa)\mathfrak{v})\times\mathcal{J}_\gamma((1-\tau)\mathfrak{e}+\tau\mathfrak{g},\kappa\mathfrak{u}+(1-\kappa)\mathfrak{v}) \\ +G_\gamma(\tau\mathfrak{e}+(1-\tau)\mathfrak{g},(1-\kappa)\mathfrak{u}+\kappa\mathfrak{v})\times\mathcal{J}_\gamma(\tau\mathfrak{e}+(1-\tau)\mathfrak{g},(1-\kappa)\mathfrak{u}+\kappa\mathfrak{v}) \\ +G_\gamma((1-\tau)\mathfrak{e}+\tau\mathfrak{g},(1-\kappa)\mathfrak{u}+\kappa\mathfrak{v})\times\mathcal{J}_\gamma((1-\tau)\mathfrak{e}+\tau\mathfrak{g},(1-\kappa)\mathfrak{u}+\kappa\mathfrak{v}) \end{bmatrix}d\tau d\kappa$$

$$+\text{ЛJ}_1{}^2\left(\tfrac{1}{2}\right)\text{ЛJ}_2{}^2\left(\tfrac{1}{2}\right)\mathcal{M}_\gamma(\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v})$$

$$\times\int_0^1\int_0^1 \tau^{\alpha-1}\kappa^{\beta-1}\begin{bmatrix} \text{ЛJ}_1(\tau)\text{ЛJ}_1(\kappa)[\text{ЛJ}_2(\tau)\text{ЛJ}_2(1-\kappa)+\text{ЛJ}_2(1-\tau)\text{ЛJ}_2(\kappa)+\text{ЛJ}_2(1-\tau)\text{ЛJ}_2(1-\kappa)] \\ +\text{ЛJ}_1(\tau)\text{ЛJ}_1(1-\kappa)[\text{ЛJ}_2(\tau)\text{ЛJ}_2(\kappa)+\text{ЛJ}_2(1-\tau)\text{ЛJ}_2(1-\kappa)+\text{ЛJ}_2(1-\tau)\text{ЛJ}_2(\kappa)] \\ +\text{ЛJ}_1(1-\tau)\text{ЛJ}_1(\kappa)[\text{ЛJ}_2(1-\tau)\text{ЛJ}_2(1-\kappa)+\text{ЛJ}_2(\tau)\text{ЛJ}_2(\kappa)+\text{ЛJ}_2(\tau)\text{ЛJ}_2(1-\kappa)] \\ +\text{ЛJ}_1(1-\tau)\text{ЛJ}_1(1-\kappa)[\text{ЛJ}_2(\tau)\text{ЛJ}_2(\kappa)+\text{ЛJ}_2(1-\tau)\text{ЛJ}_2(\kappa)+\text{ЛJ}_2(\tau)\text{ЛJ}_2(1-\kappa)] \end{bmatrix}d\tau d\kappa$$

$$+\text{ЛJ}_1{}^2\left(\tfrac{1}{2}\right)\text{ЛJ}_2{}^2\left(\tfrac{1}{2}\right)P_\gamma(\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v})$$

$$\times\int_0^1\int_0^1 \tau^{\alpha-1}\kappa^{\beta-1}\begin{bmatrix} \text{ЛJ}_1(\tau)\text{ЛJ}_1(\kappa)[\text{ЛJ}_2(1-\tau)\text{ЛJ}_2(1-\kappa)+\text{ЛJ}_2(\tau)\text{ЛJ}_2(\kappa)+\text{ЛJ}_2(\tau)\text{ЛJ}_2(1-\kappa)] \\ +\text{ЛJ}_1(\tau)\text{ЛJ}_1(1-\kappa)[\text{ЛJ}_2(1-\tau)\text{ЛJ}_2(\kappa)+\text{ЛJ}_2(\tau)\text{ЛJ}_2(1-\kappa)+\text{ЛJ}_2(\tau)\text{ЛJ}_2(\kappa)] \\ +\text{ЛJ}_1(1-\tau)\text{ЛJ}_1(\kappa)[\text{ЛJ}_2(\tau)\text{ЛJ}_2(1-\kappa)+\text{ЛJ}_2(1-\tau)\text{ЛJ}_2(\kappa)+\text{ЛJ}_2(1-\tau)\text{ЛJ}_2(1-\kappa)] \\ +\text{ЛJ}_1(1-\tau)\text{ЛJ}_1(1-\kappa)[\text{ЛJ}_2(\tau)\text{ЛJ}_2(\kappa)+\text{ЛJ}_2(1-\tau)\text{ЛJ}_2(1-\kappa)+\text{ЛJ}_2(1-\tau)\text{ЛJ}_2(\kappa)] \end{bmatrix}d\tau d\kappa$$

$$+\text{ЛJ}_1{}^2\left(\tfrac{1}{2}\right)\text{ЛJ}_2{}^2\left(\tfrac{1}{2}\right)\mathcal{N}_\gamma(\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v})$$

$$\times\int_0^1\int_0^1 \tau^{\alpha-1}\kappa^{\beta-1}\begin{bmatrix} \text{ЛJ}_1(\tau)\text{ЛJ}_1(\kappa)[\text{ЛJ}_2(\tau)\text{ЛJ}_2(\kappa)+\text{ЛJ}_2(1-\tau)\text{ЛJ}_2(1-\kappa)+\text{ЛJ}_2(1-\tau)\text{ЛJ}_2(\kappa)] \\ +\text{ЛJ}_1(\tau)\text{ЛJ}_1(1-\kappa)[\text{ЛJ}_2(\tau)\text{ЛJ}_2(1-\kappa)+\text{ЛJ}_2(1-\tau)\text{ЛJ}_2(\kappa)+\text{ЛJ}_2(1-\tau)\text{ЛJ}_2(1-\kappa)] \\ +\text{ЛJ}_1(1-\tau)\text{ЛJ}_1(\kappa)[\text{ЛJ}_2(1-\tau)\text{ЛJ}_2(\kappa)+\text{ЛJ}_2(\tau)\text{ЛJ}_2(1-\kappa)+\text{ЛJ}_2(\tau)\text{ЛJ}_2(\kappa)] \\ +\text{ЛJ}_1(1-\tau)\text{ЛJ}_1(1-\kappa)[\text{ЛJ}_2(1-\tau)\text{ЛJ}_2(1-\kappa)+\text{ЛJ}_2(\tau)\text{ЛJ}_2(\kappa)+\text{ЛJ}_2(\tau)\text{ЛJ}_2(1-\kappa)] \end{bmatrix}d\tau d\kappa$$

$$+\text{ЛJ}_1{}^2\left(\tfrac{1}{2}\right)\text{ЛJ}_2{}^2\left(\tfrac{1}{2}\right)Q_\gamma(\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v})$$

$$\times\int_0^1\int_0^1 \tau^{\alpha-1}\kappa^{\beta-1}\begin{bmatrix} \text{ЛJ}_1(\tau)\text{ЛJ}_1(\kappa)[\text{ЛJ}_2(1-\tau)\text{ЛJ}_2(\kappa)+\text{ЛJ}_2(\tau)\text{ЛJ}_2(1-\kappa)+\text{ЛJ}_2(\tau)\text{ЛJ}_2(\kappa)] \\ +\text{ЛJ}_1(\tau)\text{ЛJ}_1(1-\kappa)[\text{ЛJ}_2(1-\tau)\text{ЛJ}_2(1-\kappa)+\text{ЛJ}_2(\tau)\text{ЛJ}_2(\kappa)+\text{ЛJ}_2(\tau)\text{ЛJ}_2(1-\kappa)] \\ +\text{ЛJ}_1(1-\tau)\text{ЛJ}_1(\kappa)[\text{ЛJ}_2(\tau)\text{ЛJ}_2(\kappa)+\text{ЛJ}_2(1-\tau)\text{ЛJ}_2(1-\kappa)+\text{ЛJ}_2(1-\tau)\text{ЛJ}_2(\kappa)] \\ +\text{ЛJ}_1(\tau)\text{ЛJ}_1(1-\kappa)[\text{ЛJ}_2(\tau)\text{ЛJ}_2(1-\kappa)+\text{ЛJ}_2(1-\tau)\text{ЛJ}_2(1-\kappa)+\text{ЛJ}_2(1-\tau)\text{ЛJ}_2(\kappa)] \end{bmatrix}$$

which implies that

$$\frac{1}{\alpha\beta}G_\gamma\left(\frac{\mathfrak{e}+\mathfrak{g}}{2},\ \frac{\mathfrak{u}+\mathfrak{v}}{2}\right)\times\mathcal{J}_\gamma\left(\frac{\mathfrak{e}+\mathfrak{g}}{2},\ \frac{\mathfrak{u}+\mathfrak{v}}{2}\right)$$
$$\supseteq_{\mathbb{F}}\frac{\Gamma(\alpha)\Gamma(\beta)\text{Ꞁ}_1{}^2\left(\frac{1}{2}\right)\text{Ꞁ}_2{}^2\left(\frac{1}{2}\right)}{(\mathfrak{g}-\mathfrak{e})^\alpha(\mathfrak{v}-\mathfrak{u})^\beta}\left[\mathcal{I}^{\alpha,\ \beta}_{\mathfrak{e}^+,\mathfrak{u}^+}G_\gamma(\mathfrak{g},\mathfrak{v})\times\mathcal{J}_\gamma(\mathfrak{g},\mathfrak{v})+\mathcal{I}^{\alpha,\ \beta}_{\mathfrak{e}^+,\mathfrak{v}^-}G_\gamma(\mathfrak{g},\mathfrak{u})\times\mathcal{J}_\gamma(\mathfrak{g},\mathfrak{u})\right]$$
$$+\frac{\Gamma(\alpha)\Gamma(\beta)\text{Ꞁ}_1{}^2\left(\frac{1}{2}\right)\text{Ꞁ}_2{}^2\left(\frac{1}{2}\right)}{(\mathfrak{g}-\mathfrak{e})^\alpha(\mathfrak{v}-\mathfrak{u})^\beta}\left[\mathcal{I}^{\alpha,\ \beta}_{\mathfrak{g}^-,\mathfrak{u}^+}G_\gamma(\mathfrak{e},\mathfrak{v})\times\mathcal{J}_\gamma(\mathfrak{e},\mathfrak{v})+\mathcal{I}^{\alpha,\ \beta}_{\mathfrak{g}^-,\mathfrak{v}^-}G_\gamma(\mathfrak{e},\mathfrak{u})\times\mathcal{J}_\gamma(\mathfrak{e},\mathfrak{u})\right]$$
$$+2\text{Ꞁ}_1{}^2\left(\frac{1}{2}\right)\text{Ꞁ}_2{}^2\left(\frac{1}{2}\right)\mathcal{M}_\gamma(\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v})\int_0^1\tau^{\alpha-1}\kappa^{\beta-1}[\text{Ꞁ}_1(\tau)\text{Ꞁ}_1(\kappa)[\text{Ꞁ}_2(\tau)\text{Ꞁ}_2(1-\kappa)+\text{Ꞁ}_2(1-\tau)$$
$$\text{Ꞁ}_2(\kappa)+\text{Ꞁ}_2(1-\tau)\text{Ꞁ}_2(1-\kappa)]+\text{Ꞁ}_1(\tau)\text{Ꞁ}_1(1-\kappa)[\text{Ꞁ}_2(\tau)\text{Ꞁ}_2(\kappa)+\text{Ꞁ}_2(1-\tau)\text{Ꞁ}_2(1-\kappa)+$$
$$\text{Ꞁ}_2(1-\tau)\text{Ꞁ}_2(\kappa)]]d\tau d\kappa$$
$$+2\text{Ꞁ}_1{}^2\left(\frac{1}{2}\right)\text{Ꞁ}_2{}^2\left(\frac{1}{2}\right)P_\gamma(\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v})\int_0^1\tau^{\alpha-1}\kappa^{\beta-1}[\text{Ꞁ}_1(\tau)\text{Ꞁ}_1(\kappa)[\text{Ꞁ}_2(1-\tau)\text{Ꞁ}_2(1-\kappa)+$$
$$\text{Ꞁ}_2(\tau)\text{Ꞁ}_2(\kappa)+\text{Ꞁ}_2(\tau)\text{Ꞁ}_2(1-\kappa)]+\text{Ꞁ}_1(\tau)\text{Ꞁ}_1(1-\kappa)[\text{Ꞁ}_2(1-\tau)\text{Ꞁ}_2(\kappa)+\text{Ꞁ}_2(\tau)\text{Ꞁ}_2(1-\kappa)+$$
$$\text{Ꞁ}_2(\tau)\text{Ꞁ}_2(\kappa)]]d\tau d\kappa$$
$$+2\text{Ꞁ}_1{}^2\left(\frac{1}{2}\right)\text{Ꞁ}_2{}^2\left(\frac{1}{2}\right)\mathcal{N}_\gamma(\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v})\int_0^1\tau^{\alpha-1}\kappa^{\beta-1}[\text{Ꞁ}_1(\tau)\text{Ꞁ}_1(\kappa)[\text{Ꞁ}_2(\tau)\text{Ꞁ}_2(\kappa)+\text{Ꞁ}_2(1-\tau)$$
$$\text{Ꞁ}_2(1-\kappa)+\text{Ꞁ}_2(1-\tau)\text{Ꞁ}_2(\kappa)]+\text{Ꞁ}_1(\tau)\text{Ꞁ}_1(1-\kappa)[\text{Ꞁ}_2(\tau)\text{Ꞁ}_2(1-\kappa)+\text{Ꞁ}_2(1-\tau)\text{Ꞁ}_2(\kappa)+$$
$$\text{Ꞁ}_2(1-\tau)\text{Ꞁ}_2(1-\kappa)]]d\tau d\kappa$$
$$+2\text{Ꞁ}_1{}^2\left(\frac{1}{2}\right)\text{Ꞁ}_2{}^2\left(\frac{1}{2}\right)Q_\gamma(\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v})\int_0^1\tau^{\alpha-1}\kappa^{\beta-1}[\text{Ꞁ}_1(\tau)\text{Ꞁ}_1(\kappa)[\text{Ꞁ}_2(1-\tau)\text{Ꞁ}_2(\kappa)+\text{Ꞁ}_2(\tau)\text{Ꞁ}_2(1-\kappa)+\text{Ꞁ}_2(\tau)\text{Ꞁ}_2(\kappa)]$$
$$+\text{Ꞁ}_1(\tau)\text{Ꞁ}_1(1-\kappa)[\text{Ꞁ}_2(1-\tau)\text{Ꞁ}_2(1-\kappa)+\text{Ꞁ}_2(\tau)\text{Ꞁ}_2(\kappa)+\text{Ꞁ}_2(\tau)\text{Ꞁ}_2(1-\kappa)]]d\tau d\kappa,$$

since $\gamma\in[0,\ 1]$, then, after simplification, we reach the required conclusion. □

Remark 4. *If one assumes that* $\text{Ꞁ}(\tau)=\tau,\ \text{Ꞁ}(\kappa)=\kappa,\ \alpha=1$*, and* $\beta=1$*, then, from (68), as a result, there will be an inequality (see [69]):*

$$\begin{aligned}&4\widetilde{G}\left(\frac{\mathfrak{e}+\mathfrak{g}}{2},\frac{\mathfrak{u}+\mathfrak{v}}{2}\right)\otimes\widetilde{\mathcal{J}}\left(\frac{\mathfrak{e}+\mathfrak{g}}{2},\frac{\mathfrak{u}+\mathfrak{v}}{2}\right)\\&\supseteq_{\mathbb{F}}\frac{1}{(\mathfrak{g}-\mathfrak{e})(\mathfrak{v}-\mathfrak{u})}\int_{\mathfrak{e}}^{\mathfrak{g}}\int_{\mathfrak{u}}^{\mathfrak{v}}\widetilde{G}(x,y)\otimes\widetilde{\mathcal{J}}(x,y)dydx\oplus\frac{5}{36}\widetilde{\mathcal{M}}(\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v})\\&\oplus\frac{7}{36}\left[\widetilde{P}(\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v})\widetilde{+}\widetilde{\mathcal{N}}(\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v})\right]\oplus\frac{2}{9}\widetilde{Q}(\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v}).\end{aligned}\tag{70}$$

If $\widetilde{G}$ *is a coordinated left-UD-Ꞁ-convex function with* $\text{Ꞁ}(\tau)=\tau,\ \ \text{Ꞁ}(\kappa)=\kappa$ *and one assumes that* $\alpha=1$ *and* $\beta=1$*, then, from (68), as a result, there will be an inequality (see [59]):*

$$\begin{aligned}&4\widetilde{G}\left(\frac{\mathfrak{e}+\mathfrak{g}}{2},\frac{\mathfrak{u}+\mathfrak{v}}{2}\right)\otimes\widetilde{\mathcal{J}}\left(\frac{\mathfrak{e}+\mathfrak{g}}{2},\frac{\mathfrak{u}+\mathfrak{v}}{2}\right)\\&\leq_{\mathbb{F}}\frac{1}{(\mathfrak{g}-\mathfrak{e})(\mathfrak{v}-\mathfrak{u})}\int_{\mathfrak{e}}^{\mathfrak{g}}\int_{\mathfrak{u}}^{\mathfrak{v}}\widetilde{G}(x,y)\otimes\widetilde{\mathcal{J}}(x,y)dydx\oplus\frac{5}{36}\widetilde{\mathcal{M}}(\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v})\\&\oplus\frac{7}{36}\left[\widetilde{P}(\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v})\widetilde{+}\widetilde{\mathcal{N}}(\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v})\right]\oplus\frac{2}{9}\widetilde{Q}(\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v}).\end{aligned}\tag{71}$$

If $G_*((x,y),\ \gamma)\neq G^*((x,y),\ \gamma)$ *with* $\text{Ꞁ}(\tau)=\tau,\ \text{Ꞁ}(\kappa)=\kappa$ *and* $\gamma=1$*, then, from (68), we succeed in bringing about the upcoming inequality (see [55]):*

$$\begin{aligned}&4\ G\left(\frac{\mathfrak{e}+\mathfrak{g}}{2},\frac{\mathfrak{u}+\mathfrak{v}}{2}\right)\times\mathcal{J}\left(\frac{\mathfrak{e}+\mathfrak{g}}{2},\frac{\mathfrak{u}+\mathfrak{v}}{2}\right)\\&\supseteq\frac{1}{(\mathfrak{g}-\mathfrak{e})(\mathfrak{v}-\mathfrak{u})}\int_{\mathfrak{e}}^{\mathfrak{g}}\int_{\mathfrak{u}}^{\mathfrak{v}}G(x,y)\times\mathcal{J}(x,y)dydx+\frac{5}{36}\mathcal{M}(\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v})\\&+\frac{7}{36}[P(\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v})+\mathcal{N}(\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v})]+\frac{2}{9}Q(\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v}).\end{aligned}\tag{72}$$

If $G_*((x,y),\ \gamma) \neq G^*((x,y),\ \gamma)$ with $\gamma = 1$ and $\daleth(\tau) = \tau$, $\daleth(\kappa) = \kappa$, then, from (68), we succeed in bringing about the upcoming inequality (see [71]):

$$\begin{aligned}
&4G\left(\tfrac{\mathfrak{e}+\mathfrak{g}}{2},\ \tfrac{\mathfrak{u}+\mathfrak{v}}{2}\right) \times \mathcal{J}\left(\tfrac{\mathfrak{e}+\mathfrak{g}}{2},\ \tfrac{\mathfrak{u}+\mathfrak{v}}{2}\right) \\
&\supseteq \tfrac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\mathfrak{g}-\mathfrak{e})^{\alpha}(\mathfrak{v}-\mathfrak{u})^{\beta}} \begin{bmatrix} \mathcal{I}^{\alpha,\ \beta}_{\mathfrak{e}^+,\mathfrak{u}^+}\ G(\mathfrak{g},\mathfrak{v}) \times \mathcal{J}(\mathfrak{g},\mathfrak{v}) + \mathcal{I}^{\alpha,\ \beta}_{\mathfrak{e}^+,\mathfrak{v}^-}\ G(\mathfrak{g},\mathfrak{u}) \times \mathcal{J}(\mathfrak{g},\mathfrak{u}) \\ +\mathcal{I}^{\alpha,\ \beta}_{\mathfrak{g}^-,\mathfrak{u}^+}\ G(\mathfrak{e},\mathfrak{v}) \times \mathcal{J}(\mathfrak{e},\mathfrak{v}) + \mathcal{I}^{\alpha,\ \beta}_{\mathfrak{g}^-,\mathfrak{v}^-}\ G(\mathfrak{e},\mathfrak{u}) \times \mathcal{J}(\mathfrak{e},\mathfrak{u}) \end{bmatrix} \\
&+\left[\tfrac{\alpha}{2(\alpha+1)(\alpha+2)} + \tfrac{\beta}{(\beta+1)(\beta+2)}\left(\tfrac{1}{2} - \tfrac{\alpha}{(\alpha+1)(\alpha+2)}\right)\right]\mathcal{M}(\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v}) \\
&+\left[\tfrac{1}{2}\left(\tfrac{1}{2} - \tfrac{\alpha}{(\alpha+1)(\alpha+2)}\right) + \tfrac{\alpha}{(\alpha+1)(\alpha+2)}\tfrac{\beta}{(\beta+1)(\beta+2)}\right]P(\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v}) \\
&+\left[\tfrac{1}{2}\left(\tfrac{1}{2} - \tfrac{\beta}{(\beta+1)(\beta+2)}\right) + \tfrac{\alpha}{(\alpha+1)(\alpha+2)}\tfrac{\beta}{(\beta+1)(\beta+2)}\right]\mathcal{N}(\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v}) \\
&+\left[\tfrac{1}{4} - \tfrac{\alpha}{(\alpha+1)(\alpha+2)}\tfrac{\beta}{(\beta+1)(\beta+2)}\right]Q(\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v}).
\end{aligned} \tag{73}$$

If $G_*((x,y),\ \gamma) = G^*((x,y),\ \gamma)$ and $\mathcal{J}_*((x,y),\ \gamma) = \mathcal{J}^*((x,y),\ \gamma)$ with $\gamma = 1$ and $\daleth(\tau) = \tau$, $\daleth(\kappa) = \kappa$, then, from (68), we succeed in bringing about the upcoming inequality (see [69]):

$$\begin{aligned}
&4G\left(\tfrac{\mathfrak{e}+\mathfrak{g}}{2},\ \tfrac{\mathfrak{u}+\mathfrak{v}}{2}\right) \times \mathcal{J}\left(\tfrac{\mathfrak{e}+\mathfrak{g}}{2},\ \tfrac{\mathfrak{u}+\mathfrak{v}}{2}\right) \\
&\leq \tfrac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\mathfrak{g}-\mathfrak{e})^{\alpha}(\mathfrak{v}-\mathfrak{u})^{\beta}} \begin{bmatrix} \mathcal{I}^{\alpha,\ \beta}_{\mathfrak{e}^+,\mathfrak{u}^+} G(\mathfrak{g},\mathfrak{v}) \times \mathcal{J}(\mathfrak{g},\mathfrak{v}) + \mathcal{I}^{\alpha,\ \beta}_{\mathfrak{e}^+,\mathfrak{v}^-} G(\mathfrak{g},\mathfrak{u}) \times \mathcal{J}(\mathfrak{g},\mathfrak{u}) \\ +\mathcal{I}^{\alpha,\ \beta}_{\mathfrak{g}^-,\mathfrak{u}^+} G(\mathfrak{e},\mathfrak{v}) \times \mathcal{J}(\mathfrak{e},\mathfrak{v}) + \mathcal{I}^{\alpha,\ \beta}_{\mathfrak{g}^-,\mathfrak{v}^-} G(\mathfrak{e},\mathfrak{u}) \times \mathcal{J}(\mathfrak{e},\mathfrak{u}) \end{bmatrix}. \\
&+\left[\tfrac{\alpha}{2(\alpha+1)(\alpha+2)} + \tfrac{\beta}{(\beta+1)(\beta+2)}\left(\tfrac{1}{2} - \tfrac{\alpha}{(\alpha+1)(\alpha+2)}\right)\right]\mathcal{M}(\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v}) \\
&+\left[\tfrac{1}{2}\left(\tfrac{1}{2} - \tfrac{\alpha}{(\alpha+1)(\alpha+2)}\right) + \tfrac{\alpha}{(\alpha+1)(\alpha+2)}\tfrac{\beta}{(\beta+1)(\beta+2)}\right]P(\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v}) \\
&+\left[\tfrac{1}{2}\left(\tfrac{1}{2} - \tfrac{\beta}{(\beta+1)(\beta+2)}\right) + \tfrac{\alpha}{(\alpha+1)(\alpha+2)}\tfrac{\beta}{(\beta+1)(\beta+2)}\right]\mathcal{N}(\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v}) \\
&+\left[\tfrac{1}{4} - \tfrac{\alpha}{(\alpha+1)(\alpha+2)}\tfrac{\beta}{(\beta+1)(\beta+2)}\right]Q(\mathfrak{e},\mathfrak{g},\mathfrak{u},\mathfrak{v}).
\end{aligned} \tag{74}$$

4. Conclusions

This study makes use of fuzzy-number-valued fractional integrals to handle certain fractional integral inclusions involving the Hermite–Hadamard integral inequality via a newly defined class of coordinated *UD*-$\daleth$-convex *FNVMs*. We also look into other set inclusion connections related to the fractional Pachpatte integral inequality. Additionally, a few examples are provided to support the accuracy of the conclusions drawn in the research. We highlight the links between the results obtained here and those previously published in order to demonstrate the generic properties of the fuzzy set inclusion relations offered. Based on published works [59,68] and the bibliographies cited in them, we can confidently conclude that fuzzy-number-valued analyses are commonly used in applied analyses, particularly in the field of optimality analysis. In the integration with the fuzzy-number-valued fractional integral operators, the fuzzy *UD*-inclusion relations are somewhat interesting and need more investigation.

Author Contributions: Conceptualization, M.B.K.; validation, M.B.K.; formal analysis, E.R.N., K.H.H. and C.-C.L.; investigation, M.B.K.; resources, M.B.K., E.R.N., C.-C.L., D.-C.L. and K.H.H.; writing—original draft, M.B.K.; writing—review and editing, M.B.K., E.R.N., K.H.H., C.-C.L., D.-C.L. and H.G.Z.; visualization, M.B.K.; supervision, M.B.K. and H.G.Z.; project administration, H.G.Z. and D.-C.L. All authors have read and agreed to the published version of the manuscript.

Funding: This work was supported by the National Science and Technology Council of the Republic of China under Contract No. MOST 111-2221-E-182-048- and Chang Gung Memorial Hospital under the grant BMRPB30. The researchers also would like to acknowledge Deanship of Scientific Research, Taif University for funding this work.

Data Availability Statement: Data sharing is not applicable to this article.

Acknowledgments: The authors would like to thank Transilvania University of Brasov, Romania, for providing excellent research. The researchers also would like to acknowledge Deanship of Scientific Research, Taif University for funding this work.

Conflicts of Interest: The authors claim to have no conflicts of interest.

References

1. Chu, Y.-M.; Wang, G.-D.; Zhang, X.-H. The Schur multiplicative and harmonic convexities of the complete symmetric function. *Math. Nachr.* **2011**, *284*, 653–663. [CrossRef]
2. Chu, Y.-M.; Xia, W.-F.; Zhang, X.-H. The Schur concavity, Schur multiplicative and harmonic convexities of the second dual form of the Hamy symmetric function with applications. *J. Multivar. Anal.* **2012**, *105*, 412–421. [CrossRef]
3. Nwaeze, E.R.; Khan, M.A.; Chu, Y.M. Fractional inclusions of the Hermite-Hadamard type for m-polynomial convex interval valued functions. *Adv. Differ. Equ.* **2020**, *2020*, 507. [CrossRef]
4. Zhao, T.H.; He, Z.Y.; Chu, Y.M. On some refinements for inequalities involving zero-balanced hyper geometric function. *AIMS Math.* **2020**, *5*, 6479–6495. [CrossRef]
5. Zhao, T.H.; Wang, M.K.; Chu, Y.M. A sharp double inequality involving generalized complete elliptic integral of the first kind. *AIMS Math.* **2020**, *5*, 4512–4528. [CrossRef]
6. Dragomir, S.S.; Pearce, C.E.M. Selected topics on Hermite–Hadamard inequalities and applications. In *RGMIA*; Victoria University: Melbourne, VIC, Australia, 2000.
7. Peajcariaac, J.E.; Tong, Y.L. *Convex Functions, Partial Orderings, and Statistical Applications*; Academic Press: Cambridge, MA, USA, 1992.
8. Zhao, T.H.; Wang, M.K.; Hai, G.J.; Chu, Y.M. Landen inequalities for Gaussian hypergeometric function. *Racsam Rev. R. Acad. A* **2022**, *116*, 53. [CrossRef]
9. Wang, M.K.; Hong, M.Y.; Xu, Y.F.; Shen, Z.H.; Chu, Y.M. Inequalities for generalized trigonometric and hyperbolic functions with one parameter. *J. Math. Inequal.* **2020**, *14*, 1–21. [CrossRef]
10. Zhao, T.H.; Shi, L.; Chu, Y.M. Convexity and concavity of the modified Bessel functions of the first kind with respect to Hölder means. *Racsam Rev. R. Acad. A* **2020**, *114*, 96. [CrossRef]
11. Zhao, T.H.; Zhou, B.C.; Wang, M.K.; Chu, Y.M. On approximating the quasi-arithmetic mean. *J. Inequal. Appl.* **2019**, *2019*, 42. [CrossRef]
12. Zhao, T.H.; Wang, M.K.; Zhang, W.; Chu, Y.M. Quadratic transformation inequalities for Gaussian hyper geometric function. *J. Inequal. Appl.* **2018**, *2018*, 251. [CrossRef] [PubMed]
13. Qian, W.M.; Chu, H.H.; Wang, M.K.; Chu, Y.M. Sharp inequalities for the Toader mean of order –1 in terms of other bivariate means. *J. Math. Inequal.* **2022**, *16*, 127–141. [CrossRef]
14. Zhao, T.H.; Chu, H.H.; Chu, Y.M. Optimal Lehmer mean bounds for the nth power-type Toader mean of n = -1, 1, 3. *J. Math. Inequal.* **2022**, *16*, 157–168. [CrossRef]
15. Wang, M.-K.; Chu, Y.-M. Refinements of transformation inequalities for zero-balanced hypergeometric functions. *Acta Math. Sci.* **2017**, *37*, 607–622. [CrossRef]
16. Wang, M.-K.; Chu, Y.-M. Landen inequalities for a class of hypergeometric functions with applications. *Math. Inequal. Appl.* **2018**, *21*, 521–537. [CrossRef]
17. Wu, S.; Awan, M.U.; Noor, M.A.; Noor, K.I.; Iftikhar, S. On a new class of convex functions and integral inequalities. *J. Inequalities Appl.* **2019**, *2019*, 131. [CrossRef]
18. Kashuri, A.; Awan, M.U.; Talib, S.; Noor, M.A.; Noor, K.I. On Exponetially $-Preinvex Functions and Associated Trapezium Like Inequalities. *Appl. Anal. Discret. Math.* **2021**, *15*, 317–336. [CrossRef]
19. Wang, M.-K.; Chu, Y.-M.; Qiu, Y.-F.; Qiu, S.-L. An optimal power mean inequality for the complete elliptic integrals. *Appl. Math. Lett.* **2011**, *24*, 887–890. [CrossRef]
20. Wang, M.-K.; Chu, Y.-M.; Zhang, W. Monotonicity and inequalities involving zero-balanced hypergeometric function. *Math. Inequal. Appl.* **2019**, *22*, 601–617. [CrossRef]
21. Chu, Y.-M.; Wang, M.-K. Inequalities between arithmetic geometric, Gini, and Toader means. *Abstr. Appl. Anal.* **2012**, *2012*, 830585. [CrossRef]
22. Moore, R.E. *Interval Analysis*; Prentice-Hall: Englewood Cliffs, NJ, USA, 1966; Volume 4, pp. 8–13.
23. Breckner, W.W. Continuity of generalized convex and generalized concave set-valued functions. *Rev. D'Anal. Numer. Theor. L'Approx* **1993**, *22*, 39–51.

24. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; Elsevier: Amsterdam, The Netherlands, 2006; Volume 204.
25. Hu, X.-M.; Tian, J.-F.; Chu, Y.-M.; Lu, Y.-X. On Cauchy–Schwarz inequality for N-tuple diamond-alpha integral. *J. Inequal. Appl.* **2020**, *2020*, 8. [CrossRef]
26. Zhao, T.-H.; Chu, Y.-M.; Wang, H. Logarithmically complete monotonicity properties relating to the gamma function. *Abstr. Appl. Anal.* **2011**, *2011*, 896483. [CrossRef]
27. Chu, Y.M.; Rauf, A.; Ishtiaq, M.; Siddiqui, M.K.; Muhammad, M.H. Topological properties of polycyclic aromatic nanostars dendrimers. *Polycycl. Aromat. Compd.* **2022**, *42*, 1891–1908. [CrossRef]
28. Ashpazzadeh, E.; Chu, Y.-M.; Hashemi, M.S.; Moharrami, M.; Inc, M. Hermite multiwavelets representation for the sparse solution of nonlinear Abel's integral equation. *Appl. Math. Comput.* **2022**, *427*, 127171. [CrossRef]
29. Chu, Y.-M.; Ullah, S.; Ali, M.; Tuzzahrah, G.F.; Munir, T. Numerical investigation of Volterra integral equations of second kind using optimal homotopy asymptotic methd. *Appl. Math. Comput.* **2022**, *430*, 127304.
30. Chu, Y.-M.; Inc, M.; Hashemi, M.S.; Eshaghi, S. Analytical treatment of regularized Prabhakar fractional differential equations by invariant subspaces. *Comput. Appl. Math.* **2022**, *41*, 271. [CrossRef]
31. Chalco-Cano, Y.; Flores-Franulic, A.; Roman-Flores, H. Ostrowski type inequalities for interval-valued functions using generalized Hukuhara derivative. *Comput. Appl. Math.* **2012**, *31*, 457–472.
32. Chalco-Cano, Y.; Lodwick, W.A.; Condori-Equice, W. Ostrowski type inequalities and applications in numerical integration for interval-valued functions. *Soft Comput.* **2015**, *19*, 3293–3300. [CrossRef]
33. Costa, T.M.; Román-Flores, H. Some integral inequalities for fuzzy-interval-valued functions. *Inf. Sci.* **2017**, *420*, 110–125. [CrossRef]
34. Roman-Flores, H.; Chalco-Cano, Y.; Lodwick, W. Some integral inequalities for interval-valued functions. *Comput. Appl. Math.* **2018**, *37*, 1306–1318. [CrossRef]
35. Sharma, N.; Singh, S.K.; Mishra, S.K.; Hamdi, A. Hermite-Hadamard-type inequalities for interval-valued preinvex functions via Riemann-Liouville fractional integrals. *J. Inequalities Appl.* **2021**, *2021*, 98. [CrossRef]
36. Zhao, D.; An, T.; Ye, G.; Liu, W. New Jensen and Hermite-Hadamard type inequalities for H-convex interval-valued functions. *J. Inequalities Appl.* **2018**, *2018*, 302. [CrossRef]
37. Zhao, D.; An, T.; Ye, G.; Liu, W. Chebyshev type inequalities for interval-valued functions. *Fuzzy Sets Syst.* **2020**, *396*, 82–101. [CrossRef]
38. Sarikaya, M.Z.; Set, E.; Yaldiz, H.; Basak, N. Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities. *Math. Comput. Model.* **2013**, *57*, 2403–2407. [CrossRef]
39. Mohammed, P.O.; Sarikaya, M.Z.; Baleanu, D. On the generalized Hermite-Hadamard inequalities via the tempered fractional integrals. *Symmetry* **2020**, *12*, 595. [CrossRef]
40. Mohammed, P.O.; Brevik, I. A new version of the Hermite-Hadamard inequality for Riemann-Liouville fractional integrals. *Symmetry* **2020**, *12*, 610. [CrossRef]
41. Saeed, T.; Cătaș, A.; Khan, M.B.; Alshehri, A.M. Some New Fractional Inequalities for Coordinated Convexity over Convex Set Pertaining to Fuzzy-Number-Valued Settings Governed by Fractional Integrals. *Fractal Fract.* **2023**, *7*, 856. [CrossRef]
42. Saeed, T.; Khan, M.B.; Treanț®, S.; Alsulami, H.H.; Alhodaly, M.S. Interval Fejér-Type Inequalities for Left and Right-λ-Preinvex Functions in Interval-Valued Settings. *Axioms* **2022**, *11*, 368. [CrossRef]
43. Khan, M.B.; Cătaș, A.; Saeed, T. Generalized fractional integral inequalities for p-convex fuzzy interval-valued mappings. *Fractal Fract.* **2022**, *6*, 324. [CrossRef]
44. Kórus, P.; Valdés, J.E.N.; Bayraktar, B. Weighted Hermite–Hadamard integral inequalities for general convex functions. *Math. Biosci. Eng.* **2023**, *20*, 19929–19940. [CrossRef]
45. Akdemir, A.O.; Butt, S.I.; Nadeem, M.; Ragusa, M.A. New general variants of Chebyshev type inequalities via generalized fractional integral operators. *Mathematics* **2021**, *9*, 122. [CrossRef]
46. Set, E.; Butt, S.I.; Akdemir, A.O.; Karaoglan, A.; Abdeljawad, T. New integral inequalities for differentiable convex functions via Atangana-Baleanu fractional integral operators. *Chaos Solitons Fractals* **2021**, *143*, 110554. [CrossRef]
47. Budak, H.; Tunc, T.; Sarikaya, M. Fractional Hermite-Hadamard-type inequalities for interval-valued functions. *Proc. Am. Math. Soc.* **2020**, *148*, 705–718. [CrossRef]
48. Kara, H.; Ali, M.A.; Budak, H. Hermite-Hadamard-type inequalities for interval-valued coordinated convex functions involving generalized fractional integrals. *Math. Methods Appl. Sci.* **2021**, *44*, 104–123. [CrossRef]
49. Mohsin, B.B.; Awan, M.U.; Javed, M.Z.; Budak, H.; Khan, A.G.; Noor, M.A. Inclusions Involving Interval-Valued Harmonically Co-Ordinated Convex Functions and Raina's Fractional Double Integrals. *J. Math.* **2022**, *2022*, 5815993. [CrossRef]
50. Zhou, T.; Du, T. Certain Fractional Integral Inclusions Pertaining to Interval-Valued Exponential Trigonometric Convex Functions. *J. Math. Inequalities* **2023**, *17*, 283–314. [CrossRef]
51. Khan, M.B.; Catas, A.; Aloraini, N.; Soliman, M.S. Some New Versions of Fractional Inequalities for Exponential Trigonometric Convex Mappings via Ordered Relation on Interval-Valued Settings. *Fractal Fract.* **2023**, *7*, 223. [CrossRef]
52. Kalsoom, H.; Ali, M.A.; Idrees, M.; Agarwal, P.; Arif, M. New post quantum analogues of Hermite-Hadamard type inequalities for interval-valued convex functions. *Math. Probl. Eng.* **2021**, *2021*, 5529650. [CrossRef]
53. Bin-Mohsin, B.; Rafique, S.; Cesarano, C.; Javed, M.Z.; Awan, M.U.; Kashuri, A.; Noor, M.A. Some General Fractional Integral Inequalities Involving LR-Bi-Convex Fuzzy Interval-Valued Functions. *Fractal Fract.* **2022**, *6*, 565. [CrossRef]

54. Kara, H.; Budak, H.; Ali, M.A.; Sarikaya, M.Z.; Chu, Y.M. Weighted Hermite-Hadamard type inclusions for products of coordinated convex interval-valued functions. *Adv. Differ. Eqs.* **2021**, *2021*, 104. [CrossRef]
55. Budak, H.; Kara, H.; Ali, M.A.; Khan, S.; Chu, Y. Fractional Hermite-Hadamard-type inequalities for interval-valued coordinated convex functions. *Open Math.* **2021**, *19*, 1081–1097. [CrossRef]
56. Ali, M.A.; Budak, H.; Murtaza, G.; Chu, Y.M. Post-quantum Hermite-Hadamard type inequalities for interval-valued convex functions. *J. Inequalities Appl.* **2021**, *2021*, 84. [CrossRef]
57. Du, T.; Zhou, T. On the fractional double integral inclusion relations having exponential kernels via interval-valued coordinated convex mappings. *Chaos Solitons Fractals* **2022**, *156*, 111846. [CrossRef]
58. Abdeljawad, T.; Rashid, S.; Khan, H.; Chu, Y.M. On new fractional integral inequalities for p-convexity within interval-valued functions. *Adv. Differ. Eqs.* **2020**, *2020*, 330. [CrossRef]
59. Khan, M.B.; Mohammed, P.O.; Noor, M.A.; Abualnaja, K.M. Fuzzy integral inequalities on coordinates of convex fuzzy interval-valued functions. *Math. Biosci. Eng.* **2021**, *18*, 6552–6580. [CrossRef]
60. Bin-Mohsin, B.; Awan, M.U.; Javed, M.Z.; Khan, A.G.; Budak, H.; Mihai, M.V.; Noor, M.A. Generalized AB-Fractional Operator Inclusions of Hermite-Hadamard's Type via Fractional Integration. *Symmetry* **2023**, *15*, 1012. [CrossRef]
61. Vivas-Cortez, M.; Ramzan, S.; Awan, M.U.; Javed, M.Z.; Khan, A.G.; Noor, M.A. IV-CR-γ-Convex Functions and Their Application in Fractional Hermite-Hadamard Inequalities. *Symmetry* **2023**, *15*, 1405. [CrossRef]
62. Khan, M.B.; Santos-García, G.; Noor, M.A.; Soliman, M.S. Some new concepts related to fuzzy fractional calculus for up and down convex fuzzy-number valued functions and inequalities. *Chaos Solitons Fractals* **2022**, *164*, 112692. [CrossRef]
63. Diamond, P.; Kloeden, P. *Metric Spaces of Fuzzy Sets: Theory and Applications*; World Scientific: Singapore, 1994.
64. Bede, B. *Mathematics of Fuzzy Sets and Fuzzy Logic, Volume 295 of Studies in Fuzziness and Soft Computing*; Springer: Berlin/Heidelberg, Germany, 2013.
65. Zhang, D.; Guo, C.; Chen, D.; Wang, G. Jensen's inequalities for set-valued and fuzzy set-valued functions. *Fuzzy Sets Syst.* **2020**, *404*, 178–204. [CrossRef]
66. Goetschel, R., Jr.; Voxman, W. Elementary fuzzy calculus. *Fuzzy Sets Syst.* **1986**, *18*, 31–43. [CrossRef]
67. Lupulescu, V. Fractional calculus for interval-valued functions. *Fuzzy Sets Syst.* **2015**, *265*, 63–85. [CrossRef]
68. Allahviranloo, T.; Salahshour, S.; Abbasbandy, S. Explicit solutions of fractional differential equations with uncertainty. *Soft Comput.* **2012**, *16*, 297–302. [CrossRef]
69. Khan, M.B.; Zaini, H.G.; Macías-Díaz, J.E.; Treanțӛ, S.; Soliman, M.S. Some Fuzzy Riemann–Liouville Fractional Integral Inequalities for Preinvex Fuzzy Interval-Valued Functions. *Symmetry* **2022**, *14*, 313. [CrossRef]
70. Khan, M.B.; Santos-García, G.; Zaini, H.G.; Treanță, S.; Soliman, M.S. Some New Concepts Related to Integral Operators and Inequalities on Coordinates in Fuzzy Fractional Calculus. *Mathematics* **2022**, *10*, 534. [CrossRef]
71. Zhao, D.F.; Ali, M.A.; Murtaza, G. On the Hermite-Hadamard inequalities for interval-valued coordinated convex functions. *Ad. Differ. Equ.* **2020**, *2020*, 570. [CrossRef]
72. Budak, H.; Sarikaya, M.Z. Hermite-Hadamard type inequalities for products of two co-ordinated convex mappings via fractional integrals. *Int. J. Appl. Math. Stat.* **2019**, *58*, 11–30.
73. Khan, M.B.; Althobaiti, A.; Lee, C.-C.; Soliman, M.S.; Li, C.-T. Some New Properties of Convex Fuzzy-Number-Valued Mappings on Coordinates Using Up and Down Fuzzy Relations and Related Inequalities. *Mathematics* **2023**, *11*, 2851. [CrossRef]
74. Sarikaya, M.Z. On the Hermite-Hadamard-type inequalities for coordinated convex function via fractional integrals. Integral Transform. *Spec. Funct.* **2013**, *25*, 134–147.