

Article

Unitary Diagonalization of the Generalized Complementary Covariance Quaternion Matrices with Application in Signal Processing

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Abstract: Let \mathbb{H} denote the quaternion algebra. This paper investigates the generalized complementary covariance, which is the ϕ -Hermitian quaternion matrix. We give the properties of the generalized complementary covariance matrices. In addition, we explore the unitary diagonalization of the covariance and generalized complementary covariance. Moreover, we give the generalized quaternion unitary transform algorithm and test the performance by numerical simulation.

Keywords: quaternion algebra; covariance matrices; ϕ -Hermicity; signal processing

MSC: 15A23; 15A66



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1. Introduction

A covariance matrix describes the relationship between different dimensions, and it can be applied in stochastic modeling and principal component analysis (PCA) (e.g., [1–6]). The central idea of principal component analysis is the diagonalization of covariance matrix. Moreover, diagonalization covariance matrices play an important role in many statistical signal processing algorithms (e.g., [7–9]). The covariance matrix $\mathbf{C} = E\{\mathbf{xx}^H\}$ and pseudo-covariance matrix $\mathbf{P} = E\{\mathbf{xx}^T\}$ both compose complete second-order statistical information on the complex field [10]. Diagonalization of \mathbf{C} and \mathbf{P} is performed using eigen-decomposition and Takagi factorization, respectively. These two decompositions can decorrelate the data in its general form [11]. De Lathauwer and De Moor [12] as well as Eriksson and Koivunen [13] introduced the strong uncorrelated transformation (SUT). Ollila and Koivunen [14] presented its extension, namely, generalized uncorrelated transformation (GUT). Cheong Took et al. [15] researched the approximate uncorrelating transform (AUT) to decorrelate the mixed signal. SUT, GUT, and AUT can be used in blind separation of non-circular complex source [16], separation of signal and noise components in harmonic signal subbands [17], and so on.

Quaternion algebra is an associative and non-commutative division algebra over the real number field. The application of quaternion matrices involves many fields, such as computer science, orbital mechanics, statistics, and so on (e.g., [1,18–21]). Especially in the field of signal processing, C.C. Took and D.P. Mandic [22] proposed the quaternion widely linear (QWL) model for quaternion valued mean square error (MSE) estimation. C.C. Took et al. [23] studied the augmented second-order statistics of quaternion random signals, in [15] they investigated the approximate diagonalization of correlation matrices in widely linear signal processing, and so on. Quaternions can be used to process high-dimensional data. As such, the quaternion matrix is a new and effective tool in signal processing. The research of quaternion signal processing has involved Fourier transforms [24], neural networks (e.g., [25,26]), independent component analysis (ICA) [27], and

so on. Due to its widespread application, in linear algebra, the structure and simultaneous diagonalization of the covariance matrices of the quaternion field have received great attention in widely linear signal processing (e.g., [21,23,28]).

Because quaternion multiplication cannot be exchanged, the related theory in complex field \mathbb{C} cannot be directly extended to quaternion field \mathbb{H} . In the quaternion field, Cheong Took et al. [23] presented the standard covariance matrix $\mathbf{C}_x = E\{\mathbf{x}\mathbf{x}^H\}$, which is a Hermitian quaternion matrix, and the complementary covariance matrices $\mathbf{C}_x^\eta = E\{\mathbf{x}\mathbf{x}^{\eta H}\}$, which are η -Hermitian quaternion matrices, where $\eta \in \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$, $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are unit imaginary numbers and $(\cdot)^H$ represents the Hermitian (conjugate transpose). In recent years, some scholars have studied the η (-skew-) Hermitian (e.g., [2,29]). The diagonalization of \mathbf{C}_x can be used in eigen-decomposition and \mathbf{C}_x^η can be used in singular value decomposition [21]. As for their joint diagonalization, Enshaeifar et al. [30] presented the quaternion uncorrelating transform (QUT), which is the generalization of SUT in the quaternion field. Moreover, Min et al. [31] introduced the quaternion approximate uncorrelating transform (QAUT), which simultaneously diagonalizes all four covariance matrices associated with improper quaternion signals.

Rodman [32] presented the definition of ϕ -Hermitian quaternion matrix $\mathbf{A} = \mathbf{A}_\phi$, where ϕ is a non-standard involution, i.e., $\phi(a) = -\delta a^H \delta$, a^H is a conjugate transposition of quaternion a , and $\delta = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k} \in \mathbb{H}^{n \times n}$ is unit and purely imaginary. Obviously, when $\delta \in \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$, ϕ -Hermitian is the same as η -Hermitian, so ϕ -Hermitian is a more general case of η -Hermitian. In recent years, some scholars researched the ϕ -Hermitian quaternion matrices in many fields. For example, Aghamollaei et al. [33] studied the numerical ranges with respect to non-standard involutions ϕ on the quaternion field. They [34] also presented some quaternion matrix equations involving ϕ -Hermicity.

In this paper, we investigate the generalized complementary covariance quaternion matrices $\mathbf{C}_{x_\phi} = E\{\mathbf{x}\mathbf{x}^{\delta H}\}$, where $\delta = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k} \in \mathbb{H}^{n \times n}$ is unit and purely imaginary. It is obvious that \mathbf{C}_{x_ϕ} is a ϕ -Hermitian quaternion matrix. Furthermore, we study the simultaneous diagonalization of the standard covariance matrix and the generalized complementary covariance quaternion matrices. Moreover, we give the joint diagonalization algorithm and the numerical simulation. The main contribution of this paper is to generalize the complementary covariance quaternion matrices, which we promote from η -Hermitian matrix to ϕ -Hermitian matrix, and by comparing the separation time of mixed signals, the performance after generalization is improved.

The remainder of this paper is organized as follows. In Section 2, we review the definitions of involution, complex representation of quaternion, the ϕ -Hermitian quaternion matrix, and quaternion improperness. In Section 3, we introduce the structure of the generalized complementary covariance quaternion matrices. In Section 4, we give the conditions of unitary diagonalization of standard covariance and generalized complementary covariance quaternion matrices. In Section 5, we present the generalized quaternion unitary transform algorithm and test the performance by numerical simulation.

2. Preliminaries

2.1. Quaternion Algebra

Let $\mathbb{R}, \mathbb{C}, \mathbb{H}$ represent the real number field, complex field, and quaternion field, respectively. We know that the complex field is composed of two unit bases $\{1, \mathbf{i}\}$, and the quaternion field is composed of four bases $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$. A quaternion \mathbf{x} includes one real part and three imaginary parts, whose general expression [35] is

$$\mathbf{x} = a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k},$$

where $a_0, a_1, a_2, a_3 \in \mathbb{R}$ is a real number, and $\mathbf{i}, \mathbf{j}, \mathbf{k}$ satisfy

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1,$$

$$\mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \mathbf{ki} = -\mathbf{ik} = \mathbf{j}.$$

The real part and imaginary part of quaternion \mathbf{x} are expressed as $\mathbf{Re} \{ \mathbf{x} \} = a_0$ and $\mathbf{Im} \{ \mathbf{x} \} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, respectively. In particular, when $a_0 = 0$, it is called a pure quaternion. The conjugation of \mathbf{x} is defined as

$$\bar{\mathbf{x}} = \mathbf{x}^* = a_0 - a_1\mathbf{i} - a_2\mathbf{j} - a_3\mathbf{k},$$

and the norm is defined as

$$|\mathbf{x}| = \sqrt{\mathbf{x}^*\mathbf{x}} = \sqrt{a_0^2 + a_1^2 + a_2^2 + a_3^2}.$$

When $|\mathbf{x}| = 1$, the quaternion \mathbf{x} is called a unit quaternion.

Next, we review the definition of quaternion involution. On the quaternion field, Rodman [32] provided the definition of involution.

Definition 1 (involution [32]). *A map $\phi: \mathbb{H} \rightarrow \mathbb{H}$ is called an antiendomorphism if $\phi(xy) = \phi(y)\phi(x)$ and $\phi(x + y) = \phi(x) + \phi(y)$ for all $x, y \in \mathbb{H}$. An antiendomorphism ϕ is called an involution if $\phi(\phi(x)) = x$ for every $x \in \mathbb{H}$.*

Quaternion involution is divided into standard involution and non-standard involution [32]. In this paper, we only research the non-standard involution, whereby ϕ needs to be contented $\phi(a) = a^{\delta H} = -\delta a^H \delta$, where $\delta = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k} \in \mathbb{H}^{n \times n}$ is unit and purely imaginary, i.e., $u_1, u_2, u_3 \in \mathbb{R}$ and $u_1^2 + u_2^2 + u_3^2 = 1$.

The non-standard involuton of quaternion \mathbf{x} defined as $\mathbf{x}^{\delta H} = -\delta \mathbf{x}^H \delta$, where $\delta = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$. In particular, when $\delta \in \{ \mathbf{i}, \mathbf{j}, \mathbf{k} \}$ [21], the non-standard involution is

$$\begin{aligned} \mathbf{x}^{\mathbf{i}H} &= (\mathbf{x}^{\mathbf{i}})^H = (\mathbf{x}^H)^{\mathbf{i}} = -\mathbf{i}\mathbf{x}^H\mathbf{i} = a_0 - a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}, & \text{if } \delta = \mathbf{i}, \\ \mathbf{x}^{\mathbf{j}H} &= (\mathbf{x}^{\mathbf{j}})^H = (\mathbf{x}^H)^{\mathbf{j}} = -\mathbf{j}\mathbf{x}^H\mathbf{j} = a_0 + a_1\mathbf{i} - a_2\mathbf{j} + a_3\mathbf{k}, & \text{if } \delta = \mathbf{j}, \\ \mathbf{x}^{\mathbf{k}H} &= (\mathbf{x}^{\mathbf{k}})^H = (\mathbf{x}^H)^{\mathbf{k}} = -\mathbf{k}\mathbf{x}^H\mathbf{k} = a_0 + a_1\mathbf{i} + a_2\mathbf{j} - a_3\mathbf{k}, & \text{if } \delta = \mathbf{k}. \end{aligned} \tag{1}$$

Furthermore, Rodman [32] presented the definition of ϕ -Hermitian matrices as follows.

Definition 2 (ϕ -Hermitian [32]). *$\mathbf{A} \in \mathbb{H}^{n \times n}$ is said to be ϕ -Hermitian if $\mathbf{A} = \mathbf{A}_\phi$, where ϕ is a non-standard involution.*

2.2. Complex Representation of Quaternion and Quaternion Matrix

In this section, we review the complex representation of a quaternion and quaternion matrix [36]. A quaternion \mathbf{x} can be represented as

$$\mathbf{x} = a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} = (a_0 + a_1\mathbf{i}) + (a_2 + a_3\mathbf{i})\mathbf{j}, \tag{2}$$

Let $a_0 + a_1\mathbf{i} = \mathbf{x}_a$, $a_2 + a_3\mathbf{i} = \mathbf{x}_b$, and they are both complex numbers. Then the quaternion \mathbf{x} can be expressed as

$$\mathbf{x} = \mathbf{x}_a + \mathbf{x}_b\mathbf{j}, \tag{3}$$

which is the complex representation of quaternion \mathbf{x} . Its matrix representation [35] is as follows:

$$\hat{\mathbf{x}} = \begin{pmatrix} \mathbf{x}_a & \mathbf{x}_b \\ -\bar{\mathbf{x}}_b & \bar{\mathbf{x}}_a \end{pmatrix}, \tag{4}$$

where $\hat{\mathbf{x}}$ is a 2×2 complex matrix.

Similarly, the quaternion matrix $\mathbf{A}_{n \times n}$ can be expressed as

$$\mathbf{A} = \mathbf{A}_0 + \mathbf{A}_1\mathbf{i} + \mathbf{A}_2\mathbf{j} + \mathbf{A}_3\mathbf{k} = (\mathbf{A}_0 + \mathbf{A}_1\mathbf{i}) + (\mathbf{A}_2 + \mathbf{A}_3\mathbf{i})\mathbf{j} = \mathbf{A}_a + \mathbf{A}_b\mathbf{j}, \tag{5}$$

where $\mathbf{A}_0, \mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3 \in \mathbb{R}^{n \times n}$, $\mathbf{A}_a = \mathbf{A}_0 + \mathbf{A}_1\mathbf{i}$, $\mathbf{A}_b = \mathbf{A}_2 + \mathbf{A}_3\mathbf{i}$ are complex matrices. The above formula (5) is the complex representation of the quaternion matrix, whose matrix expression [35] is as follows

$$\hat{\mathbf{A}} = \begin{pmatrix} \mathbf{A}_a & \mathbf{A}_b \\ -\mathbf{A}_b & \mathbf{A}_a \end{pmatrix}, \tag{6}$$

where $\hat{\mathbf{A}}$ is a $2n \times 2n$ complex matrix.

Using complex representation, any quaternion matrix can be transformed from the quaternion field to the complex field; thus, many quaternion problems can be solved.

2.3. Quaternion Improperness

On the quaternion field, the improperness is characterized by the degree of irrelevance between the real part and the imaginary part. The properness of a quaternion can be divided into two types: \mathbb{H} -improperness and \mathbb{C}^α -improperness [37]. The definitions are given as follows.

Definition 3 (\mathbb{H} -properness [37]). A quaternion random vector \mathbf{x} is \mathbb{H} -proper if it is uncorrelated with its vector involutions, $\mathbf{x}^i, \mathbf{x}^j$, and \mathbf{x}^k , so that

$$\mathbf{C}_{\mathbf{x}^i} = E\{\mathbf{x}\mathbf{x}^{iH}\} = \mathbf{0}, \quad \mathbf{C}_{\mathbf{x}^j} = E\{\mathbf{x}\mathbf{x}^{jH}\} = \mathbf{0}, \quad \mathbf{C}_{\mathbf{x}^k} = E\{\mathbf{x}\mathbf{x}^{kH}\} = \mathbf{0}. \tag{7}$$

Definition 4 (\mathbb{C}^α -properness [37]). A quaternion random vector \mathbf{x} is \mathbb{C}^α -improper with respect to $\alpha = \mathbf{i}, \mathbf{j}$, or \mathbf{k} if it is correlated only with the involution \mathbf{x}^α , so that all the complementary covariances except for $\mathbf{C}_{\mathbf{x}^\alpha}$ vanish.

According to the complex representation of a quaternion, two proper complex random vectors can produce a \mathbb{C}^α -properness vector. According to the Cayley–Dickson construction, for different $\alpha, \beta \in \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$, a quaternion can be represented as $\mathbf{x} = \mathbf{z}_1 + \mathbf{z}_2\beta$, where \mathbf{z}_1 and \mathbf{z}_2 are generated by $\{1, \alpha\}$. Any quaternion vector \mathbf{x} is \mathbb{C}^α -improper if and only if the following conditions are both satisfied [37]:

- (1) \mathbf{z}_1 and \mathbf{z}_2 are proper complex vectors, which is achieved when their real and imaginary parts are uncorrelated and with the same variance.
- (2) \mathbf{z}_1 and \mathbf{z}_2 have different variances.

3. The Structure of Quaternion Covariance Matrices

3.1. Three Kinds of Structure of Quaternion Covariance Matrices

There are three kinds of quaternion covariance matrices: the pseudo-covariance matrix, the standard covariance matrix, and the complementary covariance matrix.

In recent years, there has been success in using the standard covariance matrix ($\mathbf{C}_\mathbf{X} = E\{\mathbf{X}\mathbf{X}^H\}$) and pseudo-covariance matrix ($\mathbf{P}_\mathbf{X} = E\{\mathbf{X}\mathbf{X}^T\}$) of simple expression and physical meaning in complex augmented statistics [31]. The pseudo-covariance matrix can explain the complex variable improperness. Moreover, in the complex domain, the pseudo-covariance matrix is a symmetric matrix, which can use Takagi decomposition $\mathbf{P}_\mathbf{X} = \mathbf{Q}\mathbf{\Sigma}\mathbf{Q}^T$, where \mathbf{Q} is the complex unitary matrix and $\mathbf{\Sigma}$ is the real diagonal matrix. Note that the diagonal elements are the singular values of $\mathbf{P}_\mathbf{X}$ [38]. Because quaternion multiplication cannot be exchanged, pseudo-covariance in a quaternion is asymmetrical. Given a random quaternion vector $\mathbf{X} = [x_1, x_2, \dots, x_n]^T$, the expression of the pseudo-covariance matrix is as follows:

$$\mathbf{P}_\mathbf{X} = E\{\mathbf{X}\mathbf{X}^T\} = \begin{pmatrix} E\{x_1x_1\} & E\{x_1x_2\} & \cdots & E\{x_1x_n\} \\ E\{x_2x_1\} & E\{x_2x_2\} & \cdots & E\{x_2x_n\} \\ \vdots & \vdots & \ddots & \vdots \\ E\{x_nx_1\} & E\{x_nx_2\} & \cdots & E\{x_nx_n\} \end{pmatrix}.$$

Quaternion involution plays an important role in generalized linear processing (e.g., [22,39,40]), which provides a useful basis for second-order statistics in the quaternion domain. Complete second-order statistics need to consider both the standard covariance matrix and complementary covariance matrix. The following are their respective expressions [23].

Given a random quaternion vector $\mathbf{X} = [x_1, x_2, \dots, x_n]^T$, the expression of standard covariance matrix is as follows:

$$\mathbf{C}_X = E\{\mathbf{X}\mathbf{X}^H\} = \begin{pmatrix} E\{x_1x_1^*\} & E\{x_1x_2^*\} & \cdots & E\{x_1x_n^*\} \\ E\{x_2x_1^*\} & E\{x_2x_2^*\} & \cdots & E\{x_2x_n^*\} \\ \vdots & \vdots & \ddots & \vdots \\ E\{x_nx_1^*\} & E\{x_nx_2^*\} & \cdots & E\{x_nx_n^*\} \end{pmatrix}.$$

The expression of complementary covariance matrix is as follows:

$$\mathbf{C}_{X^\eta} = E\{\mathbf{X}\mathbf{X}^{\eta H}\} = \begin{pmatrix} E\{x_1x_1^{\eta*}\} & E\{x_1x_2^{\eta*}\} & \cdots & E\{x_1x_n^{\eta*}\} \\ E\{x_2x_1^{\eta*}\} & E\{x_2x_2^{\eta*}\} & \cdots & E\{x_2x_n^{\eta*}\} \\ \vdots & \vdots & \ddots & \vdots \\ E\{x_nx_1^{\eta*}\} & E\{x_nx_2^{\eta*}\} & \cdots & E\{x_nx_n^{\eta*}\} \end{pmatrix}, \tag{8}$$

where $\eta \in \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$, x_i^* is defined as the conjugate of x_i , and the non-diagonal elements satisfy $\mathbf{C}_{X^\eta}(m, n) = (\mathbf{C}_{X^\eta}(n, m))^{\eta*}$, that is, m rows and n columns in \mathbf{C}_{X^η} are equal to the conjugate of n rows and m columns.

It is worth noting that the standard covariance matrix \mathbf{C}_X is a Hermitian quaternion matrix, i.e., $\mathbf{C}_X = \mathbf{C}_X^H$, and the complementary covariance matrix is a η -Hermitian quaternion matrix, i.e., $\mathbf{C}_{X^\eta} = (\mathbf{C}_{X^\eta})^{\eta H}$. In addition, the relationship among the three covariance matrices is as follows:

$$\mathbf{P}_X = \frac{1}{2}(\mathbf{C}_{X^i} + \mathbf{C}_{X^j} + \mathbf{C}_{X^k} - \mathbf{C}_X).$$

3.2. The Generalized Complementary Covariance Quaternion Matrix

Quaternion involution basis occupies an important place in second-order statistics. We have already given the complementary covariance matrix representation (8), where $\eta \in \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$. According to the definition of non-standard involution given above, in this section, we will give the generalized complementary covariance quaternion matrix, whose representation is as follows.

Definition 5 (Generalized complementary covariance quaternion matrix). *The expression of the generalized complementary covariance quaternion matrix is*

$$\mathbf{C}_{X_\phi} = E\{\mathbf{X}\mathbf{X}^{\delta H}\} = \begin{pmatrix} E\{x_1x_1^{\delta*}\} & E\{x_1x_2^{\delta*}\} & \cdots & E\{x_1x_n^{\delta*}\} \\ E\{x_2x_1^{\delta*}\} & E\{x_2x_2^{\delta*}\} & \cdots & E\{x_2x_n^{\delta*}\} \\ \vdots & \vdots & \ddots & \vdots \\ E\{x_nx_1^{\delta*}\} & E\{x_nx_2^{\delta*}\} & \cdots & E\{x_nx_n^{\delta*}\} \end{pmatrix}, \tag{9}$$

where $\delta = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k} \in \mathbb{H}^{n \times n}$ is unit and purely imaginary.

Note that \mathbf{C}_{X_ϕ} is ϕ -Hermitian matrix. It is obvious that when $\delta \in \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$, $\mathbf{C}_{X_\phi} = \mathbf{C}_{X^\eta}$. Thus, the complementary covariance matrix is a special case of the generalized complementary covariance quaternion matrix.

4. Unitary Diagonalization of Standard Covariance and Generalized Complementary Covariance Quaternion Matrices

In this section, we consider three cases where the Hermitian quaternion matrix and ϕ -Hermitian quaternion matrix are simultaneously unitary diagonalized: both are Her-

mitian quaternion matrices, both are ϕ -Hermitian quaternion matrices, and there is one Hermitian quaternion matrix and one ϕ -Hermitian quaternion matrix.

Firstly, we give some Lemmas:

Lemma 1 ([30]). *The standard covariance quaternion matrix (\mathbf{C}_X) is a Hermitian quaternion matrix. Its eigen-decomposition is $\mathbf{C}_X = \mathbf{Q}^H \mathbf{\Lambda}_X \mathbf{Q}$, where \mathbf{Q} is the unitary quaternion matrix, $\mathbf{\Lambda}_X$ is a real-valued diagonal matrix, and the elements on the diagonal are the eigenvalues of \mathbf{C}_X .*

Lemma 2. *The generalized complementary covariance quaternion matrix (\mathbf{C}_{X_ϕ}) is a ϕ -Hermitian quaternion matrix. Its eigen-decomposition is $\mathbf{C}_{X_\phi} = \mathbf{U} \mathbf{\Lambda} \mathbf{U}^{\delta H}$, where \mathbf{U} is the unitary quaternion matrix, $\mathbf{\Lambda}$ is a real-valued non-negative diagonal matrix, and the diagonal elements are singular values of \mathbf{C}_{X_ϕ} . $\delta = u_1 \mathbf{i} + u_2 \mathbf{j} + u_3 \mathbf{k}$ is unit and purely imaginary.*

Proof. The singular value decomposition of the ϕ -Hermitian quaternion matrix \mathbf{C}_{X_ϕ} is [32]: $\mathbf{C}_{X_\phi} = \mathbf{U} \mathbf{S} \mathbf{V}^H$, $\mathbf{C}_{X_\phi} \mathbf{C}_{X_\phi}^H$ can be represented as

$$\mathbf{C}_{X_\phi} \mathbf{C}_{X_\phi}^H = \mathbf{U} \mathbf{S}^2 \mathbf{U}^H.$$

Using the ϕ -Hermitian property, $\mathbf{C}_{X_\phi} = \mathbf{C}_{X_\phi}^{\delta H}$, we rewrite the matrix product $\mathbf{C}_{X_\phi} \mathbf{C}_{X_\phi}^H$ as

$$\mathbf{C}_{X_\phi} \mathbf{C}_{X_\phi}^H = \mathbf{V}^\delta \mathbf{S}^2 \mathbf{V}^{\delta H}.$$

On the basis of [21] Lemma 2.1, we can assume $\mathbf{U} = \mathbf{V}^\delta \mathbf{D}$, where \mathbf{D} is diagonal quaternion matrix. Therefore, we can obtain

$$\begin{aligned} \mathbf{C}_{X_\phi} &= \mathbf{U} \mathbf{S} \mathbf{V}^H \\ &= \mathbf{U} \mathbf{S} \mathbf{V}^H \mathbf{U}^\delta \mathbf{U}^{\delta H} \\ &= \mathbf{U} \mathbf{S} \left(\mathbf{V}^H \mathbf{V} \mathbf{D}^\delta \right) \mathbf{U}^{\delta H} \\ &= \mathbf{U} \left(\mathbf{S} \mathbf{D}^{\text{ffi}} \right) \mathbf{U}^{\delta H} \\ &= \mathbf{U} \mathbf{\Lambda} \mathbf{U}^{\delta H}. \end{aligned} \tag{10}$$

□

Lemma 3. *If $\mathbf{A}, \mathbf{B} \in \mathbb{H}$ are ϕ -Hermitian matrices and \mathbf{A} is a non-singular matrix, then \mathbf{A}, \mathbf{B} are simultaneously diagonalizable if and only if $\mathbf{D} = \mathbf{A}^{-1} \mathbf{B}$ is normal.*

Proof. Assuming the existence of unitary matrix \mathbf{M} , this makes

$$\mathbf{\Lambda}_a = \mathbf{M}^{\delta H} \mathbf{A} \mathbf{M}, \quad \mathbf{\Lambda}_b = \mathbf{M}^{\delta H} \mathbf{B} \mathbf{M},$$

able to be diagonalized simultaneously. We have

$$\mathbf{A}^{-1} = \mathbf{M} \mathbf{\Lambda}_a^{-1} \mathbf{M}^{\delta H}, \quad \mathbf{B} = \mathbf{M}^\delta \mathbf{\Lambda}_b \mathbf{M}^H.$$

Therefore,

$$\mathbf{A}^{-1} \mathbf{B} = \mathbf{M} \left(\mathbf{\Lambda}_a^{-1} \mathbf{\Lambda}_b \right) \mathbf{M}^H$$

is unitary diagonalized. In other words, $\mathbf{D} = \mathbf{A}^{-1} \mathbf{B}$ is normal. □

Lemma 4 ([30]). *If $\mathbf{A} = \begin{pmatrix} \mathbf{B} & \mathbf{C} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \in \mathbb{H}$, then \mathbf{A} is normal if and only if \mathbf{B} is normal and $\mathbf{C} = \mathbf{0}$.*

According to Lemmas 1–4, we will give the simultaneous diagonalization of covariance matrix conditions in the following theorem.

Theorem 1. Given the matrices $\mathbf{A}, \mathbf{B} \in \mathbb{H}$:

- (a) If \mathbf{A} and \mathbf{B} are both Hermitian quaternion matrices, then there exists a unitary matrix $\mathbf{R} \in \mathbb{H}$ that makes $\mathbf{R}^H \mathbf{A} \mathbf{R}$ and $\mathbf{R}^H \mathbf{B} \mathbf{R}$ able to be diagonalized simultaneously if and only if $\mathbf{A} \mathbf{B}$ is Hermitian quaternion matrix, i.e., $\mathbf{A} \mathbf{B} = \mathbf{B} \mathbf{A}$.
- (b) If \mathbf{A} and \mathbf{B} are both ϕ -Hermitian quaternion matrices, then there exists a unitary matrix $\mathbf{R} \in \mathbb{H}$ that makes $\mathbf{R}^{\delta H} \mathbf{A} \mathbf{R}$ and $\mathbf{R}^{\delta H} \mathbf{B} \mathbf{R}$ able to be diagonalized simultaneously if and only if $\mathbf{A} \mathbf{B}^\delta$ is normal, i.e., $\mathbf{A} \mathbf{B}^\delta \mathbf{B}^{\delta H} \mathbf{A}^H = \mathbf{B}^{\delta H} \mathbf{A}^H \mathbf{A} \mathbf{B}^\delta$, where $\delta = u_1 \mathbf{i} + u_2 \mathbf{j} + u_3 \mathbf{k}$ is unit and purely imaginary.
- (c) If \mathbf{A} is a Hermitian quaternion matrix and \mathbf{B} is a ϕ -Hermitian quaternion matrix, then there exists a unitary matrix $\mathbf{R} \in \mathbb{H}$ that makes $\mathbf{R}^H \mathbf{A} \mathbf{R}$ and $\mathbf{R}^{\delta H} \mathbf{B} \mathbf{R}$ able to be diagonalized simultaneously if and only if $\mathbf{B} \mathbf{A}$ is ϕ -Hermitian quaternion matrix, i.e., $\mathbf{B} \mathbf{A} = (\mathbf{B} \mathbf{A})^{\delta H} = \mathbf{A}^{\delta H} \mathbf{B}^{\delta H} = \mathbf{A}^\delta \mathbf{B}$, where $\delta = u_1 \mathbf{i} + u_2 \mathbf{j} + u_3 \mathbf{k}$ is unit and purely imaginary.

Proof. (a). Because \mathbf{A} and \mathbf{B} are both Hermitian quaternion matrices, matrix \mathbf{A} can be decomposed into $\mathbf{A} = \mathbf{U} \mathbf{S}_a \mathbf{U}^H$. Let $\mathbf{D} = \mathbf{S}_a^{-\frac{1}{2}} \mathbf{U}^H$, therefore

$$\mathbf{D} \mathbf{A} \mathbf{D}^H = \mathbf{I}, \quad \mathbf{D} \mathbf{B} \mathbf{D}^H = \mathbf{W} \mathbf{\Lambda}_b \mathbf{W}^H.$$

We consider $\mathbf{R} = \mathbf{W}^H \mathbf{D}$, then

$$\begin{aligned} \mathbf{R} \mathbf{A} \mathbf{R}^H &= \mathbf{W}^H \mathbf{D} \mathbf{A} \mathbf{D}^H \mathbf{W} = \mathbf{I} = \mathbf{\Lambda}_a, \\ \mathbf{R} \mathbf{B} \mathbf{R}^H &= \mathbf{W}^H (\mathbf{D} \mathbf{B} \mathbf{D}^H) \mathbf{W} = \mathbf{\Lambda}_b. \end{aligned}$$

Therefore, there exists a unitary quaternion matrix \mathbf{R} that makes $\mathbf{R}^H \mathbf{A} \mathbf{R}$ and $\mathbf{R}^H \mathbf{B} \mathbf{R}$ able to be diagonalized simultaneously. Next, we will prove the sufficiency and necessity of diagonalization, respectively.

First, we prove the sufficiency:

Assuming $\mathbf{R}^H \mathbf{A} \mathbf{R} = \mathbf{\Lambda}_a$, $\mathbf{R}^H \mathbf{B} \mathbf{R} = \mathbf{\Lambda}_b$, then $\mathbf{A} = \mathbf{R} \mathbf{\Lambda}_a \mathbf{R}^H$, $\mathbf{B} = \mathbf{R} \mathbf{\Lambda}_b \mathbf{R}^H$. Hence, we have

$$\mathbf{A} \mathbf{B} = \mathbf{R} \mathbf{\Lambda}_a \mathbf{R}^H \mathbf{R} \mathbf{\Lambda}_b \mathbf{R}^H = \mathbf{R} \mathbf{\Lambda}_b (\mathbf{R}^H \mathbf{R}) \mathbf{\Lambda}_a \mathbf{R}^H = \mathbf{B} \mathbf{A}.$$

Then, we prove the necessity:

Because \mathbf{A} and \mathbf{B} are both Hermitian quaternion matrices, i.e., $\mathbf{A} = \mathbf{A}^H$, $\mathbf{B} = \mathbf{B}^H$. According to Lemma 3, there exists unitary quaternion matrices \mathbf{U} and \mathbf{V} that make $\mathbf{A} = \mathbf{U}^H \mathbf{\Lambda}_a \mathbf{U}$, $\mathbf{B} = \mathbf{V}^H \mathbf{\Lambda}_b \mathbf{V}$. On the basis of $\mathbf{A} \mathbf{B} = \mathbf{B} \mathbf{A}$, we can obtain

$$\mathbf{A} \mathbf{B} = \mathbf{A}^H \mathbf{B}^H = (\mathbf{B} \mathbf{A})^H = \mathbf{B} \mathbf{A}. \tag{11}$$

Substituting $\mathbf{A} = \mathbf{U}^H \mathbf{\Lambda}_a \mathbf{U}$, $\mathbf{B} = \mathbf{V}^H \mathbf{\Lambda}_b \mathbf{V}$ into (11), we have

$$\mathbf{B} \mathbf{A} = \mathbf{V}^H \mathbf{\Lambda}_b \mathbf{V} \mathbf{U}^H \mathbf{\Lambda}_a \mathbf{U} = \mathbf{U}^H \mathbf{\Lambda}_a \mathbf{U} \mathbf{V}^H \mathbf{\Lambda}_b \mathbf{V} = (\mathbf{B} \mathbf{A})^H.$$

Let $\mathbf{U} = \mathbf{V} := \mathbf{R}$, and we can obtain

$$\mathbf{\Lambda}_a = \mathbf{R}^H \mathbf{A} \mathbf{R}, \quad \mathbf{\Lambda}_b = \mathbf{R}^H \mathbf{B} \mathbf{R}.$$

Therefore, there exists a unitary matrix that makes \mathbf{A} and \mathbf{B} able to be diagonalized simultaneously.

(b). If \mathbf{A} and \mathbf{B} are both ϕ -Hermitian quaternion matrices, and $\mathbf{A} \mathbf{B}^\delta$ is normal, the singular value of matrix \mathbf{A} can be decomposed into $\mathbf{A} = \mathbf{U} \mathbf{S} \mathbf{V}^H$. \mathbf{A} can also be decomposed into $\mathbf{A} = \mathbf{Q} \mathbf{S} \mathbf{Q}^{\delta H}$, where $\mathbf{Q} = \mathbf{U} (\mathbf{D}^\delta)^{\frac{1}{2}}$, $\mathbf{U} = \mathbf{V}^\delta \mathbf{D}$. Therefore, there exists $\mathbf{R} = \mathbf{Q}^H$ that makes \mathbf{A} and \mathbf{B} able to be diagonalized simultaneously. Next, we will prove the sufficiency and necessity of diagonalization, respectively.

First, we prove the sufficiency:

Assuming $\mathbf{R}^{\delta H} \mathbf{A} \mathbf{R} = \Lambda_a$, $\mathbf{R}^{\delta H} \mathbf{B} \mathbf{R} = \Lambda_b$, then, $\mathbf{A} = \mathbf{R}^\delta \Lambda_a \mathbf{R}^H$, $\mathbf{B} = \mathbf{R}^\delta \Lambda_b \mathbf{R}^H$. Hence, we have

$$\mathbf{A} \mathbf{B}^\delta = \mathbf{R}^\delta \Lambda_a \mathbf{R}^H \mathbf{R} \Lambda_b \mathbf{R}^{\delta H} = \mathbf{R}^\delta (\Lambda_a \Lambda_b) \mathbf{R}^{\delta H},$$

It is easy to know that $\mathbf{A} \mathbf{B}^\delta$ is normal.

Then, we prove the necessity:

We consider the following two cases to prove:

(i) Assuming $\mathbf{A} \mathbf{B}^\delta$ is a normal matrix, \mathbf{A} is a non-singular matrix. $\mathbf{A} \mathbf{B}^\delta = (\mathbf{A}^{-1})^{-1} \mathbf{B}^\delta$ is normal, on the basis of Lemma 3, and \mathbf{A}^{-1} and \mathbf{B}^δ can be diagonalized simultaneously.

Because \mathbf{A} and \mathbf{B} are both ϕ -Hermitian quaternion matrices, we have $\mathbf{A}^{-1} = \mathbf{R} \Lambda_a^{-1} \mathbf{R}^{\delta H}$, $\mathbf{B}^\delta = \mathbf{R} \Lambda_b \mathbf{R}^{\delta H}$. Hence,

$$\mathbf{A} = \mathbf{R}^\delta \Lambda_a \mathbf{R}^H = \mathbf{R}^\delta \Lambda_a (\mathbf{R}^\delta)^{\delta H},$$

$$\mathbf{B} = \mathbf{R}^\delta \Lambda_b \mathbf{R}^H = \mathbf{R}^\delta \Lambda_b (\mathbf{R}^\delta)^{\delta H}.$$

In other words, \mathbf{A} and \mathbf{B} can be diagonalized simultaneously.

(ii) Assuming $\mathbf{A} \mathbf{B}^\delta$ is a normal matrix, \mathbf{A} is a singular matrix, then there exists unitary matrix $\mathbf{R} \in \mathbb{H}$ that makes $\mathbf{R}^{\delta H} \mathbf{A} \mathbf{R}$ a diagonal matrix, and the column elements of \mathbf{R} can be rearranged:

$$\mathbf{R}^{\delta H} \mathbf{A} \mathbf{R} = \begin{bmatrix} \Sigma & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \mathbf{R}^{\delta H} \mathbf{B} \mathbf{R} = \begin{bmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{12}^{\delta H} & \mathbf{B}_{22} \end{bmatrix},$$

where Σ is a block diagonal matrix. The block matrix Σ , \mathbf{B}_{11} , \mathbf{B}_{22} module length of elements on the diagonal is 1. We can obtain

$$\begin{aligned} (\mathbf{R}^{\delta H} \mathbf{A} \mathbf{R}) (\mathbf{R}^{\delta H} \mathbf{B} \mathbf{R})^\delta &= \mathbf{R}^{\delta H} \mathbf{A} \mathbf{B}^\delta \mathbf{R}^\delta \\ &= \begin{bmatrix} \Sigma \mathbf{B}_{11}^\delta & \Sigma \mathbf{B}_{12}^\delta \\ \mathbf{0} & \mathbf{0} \end{bmatrix}. \end{aligned}$$

Because $\mathbf{A} \mathbf{B}^\delta$ is a normal matrix, according to Lemma 4, we have $\Sigma \mathbf{B}_{12}^\delta = \mathbf{0}$. Owing to Σ is the non-singular matrix, $\mathbf{B}_{12} = \mathbf{0}$, i.e.,

$$\mathbf{R}^{\delta H} \mathbf{A} \mathbf{R} = \begin{bmatrix} \Sigma & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \mathbf{R}^{\delta H} \mathbf{B} \mathbf{R} = \begin{bmatrix} \mathbf{B}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_{22} \end{bmatrix},$$

Therefore, if there exists unitary matrix \mathbf{R} that makes \mathbf{A} diagonalized, it can also diagonalize \mathbf{B} .

(c). Because $\mathbf{A} = \mathbf{U} \mathbf{S}_a \mathbf{U}^H$ is a Hermitian quaternion matrix and \mathbf{B} is a ϕ -Hermitian quaternion matrix, let $\mathbf{D} = \mathbf{S}_a^{\frac{1}{2}} \mathbf{U}^H$, and we have

$$\mathbf{D} \mathbf{A} \mathbf{D}^H = \mathbf{I}, \quad \mathbf{D} \mathbf{B} \mathbf{D}^{\delta H} = \mathbf{W} \Lambda_b \mathbf{W}^{\delta H}.$$

We consider $\mathbf{R} = \mathbf{W}^H \mathbf{D}$, so

$$\mathbf{R} \mathbf{A} \mathbf{R}^H = \mathbf{W}^H \mathbf{D} \mathbf{A} \mathbf{D}^H \mathbf{W} = \mathbf{I},$$

$$\mathbf{R} \mathbf{B} \mathbf{R}^{\delta H} = \mathbf{W}^H (\mathbf{D} \mathbf{B} \mathbf{D}^{\delta H}) \mathbf{W}^\delta = \Lambda_b.$$

Hence, there exists unitary matrix $\mathbf{R} \in \mathbb{H}$ that makes $\mathbf{R}^H \mathbf{A} \mathbf{R}$ and $\mathbf{R}^{\delta H} \mathbf{B} \mathbf{R}$ able to be diagonalized simultaneously. Next, we will prove the sufficiency and necessity of diagonalization, respectively.

First, we prove the sufficiency:

If $\mathbf{R}^H \mathbf{A} \mathbf{R} = \Lambda_a, \mathbf{R}^{\delta H} \mathbf{B} \mathbf{R} = \Lambda_b$, we obtain $\mathbf{A} = \mathbf{R} \Lambda_a \mathbf{R}^H, \mathbf{B} = \mathbf{R}^\delta \Lambda_b \mathbf{R}^H$. Therefore,

$$\begin{aligned} \mathbf{B} \mathbf{A} &= \mathbf{R}^\delta \Lambda_b \mathbf{R}^H \mathbf{R} \Lambda_a \mathbf{R}^H \\ &= \mathbf{R}^\eta \Lambda_a \left(\mathbf{R}^{\delta H} \mathbf{R}^\delta \right) \Lambda_b \mathbf{R}^H = \mathbf{A}^\delta \mathbf{B}, \end{aligned}$$

where $\delta = u_1 \mathbf{i} + u_2 \mathbf{j} + u_3 \mathbf{k} \in \mathbb{H}^{n \times n}$ is unit and purely imaginary.

Then, we prove the necessity:

Because \mathbf{A} and \mathbf{B} are both ϕ -Hermitian matrices, i.e., $\mathbf{A} = \mathbf{A}_\phi, \mathbf{B} = \mathbf{B}_\phi$. According to Lemma 2, there exist unitary matrices \mathbf{U} and \mathbf{V} that make $\mathbf{A} = \mathbf{U}^{\delta H} \Lambda_a \mathbf{U}, \mathbf{B} = \mathbf{V}^{\delta H} \Lambda_b \mathbf{V}$. We have

$$\mathbf{A} \mathbf{B} = \mathbf{A}^{\delta H} \mathbf{B}^{\delta H} = (\mathbf{B} \mathbf{A})^{\delta H} = \mathbf{B} \mathbf{A}. \tag{12}$$

Substituting $\mathbf{A} = \mathbf{U}^{\delta H} \Lambda_a \mathbf{U}$ and $\mathbf{B} = \mathbf{V}^{\delta H} \Lambda_b \mathbf{V}$ into (12), we can obtain

$$\mathbf{B} \mathbf{A} = \mathbf{V}^{\delta H} \Lambda_b \mathbf{V} \mathbf{U}^{\delta H} \Lambda_a \mathbf{U} = \mathbf{U}^{\delta H} \Lambda_a \mathbf{U} \mathbf{V}^{\delta H} \Lambda_b \mathbf{V} = (\mathbf{B} \mathbf{A})^{\delta H}.$$

Let $\mathbf{U} = \mathbf{V} := \mathbf{R}$, we can obtain

$$\Lambda_a = \mathbf{R}^{\delta H} \mathbf{A} \mathbf{R}, \quad \Lambda_b = \mathbf{R}^{\delta H} \mathbf{B} \mathbf{R},$$

where $\delta = u_1 \mathbf{i} + u_2 \mathbf{j} + u_3 \mathbf{k}$ is unit and purely imaginary. Therefore, the matrices \mathbf{A} and \mathbf{B} can be diagonalized simultaneously. \square

5. Generalized Quaternion Unitary Transform

5.1. The Algorithm of Generalized Quaternion Unitary Transform

Based on Lemma 1, the Hermitian quaternion matrix \mathbf{C}_X can be decomposed into $\mathbf{C}_X = \mathbf{U} \Lambda_X \mathbf{U}^H$, where \mathbf{U} is a unitary quaternion matrix and Λ_X is a real diagonal matrix. We define the whitening transformation $\mathbf{D} = \Lambda_X^{-\frac{1}{2}} \mathbf{U}^H$, let $\mathbf{s} = \mathbf{D} \mathbf{X}$, and \mathbf{s} is a ϕ -Hermitian quaternion matrix. We can obtain the covariance matrix of \mathbf{s} :

$$\mathbf{C}_s = \mathbf{D} \mathbf{C}_X = \mathbf{U} \Lambda_X^{-\frac{1}{2}} \mathbf{U}^H \mathbf{U} \Lambda_X \mathbf{U}^H \mathbf{U} \Lambda_X^{-\frac{1}{2}} \mathbf{U}^H = \mathbf{I}.$$

As a consequence, \mathbf{C}_s can be decomposed into $\mathbf{C}_{s_\phi} = \mathbf{W} \Lambda_\delta \mathbf{W}^{\delta H}$, where \mathbf{W} is a unitary quaternion matrix and Λ_δ is real diagonal matrix. We define the non-singular uncorrelated transformation $\mathbf{Q} = \mathbf{W}^H \mathbf{D}$, let $\mathbf{y} = \mathbf{Q} \mathbf{X}$. Then, \mathbf{C}_y and \mathbf{C}_{y_ϕ} can be diagonalized simultaneously:

$$\mathbf{C}_y = \mathbf{W}^H \mathbf{C}_s \mathbf{W} = \mathbf{W}^H \mathbf{I} \mathbf{W} = \mathbf{I},$$

$$\mathbf{C}_{y_\phi} = \mathbf{W}^H \mathbf{C}_{s_\phi} \mathbf{W}^\delta = \mathbf{W}^H \mathbf{W} \Lambda_\delta \mathbf{W}^{\delta H} \mathbf{W}^\delta = \Lambda_\delta.$$

We call the transformed \mathbf{Q} as the generalized quaternion uncorrelated transformation.

Using the theory of simultaneous diagonalization of covariance matrix based on Section 4, we can receive the steps of solving the generalized quaternion uncorrelated transformation matrix. First, the covariance matrix can eigen-decomposed into $\mathbf{C}_x = E\{\mathbf{x} \mathbf{x}^H\} = \mathbf{U} \Lambda_x \mathbf{U}^H$. Then, we calculate the whitening matrix $\mathbf{D} = \mathbf{V} \Lambda_x^{-\frac{1}{2}} \mathbf{V}^H$, and we can obtain the whitened data $\mathbf{s} = \mathbf{D} \mathbf{X}$. Next, we calculate the complementary covariance matrix of \mathbf{s} , i.e., $\mathbf{C}_{s_\phi} = E\{\mathbf{s} \mathbf{s}^{\delta H}\}$. It is obvious that \mathbf{C}_{s_ϕ} is a ϕ -Hermitian quaternion matrix, and we can calculate the eigen-decomposition of \mathbf{C}_{s_ϕ} : $\mathbf{C}_{s_\phi} = \mathbf{W} \Lambda_\delta \mathbf{W}^{\delta H}$, where \mathbf{W} is unitary quaternion matrix and Λ_δ is real diagonal matrix. Finally, we can gain the generalized quaternion uncorrelated transformation $\mathbf{Q} = \mathbf{W}^H \mathbf{D}$.

The following Algorithm 1 shows the specific steps of the MATLAB implementation of the generalized quaternion unitary transform algorithm for the standard covariance matrix and generalized complementary covariance matrix.

Algorithm 1: The specific steps of the MATLAB implementation of the generalized quaternion unitary transform algorithm for the standard covariance matrix and the generalized complementary covariance matrix

```

function [R,t] = GQUT(s)
if size(s,1)>size(s,2)
s = s.';
end
n = length(s);
%Generate the covariance matrix of s
Cs = (s * s')/n;
%Using singular value decomposition to calculate standard covariance matrix
[U, V] = svd(Cs);
%Obtain whitening matrix D and whitening data q
D = diag(diag(V)^(-1/2)) * U';
q = D * s;
%Calculating the phi complementary covariance matrix of generating whitening
data q
%We use 'inv' function to calculate the phi involution, note that delta = u1i + u2j + u3k
Cphi = (q * inv_ijk(q, delta'))'/n;
%Analyze the complementary covariance matrix
%Select the unit imaginary number with the greatest correlation
c = ( norm(Cphi - diag(diag(Cphi))) );
[ , g] = max(c. ');
%For the phi complementary covariance matrix, we use the Takagi factorization as
% Cphi = W * V1 * inv_ijk(W, fff')'
[U2, S2, V2] = svd(Cdelta);
P = inv_ijk(V2, delta') * U2;
W = U2 * diag(sqrt(diag(inv_ijk(P, delta'))));
%Calculate the generalized quaternion unitary matrix R and after decorrelation t
R = W' * D;
t = R * s.

```

5.2. Testing the Performance of the Generalized Quaternion Unitary Transform by Numerical Simulation

In this section, we test the performance of the generalized unitary transform for Cs decorrelation of improper quaternion signals. First, we use the complex representation of a quaternion to generate the C^k-improper quaternion signal. Then, randomly generated a third-order mixed matrix A, which obeying standard normal distribution. We can generate the mixed C^k-improper signal x = As. Each line is recorded as x₁, x₂, x₃. The 3D diagram represents the degree of correlation of mixed signals, which is shown in Figure 1. From the 3D scatter diagram, it can be seen that the signals are highly correlated.

We use the generalized quaternion unitary transform to decorrelate the mixed signals. We know that the phi-Hermitian quaternion matrix A satisfies A = delta A^H delta, where delta = u₁i + u₂j + u₃k. In this section, we consider three cases, delta in {i, j, k}, delta in {sqrt(2)/2(i + j), sqrt(2)/2(j + k), sqrt(2)/2(i + k)}, and delta = sqrt(3)/3(i + j + k), respectively.

We define non-standard involution phi(a) = -delta a* delta, where a is any quaternion. For the first case, the quaternion matrix A is a eta-Hermitian quaternion matrix. In the second case delta in {sqrt(2)/2(i + j), sqrt(2)/2(j + k), sqrt(2)/2(i + k)} [41], we can obtain:

$$\begin{aligned}
 a^{(\sqrt{2}/2)(i+j)*} &= a_0 - a_2i - a_1j + a_3k, & \text{if } \delta = \frac{\sqrt{2}}{2}(i + j); \\
 a^{(\sqrt{2}/2)(i+k)*} &= a_0 - a_3i + a_2j - a_1k, & \text{if } \delta = \frac{\sqrt{2}}{2}(i + k); \\
 a^{(\sqrt{2}/2)(j+k)*} &= a_0 + a_1i - a_3j - a_2k, & \text{if } \delta = \frac{\sqrt{2}}{2}(j + k).
 \end{aligned}$$

In the third case $\delta = \frac{\sqrt{3}}{3}(\mathbf{i} + \mathbf{j} + \mathbf{k})$, we can obtain

$$a \frac{\sqrt{3}}{3}(\mathbf{i} + \mathbf{j} + \mathbf{k})^* = a_0 - \frac{1}{3}(-a_1 + 2a_2 + 2a_3)\mathbf{i} - \frac{1}{3}(2a_1 - a_2 + 2a_3)\mathbf{j} - \frac{1}{3}(2a_1 + 2a_2 - a_3)\mathbf{k}.$$

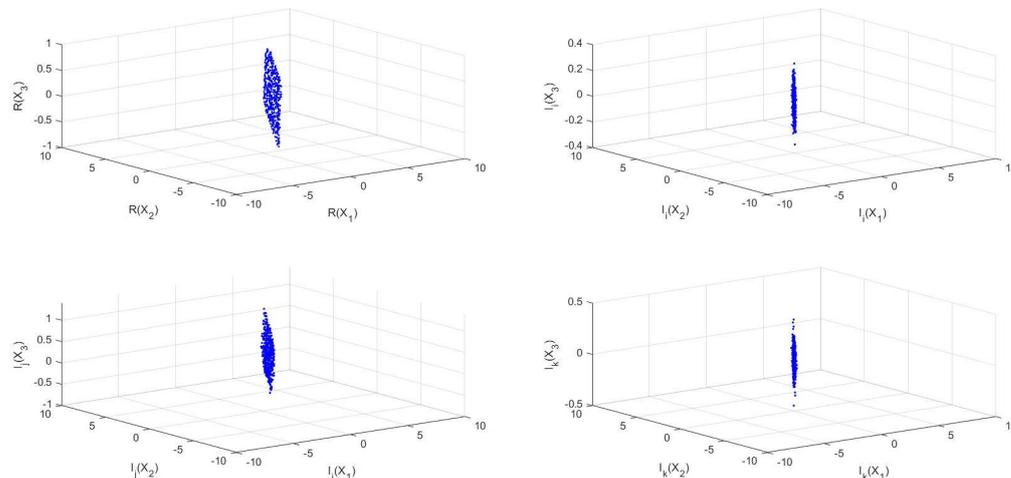


Figure 1. 3D scatter diagram of the original signal.

We use the generalized quaternion unitary transform algorithm to de-mix the signal. For the above three cases, we obtain the 3D scatter diagram after decorrelation, which is shown in Figures 2–4. Compared with Figure 1, it can be clearly seen that the signal has been de-mixed and the degree of correlation has decreased.

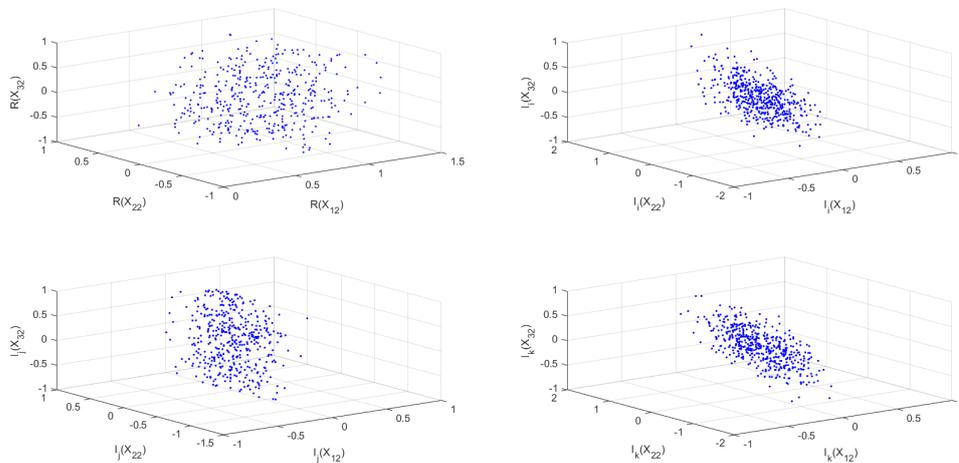


Figure 2. Generalized quaternion unitary transform algorithm to de-mix the signal ($\delta \in \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$).

Next, we will compare the CPU time for the above cases, take 150, 300, 500, 650, 800, and 1000 samples, respectively, and calculate their CPU time and draw a scatter diagram as shown in Figure 5. It can be seen from the figure that case 3, $\delta = \frac{\sqrt{3}}{3}(\mathbf{i} + \mathbf{j} + \mathbf{k})$, requires the least CPU time. Case 1, $\delta \in \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$, requires a longer time than the second case, $\delta \in \{\frac{\sqrt{2}}{2}(\mathbf{i} + \mathbf{j}), \frac{\sqrt{2}}{2}(\mathbf{j} + \mathbf{k}), \frac{\sqrt{2}}{2}(\mathbf{i} + \mathbf{k})\}$.

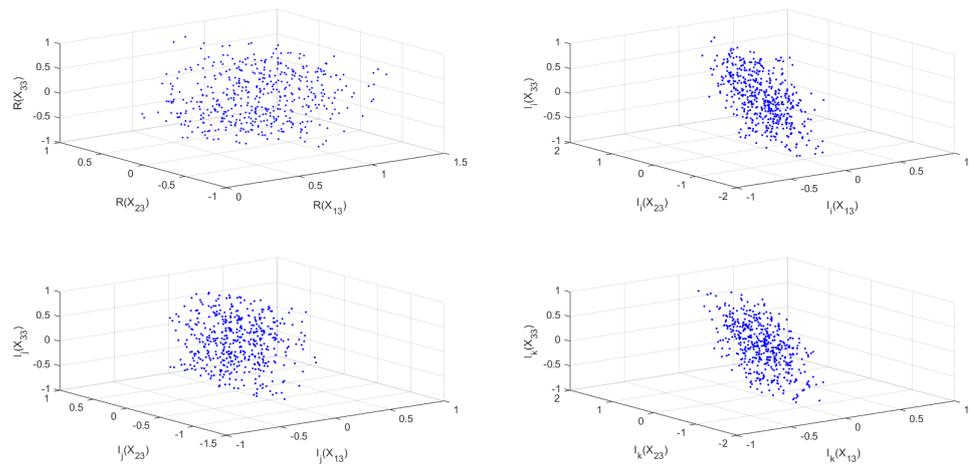


Figure 3. Generalized quaternion unitary transform algorithm to de-mix the signal ($\delta \in \{\frac{\sqrt{2}}{2}(\mathbf{i} + \mathbf{j}), \frac{\sqrt{2}}{2}(\mathbf{j} + \mathbf{k}), \frac{\sqrt{2}}{2}(\mathbf{i} + \mathbf{k})\}$).

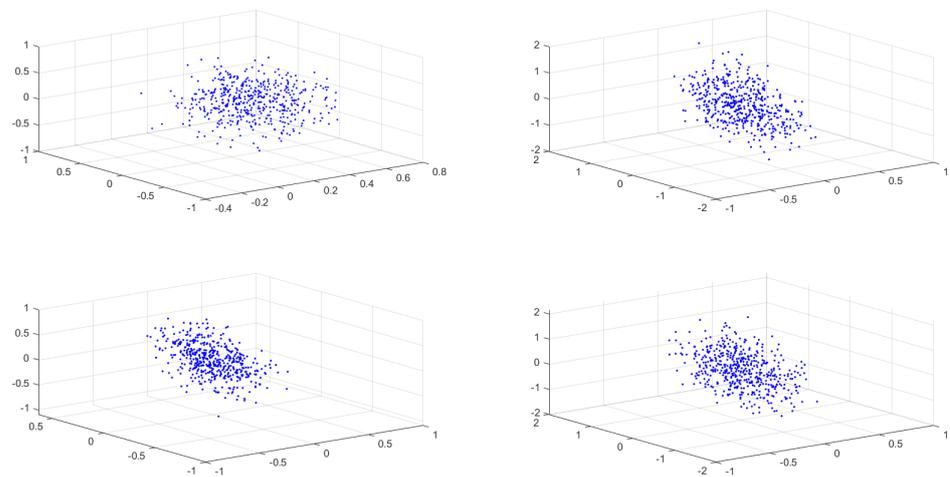


Figure 4. Generalized quaternion unitary transform algorithm to de-mix the signal ($\delta = \frac{\sqrt{3}}{3}(\mathbf{i} + \mathbf{j} + \mathbf{k})$).

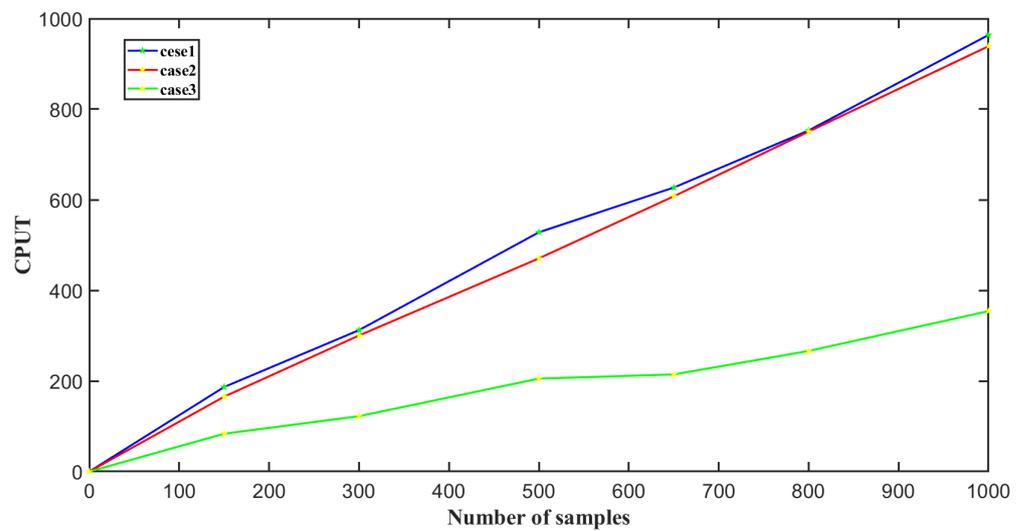


Figure 5. CPU Time.

6. Conclusions

We investigated the generalized complementary covariance quaternion matrix. Afterwards, we presented the conditions of unitary diagonalization of standard covariance and generalized complementary covariance quaternion matrices. Furthermore, we investigated the generalized quaternion unitary transform algorithm and tested its performance under three different conditions by numerical simulation. Finally, we compared the CPU time required by the algorithm in three cases. In the future, we intend to generalize this result to the field of quaternion tensors for the separation of high-dimensional signals.

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