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Hom-Lie Superalgebras in Characteristic 2

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Abstract: The main goal of this paper was to develop the structure theory of Hom-Lie superalgebras in characteristic 2. We discuss their representations, semidirect product, and α^k-derivations and provide a classification in low dimension. We introduce another notion of restrictedness on Hom-Lie algebras in characteristic 2, different from the one given by Guan and Chen. This definition is inspired by the process of the queerification of restricted Lie algebras in characteristic 2. We also show that any restricted Hom-Lie algebra in characteristic 2 can be queerified to give rise to a Hom-Lie superalgebra. Moreover, we developed a cohomology theory of Hom-Lie superalgebras in characteristic 2, which provides a cohomology of ordinary Lie superalgebras. Furthermore, we established a deformation theory of Hom-Lie superalgebras in characteristic 2 based on this cohomology.

Keywords: Hom-Lie superalgebra; modular Lie superalgebra; characteristic 2; representation; queerification; cohomology; deformation

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1. Introduction

Throughout the text, $\mathbb{K}$ stands for an arbitrary field of characteristic 2. In almost all our constructions, $\mathbb{K}$ is arbitrary. There are a few instances where $\mathbb{K}$ is required to be infinite. We will point out these instances.

1.1. Lie Superalgebras in Characteristic 2

Roughly speaking, a Lie superalgebra in characteristic 2 is a $\mathbb{Z}/2\mathbb{Z}$-graded vector space that has a Lie algebra structure on the even part and is endowed with a squaring on the odd part that satisfies a modified Jacobi identity; see Section 2.1 for a precise definition. Because we are in characteristic 2, those Lie superalgebras are sometimes confused with $\mathbb{Z}/2\mathbb{Z}$-graded Lie algebras, though they are totally different algebras due to the presence of the squaring. They can, however, be considered as a $\mathbb{Z}/2\mathbb{Z}$-graded Lie algebra by forgetting the super structure. The other way round is not always true in general.

The classification of simple Lie superalgebras into characteristic 2 is still an open and wide problem. Nevertheless, Lie superalgebras in characteristic 2 admitting a Cartan matrix were classified in [1], with the following assumption: each Lie superalgebra possesses a Dynkin diagram with only one odd node. The list of non-equivalent Cartan matrices for each Lie superalgebra is also given in [1]. Moreover, it was recently showed in [2] that each finite-dimensional simple Lie superalgebra in characteristic 2 can be obtained from a simple finite-dimensional Lie algebra in characteristic 2, hence reducing the classification to the classification of simple Lie algebras, which on its own is a very tough problem. As a matter of fact, there are plenty of (vectorial and non-vectorial) Lie superalgebras in characteristic 2 that have no analogue in other characteristics; see [2–4] and the references therein.

It is worth mentioning that the characteristic 2 case is a very tricky case, due to the presence of the squaring. It does require new ideas and techniques.

1.2. Hom-Lie Superalgebras in Characteristic 2

The first instances of Hom-type algebras appeared in the physics literature; see, for example, [5], where q-deformations of some Lie algebras of vector fields led to a structure in which the Jacobi identity is no longer satisfied. This class of algebras was formalized and studied in [6–8], where these algebras were called *Hom-Lie algebras* since the Jacobi identity is twisted by a homomorphism. The super case was considered in [9], where Hom-Lie superalgebras were introduced as a $\mathbb{Z}/2\mathbb{Z}$-graded generalization of the Hom-Lie algebras. The authors of [9] characterized Hom-Lie admissible superalgebras and proved a $\mathbb{Z}/2\mathbb{Z}$-graded version of a Hartwig–Larsson–Silvestrov Theorem, which led to the construction of a q-deformed Witt superalgebra using σ-derivations. Moreover, they derived a one-parameter family of Hom-Lie superalgebras deforming the orthosymplectic Lie superalgebra $\mathfrak{osp}(1|2)$. The cohomology of Hom-Lie superalgebras was defined in [10]. For other contributions, see, for example, [11,12] and the references therein. Notice that all these studies and results were performed over a field of characteristic 0.

1.3. The Main Results

The main purpose of this paper was to tackle the positive characteristic and provide a study of Hom-Lie superalgebras in characteristic 2. We introduce the main definitions and some key constructions, as well as a cohomology theory fitting a deformation theory. In Section 2, we recall some basic definitions and introduce Hom-Lie algebras and Hom-Lie superalgebras over fields of characteristic 2 and some related structures. We show that a Lie superalgebra in characteristic 2 and an even Lie superalgebra morphism give rise to a Hom-Lie superalgebra in characteristic 2. Moreover, we provide a classification of Hom-Lie superalgebras in characteristic 2 in low dimensions. In Section 3, we consider the representations and semidirect product of Hom-Lie superalgebras in characteristic 2. The structure map defining a Hom-Lie superalgebra in characteristic 2 allows a new type of derivation called α^k-derivations, discussed in Section 4. In Section 5, we introduce the notion of the p-structure and discuss the queerification of restricted Hom-Lie algebras in characteristic 2. Section 6 is dedicated to cohomology theory. We construct a cohomology complex of a Hom-Lie superalgebra $\mathfrak{g}$ in characteristic 2 with values in a $\mathfrak{g}$-module. This cohomology complex has no analogue in characteristic $p \neq 2$. In the last section, we provide a deformation theory of Hom-Lie superalgebras in characteristic 2 using the cohomology we constructed previously.

2. Backgrounds and Main Definitions

Let V and W be two vector spaces over $\mathbb{K}$. A map $s : V \to W$ is called a *squaring* if

$$\begin{aligned} &s(\lambda x) = \lambda^2 s(x) \quad \text{for all } \lambda \in \mathbb{K} \text{ and for all } x \in V\text{, and the map} \\ &(x, y) \mapsto s(x+y) - s(x) - s(y) \text{ is bilinear.} \end{aligned} \tag{1}$$

2.1. Lie Superalgebras in Characteristic 2

Following [4,13], a *Lie superalgebra* in characteristic 2 is a superspace $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ over $\mathbb{K}$ such that $\mathfrak{g}_{\bar{0}}$ is an ordinary Lie algebra, $\mathfrak{g}_{\bar{1}}$ is a $\mathfrak{g}_{\bar{0}}$-module made two-sided by symmetry, and on $\mathfrak{g}_{\bar{1}}$, a squaring, denoted by $s_{\mathfrak{g}} : \mathfrak{g}_{\bar{1}} \to \mathfrak{g}_{\bar{0}}$, is given. The bracket on $\mathfrak{g}_{\bar{0}}$, as well as the action of $\mathfrak{g}_{\bar{0}}$ on $\mathfrak{g}_{\bar{1}}$ are denoted by the same symbol $[\cdot,\cdot]$. For any $x, y \in \mathfrak{g}_{\bar{1}}$, their bracket is then defined by

$$[x, y] := s(x+y) - s(x) - s(y).$$

The bracket is extended to non-homogeneous elements by bilinearity. The Jacobi identity involving the squaring reads as follows:

$$[s(x), y] = [x, [x, y]] \quad \text{for any } x \in \mathfrak{g}_{\bar{1}} \text{ and } y \in \mathfrak{g}.$$

Such a Lie superalgebra in characteristic 2 will be denoted by $(\mathfrak{g}, [\cdot,\cdot], s)$.

For any Lie superalgebra $\mathfrak{g}$ in characteristic 2, its *derived algebras* are defined to be (for $i \geq 0$)

$$\mathfrak{g}^{(0)} := \mathfrak{g}, \quad \mathfrak{g}^{(i+1)} = [\mathfrak{g}^{(i)}, \mathfrak{g}^{(i)}] + \mathrm{Span}\{s(x) \mid x \in (\mathfrak{g}^{(i)})_{\bar{1}}\}.$$

A linear map $D : \mathfrak{g} \to \mathfrak{g}$ is called a *derivation* of the Lie superalgebra $\mathfrak{g}$ if, in addition to

$$D([x,y]) = [D(x), y] + [x, D(y)] \quad \text{for any } x \in \mathfrak{g}_{\bar{0}} \text{ and } y \in \mathfrak{g}, \text{ we have} \tag{2}$$

$$D(s(x)) = [D(x), x] \quad \text{for any } x \in \mathfrak{g}_{\bar{1}}. \tag{3}$$

It is worth noticing that condition (3) implies condition (2) if $x, y \in \mathfrak{g}_{\bar{1}}$.

We denote the space of all derivations of $\mathfrak{g}$ by $\mathfrak{der}(\mathfrak{g})$.

Let $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, s_{\mathfrak{g}})$ and $(\mathfrak{h}, [\cdot,\cdot]_{\mathfrak{h}}, s_{\mathfrak{h}})$ be two Lie superalgebras in characteristic 2. An even linear map $\varphi : \mathfrak{g} \to \mathfrak{h}$ is called a *morphism* (of Lie superalgebras) if, in addition to

$$\varphi([x,y]_{\mathfrak{g}}) = [\varphi(x), \varphi(y)]_{\mathfrak{h}} \quad \text{for any } x \in \mathfrak{g}_{\bar{0}} \text{ and } y \in \mathfrak{g}, \text{ we have}$$

$$\varphi(s_{\mathfrak{g}}(x)) = s_{\mathfrak{h}}(\varphi(x)) \quad \text{for any } x \in \mathfrak{g}_{\bar{1}}.$$

Therefore, morphisms in the category of Lie superalgebras in characteristic 2 preserve not only the bracket, but the squaring as well. In particular, subalgebras and ideals have to be stable under the bracket and the squaring.

An even linear map $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ is a *representation of the Lie superalgebra* $(\mathfrak{g}, [\cdot,\cdot], s)$ in the superspace V called the $\mathfrak{g}$-*module* if

$$\rho([x,y]) = [\rho(x), \rho(y)] \quad \text{for any } x, y \in \mathfrak{g}; \text{ and } \rho(s(x)) = (\rho(x))^2 \text{ for any } x \in \mathfrak{g}_{\bar{1}}. \tag{4}$$

Remark 1. *Associative superalgebras in characteristic 2 lead to Lie superalgebras in characteristic 2. The bracket is standard, and the squaring is defined by $s(x) = x \cdot x$, for every odd element x.*

2.2. Hom-Lie Algebras in Characteristic 2

A *Hom-Lie algebra* in characteristic 2 is a vector space $\mathfrak{g}$ over $\mathbb{K}$ and a map $\alpha \in \mathrm{End}(\mathfrak{g})$ together with a bracket satisfying the following conditions:

$$[x,x] = 0, \quad \alpha[x,y] = [\alpha(x), \alpha(y)] \quad \text{and} \quad [\alpha(x), [y,z]] + \circlearrowleft (x,y,z) = 0, \quad \text{for all } x, y, z \in \mathfrak{g}.$$

Such a Hom-Lie algebra will be denoted by $(\mathfrak{g}, [\cdot,\cdot], \alpha)$.

A *representation of a Hom-Lie algebra* $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \alpha)$ is a triplet $(V, [\cdot,\cdot]_V, \beta)$, where V is a vector space, $\beta \in \mathfrak{gl}(V)$, and $[\cdot,\cdot]_V$ is the action of $\mathfrak{g}$ on V such that (for all $x, y \in \mathfrak{g}$ and for all $v \in V$):

$$[\alpha(x), \beta(v)]_V = \beta([x,v]_V), \quad [[x,y]_{\mathfrak{g}}, \beta(v)]_V = [\alpha(x), [y,v]_V]_V + [\alpha(y), [x,v]_V]_V. \tag{5}$$

Writing Equation (5) using the notation of Equation (4), we put $\rho_\beta := [\cdot,\cdot]_V$ and obtain (for all $x, y \in \mathfrak{g}$):

$$\rho_\beta(\alpha(x)) \circ \beta = \beta \circ \rho(x), \quad \rho([x,y]_{\mathfrak{g}}) \circ \beta = \rho(\alpha(x))\rho(y) + \rho(\alpha(y))\rho(x). \tag{6}$$

2.3. Hom-Lie Superalgebras in Characteristic 2

Our main definition is given below. Due to the presence of the squaring, our approach to define Hom-Lie superalgebras in characteristic 2 will differ from that used in characteristics $p \neq 2$; see [9].

Definition 1. *A Hom-Lie superalgebra in characteristic 2 is a quadruple $(\mathfrak{g}, [\cdot,\cdot], s, \alpha)$ consisting of a $\mathbb{Z}/2\mathbb{Z}$-graded superspace $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ over $\mathbb{K}$, a symmetric bracket $[\cdot,\cdot]$, a squaring $s : \mathfrak{g}_{\bar{1}} \to \mathfrak{g}_{\bar{0}}$, and an even map $\alpha \in \mathrm{End}(\mathfrak{g})$ such that:*

(*i*) $(\mathfrak{g}_{\bar{0}}, [\cdot,\cdot], \alpha|_{\mathfrak{g}_{\bar{0}}})$ *is an ordinary Hom-Lie algebra;*

(*ii*) $\mathfrak{g}_{\bar{1}}$ *is a* $\mathfrak{g}_{\bar{0}}$*-module made two-sided by symmetry, where the action is still denoted by the bracket* $[\cdot,\cdot]$*;*

(*iii*) *The map*

$$\mathfrak{g}_{\bar{1}} \times \mathfrak{g}_{\bar{1}} \to \mathfrak{g}_{\bar{0}} \quad (x,y) \mapsto s(x+y) + s(x) + s(y) \tag{7}$$

is bilinear and induces the bracket on odd elements; namely, for any $x, y \in \mathfrak{g}_{\bar{1}}$*:*

$$[x,y] := s(x+y) + s(x) + s(y);$$

(*iv*) *The following three conditions hold:*

$$\begin{aligned} [s(x), \alpha(y)] &= [\alpha(x), [x,y]] \text{ for any } x \in \mathfrak{g}_{\bar{1}} \text{ and } y \in \mathfrak{g}, & (8)\\ \alpha([x,y]) &= [\alpha(x), \alpha(y)] \text{ for any } x, y \in \mathfrak{g}, & (9)\\ \alpha(s(x)) &= s(\alpha(x)) \text{ for any } x \in \mathfrak{g}_{\bar{1}}. & (10) \end{aligned}$$

Remark 2. (*i*) *The Jacobi identity on triples in* $\{\mathfrak{g}_0, \mathfrak{g}_1, \mathfrak{g}_1\}$ *and* $\{\mathfrak{g}_1, \mathfrak{g}_1, \mathfrak{g}_1\}$ *follow from condition (8). We, therefore, recover the usual definition of Hom-Lie superalgebras [9].*

(*ii*) *Since we are working over a field of characteristic 2, skew symmetry and symmetry coincide since* $-1 \equiv 1 \pmod 2$*.*

(*iii*) *We may want to consider Hom-Lie superalgebras in characteristic 2 without conditions* (9) *and* (10)*, which corresponds to the multiplicativity of the structure map* α*.*

Let $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, s_{\mathfrak{g}}, \alpha)$ and $(\mathfrak{g}', [\cdot,\cdot]_{\mathfrak{g}'}, s_{\mathfrak{g}'}, \alpha')$ be two Hom-Lie superalgebras in characteristic 2. A map $\phi : \mathfrak{g} \to \mathfrak{g}'$ is a morphism of Hom-Lie superalgebras if the following conditions are satisfied:

$$\phi([\cdot,\cdot]_{\mathfrak{g}}) = [\phi(\cdot), \phi(\cdot)]_{\mathfrak{g}'}, \quad \phi \circ s_{\mathfrak{g}} = s_{\mathfrak{g}'} \circ \phi, \quad \phi \circ \alpha = \alpha' \circ \phi. \tag{11}$$

Two Hom-Lie superalgebras $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, s_{\mathfrak{g}}, \alpha)$ and $(\mathfrak{g}', [\cdot,\cdot]_{\mathfrak{g}'}, s_{\mathfrak{g}'}, \alpha')$ are called *isomorphic* if there exists a homomorphism $\phi : \mathfrak{g} \to \mathfrak{g}'$ as in (11) that it is bijective.

Let $(\mathfrak{g}, [\cdot,\cdot], s, \alpha)$ be a Hom-Lie superalgebra in characteristic 2. Let I be a subset of $\mathfrak{g}$. The set I is called an ideal of $\mathfrak{g}$ if and only if I is closed under addition and scalar multiplication, together with

$$[I, \mathfrak{g}] \subseteq I, \ \alpha(I) \subseteq I \text{ and } s(x) \in I \text{ whenever } x \in I \cap \mathfrak{g}_{\bar{1}}.$$

In particular, if the ideal I is homogeneous, namely $I = I \cap \mathfrak{g}_{\bar{0}} \oplus I \cap \mathfrak{g}_{\bar{1}} = I_{\bar{0}} \oplus I_{\bar{1}}$, then the condition involving the squaring reads $s(x) \in I_{\bar{0}}$ for all $x \in I_{\bar{1}}$. In addition, the superspace $\mathfrak{g}/I$ is also a Hom-Lie superalgebra in characteristic 2. The bracket and the squaring are defined as follows:

$$\begin{aligned} [x+I, y+I] &:= [x,y] + I \quad \text{for all } x, y \in \mathfrak{g},\\ s(x+I) &:= s(x) + I \quad \text{for all } x \in \mathfrak{g}_{\bar{1}}, \end{aligned}$$

while the twist map $\tilde{\alpha}$ on $\mathfrak{g}/I$ is defined by

$$\tilde{\alpha}(x+I) = \alpha(x) + I \quad \text{for all } x \in \mathfrak{g}.$$

We will only show that the squaring is well-defined. Suppose that $\tilde{x} - x = i \in I_{\bar{1}}$; we have

$$s(\tilde{x}) = s(x+i) = s(x) + s(i) + [x,i] = s(x) \mod (I).$$

In the following proposition, we will show that an ordinary Lie superalgebra together with a morphism gives rise to a Hom-Lie superalgebra structure on the underlying vector space.

Proposition 1. *Let $(\mathfrak{g}, [\cdot,\cdot], s)$ be a Lie superalgebra in characteristic 2, and let $\alpha : \mathfrak{g} \to \mathfrak{g}$ be an even superalgebra morphism. Then, $(\mathfrak{g}, [\cdot,\cdot]_\alpha, s_\alpha, \alpha)$, where $[\cdot,\cdot]_\alpha = \alpha \circ [\cdot,\cdot]$ and $s_\alpha = \alpha \circ s$, is a Hom-Lie superalgebra in characteristic 2.*

Proof. The first part of the proof is given in [9]. We have to check Equations (1) and (8). Indeed, let $\lambda \in \mathbb{K}$, and let $x \in \mathfrak{g}_{\bar{1}}$. We have

$$s_\alpha(\lambda x) = \alpha(s(\lambda x)) = \alpha(\lambda^2 s(x)) = \lambda^2 \alpha(s(x)) = \lambda^2 s_\alpha(x).$$

On the other hand, for any $x \in \mathfrak{g}_{\bar{1}}$ and $y \in \mathfrak{g}$, we have

$$\begin{aligned} &[s_\alpha(x), \alpha(y)]_\alpha = \alpha([\alpha(s(x)), \alpha(y)]) = \alpha^2([s(x), y]) = \alpha^2([x,[x,y]]) \\ &= \alpha([\alpha(x), \alpha([x,y])]) = [\alpha(x), [x,y]_\alpha]_\alpha. \quad \square \end{aligned}$$

More generally, let $(\mathfrak{g}, [\cdot,\cdot], s, \alpha)$ be a Hom-Lie superalgebra in characteristic 2, and let $\beta : \mathfrak{g} \to \mathfrak{g}$ be an even weak superalgebra morphism (the third condition of (11) is not necessarily satisfied). Then, $(\mathfrak{g}, [\cdot,\cdot]_\beta := \beta \circ [\cdot,\cdot], s_\beta := \beta \circ s, \alpha \circ \beta)$ is a Hom-Lie superalgebra in characteristic 2. The proof is similar to that of Proposition 1.

Example 1. Consider the ortho-orthogonal Lie superalgebra $\mathfrak{g} := \mathfrak{oo}^{(1)}_{III}(1|2)$ (see [1,4]) spanned by the even vectors h, x_2, y_2 and the odd vectors x_1, y_1 with the non-zero brackets:

$$[x_1, y_1] = [x_2, y_2] = h, \quad [h, x_1] = x_1, \quad [h, y_1] = y_1, \quad [x_2, y_1] = x_1, \quad [y_2, x_1] = y_1,$$

and the squaring:

$$s(x_1) = x_2, \quad s(y_1) = y_2.$$

Let us define the map α on the vector space underlying $\mathfrak{oo}^{(1)}_{III}(1|2)$:

$$\begin{aligned} &\alpha(x_1) = \delta_1 x_1 + \delta_2 y_1, && \alpha(y_1) = \varepsilon_1 x_1 + \varepsilon_2 y_1, \quad \alpha(x_2) = \lambda_1 h + \lambda_2 x_2 + \lambda_3 y_2, \\ &\alpha(y_2) = \beta_1 h + \beta_2 x_2 + \beta_3 y_2, && \alpha(h) = \gamma_1 h. \end{aligned}$$

A direct computation shows that the map α is a morphism of Lie superalgebras if and only if (where we have put for simplicity $T := 1 + \delta_2\varepsilon_1 + \delta_1\varepsilon_2$):

$$\gamma_1 = (1+T)^2, \quad \beta_1 = \varepsilon_1\varepsilon_2, \quad \beta_2 = \varepsilon_1^2, \quad \beta_3 = \varepsilon_2^2, \quad \lambda_1 = \delta_1\delta_2, \quad \lambda_2 = \delta_1^2, \quad \lambda_3 = \delta_2^2;$$

together with

$$\varepsilon_1 T = \varepsilon_1 T^2 = \varepsilon_2 T = \varepsilon_2 T^2 = \delta_1 T = \delta_1 T^2 = \delta_2 T = \delta_2 T^2 = T(1+T) = 0. \qquad (12)$$

The only solutions to Equation (12) that do not produce the zero map are given by $T = 0$.

We can, therefore, construct a Hom-Lie superalgebra by means of the map α, depending on three parameters, as in Proposition 1. So, we have

$$\begin{aligned} &\alpha(x_1) = \delta_1 x_1 + \delta_2 y_1, && \alpha(y_1) = \varepsilon_1 x_1 + \varepsilon_2 y_1, \quad \alpha(x_2) = \delta_1\delta_2 h + \delta_1^2 x_2 + \delta_2^2 y_2, \\ &\alpha(y_2) = \varepsilon_1\varepsilon_2 h + \varepsilon_1^2 x_2 + \varepsilon_2^2 y_2, && \alpha(h) = h. \end{aligned}$$

such that $\begin{pmatrix} \varepsilon_2 & \varepsilon_1 \\ \delta_2 & \delta_1 \end{pmatrix} \in SL_2(\mathbb{K})$.

In particular, we have the following Hom-Lie superalgebra in characteristic 2, which we denote by $\mathfrak{oo}_{III}^{(1)}(1|2)_\alpha$, defined by the brackets:

$$[x_1, y_1]_\alpha = [x_2, y_2]_\alpha = h,\ [h, x_1]_\alpha = x_1,\ [h, y_1]_\alpha = \varepsilon x_1 + y_1,$$
$$[x_2, y_1]_\alpha = x_1,\ [y_2, x_1]_\alpha = \varepsilon x_1 + y_1,$$

with the corresponding squaring:

$$s(x_1) = x_2, \quad s(y_1) = \varepsilon h + \varepsilon^2 x_2 + y_2,$$

and the twist map:

$$\alpha(x_1) = x_1, \quad \alpha(y_1) = \varepsilon x_1 + y_1, \quad \alpha(x_2) = x_2, \quad \alpha(y_2) = \varepsilon h + \varepsilon^2 x_2 + y_2, \quad \alpha(h) = h,$$

where ε is a parameter in $\mathbb{K}$. We recover the Lie superalgebra $\mathfrak{oo}_{III}^{(1)}(1|2)$ for $\varepsilon = 0$.

2.4. *The Classification In Low Dimensions*

Let us assume here that the field $\mathbb{K}$ is infinite (for instance, algebraically closed). For the classification of Hom-Lie algebras and superalgebras in low dimensions, see [14–20].

2.4.1. The Case $\mathrm{sdim}(\mathfrak{g}) = 1|1$

Assume that $\mathfrak{g}_{\bar 0} = \mathrm{Span}\{e\}$ and $\mathfrak{g}_{\bar 1} = \mathrm{Span}\{f\}$. We set

$$\alpha(e) = \lambda_1 e, \quad \alpha(f) = \lambda_2 f, \quad s_{\mathfrak{g}}(f) = \rho\, e, \quad [e, e] = 0, \quad [e, f] = \gamma f.$$

It follows that $[f, f] = s(2f) - 2s(f) = 2s(f) = 2\rho e = 0$. Calculations on the conditions lead to

$$\rho\gamma\lambda_2 = 0, \quad \lambda_2\gamma = \lambda_2\lambda_1\gamma, \quad \rho\lambda_1 = \rho\lambda_2^2.$$

These are all Hom-Lie superalgebras up to an isomorphism:

(i) Abelian: the twist is given by $\alpha(e) = \lambda_1 e$, $\alpha(f) = \lambda_2 f$, where $(\lambda_1, \lambda_2) \neq (0, 0)$.

(ii) $[e, f] = f, s(f) = 0$: there are two twists given by:

$$\alpha_1(e) = e,\ \alpha_1(f) = \lambda_2 f, \text{ where } \lambda_2 \neq 0, \quad \alpha_2(e) = \lambda_1 e,\ \alpha_2(f) = 0, \text{ where } \lambda_1 \neq 0.$$

(iii) $[e, f] = 0, s(f) = e$: the twist is given by $\alpha(e) = \lambda^2 e, \alpha(f) = \lambda f$, where $\lambda \neq 0$.

As the field $\mathbb{K}$ is infinite, we have a family of Hom-Lie superalgebras.

2.4.2. The Case $\mathrm{sdim}(\mathfrak{g}) = 1|2$

Assume that $\mathfrak{g}_{\bar 0} = \mathrm{Span}\{e\}$ and $\mathfrak{g}_{\bar 1} = \mathrm{Span}\{f_1, f_2\}$. We define the brackets as (where $a_i, b_i \in \mathbb{K}$ for $i, j = 1, 2$):

$$[e, f_1] = a_1 f_1 + a_2 f_2, \quad [e, f_2] = b_1 f_1 + b_2 f_2,$$

and finally, the squaring as (where $\rho_i \in \mathbb{K}$ for $i = 1, 2, 3$):

$$s(f_1) = \rho_1\, e, \quad s(f_2) = \rho_2\, e, \quad s(f_1 + f_2) = \rho_3\, e.$$

Let us consider a linear map α by which we will construct the Hom-structure. As α preserves the $\mathbb{Z}/2\mathbb{Z}$-grading, and by using the Jordan decomposition, we distinguish two cases:

Case 1: Suppose that α is given by (where $s, t_1, r_2 \in \mathbb{K}$):

$$\alpha(e) = se, \quad \alpha(f_1) = t_1 f_1, \qquad \alpha(f_2) = r_2 f_2.$$

A direct computation shows that there are only the following sub-cases:

Sub-case 1a: We have $\rho_3 = \rho_1 + \rho_2,\ \rho_1 \neq 0,\ s = t_1^2,\ \rho_2\,(s + r_2^2) = 0$ and $a_i = b_i = 0$ for $i = 1, 2$. Here are the two possible cases:

$$\rho_3 = \rho_1 + \rho_2,\ \rho_1 \neq 0,\ \rho_2 = 0,\ s = t_1^2,\ a_1 = b_1 = a_2 = b_2 = 0,\ r_2 \text{ arbitrary; or}$$
$$\rho_3 = \rho_1 + \rho_2,\ \rho_1, \rho_2 \neq 0,\ s = t_1^2,\ t_1 = r_2,\ a_1 = b_1 = a_2 = b_2 = 0.$$

Sub-case 1b: We have $\rho_1 = \rho_2 = \rho_3 = 0$ together with

$$b_1(t_1 + sr_2) = 0,\ b_2r_2(1 + s) = 0,\ a_1t_1(1 + s) = 0,\ a_2(r_2 + st_1) = 0.$$

We can disregard this case, because it produces a Lie algebra instead of a Lie superalgebra.

Sub-case 1c: We have $\rho_1 + \rho_2 + \rho_3 \neq 0,\ \rho_1 \neq 0$, together with

$$s = t_1^2 = t_1r_2,\ \rho_2t_1^2 = r_2^2\rho_2,\ a_1 = b_1 = a_2 = b_2 = 0.$$

Here are the two possible cases:

$$\rho_1 + \rho_2 + \rho_3 \neq 0,\ \rho_1 \neq 0,\ \rho_2 = 0,\ s = t_1^2 = t_1r_2,\ r_2 \neq 0,\ a_1 = b_1 = a_2 = b_2 = 0;\ \text{or}$$
$$\rho_1 + \rho_2 + \rho_3 \neq 0,\ \rho_1, \rho_2 \neq 0,\ s = t_1^2,\ t_1 = r_2 \neq 0,\ a_1 = b_1 = a_2 = b_2 = 0.$$

Case 2: Suppose that α is given by (where $s, t_1 \in \mathbb{K}$):

$$\alpha(e) = se, \quad \alpha(f_1) = t_1f_1, \qquad \alpha(f_2) = f_1 + t_1f_2.$$

A direct computation shows that there are only the following sub-cases:

Subcase 2a: We have $\rho_1, \rho_2 \neq 0$, but ρ_3 arbitrary, together with

$$a_1 = a_2 = b_1 = b_2 = 0,\ s = t_1^2,\ \rho_1(1 + t_1) = t_1(\rho_2 + \rho_3).$$

Subcase 2b: We have $\rho_1, \rho_3 \neq 0$, but $\rho_2 = 0$ arbitrary, together with

$$a_1 = a_2 = b_1 = b_2 = 0,\ s = t_1^2,\ \rho_1(1 + t_1) = t_1\rho_3.$$

Subcase 2c: We have $\rho_1 \neq 0$, but $\rho_2 = \rho_3 = 0$, together with

$$a_1 = a_2 = b_1 = b_2 = 0,\ s = 1,\ t_1 = 1.$$

Subcase 2d: We have $\rho_1 = 0,\ \rho_2 \neq 0$, but ρ_3 arbitrary, together with

$$a_1 = a_2 = b_1 = b_2 = 0,\ s = t_1^2,\ t_1(\rho_2 + \rho_3) = 0.$$

Subcase 2: We have $\rho_1 = \rho_2 = 0$, but $\rho_3 \neq 0$, together with

$$a_1 = a_2 = b_1 = b_2 = 0,\ s = 0,\ t_1 = 0.$$

The tables below summarize our finding. We find it convenient to order the Hom-Lie superalgebras into two groups: (i) type I comprises those for which the $\mathfrak{g}_0$-module structure on $\mathfrak{g}_{\bar{1}}$ is trivial; (ii) type II comprises those for which the $\mathfrak{g}_0$-module structure on $\mathfrak{g}_{\bar{1}}$ is not trivial.

Remark 3. *We do not explore the possibility of isomorphisms between the Hom-Lie superalgebras in Tables 1–4.*

Table 1. Type I (i.e., $[\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{1}}] = \{0\}$) with $\alpha(e) = se,\ \alpha(f_1) = t_1 f_1,\ \alpha(f_2) = r_2 f_2$.

The HLSA	The Squaring s	The Conditions
A_1	$s(f_1) = \rho_1 e,$ $s(f_2) = 0,$ $s(f_1 + \lambda f_2) = \rho_1 e$	$\rho_1 \neq 0,\ s = t_1^2,$ r_2 arbitrary
A_2	$s(f_1) = \rho_1 e,$ $s(f_2) = \rho_2 e,$ $s(f_1 + \lambda f_2) = (\rho_1 + \lambda^2 \rho_2) e$	$\rho_1, \rho_2 \neq 0,$ $s = t_1^2, r_2 = t_1$
A_3	$s(f_1) = \rho_1 e,$ $s(f_2) = 0,$ $s(f_1 + \lambda f_2) = ((1+\lambda)\rho_1 + \lambda \rho_3) e$	$\rho_1 \neq 0,\ \rho_1 + \rho_3 \neq 0,$ $s = t_1^2 = t_1 r_2,\ r_2 \neq 0$
A_4	$s(f_1) = \rho_1 e,$ $s(f_2) = \rho_2 e,$ $s(f_1 + \lambda f_2) = \lambda(\rho_1 + (1+\lambda)\rho_2 + \rho_3) e$ $+\rho_1 e$	$\rho_1, \rho_2 \neq 0,\ \rho_1 + \rho_2 + \rho_3 \neq 0,$ $s = t_1^2,\ t_1 = r_2,\ r_2 \neq 0$

Table 2. Type I (i.e. $[\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{1}}] = \{0\}$) with $\alpha(e) = se,\ \alpha(f_1) = t_1 f_1,\ \alpha(f_2) = f_1 + t_1 f_2$.

The HLSA	The Squaring s	The Conditions
A_5	$s(f_1) = \rho_1 e,$ $s(f_2) = \rho_2 f_2,$ $s(f_1 + \lambda f_2) = \lambda(\rho_1 + (1+\lambda)\rho_2) e$ $+(\lambda \rho_3 + \rho_1) e$	$\rho_1, \rho_2 \neq 0,\ \rho_3 = \dfrac{1+t_1}{t_1}\rho_1 + \rho_2,$ $s = t_1^2$
A_6	$s(f_1) = \rho_1 e,$ $s(f_2) = 0,$ $s(f_1 + \lambda f_2) = (\lambda(\rho_1 + \rho_3) + \rho_1) e$	$\rho_1, \rho_3 \neq 0,\ \rho_3 = \dfrac{1+t_1}{t_1}\rho_1,$ $s = t_1^2$
A_7	$s(f_1) = \rho_1 e,$ $s(f_2) = 0,$ $s(f_1 + \lambda f_2) = \rho_1(1+\lambda) e$	$\rho_1 \neq 0,$ $s = t_1 = 1$
A_8	$s(f_1) = 0,$ $s(f_2) = \rho_2 e,$ $s(f_1 + \lambda f_2) = \lambda(\rho_2 + \rho_3 + \lambda \rho_2) e$	$\rho_2 \neq 0,\ \rho_3$ arbitrary, $s = t_1 = 0,$
A_9	$s(f_1) = 0,$ $s(f_2) = \rho_2 e,$ $s(f_1 + \lambda f_2) = \lambda^2 \rho_2 e$	$\rho_2 \neq 0,$ $s = t_1^2,\ t_1 \neq 0$
A_{10}	$s(f_1) = 0,$ $s(f_2) = 0,$ $s(f_1 + \lambda f_2) = \lambda \rho_3 e$	$\rho_3 \neq 0,$ $s = t_1 = 0$

Table 3. Type II (i.e., $[\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{1}}] \neq \{0\}$) with $\alpha(e) = se,\ \alpha(f_1) = t_1 f_1,\ \alpha(f_2) = r_2 f_2$.

The HLSA	The Squaring s	$[\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{1}}]$	The Conditions
B_1	$s(f_1) = 0,\ s(f_2) = 0,\ s(f_1 + \lambda f_2) = 0$	$[e, f_1] = a_1 f_1 + a_2 f_2,\ [e, f_2] = b_1 f_1 + b_2 f_2$	$s = 1$, $t_1 = r_2$, a_1, a_2, b_1, b_2 arbitrary
B_2	$s(f_1) = 0,\ s(f_2) = 0,\ s(f_1 + \lambda f_2) = 0$	$[e, f_1] = a_1 f_1,\ [e, f_2] = b_2 f_2$	$s = 1$, $t_1 \neq r_2$, a_1, b_2 arbitrary
B_3	$s(f_1) = 0,\ s(f_2) = 0,\ s(f_1 + \lambda f_2) = 0$	$[e, f_1] = a_1 f_1 + a_2 f_2,\ [e, f_2] = b_1 f_1 + b_2 f_2$	$s \neq 0, 1$, $t_1 = r_2 = 0$, a_1, a_2, b_1, b_2 arbitrary
B_4	$s(f_1) = 0,\ s(f_2) = 0,\ s(f_1 + \lambda f_2) = 0$	$[e, f_1] = 0,\ [e, f_2] = b_2 f_2$	$s \neq 0, 1$, $t_1 \neq 0, r_2 = 0$, b_2 arbitrary
B_5	$s(f_1) = 0,\ s(f_2) = 0,\ s(f_1 + \lambda f_2) = 0$	$[e, f_1] = 0,\ [e, f_2] = b_1 f_1$	$s \neq 0, 1$, $t_1 = rs_2$, $b_1 \neq 0$ and arbitrary
B_6	$s(f_1) = 0,\ s(f_2) = 0,\ s(f_1 + \lambda f_2) = 0$	$[e, f_1] = a_2 f_2,\ [e, f_2] = 0$	$s \neq 0, 1$, $t_1 \neq rs_2$, $a_2 \neq 0$ and arbitrary
B_7	$s(f_1) = 0,\ s(f_2) = 0,\ s(f_1 + \lambda f_2) = 0$	$[e, f_1] = a_2 f_2,\ [e, f_2] = b_2 f_2$	$s \neq 0, 1$, $r_2 = 0$, a_2, b_2 arbitrary

Table 4. Type II (i.e., $[\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{1}}] \neq \{0\}$) with $\alpha(e) = se,\ \alpha(f_1) = t_1 f_1,\ \alpha(f_2) = f_1 + t_1 f_2$.

The HLSA	The Squaring s	$[\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{1}}]$	The Conditions
B_8	$s(f_1) = 0,\ s(f_2) = 0,\ s(f_1 + \lambda f_2) = 0$	$[e, f_1] = a_1 f_1,\ [e, f_2] = b_1 f_1 + a_1 f_2$	$s = 1$, $t_1 = 0$, a_1, b_1 arbitrary
B_9	$s(f_1) = 0,\ s(f_2) = 0,\ s(f_1 + \lambda f_2) = 0$	$[e, f_1] = 0,\ [e, f_2] = b_1 f_1$	$s \neq 1$, $t_1 = 0$, $b_1 \neq 0$ and arbitrary
B_{10}	$s(f_1) = 0,\ s(f_2) = 0,\ s(f_1 + \lambda f_2) = 0$	$[e, f_1] = a_1 f_1,\ [e, f_2] = a_1 f_2$	$s = 1$, $t_1 \neq 0$, $a_1 \neq 0$ and arbitrary

3. Representations and Semidirect Product

Definition 2. *A representation of a Hom-Lie superalgebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, s_{\mathfrak{g}}, \alpha)$ is a triple $(V, [\cdot,\cdot]_V, \beta)$, where V is a superspace, β is an even map in $\mathfrak{gl}(V)$, and $[\cdot,\cdot]_V$ is the action of $\mathfrak{g}$ on V such that*

$$\begin{aligned} [\alpha(x), \beta(v)]_V &= \beta([x, v]_V) \ \text{ for any } x \in \mathfrak{g} \text{ and } v \in V, \\ [[x, y]_{\mathfrak{g}}, \beta(v)]_V &= [\alpha(x), [y, v]_V]_V + [\alpha(y), [x, v]_V]_V \ \text{ for any } x, y \in \mathfrak{g} \text{ and } v \in V, \\ [s_{\mathfrak{g}}(x), \beta(v)]_V &= [\alpha(x), [x, v]_V]_V \ \text{ for any } x \in \mathfrak{g}_{\bar{1}} \text{ and } v \in V. \end{aligned} \tag{13}$$

We say that V is a $\mathfrak{g}$-module.

Sometimes, it is more convenient to use the notation $\rho_\beta = [\cdot,\cdot]_V$ and write:

$$\begin{aligned} \rho_\beta \circ \alpha(x) &= \beta \circ \rho_\beta(x) \ \text{ for any } x \in \mathfrak{g}, \\ \rho_\beta([x,y]_\mathfrak{g}) \circ \beta &= \rho_\beta(\alpha(x))\rho(y) + \rho_\beta(\alpha(y))\rho(x) \ \text{ for any } x, y \in \mathfrak{g}, \\ \rho_\beta \circ s_\mathfrak{g}(x) \circ \beta &= \rho_\beta(\alpha(x)) \circ \rho_\beta(x) \ \text{ for any } x \in \mathfrak{g}_{\bar{1}}. \end{aligned} \tag{14}$$

Theorem 1. *Let $(\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g}, s_\mathfrak{g}, \alpha)$ be a Hom-Lie superalgebra and $(V, [\cdot,\cdot]_V, \beta)$ be a representation. With the above notation, we define a Hom-Lie superalgebra structure on the superspace $\mathfrak{g} \oplus V = (\mathfrak{g}_{\bar{0}} + V_{\bar{0}}) \oplus (\mathfrak{g}_{\bar{1}} + V_{\bar{1}})$, where the bracket is defined by*

$$[x+v, y+w]_{\mathfrak{g}\oplus V} = [x,y]_\mathfrak{g} + [x,w]_V + [y,v]_V \text{ for any } x, y \in \mathfrak{g} \text{ and } v, w \in V,$$

the squaring $s_{\mathfrak{g}+V} : \mathfrak{g}_{\bar{1}} + V_{\bar{1}} \to \mathfrak{g}_{\bar{0}} + V_{\bar{0}}$ is defined by

$$s_{\mathfrak{g}+V}(x+v) = s_\mathfrak{g}(x) + [x,v]_V \text{ for any } x \in \mathfrak{g}_{\bar{1}} \text{ and } v \in V_{\bar{1}},$$

and the structure map $\alpha_{\mathfrak{g}\oplus V} : \mathfrak{g} \oplus V \to \mathfrak{g} \oplus V$ is defined by

$$\alpha_{\mathfrak{g}\oplus V}(x+v) = \alpha(x) + \beta(v) \text{ for any } x \in \mathfrak{g} \text{ and } v \in V.$$

The Hom-Lie superalgebra $(\mathfrak{g} \oplus V, [\cdot,\cdot]_{\mathfrak{g}\oplus V}, s_{\mathfrak{g}+V}, \alpha_{\mathfrak{g}\oplus V})$ is called the semidirect product of $(\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g}, s_\mathfrak{g}, \alpha)$ by the representation $(V, [\cdot,\cdot]_V, \beta)$.

Proof. Checking Axioms (i) and (ii) of Definition 1 is routine; we can refer to [9]. We should check the conditions relative to the squaring. Let us first check that the map $s_\mathfrak{g} \oplus V$ is indeed a squaring. We will check only the first condition. For all $x + v \in \mathfrak{g}_{\bar{1}} \oplus V_{\bar{1}}$ and for all $\lambda \in \mathbb{K}$, we have

$$s_{\mathfrak{g}\oplus V}(\lambda(x+v)) = s_\mathfrak{g}(\lambda x) + [\lambda x, \lambda v]_V = \lambda^2 s_\mathfrak{g}(x) + \lambda^2 [x,v]_V = \lambda^2 s_{\mathfrak{g}\oplus V}(x+v).$$

Now, for all $x + v \in \mathfrak{g}_{\bar{1}} \oplus V_{\bar{1}}$ and for all $y + w \in \mathfrak{g} \oplus V$, we have

$$\begin{aligned} &[s_{\mathfrak{g}\oplus V}(x+v), \alpha_{\mathfrak{g}\oplus V}(y+w)]_{\mathfrak{g}\oplus V} = [s_\mathfrak{g}(x) + [x,v]_V, \alpha(y) + \beta(w)]_{\mathfrak{g}\oplus V} \\ &= [s_\mathfrak{g}(x), \alpha(y)]_\mathfrak{g} + [s_\mathfrak{g}(x), \beta(w)]_V + [\alpha(y), [x,v]_V]_V \\ &= [\alpha(x), [x,y]_\mathfrak{g}]_\mathfrak{g} + [\alpha(x), [x,w]_V]_V + [\alpha(y), [x,v]_V]_V. \end{aligned}$$

On the other hand,

$$\begin{aligned} &[\alpha_{\mathfrak{g}\oplus V}(x+v), [x+v, y+w]_{\mathfrak{g}\oplus V}]_{\mathfrak{g}\oplus V} = [\alpha(x) + \beta(v), [x,y]_\mathfrak{g} + [x,w]_V + [y,v]_V]_{\mathfrak{g}\oplus V} \\ &= [\alpha(x), [x,y]_\mathfrak{g}]_\mathfrak{g} + [\alpha(x), [x,w]_V + [y,v]_V]_V + [[x,y]_\mathfrak{g}, \beta(v)]_V \\ &= [\alpha(x), [x,y]_\mathfrak{g}]_\mathfrak{g} + [\alpha(x), [x,w]_V + [y,v]_V]_V + [\alpha(x), [y,v]_V]_V + [\alpha(y), [x,v]_V]_V \\ &= [\alpha(x), [x,y]_\mathfrak{g}]_\mathfrak{g} + [\alpha(x), [x,w]_V]_V + [\alpha(y), [x,v]_V]_V. \end{aligned}$$

Therefore, Equation (8) is satisfied. Now,

$$\begin{aligned} &\alpha_{\mathfrak{g}\oplus V}(s_{\mathfrak{g}\oplus V}(x+v)) = \alpha_{\mathfrak{g}\oplus V}(s_\mathfrak{g}(x) + [x,v]_V) = \alpha(s_\mathfrak{g}(x)) + \beta([x,v]_V) \\ &= \alpha(s_\mathfrak{g}(x)) + [\alpha(x), \beta(v)]_V = s_\mathfrak{g}(\alpha(x)) + [\alpha(x), \beta(v)]_V = s_{\mathfrak{g}\oplus V}(\alpha(x) + \beta(v)) \\ &= s_{\mathfrak{g}\oplus V}(\alpha_{\mathfrak{g}\oplus V}(x+v)). \end{aligned}$$

Therefore, Equation (10) is satisfied. $\square$

In the following proposition, we show how to twist a Lie superalgebra and its representation into a Hom-Lie superalgebra together with a representation in characteristic 2.

Proposition 2. *Let $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, s_{\mathfrak{g}})$ be a Lie superalgebra and (V, ρ) a representation. Let $\alpha : \mathfrak{g} \to \mathfrak{g}$ be an even superalgebra morphism and $\beta \in \mathfrak{gl}(V)$ be a linear map such that $\rho(\alpha(x)) \circ \beta = \beta \circ \rho(x)$. Then, $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g},\alpha}, s_{\mathfrak{g},\alpha}, \alpha)$, where $[\cdot,\cdot]_{\mathfrak{g},\alpha} = \alpha \circ [\cdot,\cdot]_{\mathfrak{g}}$ and $s_{\mathfrak{g},\alpha} = \alpha \circ s_{\mathfrak{g}}$, is a Hom-Lie superalgebra and (V, ρ_β, β), where $\rho_\beta = \beta \circ \rho$, is a representation.*

Proof. We have already proven in Proposition 1 that $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g},\alpha}, s_{\mathfrak{g},\alpha}, \alpha)$ is a Hom-Lie superalgebra. Let us check that (V, ρ_β, β) is a representation with respect to $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g},\alpha}, s_{\mathfrak{g},\alpha}, \alpha)$. Indeed, the first condition is provided by the hypothesis, while the second and the third ones are straightforward. Let us check the last one. For any $x \in \mathfrak{g}_{\bar{1}}$ and $v \in V$, we have

$$[s_{\mathfrak{g},\alpha}(x), \beta(v)]_{V,\beta} = \beta([\alpha(s_{\mathfrak{g}}(x)), \beta(v)]_V) = \beta^2([s_{\mathfrak{g}}(x), v]_V),$$

and

$$[\alpha(x), [x, v]_{V,\beta}]_{V,\beta} = \beta[\alpha(x), \beta([x, v]_V]_V) = \beta^2([x, [x, v]_V]_V).$$

The equality follows from the fact that $[s_{\mathfrak{g}}(x), v]_V = [x, [x, v]_V]_V$. $\square$

Example 2. *The classification of irreducible modules over $\mathfrak{oo}^{(1)}_{III}(1|2)$ having the highest weight vectors was carried out in [21]. We will borrow here the simplest example. Consider the Hom-Lie superalgebra $\mathfrak{oo}^{(1)}_{III}(1|2)$ with the twist α given as in Example 1. We consider the $\mathfrak{oo}^{(1)}_{III}(1|2)$-module M with basis: (even | odd)*

$$m_1, m_3 \quad | \quad m_2.$$

The vector m_1 is the highest weight vector with weight $(m_1) = (1)$. The map β is given as follows:

$$\beta(m_1) = \delta_1 m_1 + \delta_2 m_3, \quad \beta(m_3) = \varepsilon_1 m_1 + \varepsilon_2 m_3, \quad \beta(m_2) = m_2,$$

where the coefficients $\delta_1, \delta_2, \varepsilon_1, \varepsilon_2$ are given as in Example 1.

Here, we will introduce another point of view concerning the representations of Hom-Lie superalgebras in characteristic 2, inspired by [22].

Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a vector superspace, and let $\beta \in GL(V)$ be an even map. We will define a bracket on $\mathfrak{gl}(V)$, as well as a product as follows (where β^{-1} is the inverse of β):

$$[f, g]_{\mathfrak{gl}(V)} := \beta \circ f \circ \beta^{-1} g \circ \beta^{-1} + \beta \circ g \circ \beta^{-1} f \circ \beta^{-1} \quad \text{for all } f, g \in \mathfrak{gl}(V), \tag{15}$$

$$s_{\mathfrak{gl}(V)}(f) := \beta \circ f \circ \beta^{-1} f \circ \beta^{-1} \quad \text{for all } f \in \mathfrak{gl}(V)_{\bar{1}}. \tag{16}$$

Obviously, $s_{\mathfrak{gl}(V)}(\lambda f) = \lambda^2 s_{\mathfrak{gl}(V)}(f)$ for all $\lambda \in \mathbb{K}$ and for all $f \in \mathfrak{gl}(V)_{\bar{1}}$. Now, the map:

$$(f, g) \mapsto s_{\mathfrak{gl}(V)}(f + g) + s_{\mathfrak{gl}(V)}(f) + s_{\mathfrak{gl}(V)}(g) = \beta \circ f \circ \beta^{-1} g \circ \beta^{-1} + \beta \circ g \circ \beta^{-1} f \circ \beta^{-1}$$

is obviously bilinear on $\mathfrak{gl}(V)_{\bar{1}}$ as well.

Denote by $\mathrm{Ad}_\beta : \mathfrak{gl}(V) \to \mathfrak{gl}(V)$ the adjoint action on $\mathfrak{gl}(V)$, i.e., $\mathrm{Ad}_\beta(f) = \beta \circ f \circ \beta^{-1}$.

Proposition 3. *The brackets and the squaring defined in Equations (15) and (16) make $(\mathfrak{gl}(V), [\cdot,\cdot]_{\mathfrak{gl}(V)}, s_{\mathfrak{gl}(V)}, \mathrm{Ad}_\beta)$ a Hom-Lie superalgebra in characteristic 2.*

Proof. The map Ad_β is invertible with inverse $\mathrm{Ad}_{\beta^{-1}}$. Let us check the multiplicativity conditions:

$$\begin{aligned}
&[\mathrm{Ad}_\beta(f), \mathrm{Ad}_\beta(g)]_{\mathfrak{gl}(V)} = [\beta \circ f \circ \beta^{-1}, \beta \circ g \circ \beta^{-1}]_{\mathfrak{gl}(V)} \\
&= \beta \circ (\beta \circ f \circ \beta^{-1}) \circ \beta^{-1}(\beta \circ g \circ \beta^{-1}) \circ \beta^{-1} + \beta \circ (\beta \circ g \circ \beta^{-1}) \circ \beta^{-1}(\beta \circ f \circ \beta^{-1}) \circ \beta^{-1} \\
&= \beta \circ (\beta \circ f \circ \beta^{-1} \circ g \circ \beta^{-1}) \circ \beta^{-1} + \beta \circ (\beta \circ g \circ \beta^{-1} \circ f \circ \beta^{-1}) \circ \beta^{-1} \\
&= \mathrm{Ad}_\beta([f,g]_{\mathfrak{gl}(V)}).
\end{aligned}$$

Similarly,

$$\begin{aligned}
&s_{\mathfrak{gl}(V)}(\mathrm{Ad}_\beta(f)) = s_{\mathfrak{gl}(V)}(\beta \circ f \circ \beta^{-1}) = \beta \circ (\beta \circ f \circ \beta^{-1}) \circ \beta^{-1} \circ (\beta \circ f \circ \beta^{-1}) \circ \beta^{-1} \\
&= \beta \circ (\beta \circ f \circ \beta^{-1} \circ f \circ \beta^{-1}) \circ \beta^{-1} = \mathrm{Ad}_\beta(s_{\mathfrak{gl}(V)}).
\end{aligned}$$

For the Jacobi identity, let us just deal with the squaring. The LHS of the Jacobi identity reads (for all $f \in \mathfrak{gl}(V)_{\bar{1}}$ and for all $g \in \mathfrak{gl}(V)$)

$$\begin{aligned}
[s_{\mathfrak{gl}(V)}(f), \mathrm{Ad}_\beta(g)]_{\mathfrak{gl}(V)} &= \beta \circ s_{\mathfrak{gl}(V)}(f) \circ \beta^{-1} \circ \beta \circ g \circ \beta^{-1} \circ \beta^{-1} + \\
&\quad + \beta \circ \beta \circ g \circ \beta^{-1} \circ \beta^{-1} \circ s_{\mathfrak{gl}(V)}(f) \circ \beta^{-1} \\
&= \beta^2 \circ (f \circ \beta^{-1} \circ f \circ \beta^{-1} \circ g + g \circ \beta^{-1} \circ f \circ \beta^{-1} \circ f) \circ \beta^{-2}.
\end{aligned}$$

The RHS reads

$$\begin{aligned}
[\mathrm{Ad}_\beta(f), [f,g]_{\mathfrak{gl}(V)}]_{\mathfrak{gl}(V)} &= \beta^2 \circ f \circ \beta^{-2} \circ [f,g]_{\mathfrak{gl}(V)} \circ \beta^{-1} + \beta \circ [f,g]_{\mathfrak{gl}(V)} \circ f \circ \beta^{-1} \circ \beta^{-1} \\
&= \beta^2 \circ f \circ \beta^{-2} \circ (\beta \circ f \circ \beta^{-1} g \circ \beta^{-1} + \beta \circ g \circ \beta^{-1} f \circ \beta^{-1}) \circ \beta^{-1} \\
&\quad + \beta \circ (\beta \circ f \circ \beta^{-1} g \circ \beta^{-1} + \beta \circ g \circ \beta^{-1} f \circ \beta^{-1}) \circ f \circ \beta^{-2} \\
&= \beta^2 \circ (f \circ \beta^{-1} \circ f \circ \beta^{-1} \circ g + g \circ \beta^{-1} \circ f \circ \beta^{-1} \circ f) \circ \beta^{-2}. \quad \square
\end{aligned}$$

Theorem 2. *Let $(\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g}, s_\mathfrak{g}, \alpha)$ be a Hom-Lie superalgebra in characteristic 2. Let V be a vector superspace, and let $\beta \in GL(V)$ be even. Then, the map $\rho_\beta : \mathfrak{g} \to \mathfrak{gl}(V)$ is a representation of $(\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g}, s_\mathfrak{g}, \alpha)$ on V with respect to β if and only if the map $\rho_\beta : (\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g}, s_\mathfrak{g}, \alpha) \to (\mathfrak{gl}(V), [\cdot,\cdot]_{\mathfrak{gl}(V)}, s_{\mathfrak{gl}(V)}, \mathrm{Ad}_\beta)$ is a morphism of Hom-Lie superalgebras.*

Proof. Let us only prove one direction. Suppose that $\rho_\beta : \mathfrak{g} \to \mathfrak{gl}(V)$ is a representation of $(\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g}, s_\mathfrak{g}, \alpha)$ on V with respect to β. Since $\rho_\beta(\alpha(x)) \circ \beta = \beta \circ \rho(x)$, for all $f \in \mathfrak{g}$, it follows that

$$\rho_\beta(x) \circ \alpha = \beta \circ \rho(x) \circ \beta^{-1} = \mathrm{Ad}_\beta \circ \rho_\beta(x).$$

Now,

$$\begin{aligned}
\rho_\beta([x,y]_\mathfrak{g}) &= \rho_\beta(\alpha(x)) \circ \rho(y) \circ \beta^{-1} + \rho_\beta(\alpha(y)) \circ \rho(x) \circ \beta^{-1} \\
&= \rho_\beta(\alpha(x)) \circ \beta \circ \beta^{-1} \circ \rho_\beta(y) \circ \beta^{-1} + \rho_\beta(\alpha(y)) \circ \beta \circ \beta^{-1} \circ \rho_\beta(x) \circ \beta^{-1} \\
&= \beta \circ \rho_\beta(x) \circ \beta^{-1} \circ \rho_\beta(y) \circ \beta^{-1} + \beta \circ \rho_\beta(y) \circ \beta^{-1} \circ \rho_\beta(x) \circ \beta^{-1} \\
&= [\rho_\beta(x), \rho_\beta(y)]_{\mathfrak{gl}(V)}
\end{aligned}$$

For the squaring, we have

$$\begin{aligned}
\rho_\beta(s_\mathfrak{g}(x)) &= \rho_\beta(\alpha(x)) \circ \rho_\beta(x) \circ \beta^{-1} \\
&= \beta \circ \rho_\beta(x) \circ \beta^{-1} \circ \rho_\beta(x) \circ \beta^{-1} \\
&= s_{\mathfrak{gl}(V)}(\rho_\beta(x)).
\end{aligned}$$

It follows that ρ_β is a homomorphism of Hom-Lie superalgebras in characteristic 2. □

Corollary 1. *Let* $(\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g}, s_\mathfrak{g}, \alpha)$ *be a Hom-Lie superalgebra in characteristic 2. Then, the adjoint representation* $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$*, which is defined by* $\mathrm{ad}_x(y) = [x,y]_\mathfrak{g}$*, is a morphism from* $(\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g}, \alpha)$ *to* $(\mathfrak{gl}(\mathfrak{g}), [\cdot,\cdot]_{\mathfrak{gl}(\mathfrak{g})}, s_{\mathfrak{gl}(\mathfrak{g})}, \mathrm{Ad}_\alpha)$*.*

4. α^k-Derivations

Let $(\mathfrak{g}, [\cdot,\cdot], s, \alpha)$ be a Hom-Lie superalgebra in characteristic 2. We denote by α^k the k-times composition of α, where α^0 is the identity map. We will need the following linear map:

$$\mathrm{ad}_{\alpha^{s,k}}(x) : y \mapsto [\alpha^s(x), \alpha^k(y)]. \tag{17}$$

Definition 3. *A linear map* $D : \mathfrak{g} \to \mathfrak{g}$ *is called an* α^k*-derivation of the Hom-Lie superalgebra* $\mathfrak{g}$ *if*

$$\begin{aligned} D \circ \alpha &= \alpha \circ D, \text{ namely } D \text{ and } \alpha \text{ commutes.} && (18)\\ D([x,y]) &= [D(x), \alpha^k(y)] + [\alpha^k(x), D(y)] \quad \text{for any } x \in \mathfrak{g}_{\bar{0}} \text{ and } y \in \mathfrak{g}\,. && (19)\\ D(s(x)) &= [D(x), \alpha^k(x)] \quad \text{for any } x \in \mathfrak{g}_{\bar{1}}. && (20) \end{aligned}$$

Remark 4. *Notice that condition (20) implies condition (19) if* $x, y \in \mathfrak{g}_{\bar{1}}$*.*

Let us give an example. Let $x \in \mathfrak{g}$ such that $\alpha(x) = x$. The linear map $\mathrm{ad}_{\alpha^{0,k}}(x) : y \mapsto [x, \alpha^k(y)]$ (see Equation (17)) is an α^k-derivation. Let us just check the condition related to the squaring. Indeed,

$$\mathrm{ad}_{\alpha^{0,k}}(x)(s(y)) = [x, \alpha^k(s(y))] = [x, s(\alpha^k(y))] = [[x, \alpha^k(y)], \alpha^k(y)] = [\mathrm{ad}_{\alpha^{0,k}}(x)(y), \alpha^k(y)].$$

Let us denote the space of α^k-derivations by $\mathfrak{der}^\alpha(\mathfrak{g})$. We have the following proposition.

Proposition 4. *The space* $\mathfrak{der}^\alpha(\mathfrak{g})$ *can be endowed with a Lie superalgebra structure in characteristic 2. The bracket is the usual commutator, and the squaring is given by*

$$s_{\mathfrak{der}^\alpha(\mathfrak{g})}(D) := D^2 \quad \text{for all } D \in \mathfrak{der}^\alpha_{\bar{1}}(\mathfrak{g}).$$

Proof. As we did before, we only prove the requirements when the squaring is involved. Let us first show that D^2 is an α^{2k}-derivation. Checking the bracket is routine. For the squaring, we have (for all $x \in \mathfrak{g}_{\bar{1}}$):

$$\begin{aligned} D^2(s_\mathfrak{g}(x)) &= D([D(x), \alpha^k(x)]_\mathfrak{g}) = [D^2(x), \alpha^k(\alpha^k(x))]_\mathfrak{g} + [\alpha^k(D(x)), D(\alpha^k(x))]_\mathfrak{g}\\ &= [D^2(x), \alpha^{2k}(x)]_\mathfrak{g} + [\alpha^k(D(x)), \alpha^k(D(x))]_\mathfrak{g} = [D^2(x), \alpha^{2k}(x)]_\mathfrak{g}. \end{aligned}$$

Before we proceed with the proof, let us re-denote the space $\mathfrak{der}^\alpha(\mathfrak{g})$ by $\mathfrak{h}$ for simplicity. Now, for all $D \in \mathfrak{h}_{\bar{1}}$ and for all $E \in \mathfrak{h}_{\bar{1}}$, we have (for all $x \in \mathfrak{g}$):

$$[s_\mathfrak{h}(D), E]_\mathfrak{h}(x) = [D^2, E]_\mathfrak{h}(x) = D^2 \circ E(x) + E \circ D^2(x).$$

On the other hand,

$$\begin{aligned} [D, [D, E]_\mathfrak{h}]_\mathfrak{h}(x) &= [D, D \circ E + E \circ D]_\mathfrak{h}(x)\\ &= D \circ (D \circ E + E \circ D)(x) + (D \circ E + E \circ D) \circ D(x)\\ &= D^2 \circ E(x) + E \circ D^2(x). \end{aligned}$$

Therefore, $[s_\mathfrak{h}(D), E]_\mathfrak{h} = [D, [D, E]_\mathfrak{h}]_\mathfrak{h}$. □

The space $\mathfrak{der}^\alpha(\mathfrak{g})$ is actually graded as $\mathfrak{der}^\alpha(\mathfrak{g}) = \oplus \mathfrak{der}^\alpha_k(\mathfrak{g})$, where $\mathfrak{der}^\alpha_k(\mathfrak{g})$ is the space of α^k-derivations, where k is fixed. Indeed, we have

$$[\mathfrak{der}^\alpha_k(\mathfrak{g}), \mathfrak{der}^\alpha_l(\mathfrak{g})] \subseteq \mathfrak{der}^\alpha_{k+l}(\mathfrak{g}) \quad \text{and} \quad s(\mathfrak{der}^\alpha_k(\mathfrak{g})_{\bar{1}}) \subseteq \mathfrak{der}^\alpha_{2k}(\mathfrak{g}).$$

Example 3. *We will describe all α^k-derivations of the Hom-Lie superalgebra $\mathfrak{oo}^{(1)}_{III}(1|2)_\alpha$ introduced in Example 1. First, observe that*

$$\alpha^{2k} = \alpha^0 = \mathrm{Id}, \quad \alpha^{2k+1} = \alpha, \quad \textit{for all } k \geq 0.$$

The case of α^0-derivations:

$$\begin{array}{llcl} \textit{(Even)} & D^0_1 & = & h_1 \otimes y_2^* + x_1 \otimes y_1^*, \\ \textit{(Even)} & D^0_2 & = & x_1 \otimes x_1^* + y_1 \otimes y_1^*, \\ \textit{(Odd)} & D^0_3 & = & x_1 \otimes h_1^* + h_1 \otimes y_1^* + y_1 \otimes y_2^*. \end{array}$$

The case of α-derivations:

$$\begin{array}{llcl} \textit{(Even)} & D^1_1 & = & h_1 \otimes y_2^* + x_1 \otimes y_1^*, \\ \textit{(Even)} & D^1_2 & = & \epsilon\, x_1 \otimes y_1^* + x_1 \otimes x_1^* + y_1 \otimes y_1^*, \\ \textit{(Odd)} & D^1_3 & = & \epsilon\, x_1 \otimes y_2^* + x_1 \otimes h_1^* + h_1 \otimes y_1^* + y_1 \otimes y_2^*. \end{array}$$

5. *p*-Structures and Queerification of Hom-Lie Algebras in Characteristic 2

We will first introduce the concept of p-structures on Hom-Lie algebras. In the case of Lie algebras, the definition is due to Jacobson [23]. Roughly speaking, one requires the existence of an endomorphism on the modular Lie algebra that resembles the pth power mapping $x \mapsto x^p$ in associative algebras. In the case of Hom-Lie algebra, there is a definition proposed in [24], but it turns out that this definition is not appropriate to queerify a restricted Hom-Lie algebras in characteristic 2, as done in [2] in the case of ordinary restricted Lie algebras. Here, we will give an alternative definition and justify the construction.

Definition 4. *Let $\mathfrak{g}$ be a Hom-Lie algebra in characteristic p with a twist α. A mapping $[p]_\alpha : \mathfrak{g} \to \mathfrak{g}$, $a \mapsto a^{[p]_\alpha}$ is called a p-structure of $\mathfrak{g}$, and $\mathfrak{g}$ is said to be restricted if:*

(*R1*) $\mathrm{ad}(x^{[p]_\alpha}) \circ \alpha^{p-1} = \mathrm{ad}(\alpha^{p-1}(x)) \circ \mathrm{ad}(\alpha^{p-2}(x)) \circ \cdots \circ \mathrm{ad}(x)$ *for all $x \in \mathfrak{g}$;*

(*R2*) $(\lambda x)^{[p]_\alpha} = \lambda^p x^{[p]_\alpha}$ *for all $x \in \mathfrak{g}$ and for all $\lambda \in \mathbb{K}$;*

(*R3*) $(x+y)^{[p]_\alpha} = x^{[p]_\alpha} + y^{[p]_\alpha} + \sum_{1 \leq i \leq p-1} s_i(x,y)$, *where $s_i(x,y)$ can be obtained from*

$$\mathrm{ad}(\alpha^{p-2}(\lambda x + y)) \circ \mathrm{ad}(\alpha^{p-3}(\lambda x + y)) \circ \cdots \circ \mathrm{ad}(\lambda x + y)(x) = \sum_{1 \leq i \leq p-1} i s_i(x,y) \lambda^{i-1}.$$

Let us exhibit this p-structure in the case where $p = 2$. The conditions (R2) and (R3) read, respectively, as

$$[x^{[2]_\alpha}, \alpha(y)] = [\alpha(x), [x,y]] \;\text{ and }\; (x+y)^{[2]_\alpha} = x^{[2]_\alpha} + y^{[2]_\alpha} + [x,y].$$

Proposition 5. *Twisting with a morphism α an ordinary Lie algebra with a p-structure gives rise to a Hom-Lie algebra with a p-structure. More precisely, given an ordinary Lie algebra $(\mathfrak{g}, [\cdot,\cdot])$*

with a p-structure and a Lie algebra morphism α, then $(\mathfrak{g}, [\cdot,\cdot]_\alpha, \alpha)$, where $[\cdot,\cdot]_\alpha := \alpha \circ [\cdot,\cdot]$, is a Hom-Lie algebra with a p-structure given by

$$x^{[p]_\alpha} := \alpha^{p-1}(x^{[p]}).$$

Proof. It was shown in [25] that, if $(\mathfrak{g}, [\cdot,\cdot])$ is an ordinary Lie algebra, then $(\mathfrak{g}, [\cdot,\cdot]_\alpha)$, where $[\cdot,\cdot]_\alpha := \alpha \circ [\cdot,\cdot]$ is a Hom-Lie algebra. Now, let us show that the map $[p]_\alpha$ defines a p-structure on the Hom-Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g})$. Indeed, let us check Axiom (R1). The LHS reads

$$\mathrm{ad}(x^{[p]_\alpha}) \circ \alpha^{p-1}(y) = [x^{[p]_\alpha}, \alpha^{p-1}(y)]_\alpha = \alpha([\alpha^{p-1}(x^{[p]}), \alpha^{p-1}(y)]) = \alpha^p([x^{[p]}, y]).$$

The RHS reads

$$\mathrm{ad}(\alpha^{p-1}(x)) \circ \mathrm{ad}(\alpha^{p-2}(x)) \circ \cdots \circ \mathrm{ad}(x)(y) = [\alpha^{p-1}(x), [\alpha^{p-2}(x), \ldots, [x,y]_\alpha]_\alpha$$

$$= \alpha([\alpha^{p-1}(x), \alpha([\alpha^{p-2}(x), [\ldots, \alpha([x,y]))])]) = \alpha^p([x,[x,\ldots,[x,y]]) = \alpha^p([x^{[p]}, y]).$$

Axiom (R2) is obviously satisfied. Let us check Axiom (R3). Indeed,

$$(x+y)^{[p]_\alpha} = \alpha^{p-1}((x+y)^{[p]}) = \alpha^{p-1}\left(x^{[p]} + y^{[p]} + \sum_{1\leq i\leq p-2} s_i(x,y)\right)$$

$$= x^{[p]_\alpha} + y^{[p]_\alpha} + \left(\sum_{1\leq i\leq p-2} \alpha^{p-1}(s_i(x,y))\right).$$

Now,

$$\begin{aligned}
&\mathrm{ad}(\alpha^{p-2}(\lambda x + y)) \circ \mathrm{ad}(\alpha^{p-3}(\lambda x + y)) \circ \cdots \circ \mathrm{ad}(\lambda x + y)(x) \\
&= [\alpha^{p-2}(\lambda x + y), [\alpha^{p-3}(\lambda x + y), \ldots, [\lambda x + y, x]_\alpha]_\alpha \\
&= \alpha([\alpha^{p-2}(\lambda x + y), \alpha([\alpha^{p-3}(\lambda x + y), [\ldots, \alpha([\lambda x + y, x]))])]) \\
&= \alpha^{p-1}([\lambda x + y, [\lambda x + y, [\ldots, [\lambda x + y, x]]) \\
&= \alpha^{p-1}\left(\sum_{1\leq i\leq p-1} i s_i(x,y)\lambda^{i-1}\right) = \sum_{1\leq i\leq p-1} i\alpha^{p-1}(s_i(x,y))\lambda^{i-1}.
\end{aligned}$$

The proof is now complete. □

Proposition 6. *Let $\mathfrak{g}$ be a restricted Hom-Lie algebra in characteristic 2 with a twist map α. On the superspace $\mathfrak{h} := \mathfrak{g} \oplus \Pi(\mathfrak{g})$, where $\Pi(\mathfrak{g})$ is copy of $\mathfrak{g}$ whose elements are odd, there exists a Hom-Lie superalgebra structure defined as follows (for all $x, y \in \mathfrak{g}$):*

$$[x,y]_\mathfrak{h} := [x,y]_\mathfrak{g}, \quad [\Pi(x), y]_\mathfrak{h} := \Pi([x,y]_\mathfrak{g}), \quad s_\mathfrak{h}(\Pi(x)) = x^{[2]_\alpha}.$$

Proof. Let us check that the map $s_\mathfrak{h}$ is indeed a squaring on $\mathfrak{h}$. The condition $s_\mathfrak{h}(\lambda\Pi(x)) = \lambda^2 s_\mathfrak{h}(\Pi(x))$, for all $\lambda \in \mathbb{K}$ and for all $x \in \mathfrak{g}$, is an immediate consequence of condition (R2). Moreover, the map

$$(\Pi(x), \Pi(y) \mapsto s_\mathfrak{h}(\Pi(x) + \Pi(y)) + s_\mathfrak{h}(\Pi(x)) + s_\mathfrak{h}(\Pi(y)) = (x+y)^{[2]_\alpha} + x^{[2]_\alpha} + y^{[2]_\alpha} = [x,y]_\mathfrak{g}$$

is obviously bilinear because it coincides with the Lie bracket on $\mathfrak{g}$.

Let us check the Jacobi identity involving the squaring. Indeed, for all $y \in \mathfrak{h}_{\bar{0}}$ and for all $\Pi(x) \in \mathfrak{h}_{\bar{1}}$, we have

$$[s_{\mathfrak{h}}(\Pi(x)), \alpha(y)]_{\mathfrak{h}} = [x^{[2]_\alpha}, \alpha(y)]_{\mathfrak{h}} = [x^{[2]_\alpha}, \alpha(y)]_{\mathfrak{g}} = [\alpha(x), [x, y]_{\mathfrak{g}}]_{\mathfrak{g}}.$$

On the other hand,

$$\begin{aligned}&[\alpha(\Pi(x)), [\Pi(x), y]_{\mathfrak{h}}]_{\mathfrak{h}} = [\Pi(\alpha(x)), \Pi([x, y]_{\mathfrak{g}})]_{\mathfrak{h}} = \Pi([\Pi(\alpha(x)), [x, y]_{\mathfrak{g}}]_{\mathfrak{h}})\\ &= \Pi^2([\alpha(x), [x, y]_{\mathfrak{g}}]_{\mathfrak{h}}) = [\alpha(x), [x, y]_{\mathfrak{g}}]_{\mathfrak{g}}.\end{aligned}$$

For all $\Pi(y) \in \mathfrak{h}_{\bar{1}}$ and for all $\Pi(x) \in \mathfrak{h}_{\bar{1}}$, we have

$$[s_{\mathfrak{h}}(\Pi(x)), \alpha(\Pi(y))]_{\mathfrak{h}} = [x^{[2]_\alpha}, \alpha(\Pi(y))]_{\mathfrak{h}} = \Pi([x^{[2]_\alpha}, \alpha(y)]_{\mathfrak{g}}) = \Pi([\alpha(x), [x, y]_{\mathfrak{g}}]_{\mathfrak{g}}).$$

On the other hand,

$$\begin{aligned}&[\alpha(\Pi(x)), [\Pi(x), \Pi(y)]_{\mathfrak{h}}]_{\mathfrak{h}} = [\Pi(\alpha(x)), s_{\mathfrak{h}}(\Pi(x) + \Pi(y)) + s_{\mathfrak{h}}(\Pi(x)) + s_{\mathfrak{h}}(\Pi(x)]_{\mathfrak{h}}\\ &= [\Pi(\alpha(x)), (x + y)^{[2]_\alpha} + x^{[2]_\alpha} + y^{[2]_\alpha}]_{\mathfrak{h}} = [\Pi(\alpha(x)), [x, y]_{\mathfrak{g}}]_{\mathfrak{h}} = \Pi([\alpha(x), [x, y]_{\mathfrak{g}}]_{\mathfrak{g}}). \quad \square\end{aligned}$$

Proposition 7. *Let $\mathfrak{g}$ be a restricted Lie algebra in characteristic 2 and $\mathfrak{h} := \mathfrak{g} \oplus \Pi(\mathfrak{g})$ be its queerification (see [2]), defined as follows (for all $x, y \in \mathfrak{g}$):*

$$[x, y]_{\mathfrak{h}} := [x, y]_{\mathfrak{g}}, \quad [\Pi(x), y]_{\mathfrak{h}} := \Pi([x, y]_{\mathfrak{g}}), \quad s_{\mathfrak{h}}(\Pi(x)) = x^{[2]}.$$

Let $\alpha : \mathfrak{g} \to \mathfrak{g}$ be a Lie algebra morphism. Let us extend it to $\tilde{\alpha}$ on $\mathfrak{h}$ by declaring $\alpha(\Pi(x)) := \Pi(\alpha(x))$ for all $x \in \mathfrak{g}$. Then, twisting the Lie superalgebra $\mathfrak{h}$ along $\tilde{\alpha}$ is exactly the queerification of the Hom-Lie algebra $\mathfrak{g}_\alpha$ obtained by twisting $\mathfrak{g}$ along α. Namely,

$$\mathfrak{h}_{\tilde{\alpha}} = (\mathfrak{g} \oplus \Pi(\mathfrak{g}))_{\tilde{\alpha}} = \mathfrak{g}_\alpha \oplus \Pi(\mathfrak{g}_\alpha).$$

Proof. Let $x, y \in \mathfrak{g}$. We have

$$[x, y]_{\mathfrak{h}_{\tilde{\alpha}}} = \tilde{\alpha}([x, y]_{\mathfrak{h}}) = \alpha([x, y]_{\mathfrak{g}}).$$

On the other hand,

$$[x, y]_{\mathfrak{g}_\alpha \oplus \Pi(\mathfrak{g}_\alpha)} = [x, y]_{\mathfrak{g}_\alpha} = \alpha([x, y]_{\mathfrak{g}}).$$

Similarly, one can easily prove that

$$[\Pi(x), y]_{\mathfrak{h}_{\tilde{\alpha}}} = [\Pi(x), y]_{\mathfrak{g}_\alpha \oplus \Pi(\mathfrak{g}_\alpha)}.$$

Let us only prove that their squarings coincide. Indeed, for all $x \in \mathfrak{g}$, we have

$$s_{\mathfrak{h}_{\tilde{\alpha}}}(\Pi(x)) = \alpha \circ s_{\mathfrak{h}}(\Pi(x)) = \alpha(x^{[2]}).$$

On the other hand,

$$s_{\mathfrak{g}_\alpha \oplus \Pi(\mathfrak{g}_\alpha)}(\Pi(x)) = x^{[2]_\alpha} = \alpha(x^{[2]}). \quad \square$$

6. Cohomology and Deformations of Finite-Dimensional Hom-Lie Superalgebras

6.1. Cohomology of Ordinary Lie Superalgebras in Characteristic 2

In this section, we define a cohomology theory of Lie superalgebras in characteristic 2. The first instances can be found in [26]. Let $\mathfrak{g}$ be a Lie superalgebra in characteristic 2 and M be a $\mathfrak{g}$-module. Let us introduce a map:

$$\mathfrak{p} : \mathfrak{g}_{\bar{1}} \times \wedge^n \mathfrak{g} \to M, \tag{21}$$

with the following properties:

(i) $\mathfrak{p}(\lambda x, z) = \lambda^2 \mathfrak{p}(x, z)$ for all $x \in \mathfrak{g}_{\bar{1}}$, for all $z \in \wedge^n \mathfrak{g}$ and for all $\lambda \in \mathbb{K}$.

(ii) For all $x \in \mathfrak{g}_{\bar{1}}$, the map $z \mapsto \mathfrak{p}(x, z)$ is multi-linear.

For $n = 0$, the map $\mathfrak{p}$ should be understood as a quadratic form on $\mathfrak{g}_{\bar{1}}$ with values in M. We are now ready to define the space of cochains on $\mathfrak{g}$ with values in M. We set ($n > 1$)

$$\begin{array}{lll} XC^{-1}(\mathfrak{g}; M) & := & \{0\}, \\ XC^{0}(\mathfrak{g}; M) & := & M, \\ XC^{1}(\mathfrak{g}; M) & := & \{c \mid \text{where } c : \mathfrak{g} \to M \text{ is linear}\}, \\ XC^{n}(\mathfrak{g}; M) & := & \{(c, \mathfrak{p}) \mid \text{where } c : \wedge^n \mathfrak{g} \to M \text{ is a multi-linear map and} \\ & & \mathfrak{p} : \mathfrak{g}_{\bar{1}} \times \wedge^{n-2} \mathfrak{g} \to M \text{ is a map as in (21) such that} \\ & & \mathfrak{p}(x+y, z) + \mathfrak{p}(x, z) + \mathfrak{p}(y, z) = c(x, y, z) \\ & & \text{for all } x, y \in \mathfrak{g}_{\bar{1}} \text{ and } z \in \wedge^{n-2} \mathfrak{g}\}. \end{array} \tag{22}$$

We define the *differential* $\mathfrak{d}^{-1} : XC^{-1}(\mathfrak{g}, M) \to XC^{0}(\mathfrak{g}, M)$ to be the trivial map. The *differential* $\mathfrak{d}^0$ is given by

$$\mathfrak{d}^0 : XC^0(\mathfrak{g}, M) \to XC^1(\mathfrak{g}, M) \quad m \mapsto \mathfrak{d}^0(m),$$

where $\mathfrak{d}^0(m)(x) = x \cdot m$. The *differential* $\mathfrak{d}^1$ is given by

$$\mathfrak{d}^1 : XC^1(\mathfrak{g}, M) \to XC^2(\mathfrak{g}, M) \quad c \mapsto (dc, \mathfrak{q}),$$

where

$$\begin{array}{lll} dc(x, z) & = & c([x, z]) + x \cdot c(z) + z \cdot c(x) \quad \text{for all } x, z \in \mathfrak{g}; \\ \mathfrak{q}(x) & = & c(s(x)) + x \cdot c(x) \quad \text{for all } x \in \mathfrak{g}_{\bar{1}}. \end{array} \tag{23}$$

Now, for $n \geq 2$, the *differential* $\mathfrak{d}^n$ is given by

$$\mathfrak{d}^n : XC^n(\mathfrak{g}, M) \to XC^{n+1}(\mathfrak{g}, M) \quad (c, \mathfrak{p}) \mapsto (d^n c, d^n \mathfrak{p}),$$

where

$$\begin{array}{lll} d^n c(z_1, \ldots, z_{n+1}) & = & \sum\limits_{1 \leq i \leq n+1} z_i \cdot c(z_1, \ldots, \hat{z}_i, \ldots, z_{n+1}) \\ & & + \sum\limits_{1 \leq i < j \leq n+1} c([z_i, z_j], z_1, \ldots, \hat{z}_i, \ldots, \hat{z}_j, \ldots z_{n+1}), \\ d^n \mathfrak{p}(x, z_1, \ldots, z_{n-1}) & = & x \cdot c(x, z_1, \ldots, z_{n-1}) + \sum\limits_{1 \leq i \leq n-1} z_i \cdot \mathfrak{p}(x, z_1, \ldots, \hat{z}_i, \ldots, z_{n-1}) \\ & & + c(s(x), z_1, \ldots, z_{n-1}) + \sum\limits_{1 \leq i \leq n-1} c([x, z_i], x, z_1, \ldots, \hat{z}_i, \ldots, z_{n-1}) \\ & & + \sum\limits_{1 \leq i < j \leq n-1} \mathfrak{p}(x, [z_i, z_j], z_1, \ldots, \hat{z}_i, \ldots, \hat{z}_j, \ldots, z_{n-1}). \end{array} \tag{24}$$

Theorem 3. *The maps $\mathfrak{d}^n$ are well-defined. Moreover, for all integers n,*

$$\mathfrak{d}^{n+1} \circ \mathfrak{d}^n = 0.$$

Hence, the pair $(XC^*(\mathfrak{g}, M), \mathfrak{d}^*)$ defines a cohomology complex for Lie superalgebras in characteristic 2.

The proof of the theorem will be given next when considering the cohomology of Hom-Lie superalgebras that reduce to ordinary Lie superalgebras when the structure map is the identity.

6.2. Elucidation for $n = 2, 3$

Let us first exhibit the sets of cochains in the case where $n = 2, 3$.

If $v \in M$ and $a, b \in \mathfrak{g}_{\bar{1}}^*$, we can define the cochain $(v \otimes a \wedge b, \mathfrak{q}) \in XC^2(\mathfrak{g}, M)$ such that the quadratic form is $\mathfrak{q}(x) = a(x)b(x)\, v$ for all $x \in \mathfrak{g}_{\bar{1}}$. The polar form associated with $\mathfrak{q}$ is

$$B_{\mathfrak{q}}(x, y) = (a(x)b(y) + a(y)b(x))\, v \text{ for all } x, y \in \mathfrak{g}_{\bar{1}}.$$

Recall that, to each quadratic form $\mathfrak{q}$ with values in a space M, its polar form is the bilinear form with the values in M given by: $B_{\mathfrak{q}}(x, y) := \mathfrak{q}(x + y) + \mathfrak{q}(x) + \mathfrak{q}(y)$.)

In particular, we can define the cochain $c = v \otimes a \wedge a$, where $q(x) = v(a(x))^2$ for all $x \in \mathfrak{g}_{\bar{1}}$ and $c(x, y) = 0$ for all $x, y \in \mathfrak{g}$.

Similarly, if $v \in M$ and $a, b \in \mathfrak{g}_{\bar{1}}^*$, but $c \in \mathfrak{g}$, we can define the cochain $(v \otimes a \wedge b \wedge c, \mathfrak{p}) \in XC^2(\mathfrak{g}, M)$ such that the map $\mathfrak{p}$ is

$$\mathfrak{p}(x, z) = (a(x)b(x)c(z) + a(z)b(x)c(x) + a(x)b(z)c(x))\, v \quad \text{for all } x \in \mathfrak{g}_{\bar{1}} \text{ and } z \in \mathfrak{g}$$

Now, a direct computation shows that

$$\begin{aligned}
\mathfrak{p}(x + y, z) + \mathfrak{p}(x, z) + \mathfrak{p}(y, z) &= (a(x)b(y)c(z) + a(y)b(x)c(z) + a(z)b(x)c(y))\, v \\
&\quad + (a(z)b(y)c(x) + b(z)a(x)c(y) + b(z)a(y)c(x))v \\
&= v \otimes (a \wedge b \wedge c)(x, y, z).
\end{aligned}$$

A one-cocycle c on $\mathfrak{g}$ with values in an $\mathfrak{g}$-module M must satisfy the following conditions:

$$x \cdot c(z) + z \cdot c(x) + c([x, z]) = 0 \quad \text{for all } x, z \in \mathfrak{g}, \tag{25}$$

$$x \cdot c(x) + c(s(x)) = 0 \quad \text{for all } x \in \mathfrak{g}_{\bar{1}}. \tag{26}$$

A two-cocycle $(c, \mathfrak{q})$ on $\mathfrak{g}$ with values in M must satisfy the following conditions:

$$0 = x \cdot c(y, z) + c([x, y], z) + \circlearrowleft (x, y, z) \quad \text{for all } x, y, z \in \mathfrak{g}, \tag{27}$$

$$0 = x \cdot c(x, z) + z \cdot \mathfrak{q}(x) + c(s(x), z) + c([x, z], x] \tag{28}$$

for all $x \in \mathfrak{g}_{\bar{1}}$ and for all $z \in \mathfrak{g}$.

6.3. Cohomology of Hom-Lie Superalgebras in Characteristic 2

Let $(\mathfrak{g}, [\cdot, \cdot], s, \alpha)$ be a Hom-Lie superalgebra in characteristic 2 and (M, β) be a $\mathfrak{g}$-module; see Definition 2. The space of n-cochains is defined similarly to (22) with a slight difference with respect to the degree 0 space and an extra condition, that is

$$\beta \circ c = c \circ (\alpha \wedge \cdots \wedge \alpha), \quad \text{and} \quad \beta \circ \mathfrak{p} = \mathfrak{p} \circ (\alpha \wedge \cdots \wedge \alpha). \tag{29}$$

$$\begin{aligned}
XC_\alpha^{-1}(\mathfrak{g}; M) &:= \{0\}, \\
XC_\alpha^{0}(\mathfrak{g}; M) &:= \{m \in M \mid \beta(m) = m \text{ and } \alpha(x) \cdot (y \cdot m) = x \cdot (y \cdot m) \text{ for all } x, y \in \mathfrak{g}\}, \\
XC_\alpha^{1}(\mathfrak{g}; M) &:= \{c \mid \text{where } c : \mathfrak{g} \to M \text{ is linear and satisfies Equation (29)}\}, \\
XC_\alpha^{n}(\mathfrak{g}; M) &:= \{(c, \mathfrak{p}) \mid \text{where } c : \wedge^n \mathfrak{g} \to M \text{ is a multi-linear map satisfying Equation (29) and} \\
&\quad\ \mathfrak{p} : \mathfrak{g}_{\bar{1}} \times \wedge^{n-2}\mathfrak{g} \to M \text{ is a map as in (21) satisfying Equation (29) such that} \\
&\quad\ \mathfrak{p}(x + y, z) + \mathfrak{p}(x, z) + \mathfrak{p}(y, z) = c(x, y, z) \text{ for all } x, y \in \mathfrak{g}_{\bar{1}} \text{ and } z \in \wedge^{n-2}\mathfrak{g}\}.
\end{aligned} \tag{30}$$

One-cochains are just linear functions c on $\mathfrak{g}$ with values in an $\mathfrak{g}$-module M such that $\beta \circ c = c \circ \alpha$. Let us define the *differentials* in our context. First, let us define $\mathfrak{d}_\alpha^0$ and $\mathfrak{d}_\alpha^1$.

$$\mathfrak{d}_\alpha^0 : XC_\alpha^0(\mathfrak{g}, M) \to XC_\alpha^1(\mathfrak{g}, M) \quad m \mapsto d_\alpha^0 m,$$

where $d_\alpha^0 m(x) = x \cdot m$ for all $x \in \mathfrak{g}$. Additionally,

$$\mathfrak{d}_\alpha^1 : XC_\alpha^1(\mathfrak{g}, M) \to XC_\alpha^2(\mathfrak{g}, M) \quad c \mapsto (d_\alpha^1 c, \mathfrak{q}),$$

where

$$\begin{array}{rcll} d_\alpha^1 c(x,z) & = & c([x,z]) + x \cdot c(z) + y \cdot c(x) & \text{for all } x, z \in \mathfrak{g}; \\ \mathfrak{q}(x) & = & c(s(x)) + x \cdot c(x) & \text{for all } x \in \mathfrak{g}_{\bar{1}}. \end{array} \tag{31}$$

Note that these definitions are consistent as, shown by the following lemma.

Proposition 8. *The differentials $\mathfrak{d}_\alpha^0$ and $\mathfrak{d}_\alpha^1$ are indeed well-defined; namely, Im $(\mathfrak{d}_\alpha^0) \subseteq XC_\alpha^1(\mathfrak{g}, M)$ and Im $(\mathfrak{d}_\alpha^1) \subseteq XC_\alpha^2(\mathfrak{g}, M)$.*

Proof. Let us first deal with $\mathfrak{d}_\alpha^0$. We have

$$d_\alpha^0 m(\alpha(x)) = \alpha(x) \cdot m = \alpha(x) \cdot \beta(m) = \beta(x \cdot m) = \beta(d_\alpha^0 m(x)).$$

Therefore, Equation (29) is satisfied. Let us now deal deal with $\mathfrak{d}_\alpha^1$. We will only prove that $\mathfrak{q}$ satisfies Equation (29). Indeed,

$$\begin{array}{rcl} \mathfrak{q}(\alpha(x)) & = & c(s(\alpha(x))) + \alpha(x) \cdot c(\alpha(x)) = c(\alpha(s(x))) + \alpha(x) \cdot \beta(c(x)) \\ & = & \beta(c(s(x))) + \beta(x \cdot c(x)) = \beta(\mathfrak{q}(x)). \end{array}$$

On the other hand, we have

$$\begin{array}{rcl} \mathfrak{q}(x+y) + \mathfrak{q}(x) + \mathfrak{q}(y) & = & c(s(x+y)) + (x+y) \cdot c(x+y) + c(s(x)) \\ & & + x \cdot c(x) + c(s(y)) + y \cdot c(y) \\ & = & c([x,y]) + y \cdot c(x) + x \cdot c(y) = d_\alpha^1 c(x,y). \quad \square \end{array}$$

A one-cocycle c on $\mathfrak{g}$ with values in an $\mathfrak{g}$-module M must satisfy the following conditions:

$$x \cdot c(y) + y \cdot c(x) + c([x,y]) = 0 \quad \text{for all } x, y \in \mathfrak{g}, \tag{32}$$

$$x \cdot c(x) + c(s(x)) = 0 \quad \text{for all } x \in \mathfrak{g}_{\bar{1}}. \tag{33}$$

The space of all one-cocycles is denoted by $Z_\alpha^1(\mathfrak{g}; M)$.

Now, for $n \geq 2$, the *differential* $\mathfrak{d}_\alpha^n$ is given by

$$\mathfrak{d}^n : XC_\alpha^n(\mathfrak{g}, M) \to XC_\alpha^{n+1}(\mathfrak{g}, M) \quad (c, \mathfrak{p}) \mapsto (d_\alpha^n c, d_\alpha^n \mathfrak{p}),$$

where

$$d^n_\alpha c(z_1,\dots,z_{n+1}) = \sum_{1\le i\le n+1} \alpha^{n-1}(z_i)\cdot c(z_1,\dots,\hat{z}_i,\dots,z_{n+1})$$

$$+\sum_{1\le i<j\le n+1} c([z_i,z_j],\alpha(z_1),\dots,\hat{z}_i,\dots,\hat{z}_j,\dots\alpha(z_{n+1})),$$

$$d^n_\alpha \mathfrak{p}(x,z_1,\dots,z_{n-1}) = \alpha^{n-1}(x)\cdot c(x,z_1,\dots,z_{n-1}) + c(s(x),\alpha(z_1),\dots,\alpha(z_{n-1}))$$

$$+\sum_{1\le i\le n-1} \alpha^{n-1}(z_i)\cdot \mathfrak{p}(x,z_1,\dots,\hat{z}_i,\dots,z_{n-1})$$

$$+\sum_{1\le i\le n-1} c([x,z_i],\alpha(x),\alpha(z_1),\dots,\hat{z}_i,\dots,\alpha(z_{n-1}))$$

$$+\sum_{1\le i<j\le n-1} \mathfrak{p}(\alpha(x),[z_i,z_j],\alpha(z_1),\dots,\hat{z}_i,\dots,\hat{z}_j,\dots,\alpha(z_{n-1})).$$

In particular, for $n=2$, the *differential* is given by

$$\mathfrak{d}^2_\alpha : XC^2_\alpha(\mathfrak{g},M)\to XC^3_\alpha(\mathfrak{g},M)\quad (c,\mathfrak{p})\mapsto (d^2_\alpha c, d^2_\alpha \mathfrak{p}),$$

where

$$\begin{aligned} d^2_\alpha c(z_1,z_2,z_3) &= \alpha(z_1)\cdot c(z_2,z_3) + c([z_1,z_2],\alpha(z_3)) + \circlearrowleft (z_1,z_2,z_3) \ \text{ for all } z_1,z_2,z_3\in\mathfrak{g};\\ d^2_\alpha \mathfrak{p}(x,z_1) &= \alpha(x)\cdot c(x,z_1) + \alpha(z_1)\cdot\mathfrak{p}(x) + c(s(x),\alpha(z_1)) + c([x,z_1],\alpha(x))\\ &\quad \text{for all } x\in\mathfrak{g}_{\bar{1}}, \text{ and for all } z_1\in\mathfrak{g}. \end{aligned}$$

A two-cocycle is two-tuple $(c,\mathfrak{p})$ satisfying the following conditions:

$$0 = \alpha(z_3)\cdot c(z_1,z_2) + c([z_1,z_2],\alpha(z_3)) + \circlearrowleft (z_1,z_2,z_3) \quad \text{for all } z_1,z_2,z_3\in\mathfrak{g}, \tag{34}$$

$$0 = \alpha(x)\cdot c(x,z_1) + \alpha(z_1)\cdot\mathfrak{p}(x) + c(s(x),\alpha(z_1)) + c([x,z_1],\alpha(x)] \tag{35}$$

$$\text{for all } x\in\mathfrak{g}_{\bar{1}} \text{ and for all } z_1\in\mathfrak{g},$$

The first step here is to show that the map $\mathfrak{d}^n_\alpha$ is well-defined, for every twist α. By doing so, we give a proof to the first part of Theorem 3 in the case where $\alpha = \mathrm{id}$.

Proposition 9. *The maps $\mathfrak{d}^n_\alpha$ are well-defined; namely, Im $(\mathfrak{d}^n_\alpha)\subseteq XC^{n+1}_\alpha(\mathfrak{g},M)$.*

Proof. For all $x,y\in\mathfrak{g}_{\bar{1}}$ and for all $z_1,\dots,z_n\in\mathfrak{g}$, we have

$$
\begin{aligned}
& d^n_\alpha \mathfrak{p}(x+y, z_1, \ldots, z_n) \\
& = \alpha^{n-1}(x+y) \cdot c(x+y, z_1, \ldots, z_{n-1}) + \sum_{1 \le i \le n-1} \alpha^{n-1}(z_i) \cdot \mathfrak{p}(x+y, z_1, \ldots, \hat{z_i}, \ldots, z_{n-1}) \\
& + c(s(x+y), \alpha(z_1), \ldots, \alpha(z_{n-1})) + \sum_{1 \le i \le n-1} c([x+y, z_i], \alpha(x+y), \alpha(z_1), \ldots, \hat{z_i}, \ldots, \alpha(z_{n-1})) \\
& + \sum_{1 \le i < j \le n-1} \mathfrak{p}(\alpha(x+y), [z_i, z_j], \alpha(z_1), \ldots, \hat{z_i}, \ldots, \hat{z_j}, \ldots, \alpha(z_{n-1})). \\
& = d^n_\alpha \mathfrak{p}(x, z_1, \ldots, z_n) + d^n_\alpha \mathfrak{p}(y, z_1, \ldots, z_n) + \alpha^{n-1}(x) \cdot c(y, z_1, \ldots, z_{n-1}) \\
& + \alpha^{n-1}(y) \cdot c(x, z_1, \ldots, z_{n-1}) + \sum_{1 \le i \le n-1} \alpha^{n-1}(z_i) \cdot c(x, y, z_1, \ldots, \hat{z_i}, \ldots, z_{n-1}) \\
& + c([x,y], \alpha(z_1), \ldots, \alpha(z_{n-1})) + \sum_{1 \le i \le n-1} c([x, z_i], \alpha(y), \alpha(z_1), \ldots, \hat{z_i}, \ldots, \alpha(z_{n-1})) \\
& + \sum_{1 \le i \le n-1} c([y, z_i], \alpha(x), \alpha(z_1), \ldots, \hat{z_i}, \ldots, \alpha(z_{n-1})) \\
& + \sum_{1 \le i < j \le n-1} c(\alpha(x), \alpha(y), [z_i, z_j], \alpha(z_1), \ldots, \hat{z_i}, \ldots, \hat{z_j}, \ldots, \alpha(z_{n-1})) \\
& = d^n_\alpha \mathfrak{p}(x, z_1, \ldots, z_n) + d^n_\alpha \mathfrak{p}(y, z_1, \ldots, z_n) + d^n_\alpha c(x, y, z_1, \ldots, z_n),
\end{aligned}
$$

where we have used the fact that $s(x+y) = s(x) + s(y) + [x,y]$ and

$$\mathfrak{p}(x+y, z_1, \ldots, z_{n-1}) + \mathfrak{p}(x, z_1, \ldots, z_{n-1}) + \mathfrak{p}(y, z_1, \ldots, z_{n-1}) \quad = \quad c(x, y, z_1, \ldots, z_{n-1}).$$

□

Theorem 4. *For all $n \ge 1$, we have $\mathfrak{d}^n_\alpha \circ \mathfrak{d}^{n-1}_\alpha = 0$. Hence, the pair $(XC^*_\alpha(\mathfrak{g}, M), \mathfrak{d}^*_\alpha)$ defines a cohomology complex for Hom-Lie superalgebras in characteristic 2.*

In order to prove this theorem, we will need the following lemma.

Lemma 1. *If $(c, \mathfrak{p}) \in XC^n_\alpha(\mathfrak{g}, M)$, then:*

(*i*) $\alpha^{n-1}(x) \cdot (\alpha^{n-2}(x) \cdot c(z_1, \ldots, z_n)) = \alpha^{n-2}(s(x)) \cdot c(\alpha(z_1), \ldots, \alpha(z_n))$ *for all $x \in \mathfrak{g}_{\bar{1}}$ and for all $z_1, \ldots, z_n \in \mathfrak{g}$.*

(*ii*) $\alpha(z_i) \cdot (z_j \cdot c(z_1, \ldots, z_n)) + \alpha(z_j) \cdot (z_i \cdot c(z_1, \ldots, z_n)) = [z_i, z_j] \cdot c(\alpha(z_1), \ldots, \alpha(z_n))$ *for all $z_1, \ldots, z_n \in \mathfrak{g}$.*

Proof. Let us only prove Part (i). Using the fact that $\beta \circ c = c \circ (\alpha \wedge \cdots \wedge \alpha)$, we obtain

$$
\begin{aligned}
& \alpha^{n-1}(x) \cdot (\alpha^{n-2}(x) \cdot c(z_1, \ldots, z_n)) = s(\alpha^{n-2}(x)) \cdot \beta(c(z_1, \ldots, z_n)) \\
& = \alpha^{n-2}(s(x)) \cdot (c(\alpha(z_1), \ldots, \alpha(z_n))). \quad \square
\end{aligned}
$$

Proof of Theorem 4. Let us first show that $\mathfrak{d}^1_\alpha \circ \mathfrak{d}^0_\alpha = 0$. Indeed, for all $x, y \in \mathfrak{g}$ and $m \in XC^0_\alpha(\mathfrak{g}; M)$, we have

$$
\begin{aligned}
d^1_\alpha \circ d^0_\alpha m(x,y) &= x \cdot d^0_\alpha m(y) + y \cdot d^0_\alpha m(x) + d^0_\alpha m([x,y]) = x \cdot (y \cdot m) + y \cdot (x \cdot m) + [x,y] \cdot m \\
&= \alpha(x) \cdot (y \cdot m) + \alpha(y) \cdot (x \cdot m) + [x,y] \cdot \beta(m) = 0.
\end{aligned}
$$

On the other hand, for all $x \in \mathfrak{g}_{\bar{1}}$ and $m \in XC^0_\alpha(\mathfrak{g}; M)$, we have

$$\mathfrak{q}(x) = d^0_\alpha m(s(x)) + x \cdot d^0_\alpha m(x) = s(x) \cdot m + x \cdot (x \cdot m) = s(x) \cdot \beta(m) + \alpha(x) \cdot (x \cdot m) = 0.$$

Let us now show that $\mathfrak{d}^n_\alpha \circ \mathfrak{d}^{n-1}_\alpha = 0$ for all $n > 1$. To show that $d^n_\alpha \circ d^{n-1}_\alpha(c) = 0$ is routine, see for instance [10]. Let us show that $d^n_\alpha \circ d^{n-1}_\alpha(\mathfrak{p}) = 0$. This would imply that $\mathfrak{d}^n_\alpha \circ \mathfrak{d}^{n-1}_\alpha(c, \mathfrak{p}) = 0$. Actually, the computation is very cumbersome, so we will break it into small pieces. First, we compute:

$$d^n_\alpha \circ d^{n-1}_\alpha(\mathfrak{p})(x, z_1, \dots, z_{n-1}) = \alpha^{n-1}(x) \cdot d^{n-1}_\alpha c(x, z_1, \dots, z_{n-1})$$

$$+ \sum_{i=1}^{n-1} \alpha^{n-1}(z_i) d^{n-1}_\alpha \mathfrak{p}(x, z_1, \dots, \widehat{z_i}, \dots, z_{n-1}) + d^{n-1}_\alpha c(s(x), \alpha(z_1), \dots, \alpha(z_{n-1}))$$

$$+ \sum_{i=1}^{n-1} d^{n-1}_\alpha c([x, z_i], \alpha(x), \alpha(z_1), \dots, \widehat{\alpha(z_i)}, \dots, \alpha(z_{n-1}))$$

$$+ \sum_{1 \le i < j \le n-1} d^{n-1}_\alpha \mathfrak{p}(\alpha(x), [z_i, z_j], \alpha(z_1), \dots \widehat{\alpha(z_i)}, \dots, \widehat{\alpha(z_j)}, \dots, \alpha(z_{n-1})).$$

There are five terms in the expression above. We will compute each term separately.

$$\underline{\text{Part 1:}} \sum_{1 \le i \le n-1} \alpha^{n-1}_\alpha(z_i) d^{n-1} \mathfrak{p}(x, z_1, \dots, \widehat{z_i}, \dots, z_{n-1}) =$$

$$+ \sum_{1 \le i \le n-1} \alpha^{n-1}(z_i) \cdot [\alpha^{n-2}(x) \cdot c(x, z_1, \dots, \widehat{z_i}, \dots, z_{n-1})$$

$$+ \sum_{1 \le j \le n-2} \alpha^{n-2}(z_j) \mathfrak{p}(x, z_1, \dots, \widehat{z_i}, \dots, \widehat{z_j}, \dots, z_{n-1}) + c(s(x), \alpha(z_1), \dots, \widehat{\alpha(z_i)}, \dots, \alpha(z_{n-1}))$$

$$+ \sum_{1 \le j \le n-2} c([x, z_i], \alpha(x), \alpha(z_1), \dots, \widehat{\alpha(z_i)}, \dots, \widehat{\alpha(z_j)}, \dots, \alpha(z_{n-1}))$$

$$+ \sum_{1 \le l < j \le n-2} \mathfrak{p}(\alpha(x), [z_l, z_j], \alpha(z_1), \dots, \widehat{\alpha(z_l)}, \dots, \widehat{\alpha(z_j)}, \dots, \widehat{\alpha(z_i)}, \dots, \alpha(z_{n-1}))].$$

$$\underline{\text{Part 2:}}\ \alpha^{n-1}(x) d^{n-1}_\alpha c(x, z_1, \dots, z_{n-1}) = \alpha^{n-1}(x) \cdot [\alpha^{n-2}(x) c(z_1, \dots, z_{n-1})$$

$$+ \sum_{1 \le i \le n-1} \alpha^{n-2}(z_i) c(x, z_1, \dots, \widehat{z_i}, \dots, z_{n-1}) + \sum_{1 \le i \le n-1} c([x, z_i], \alpha(z_1), \dots, \widehat{\alpha(z_i)}, \dots, \alpha(z_{n-1}))$$

$$+ \sum_{1 \le i < j \le n-1} c([z_i, z_j], \alpha(x), \alpha(z_1), \dots, \widehat{\alpha(z_i)}, \dots, \widehat{\alpha(z_j)}, \dots, \alpha(z_{n-1}))].$$

$$\underline{\text{Part 3:}}\ d^{n-1}_\alpha c(s(x), \alpha(z_1), \dots, \alpha(z_{n-1})) = \alpha^{n-2}(s(x)) \cdot c(\alpha(z_1), \dots, \alpha(z_{n-1}))$$

$$+ \sum_{1 \le i \le n-1} \alpha^{n-1}(z_i) \cdot c(s(x), \alpha(z_1), \dots, \widehat{\alpha(z_i)}, \dots, \alpha(z_{n-1}))$$

$$\sum_{1 \le i < j \le n-1} c([\alpha(z_i), \alpha(z_j)], \alpha(s(x)), \alpha^2(z_1), \dots, \widehat{\alpha^2(z_i)}, \dots, \widehat{\alpha^2(z_j)}, \dots, \alpha^2(z_{n-1}))$$

$$+ \sum_{1 \le i \le n-1} c([s(x), \alpha(z_i)], \alpha^2(z_1), \dots, \widehat{\alpha^2(z_i)}, \dots, \alpha^2(z_{n-1})).$$

$$\underline{\text{Part 4:}} \sum_{1\leq i\leq n-1} d_{\alpha}^{n-1}c([x,z_i],\alpha(x),\alpha(z_1),\ldots,\widehat{\alpha(z_i)},\ldots,\alpha(z_{n-1}))$$

$$= \sum_{1\leq i\leq n-1} [\alpha^{n-2}([x,z_i])\cdot c(\alpha(x),\alpha(z_1),\ldots,\widehat{\alpha(z_i)},\ldots,\alpha(z_{n-1}))$$

$$+\alpha^{n-1}(x)\cdot c([x,z_i],\alpha(z_1),\ldots,\widehat{\alpha(z_i)},\ldots,\alpha(z_{n-1}))$$

$$+\sum_{1\leq l\leq n-2} \alpha^{n-1}(z_l)\cdot c([x,z_i],\alpha(x),\alpha(z_1),\ldots,\widehat{\alpha(z_l)},\ldots,\widehat{\alpha(z_i)},\ldots,\alpha(z_{n-1}))$$

$$+c([[x,z_i],\alpha(x)],\alpha^2(z_1),\ldots,\widehat{\alpha^2(z_i)},\ldots,\alpha^2(z_{n-1}))$$

$$+\sum_{1\leq j\leq n-1} c([[x,z_i],\alpha(z_j)],\alpha^2(x),\alpha^2(z_1),\ldots,\widehat{\alpha^2(z_i)},\ldots,\widehat{\alpha^2(z_j)},\ldots\alpha^2(z_{n-1}))$$

$$+\sum_{1\leq l\leq n-1} c([\alpha(x),\alpha(z_l)],\alpha([x,z_i]),\alpha^2(z_1),\ldots,\widehat{\alpha^2(z_i)},\ldots,\widehat{\alpha^2(z_l)},\ldots\alpha^2(z_{n-1}))$$

$$+\sum_{1\leq u<v\leq n-1} c(\alpha([x,z_i]),\alpha^2(x),[\alpha(z_u),\alpha(z_v)],\alpha^2(z_1),\ldots,\widehat{\alpha^2(z_i)},\ldots,\widehat{\alpha^2(z_u)},\ldots\widehat{\alpha^2(z_v)},\ldots\alpha^2(z_{n-1}))].$$

$$\underline{\text{Part 5:}} \sum_{1\leq i<j\leq n-1} d_{\alpha}^{n-1}\mathfrak{p}(\alpha(x),[z_i,z_j],\alpha(z_1),\ldots,\widehat{\alpha(z_i)},\ldots,\widehat{\alpha(z_j)},\ldots,\alpha(z_{n-1}))$$

$$= \sum_{1\leq i<j\leq n-1} [\alpha^{n-1}(x)\cdot c(\alpha(x),[z_i,z_j],\alpha(z_1),\ldots,\widehat{\alpha(z_i)},\ldots,\widehat{\alpha(z_j)},\ldots,\alpha(z_{n-1}))$$

$$+\sum_{1\leq l\leq n-1} \alpha^{n-1}(z_l)\mathfrak{p}(\alpha(x),[z_i,z_j],\alpha(z_1),\ldots,\widehat{\alpha(z_i)},\ldots,\widehat{\alpha(z_j)},\ldots,\widehat{\alpha(z_l)},\ldots,\alpha(z_{n-1}))$$

$$+\alpha^{n-2}([z_i,z_j])\cdot\mathfrak{p}(x,\alpha(z_1),\ldots,\widehat{\alpha(z_i)},\ldots,\widehat{\alpha(z_j)},\ldots,\alpha(z_{n-1}))$$

$$+c(s(\alpha(x)),\alpha([z_i,z_j]),\alpha^2(z_1),\ldots,\widehat{\alpha(z_i)},\ldots,\widehat{\alpha(z_j)},\ldots,\alpha(z_{n-1}))$$

$$\sum_{1\leq l\leq n-1} c([\alpha(x),\alpha(z_l)],\alpha([z_i,z_j]),\alpha^2(x),\alpha^2(z_1),\ldots,\widehat{\alpha^2(z_i)},\ldots,\widehat{\alpha^2(z_j)},\ldots,\alpha^2(z_{n-1}))$$

$$+c([\alpha(x),[z_i,z_j]],\alpha^2(z_1),\ldots\widehat{\alpha^2(z_i)},\ldots,\widehat{\alpha^2(z_j)},\ldots,\widehat{\alpha^2(z_l)},\ldots,\alpha^2(z_{n-1}))$$

$$+\sum_{1\leq l\leq n-1} \mathfrak{p}(\alpha^2(x),[\alpha(z_l),[z_i,z_j]],\alpha^2(z_1),\ldots\widehat{\alpha^2(z_i)},\ldots,\widehat{\alpha^2(z_j)},\ldots,\widehat{\alpha^2(z_l)},\ldots\alpha^2(z_{n-1}))$$

$$+\sum_{1\leq u<v\leq n-1} T_{ijuv}],$$

where

$$T_{ijuv} = \mathfrak{p}(\alpha^2(x),\alpha([z_i,z_j]),[\alpha(z_u),\alpha(z_v)],\alpha^2(z_1),\ldots\widehat{\alpha^2(z_i)},\ldots,\widehat{\alpha^2(z_j)},\ldots,\widehat{\alpha^2(z_u)},\ldots,\widehat{\alpha^2(z_v)},\ldots).$$

Now, using Lemma 1, a direct computation shows that

$$\text{Part 1} + \text{Part 2} + \text{Part 3} + \text{Part 4} + \text{Part 5} = 0. \quad \square$$

Now, we are ready to define a cohomology of Hom-Lie superalgebras in characteristic 2. The kernel of the map $\mathfrak{d}_{\alpha}^{n}$, denoted by $Z_{\alpha}^{n}(\mathfrak{g};M)$, is the space of n-cocycles. The range of the map $\mathfrak{d}_{\alpha}^{n-1}$, denoted by $B_{\alpha}^{n}(\mathfrak{g};M)$, is the space of coboundaries.

We define the nth cohomology space as

$$\mathrm{H}_{\alpha}^{n}(\mathfrak{g};M) := Z_{\alpha}^{n}(\mathfrak{g};M)/B_{\alpha}^{n}(\mathfrak{g};M).$$

Remark 5. *The cohomology defined above coincides when $\alpha = \mathrm{id}_{\mathfrak{g}}$ and $\beta = \mathrm{id}_M$, with the cohomology of Lie superalgebras in characteristic 2 defined in the previous section.*

Example 4. *We compute the second cohomology of the Hom-Lie superalgebra $\mathfrak{oo}_{III}^{(1)}(1|2)_\alpha$ defined in Example 1. We will assume here that the field $\mathbb{K}$ is infinite:*

(i) The cohomology space $\mathrm{H}^2_\alpha(\mathfrak{oo}_{III}^{(1)}(1|2)_\alpha; \mathbb{K})$ is trivial. Recall that, in this case, the map $\beta = \mathrm{Id}$.

Let us first show that cocycles of the form $(0, \mathfrak{p})$ are necessarily trivial. In fact, the condition $\mathfrak{p} = \mathfrak{p} \circ \alpha$ and $\varepsilon \neq 0$ imply that

$$\mathfrak{p}(x_1) = 0 \quad \textit{and} \quad \mathfrak{p}(y_1) = m \ \ \textit{(arbitrary)}.$$

Choose $B_m = m\, y_2^*$, where $m \in \mathbb{K}$. A direct computation shows that $d^2_\alpha B_m = 0$. Let us compute the corresponding $\mathfrak{q}_m$. Indeed,

$$\mathfrak{q}_m(x_1) = B_m(s(x_1)) = B_m(x_2) = 0,$$

and

$$\mathfrak{q}_m(y_1) = B_m(s(y_1)) = B_m(\varepsilon h_1 + \varepsilon^2 x_2 + y_2) = m.$$

It follows that $(0, \mathfrak{p}) = (d^2_\alpha B_m, \mathfrak{q}_m)$, and hence, its cohomology class is trivial.

Let us now describe two-cocycles of the form $(c, \mathfrak{p})$. A direct computation shows that

$$c_1 = x_1^* \wedge y_1^* + x_2^* \wedge y_2^*, \quad c_2 = h_1^* \wedge y_1^* + x_1^* \wedge y_2^*,$$

are the only cochains verifying both conditions $c_i \circ (\alpha \wedge \alpha) = c_i$ and $d^2_\alpha c_i = 0$ for $i = 1, 2$. Let us describe the corresponding $\mathfrak{p}$s. We have

$$\begin{array}{rclrcll} \mathfrak{p}_1(x_1) & = & \varepsilon^{-1}, & \mathfrak{p}_1(y_1) & = & m_1 & \textit{(arbitrary)} \\ \mathfrak{p}_2(x_1) & = & 0 & \mathfrak{p}_2(y_1) & = & m_2 & \textit{(arbitrary)} \end{array}$$

We then obtain that $\mathfrak{d}^2_\alpha(c_1, \mathfrak{p}_1) = \mathfrak{d}^2_\alpha(c_2, \mathfrak{p}_2) = 0$.

Let us now describe the coboundaries. Choose $b_1 = h_1^* + \varepsilon^{-1} x_2^*$. It follows that

$$d^1_\alpha b_1 = x_1^* \wedge y_1^* + x_2^* \wedge y_2^*.$$

Now,

$$\mathfrak{q}_1(x_1) = b_1(s(x_1)) = b_1(x_2) = \varepsilon^{-1},$$

and

$$\mathfrak{q}_1(y_1) = b_1(s(y_1)) = b_1(\varepsilon h + \varepsilon^2 x_2 + y_2) = 0.$$

Choose $b_2 = y_1^*$. A direct computation shows that

$$d^1_\alpha b_2 = h_1^* \wedge y_1^* + x_1^* \wedge y_2^*, \quad \textit{and } \mathfrak{q}_2 \equiv 0.$$

It follows that

$$(c_1, \mathfrak{p}_1) = (d^2_\alpha b_1, \mathfrak{q}_1) + (d^2_\alpha B_{m_1} = 0, \mathfrak{q}_{m_1}), \quad \textit{and} \quad (c_2, \mathfrak{p}_2) = (d^2_\alpha b_2, \mathfrak{q}_2) + (d^2_\alpha B_{m_2} = 0, \mathfrak{q}_{m_2}).$$

Therefore, the cohomology space $\mathrm{H}^2_\alpha(\mathfrak{oo}_{III}^{(1)}(1|2)_\alpha; \mathbb{K})$ is trivial.

(ii) Let us now compute the cohomology space: $\mathrm{H}^2_\alpha(\mathfrak{oo}_{III}^{(1)}(1|2)_\alpha; \mathfrak{oo}_{III}^{(1)}(1|2)_\alpha)$. Recall that, in the case where $\alpha = \mathrm{Id}$, this cohomology space has only two non-trivial two-cocycles.

The case where $\varepsilon \neq 1$: the space $\mathrm{H}^2_\alpha(\mathfrak{oo}_{III}^{(1)}(1|2)_\alpha; \mathfrak{oo}_{III}^{(1)}(1|2)_\alpha$ is generated by the non-trivial two-cocycles:

$$(c_4, \mathfrak{p}_4), \quad (c_5, \mathfrak{p}_5), \quad (c_9, \mathfrak{p}_9), \quad (c_{10}, \mathfrak{p}_{10}), \quad (c_{11}, \mathfrak{p}_{11}), \quad (c_{12}, \mathfrak{p}_{12}),$$

where

$$
\begin{aligned}
c_4 &= \varepsilon x_1 \otimes h_1^* \wedge x_1^* + \varepsilon^2 x_1 \otimes h_1^* \wedge y_1^* + x_1 \otimes x_1^* \wedge x_2^* + \varepsilon^2 x_1 \otimes x_1^* \wedge y_2^* + \varepsilon^2 x_2 \otimes x_1^* \wedge y_1^* \\
&\quad \varepsilon^2 x_2 \otimes x_2^* \wedge y_2^* + \varepsilon y_1 \otimes h_1^* \wedge y_1^* + y_1 \otimes x_2^* \wedge y_1^*, \\
c_5 &= h_1 \otimes h_1^* \wedge y_2^* + x_1 \otimes h_1^* \wedge y_1^* + x_1 \otimes x_1^* \wedge y_2^*, \\
c_9 &= h_1 \otimes h_1^* \wedge x_1^* + \varepsilon h_1 \otimes h_1^* \wedge y_1^* + \varepsilon x_1 \otimes x_1^* \wedge y_1^* + \varepsilon x_2 \otimes h_1^* \wedge x_1^* + \varepsilon^2 x_2 \otimes h_1^* \wedge y_1^* \\
&\quad + \varepsilon^2 x_2 \otimes x_1^* \wedge y_2^* + \varepsilon x_2 \otimes x_2^* \wedge y_1^* + y_2 \otimes h_1^* \wedge y_1^* + y_2 \otimes x_1^* \wedge y_2^*, \\
c_{10} &= h_1 \otimes x_1^* \wedge y_1^*, \\
c_{11} &= h_1 \otimes x_2^* \wedge y_2^* + x_1 \otimes x_1^* \wedge y_2^* + x_1 \otimes x_2^* \wedge y_1^* + \varepsilon x_2 \otimes x_1^* \wedge y_1^* + \varepsilon x_2 \otimes x_2^* \wedge y_2^*, \\
c_{12} &= x_1 \otimes y_1^* \wedge x_1^* + \varepsilon x_1 \otimes h_1^* \wedge y_1^* + \varepsilon x_2 \otimes x_1^* \wedge y_1^* + \varepsilon x_2 \otimes x_2^* \wedge y_2^* \\
&\quad + y_1 \otimes h_1^* \wedge y_1^* + y_1 \otimes x_1^* \wedge y_2^*,
\end{aligned}
$$

and

$$
\begin{array}{rclrcl}
\mathfrak{p}_4(x_1) &=& h_1, & \mathfrak{p}_4(y_1) &=& \varepsilon^2 h_1 + \varepsilon^3 x_2 + \varepsilon y_2, \\
\mathfrak{p}_5(x_1) &=& 0, & \mathfrak{p}_5(y_1) &=& 0, \\
\mathfrak{p}_9(x_1) &=& x_1, & \mathfrak{p}_9(y_1) &=& 0, \\
\mathfrak{p}_{10}(x_1) &=& x_2, & \mathfrak{p}_{10}(y_1) &=& \varepsilon h_1 + \varepsilon^2 x_2 + y_2, \\
\mathfrak{p}_{11}(x_1) &=& x_2, & \mathfrak{p}_{11}(y_1) &=& 0, \\
\mathfrak{p}_{12}(x_1) &=& x_2, & \mathfrak{p}_{12}(y_1) &=& 0.
\end{array}
$$

The case where $\varepsilon = 1$: the space $\mathrm{H}^2_\alpha(\mathfrak{oo}^{(1)}_{III}(1|2)_\alpha; \mathfrak{oo}^{(1)}_{III}(1|2)_\alpha)$ is generated by the non-trivial two-cocycles:

$$(c_4, \mathfrak{p}_4), \quad (c_5, \mathfrak{p}_5), \quad (c_{10}, \mathfrak{p}_{10}), \quad (c_{11}, \mathfrak{p}_{11}).$$

6.4. Deformations of Hom-Lie Superalgebras

The deformation theory of Hom-Lie superalgebras in characteristic 2 will be discussed here. As a result, we also cover the Lie case, namely $\alpha = \mathrm{Id}_\mathfrak{g}$. Over a field of characteristic 0, the study was carried out in [10,27,28].

Let $(\mathfrak{g}, [\cdot,\cdot], s, \alpha)$ be a Hom-Lie superalgebra over a field $\mathbb{K}$ of characteristic 2. A deformation of $\mathfrak{g}$ is a family of Hom-Lie superalgebras $\mathfrak{g}_t$ specializing in $\mathfrak{g}$ when the even parameter t equals 0 and where the Hom-Lie superalgebra structure is defined on the tensor product $\mathfrak{g} \otimes \mathbb{K}[[t]]$ when $\mathfrak{g}$ is finite-dimensional. The bracket in the deformed Hom-Lie superalgebra $\mathfrak{g}_t$ is a $\mathbb{K}[[t]]$-bilinear map of the form (for all $x, y \in \mathfrak{g}$):

$$[x,y]_t = [x,y] + \sum_{i \geq 1} c_i(x,y) t^i,$$

while the squaring s_t, with respect to $\mathbb{K}[[t]]$, on the Hom-Lie superalgebra $\mathfrak{g}_t$ is given by (for all $x \in \mathfrak{g}_{\bar{1}}$):

$$s_t(x) = s(x) + \sum_{i \geq 1} \mathfrak{p}_i(x) t^i,$$

where $(c_i, \mathfrak{p}_i) \in XC^2_\alpha(\mathfrak{g}; \mathfrak{g})$ such that c_i is an even map and $\mathfrak{p}_i : \mathfrak{g}_{\bar{1}} \to \mathfrak{g}_{\bar{0}}$, for all $i \geq 1$. We will assume that $c_0(\cdot,\cdot) = [\cdot,\cdot]$ and $\mathfrak{p}_0(\cdot) = s(\cdot)$.

According to deformation theory, we call a deformation *infinitesimal* if the bracket $[\cdot,\cdot]_t$ and the squaring $s_t(\cdot)$ define a Hom-Lie superalgebra structure $\mod (t^2)$ (degree 1), that is $[\cdot,\cdot]_t = [\cdot,\cdot] + c_1(\cdot,\cdot)t$ and $s_t(\cdot) = s(\cdot) + \mathfrak{p}_1(\cdot)t$. A deformation is said to be of *order n* if the bracket $[\cdot,\cdot]_t$ and the squaring $s_t(\cdot)$ define a Hom-Lie superalgebra structure $\mod (t^{n+1})$, that is

$$[\cdot,\cdot]_t = [\cdot,\cdot] + \sum_{1 \leq i \leq n} c_i(\cdot,\cdot) t^i \text{ and } s_t(\cdot) = s(\cdot) + \sum_{1 \leq i \leq n} \mathfrak{p}_i(\cdot) t^i.$$

Afterwards, one tries to extend a deformation of order n to a deformation of order $n+1$. All obstructions are cohomological, as we will see.

Theorem 5. *Let $(\mathfrak{g}, [\cdot,\cdot], s, \alpha)$ be a Hom-Lie superalgebra in characteristic 2 and $(\mathfrak{g}_t, [\cdot,\cdot]_t, s_t, \alpha)$ be a deformation. Assume that $(c_1, \mathfrak{p}_1) \neq (0,0)$. Then:*
(i) $(c_1, \mathfrak{p}_1)$ is a two-cocycle, i.e., $(c_1, \mathfrak{p}_1) \in Z^2_\alpha(\mathfrak{g};\mathfrak{g})$.
(ii) For $n > 1$, consider the following maps:

$$\begin{array}{lcl} C_n(x,y,z) & := & \sum\limits_{\substack{i+j=n\\ i,j<n}} c_i(c_j(x,y),\alpha(z)) + \circlearrowleft (x,y,z), \quad \textit{for all } x,y,z \in \mathfrak{g}, \\ Q_n(x,y) & := & \sum\limits_{\substack{i+j=n\\ i,j<n}} c_i(\mathfrak{p}_j(x),\alpha(y)) + \sum\limits_{\substack{i+j=n\\ i,j<n}} c_i(c_j(x,y),\alpha(x)), \quad \textit{for all } x \in \mathfrak{g}_{\bar{1}} \textit{ and } y \in \mathfrak{g}. \end{array}$$

A deformation of order $n-1$ can be extended to a deformation of order n if and only there exists $(c_n, \mathfrak{p}_n)$:

$$(C_n, Q_n) = \mathfrak{d}^2_\alpha(c_n, \mathfrak{p}_n).$$

Proof. (i) Checking that c_1 satisfies the condition (34) is routine; see [10]. Let us deal with the squaring s_t. We have

$$\begin{aligned} &[s_t(x),\alpha(y)]_t = [s(x) + \sum_{i\geq 1} \mathfrak{p}_i(x)t^i, \alpha(y)]_t \\ &= [s(x),\alpha(y)] + \sum_{i\geq 1} c_i(s(x),\alpha(y))t^i + \sum_{i\geq 1}[\mathfrak{p}_i(x),\alpha(y)]t^i + \sum_{i,j\geq 1} c_j(\mathfrak{p}_i(x),\alpha(y))t^{i+j}. \end{aligned} \tag{36}$$

On the other hand,

$$\begin{array}{rcl} [\alpha(x),[x,y]_t]_t & = & [\alpha(x),[x,y] + \sum\limits_{i\geq 1} c_i(x,y)t^i]_t \\ & = & [\alpha(x),[x,y]] + \sum\limits_{i\geq 1} c_i(\alpha(x),[x,y])t^i + \sum\limits_{i\geq 1}[\alpha(x),c_i(x,y)]t^i + \\ & & + \sum_{i,j\geq 1} c_i(\alpha(x),c_j(x,y))t^{i+j}. \end{array} \tag{37}$$

Collecting the coefficient of t in the condition $[s_t(x),\alpha(y)]_t = [\alpha(x),[x,y]_t]_t$, we obtain

$$c_1(s(x),\alpha(y)) + [\alpha(y),\mathfrak{p}_i(x)] + c_1(\alpha(x),[x,y]) + [\alpha(x),c_1(x,y)] = 0,$$

which corresponds to Condition (35). Therefore, $(c_1, \mathfrak{p}_1)$ is a two-cocycle on $\mathfrak{g}$ with values in the adjoint representation.

(ii) Let us first show that the pair (C_n, Q_n) is a cochain in $XC^3(\mathfrak{g},\mathfrak{g})$. Indeed,

$$\begin{aligned} &Q_n(x_1+x_2,y) = \sum_{i,j} c_i(\mathfrak{p}_j(x_1+x_2),\alpha(y)) + c_i(c_j(x_1+x_2,y),\alpha(x_1+x_2)) \\ &= \sum_{i,j} c_i(\mathfrak{p}_j(x_1) + \mathfrak{p}_j(x_2) + c_j(x_1,x_2),\alpha(y)) + c_i(c_j(x_1,y) + c_j(x_2,y),\alpha(x_1)+\alpha(x_2)) \\ &= Q_n(x_1,y) + Q_n(x_2,y) + \sum_{i,j}(c_i(c_j(x_1,x_2),\alpha(y)) + c_i(c_j(x_1,y),\alpha(x_2)) + c_i(c_j(x_2,y),\alpha(x_1))) \\ &= Q_n(x_1,y) + Q_n(x_2,y) + C_n(x_1,x_2,y). \end{aligned}$$

Collecting the coefficients of t^n in (34) leads to $C_n = d^2_\alpha c_n$; see [10]. Let us deal with the squaring. Consider the coefficient of t^n in the condition $[s_t(x),\alpha(y)]_t = [\alpha(x),[x,y]_t]_t$, and using Equations (36) and (37), we obtain (for all $x \in \mathfrak{g}_{\bar{1}}$ and $y \in \mathfrak{g}$):

$$c_n(s(x),\alpha(y)) + [\mathfrak{p}_n(x),\alpha(y)] + \sum_{\substack{i+j=n\\ i,j<n}} c_j(\mathfrak{p}_i(x),\alpha(y)) = c_n(\alpha(x),[x,y]) + [\alpha(x),c_n(x,y)] + \sum_{\substack{i+j=n\\ i,j<n}} c_i(\alpha(x),c_j(x,y)).$$

Let us rewrite this expression. We obtain

$$d^2_\alpha \mathfrak{p}_n(x,y) = \sum_{\substack{i+j=n\\ i,j<n}} c_i(\alpha(x),c_j(x,y)) + \sum_{\substack{i+j=n\\ i,j<n}} c_j(\mathfrak{p}_i(x),\alpha(y)).$$

Therefore, $(C_n, Q_n) = (d^2_\alpha c_n, d^2_\alpha \mathfrak{p}_n) = \mathfrak{d}^2_\alpha(c_n,\mathfrak{p}_n)$. $\square$

Now, we discuss equivalent deformations.

Definition 5. *Let $(\mathfrak{g},[\cdot,\cdot],s,\alpha)$ be a Hom-Lie superalgebra in characteristic 2. Let $(\mathfrak{g}_t,[\cdot,\cdot]_t,s_t,\alpha)$ and $(\tilde{\mathfrak{g}}_t,\widetilde{[\cdot,\cdot]}_t,\tilde{s}_t,\alpha)$ be two deformations of $\mathfrak{g}$, such that $\widetilde{[\cdot,\cdot]}_0 = [\cdot,\cdot]_0 = [\cdot,\cdot]$ and $\tilde{s}_0 = s_0 = s$. We say that the two deformations $\mathfrak{g}_t$ and $\tilde{\mathfrak{g}}_t$ are equivalent if there exists a $\mathbb{K}[[t]]$-linear map $\tau : \mathfrak{g}_t \to \tilde{\mathfrak{g}}_t$ of the form $\tau(a) = \mathrm{id}_\mathfrak{g}(a) + \sum_{i\geq 1}\tau_i(a)t^i$ for all $a \in \mathfrak{g}$, which is an isomorphism of the Hom-Lie superalgebras.*

Theorem 6. *Two one-parameter formal deformations $\mathfrak{g}_t$ and $\tilde{\mathfrak{g}}_t$ of $\mathfrak{g}$ given by the collections $(c,\mathfrak{p})$ and $(\tilde{c},\tilde{\mathfrak{p}})$ are equivalent through an isomorphism of the form $\tau = \mathrm{id}_\mathfrak{g} + \sum_{i\geq 1} t^i\tau_i$ if and only if τ links $(c,\mathfrak{p})$ and $(\tilde{c},\tilde{\mathfrak{p}})$ by the following formulae (for all $n > 0$):*

$$\sum_{i+j=n} \tau_i(\tilde{c}_j(x,y)) = \sum_{i+j+k=n} c_i(\tau_j(x),\tau_k(y)), \quad \text{for all } x,y \in \mathfrak{g}, \tag{38}$$

and (for all $x \in \mathfrak{g}_{\bar{1}}$):

$$\sum_{i+j=n} \tau_i(\tilde{\mathfrak{p}}_j(x)) = \sum_{i+j=n} c_i(x,\tau_j(x)) + \sum_{2i+j=n} \mathfrak{p}_j(\tau_i(x)) + \sum_{\substack{u+v+j=n\\ 1\leq u<v\\ 1\leq j}} c_j([\tau_u(x),\tau_v(x)]). \tag{39}$$

In particular, if $n = 1$, we obtain

$$\begin{aligned}\tilde{c}_1(x,y) &= c_1(x,y) + \tau_1([x,y]) + [x,\tau_1(y)] + [y,\tau_1(x)], \quad \text{for all } x,y \in \mathfrak{g},\\ \tilde{\mathfrak{p}}_1(x) &= \mathfrak{p}_1(x) + \tau_1(s(x)) + [x,\tau_1(x)], \quad \text{for all } x \in \mathfrak{g}_{\bar{1}}.\end{aligned}$$

Hence, $(\tilde{c}_1,\tilde{\mathfrak{p}}_1)$ and $(c_1,\mathfrak{p}_1)$ are in the same cohomology class.

Proof. Checking Equation (38) is routine; see [10]. Let us check Equation (39).

We have

$$\tau(\tilde{s}_t(x)) = \tau\left(s(x) + \sum_{i\geq 1}\tilde{\mathfrak{p}}_i(x)t^i\right) = s(x) + \sum_{i\geq 1}\tilde{\mathfrak{p}}_i(x)t^i + \sum_{i,j\geq 1}\tau_j(\tilde{\mathfrak{p}}_i(x))t^{i+j} + \sum_{j\geq 1}\tau_j(s(x))t^j$$
$$= \sum_{i,j\geq 0}\tau_j(\tilde{\mathfrak{p}}_i(x))t^{i+j}$$

On the other hand, we obtain

$$\begin{aligned}
s_t(\tau(x)) &= s\left(x+\sum_{i\geq 1}\tau_i(x)t^i\right)+\sum_{i\geq 1}\mathfrak{p}_i\left(x+\sum_{i\geq 1}\tau_i(x)t^i\right)t^j \\
&= s(x)+\sum_{i\geq 1}s(\tau_i(x))t^{2i}+\sum_{i\geq 1}[x,\tau_i(x)]t^i+\sum_{i\geq 1}\mathfrak{p}_j\left(x+\sum_{i\geq 1}\tau_i(x)t^i\right)t^j \\
&= s(x)+\sum_{i\geq 1}s(\tau_i(x))t^{2i}+\sum_{i\geq 1}[x,\tau_i(x)]t^i+\sum_{i\geq 1}\mathfrak{p}_j(x)t^j+\sum_{i\geq 1}\mathfrak{p}_j(\tau_i(x))t^{2i+j} \\
&\quad +\sum_{i,j\geq 1}c_j(x,\tau_i(x))t^{i+j}+\sum_{1\leq j}\sum_{1\leq u<v}c_j(\tau_u(x),\tau_v(x))t^{j+u+v} \\
&= \sum_{i,j\geq 0}\mathfrak{p}_j(\tau_i(x))t^{2i+j}++\sum_{i,j\geq 0}c_j(x,\tau_i(x))t^{i+j}+\sum_{1\leq j}\sum_{1\leq u<v}c_j(\tau_u(x),\tau_v(x))t^{j+u+v}
\end{aligned}$$

The result follows by identification. □

As a consequence, we have that, when the second cohomology group is trivial, the Lie superalgebra in characteristic 2 has no non-trivial deformation. Such Lie superalgebras in characteristic 2 are called rigid.

Corollary 2. *Infinitesimal deformations over $\mathbb{K}[[t]]/(t^2)$ are classified by element $(c,\mathfrak{p})$ of the cohomology group $\mathrm{H}^2_\alpha(\mathfrak{g};\mathfrak{g})$, where c is even and $Im(\mathfrak{p})\subseteq\mathfrak{g}_0$.*

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References

1. Bouarroudj, S.; Grozman, P.; Leites, D. Classification of finite-dimensional modular Lie superalgebras with indecomposable Cartan matrix. *SIGMA Symmetry Integrability Geom. Methods Appl.* **2009**, *5*, 63. [CrossRef]
2. Bouarroudj, S.; Lebedev, A.; Leites, D.; Shchepochkina, I. Classifications of simple Lie superalgebras in characteristic 2. *Int. Math. Res. Not.* **2023**, *1*, 54–94. [CrossRef]
3. Krutov, A.; Lebedev, A. On gradings modulo 2 of simple Lie algebras in characteristic 2. *SIGMA Symmetry Integr. Geom. Methods Appl.* **2018**, *14*, 130. [CrossRef]
4. Lebedev, A. Analogs of the orthogonal, Hamiltonian, Poisson, and contact Lie superalgebras in characteristic 2. *J. Nonlinear Math. Phys*. **2010**, *17*, 217–251. [CrossRef]
5. Chaichian, M.; Kulish, K.; Lukierski, J. q-Deformed Jacobi identity, q-oscillators and q-deformed infinite-dimensional algebras. *Phys. Lett. B* **1990**, *237*, 401–406. [CrossRef]
6. Hartwig, J.T.; Larsson, D.; Silvestrov, S. Deformations of Lie algebras using σ-derivations. *J. Algebra* **2006**, *295*, 314–361. [CrossRef]
7. Larsson, D.; Silvestrov, S.D. Quasi-hom-Lie algebras, central extensions and 2-cocycle-like identities. *J. Algebra* **2005**, *288*, 321–344. [CrossRef]
8. Larsson, D.; Silvestrov, S. Graded quasi-Lie algebras. *Czech. J. Phys.* **2005**, *55*, 1473–1478. [CrossRef]
9. Ammar, F.; Makhlouf, A. Hom-Lie superalgebras and Hom-Lie admissible superalgebras. *J. Algebra* **2010**, *324*, 1513–1528. [CrossRef]
10. Ammar, F.; Makhlouf, A.; Saadaoui, N. Cohomology of Hom-Lie superalgebras and q-deformed Witt superalgebra. *Czechoslovak Math. J.* **2013**, *63*, 721–761. [CrossRef]
11. Albuquerque, H.; Barreiro, E.; Calderón, A.J.; Sànchez, J.M. On split regular Hom-Lie superalgebras. *J. Geom. Phys.* **2018**, *128*, 1–11. [CrossRef]
12. Hou, Y.; Tang, L.; Chen, L. Product and complex structures on hom-Lie superalgebras. *Comm. Algebra* **2021**, *49*, 3685–3707. [CrossRef]

13. Bahturin, Y.A.; Mikhalev, A.A.; Petrogradsky, V.M.; Zaicev, M.V. Infinite-Dimensional Lie Superalgebras. In *De Gruyter Expositions in Mathematics Book 7*; Walter de Gruyter & Co.: Berlin, Germany, 1992; 250p.
14. Garcia-Delgado, R.; Salgado, G.; Sanchez-Valenzuela, O.A. On 3-dimensional complex Hom-Lie algebras. *J. Algebra* **2020**, *555*, 361–385. Corrigendum in *J. Algebra* **2020**, *562*, 286–289. [CrossRef]
15. Li, X.; Li, Y. Classification of 3-dimensional multiplicative Hom-Lie algebras. *J. Xinyang Norm. Univ. Nat. Sci. Ed.* **2012**, *25*, 455–475.
16. Makhlouf, A.; Silvestrov, S.D. Hom-algebra structures. *J. Gen. Lie Theory Appl.* **2008**, *2*, 51–64. [CrossRef]
17. Ongong'a, E.; Richter, J.; Silvestrov S. Classification of 3-dimensional Hom-Lie algebras. *IOP Conf. Ser. J. Physics Conf. Ser.* **2019**, *1194*, 012084. [CrossRef]
18. Ongong'a, E.; Richter, J. Silvestrov, S. Classification of Low-Dimensional Hom-Lie Algebras, Classification of Low-Dimensional Hom-Lie Algebras. In *Algebraic Structures and Applications, SPAS 2017*; Silvestrov, S., Malyarenko, A., Rancic, M., Eds.; Springer Proceedings in Mathematics & Statistics; Springer: Cham, Switzerland, 2020; Volume 317. [CrossRef]
19. Remm, E. 3-Dimensional Skew-symmetric Algebras and the Variety of Hom-Lie Algebras. *Algebra Colloq.* **2018**, *25*, 547–566. [CrossRef]
20. Wang, C.; Zhang, Q.; Wei, Z.; A classification of low dimensional multiplicative Hom-Lie superalgebras. *Open Math.* **2016**, *14*, 613–628. [CrossRef]
21. Bouarroudj, S.; Grozman, P.; Krutov, A.; Leites, D. Irreducible modules over exceptional modular Lie superalgebras with indecomposable Cartan matrix. *In preparation*.
22. Sheng, Y.; Xiong, Z. On Hom-Lie algebras, Linear and Multilinear. *Algebra* **2015**, *63*, 2379–2395.
23. Jacobson, N. *Lie Algebras*; Interscience: New York, NY, USA, 1962.
24. Guan, B.; Chen L. Restricted hom-Lie algebras. *Hacet. J. Math. Stat.* **2015**, *44*, 823–837. [CrossRef]
25. Yau, D. Hom-algebras and homology. *J. Lie Theory* **2009**, *19*, 409–421.
26. Bouarroudj, S.; Grozman, P.; Lebedev, A.; Leites, D. Derivations and central extensions of simple modular Lie algebras and superalgebras. *SIGMA Symmetry Integr. Geom. Methods Appl.* **2023**, *19*, 1858.
27. Fuks, D. *Cohomology of Infinite Dimensional Lie Algebras*; Consultants Bureau: New York, NY, USA, 1986.
28. Tripathy, K.C.; Patra, M.K.; Cohomology theory and deformations of $\mathbb{Z}_2$-graded Lie algebras. *J. Math. Phys.* **1990** , *31*, 2822. [CrossRef]

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