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On a Non Local Initial Boundary Value Problem for a Semi-Linear Pseudo-Hyperbolic Equation in the Theory of Vibration

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Abstract: This research article addresses a nonclassical initial boundary value problem characterized by a non-local constraint within the framework of a pseudo-hyperbolic equation. Employing rigorous analytical techniques, the paper establishes the existence, uniqueness, and continuous dependence of a strong solution to the problem at hand. With respect to the associated linear problem, the uniqueness of its solution is ascertained through an energy inequality, which provides an a priori bound for the solution. Moreover, the solvability of this linear problem is verified by proving that the operator range engendered by the problem is indeed dense. Extending the analysis to the nonlinear problem, an iterative methodology is utilized. This approach is predicated on the insights gained from the linear problem and facilitates the demonstration of both the existence and uniqueness of a solution for the nonlinear problem under study. Consequently, the paper contributes a robust mathematical framework for solving both linear and nonlinear variants of complex initial boundary value problems with non-local constraints.

Keywords: pseudo-hyperbolic equation; existence and uniqueness; iterative process; non local constraint

MSC: primary 35L82; secondary 35L20

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1. Introduction

Over the past several decades, mixed non-local problems encompassing parabolic, hyperbolic, and pseudo-hyperbolic partial differential equations have attracted significant scholarly attention. These investigations are primarily motivated by advancements in modern physics and technological sciences, serving as foundational models for a diverse range of physical and biological phenomena. Traditional boundary value problems, characterized by classical conditions such as Dirichlet and Neumann, are adept at describing a multitude of physical situations. However, certain complex phenomena necessitate the use of non-classical boundary value problems, which employ non-local conditions like integral constraints. In these non-classical scenarios, data are not directly measurable on the boundary; rather, one is confined to understanding the average value of the solution over the domain. The applicability of non-local mixed problems is extensive, ranging from medical sciences to thermoelasticity and control theory, as illustrated by seminal works in the field (Cannon [1], Shi [2], Capasso–Kunisch [3], Cannon-Van der Hoek [4], and Day [5]). The landscape of research on different types of partial differential equations with non-local conditions has been well-documented. For instance, the treatment of second-order parabolic equations with non-local constraints has been elaborated by Kartynnik [6], Friedman [7], Mesloub and Mansour [8], and Lin [9], among others. Similarly, the study of hyperbolic and pseudo-parabolic equations with one integral condition or with purely integral conditions has been systematically addressed (Mesloub and Bouziani [10,11], Ciegis [12], Fairweather [13], Goolin-Ionkin [14]). Further advancements in both one-dimensional and multi-dimensional spaces have been discussed in the works of Wei et al. [15], Madsen and Schaffer [16], Duruk et al. [17], Guido Schneider [18], Wayne and Wright [19] and Nwogu [20], to name a few. In the present study, the focus is on one-dimensional longitudinal vibrations of a rigid rod characterized by a non-uniform cross-section. The rod is anchored at one end while the other is subject to a non-local integral constraint. Generally, longitudinal vibrations of bars are described in the realm of mathematical physics via classical models based on the wave equation, under the assumption of a slender and elongated bar. However, more generalized theories have emerged to incorporate the effects of lateral motion in thicker bars. Such models necessitate higher-order derivatives in the equations of motion. Notable among these is Rayleigh's 1894 generalization, which considered the effects of lateral motion while disregarding shear stress. Bishop further expanded this theory in 1952, resulting in the Rayleigh–Bishop model characterized by a fourth-order partial differential equation devoid of a fourth-order time derivative. The remainder of this paper is structured as follows: Section 2 delineates the problem under investigation; Section 3 provides proof for the uniqueness of the solution corresponding to the associated linear problem; Section 4 discusses the existence of such a solution; and Section 5 is dedicated to establishing the solvability of the nonlinear problem at hand.

2. Problem Statement

In the domain $\mathcal{D} = (0,\mu) \times (0,T)$, with $\mu < \infty$ and $T < \infty$, we consider the following non local initial boundary value problem for a pseudo-hyperbolic nonlinear equation

$$\begin{cases} \mathcal{L}u = \partial^2 u/\partial t^2 - \partial/\partial x(R(x,t)\partial u/\partial x) - \delta\partial^2/\partial t^2(\partial^2 u/\partial x^2) \\ \qquad = g(x,t,u),\ \forall (x,t) \in \mathcal{D} \\ \ell_1 u = u(x,0) = H_1(x), \qquad x \in (0,\mu), \\ \ell_2 u = u_t(x,0) = H_2(x), \qquad x \in (0,\mu), \\ u(0,t) = 0, \quad \int_0^\mu xu(x,t)dx = 0,\ \ t \in (0,T). \end{cases} \tag{1}$$

where δ is a strictly positive constant and the function $R(x,t)$ and its derivatives satisfy the conditions:

$$T_1 \ : \ R_0 \le R(x,t) \le R_1, 0 < R_t(x,t) \le R_2, R_x(x,t) \le R_3,\ \text{for any } (x,t) \in \overline{\mathcal{D}},$$
$$T_2 \ : \ R_{tt}(x,t) \le R_4,\ R_{xt}(x,t) \le R_5,\ \text{for any } (x,t) \in \overline{\mathcal{D}}.$$

We shall assume that the function g is Lipschitzian in $\mathcal{D}$, that is, there exists a positive constant $d > 0$, such that

$$|g(x,t,u_1) - g(x,t,u_2)| \le d(|u_1 - u_2|), \tag{2}$$

for all $(x,t) \in \mathcal{D}$.

We use the simple notations: u_x for $\partial u/\partial x$, and u_{xx} for $\partial^2 u/\partial x^2, \ldots$.

The following function spaces are needed for the study of the posed problem.

Let $L^2(0,T;L^2(0,\mu)) = L^2(\mathcal{D})$, be the usual Hilbert space of square integrable functions on $\mathcal{D}$, and $H^1(0,T;L^2(0,\mu))$ be the standard Hilbert space of functions $u \in L^2(0,T;L^2(0,\mu))$ such that $\partial u/\partial t \in L^2(0,T;L^2(0,\mu))$. To problem (1), we assign the operator $\wp = (\mathcal{L},\ell_1,\ell_2)$ with domain of definition

$$D(\wp) = \begin{cases} u \in L^2(\mathcal{D}), u_t, u_x, u_{tx}, u_{tt}, u_{ttx} \in L^2(\mathcal{D}) \\ u(0,t) = 0, \quad \int_0^\mu xu(x,t)dx = 0,\ \ t \in (0,T). \end{cases}$$

The operator $\wp$ acts on a Banach space B into a Hilbert space H, where B is the set of functions $u \in L^2(\mathcal{D})$ verifying boundary conditions in (1) and having the norm

$$\begin{aligned}\|u\|_B^2 &= \|u\|^2_{H^1(0,T;L^2(0,\mu))}\\ &= \sup_{0\le\tau\le T}\left(\|u(.,\tau)\|^2_{L^2(0,\mu)}+\|u_t(.,\tau)\|^2_{L^2(0,\mu)}\right).\end{aligned}$$

and H is the Hilbert space of vector valued functions $\mathcal{H}=(g,H_1,H_2)\in L^2(\mathcal{D})\times L^2(0,\mu)\times L^2(0,\mu)$ with finite norm

$$\|\mathcal{H}\|_H^2=\|g\|^2_{L^2(\mathcal{D})}+\|H_1\|^2_{L^2(0,\mu)}+\|H_2\|^2_{L^2(0,\mu)}.$$

We first consider the linear problem associated to problem (1), that is when $g(x,t,u)$ is replaced by $g(x,t)$.

3. Uniqueness of Solution for the Associated Linear Problem

Theorem 1. *If the assumption T_1 is satisfied, then for any function $u\in D(\wp)$ there exists a constant $c>0$ independent of u such that we have the a priori estimate*

$$\|u\|_B\le c\|\wp u\|_H. \tag{3}$$

Proof. Consider the identity

$$\begin{aligned}&(\mathcal{L}u,\mathcal{N}u)_{L^2(\mathcal{D}^\tau)}\\ &=\left(u_{tt},x\int_x^\mu u_t(\xi,t)d\xi\right)_{L^2(\mathcal{D}^\tau)}-\left((R(x,t)u_x)_x,x\int_x^\mu u_t(\xi,t)d\xi\right)_{L^2(\mathcal{D}^\tau)}\\ &-\delta\left(u_{ttxx},x\int_x^\mu u_t(\xi,t)d\xi\right)_{L^2(\mathcal{D}^\tau)}-\left(u_{tt},\int_x^\mu \xi u_t(\xi,t)d\xi\right)_{L^2(\mathcal{D}^\tau)}\\ &+\left((R(x,t)u_x)_x,\int_x^\mu \xi u_t(\xi,t)d\xi\right)_{L^2(\mathcal{D}^\tau)}+\delta\left(u_{ttxx},\int_x^\mu \xi u_t(\xi,t)d\xi\right)_{L^2(\mathcal{D}^\tau)}.\end{aligned} \tag{4}$$

where $D^\tau=(0,\mu)\times(0,\tau)$, $0\le\tau\le T$, and

$$\mathcal{N}u=\int_x^\mu(x-\xi)u_t(\xi,t)d\xi.$$

By using the notations $\mathcal{J}_x v=\int_x^\mu v(\xi,t)d\xi$, and $\mathcal{J}_x(\xi v)=\int_x^\mu \xi v(\xi,t)d\xi$, and boundary and initial conditions, successive integration by parts of the terms on the right hand side of (4) give

$$\begin{aligned}&\left\|R^{\frac12}u(.,\tau)\right\|^2_{L^2(0,\mu)}+\|\mathcal{J}_x u_t(.,\tau)\|^2_{L^2(0,\mu)}+\delta\|u_t(.,\tau)\|^2_{L^2(0,\mu)}\\ &=\left\|R^{\frac12}H_1\right\|^2_{L^2(0,\mu)}+\|\mathcal{J}_x H_2\|^2_{L^2(0,\mu)}\\ &+\delta\|\varphi_2\|^2_{L^2(0,\mu)}+\left\|\sqrt{R_t}u\right\|^2_{L^2(\mathcal{D}^\tau)}+2(\mathcal{L}u,x\mathcal{J}_x u_t)_{L^2(\mathcal{D}^\tau)}\\ &-2(\mathcal{L}u,\mathcal{J}_x(\xi u_t))_{L^2(\mathcal{D}^\tau)}+2(R_x(x,t)u,\mathcal{J}_x u_t)_{L^2(\mathcal{D}^\tau)}.\end{aligned} \tag{5}$$

By evoking conditions T_1 and Cauchy-ε inequality $AB\le\frac{\varepsilon}{2}A^2+\frac{1}{2\varepsilon}B^2$ which holds for positive ε and for arbitrary constants A and B, in Equation (5) and then applying Gronwall's lemma (See Lemma 3.3, [21]) to the resulted inequality, we obtain

$$\begin{aligned}&\|u(.,\tau)\|^2_{L^2(0,\mu)}+\|u_t(.,\tau)\|^2_{L^2(0,\mu)}+\|\mathcal{J}_x u_t(.,\tau)\|^2_{L^2(0,\mu)}\\ \le\ & k_1\exp(k_2\tau)\left(\|g\|^2_{L^2(\mathcal{D}^\tau)}+\|H_1\|^2_{L^2(0,\mu)}+\|H_2\|^2_{L^2(0,\mu)}\right)\\ & k_1\exp(k_2 T)\left(\|g\|^2_{L^2(\mathcal{D})}+\|H_1\|^2_{L^2(0,\mu)}+\|H_2\|^2_{L^2(0,\mu)}\right),\end{aligned} \tag{6}$$

where

$$k_1 = \frac{\max\left(\frac{R_1}{2}, \frac{\mu^2+2\delta}{4}\right)}{\min(1, R_0, \delta)}, \; k_2 = \frac{\max\left(\frac{R_2+R_3^2}{2}, \frac{\mu^2+1}{2}, \frac{\mu^4}{4}\right)}{\min(1, R_0, \delta)}.$$

By discarding the last term on the left-hand side of (6), and taking the upper bound with respect to τ over $[0, T]$, we obtain the desired estimate (3) with $c = \sqrt{2k_1}\exp(k_2 \frac{T}{2})$. □

4. Existence of Solution for the Associated Linear Problem

Since we only know that the range (of the operator $\wp$), $R(\wp) \subset H$, we extend $\wp$ in a way that the a priori estimate (3) holds for the extension $\overline{\wp}$ and its range $R(\overline{\wp})$ coincides with the whole space H. It is straightforward to show that the operator $\wp$ has a closure, hence the following lemma.

Lemma 1. *The operator $\wp : B \to H$ is closable and admits a closure $\overline{\wp}$.*

Proof. The proof can be carried out in the same manner as in Ref. [11].

We define the strong solution of problem (1) as the solution of the operator equation $\overline{\wp}u = \mathcal{H} = (g, H_1, H_2)$ for all u in the domain $D(\overline{\wp})$ of the unbounded operator $\wp$. By passing to the limit, the estimate (3) can be extended to strong solutions, that is $\|u\|_B \leq c\|\overline{\wp}u\|_H$ for all u in $D(\overline{\wp})$, and hence we have the following Corollary. □

Corollary 1. *The set $R(\overline{\wp})$ is closed in H and $R(\overline{\wp}) = \overline{R(\wp)}$.*

Theorem 2. *Problem (1) admits a unique strong solution $u = \wp^{-1}(g, H_1, H_2) = \overline{\wp^{-1}}(g, H_1, H_2)$ depending continuously on $g \in L^2(\mathcal{D})$, $H_1 \in L^2(0, \mu)$, and $H_2 \in L^2(0, \mu)$, satisfying the a priori bound $\|u\|_B \leq c\|\wp u\|_H$, where c is a positive constant independent of u.*

Proof. We must show that $\overline{\wp}$ is one to one (injective), since it follows from Corollary 1 that to prove the existence of the strong solution, it suffices to show that $\overline{R(\wp)} = H$. ($R(\wp)$ is dense in H). Let us first prove this density in the special case given by the following theorem. □

Theorem 3. *Let $D_0(\wp)$ be the set of all functions u in $D(\wp)$ such that $u(x, 0)$ and $u_t(x, 0)$ vanish in the neighborhood of $t = 0$. If for $G \in L^2(\mathcal{D})$ and for all u in $D_0(\wp)$, we have*

$$(\mathcal{L}u, G)_{L^2(\mathcal{D})} = 0, \tag{7}$$

then G vanishes almost everywhere in $\mathcal{D}$.

Proof of Theorem 3. Define the function: $h(x,t) = \int_t^T G(x,\tau)d\tau$, and let y_{tt} be the solution of

$$R(\sigma, t)\int_x^\mu (x - \xi)y_{tt}(\xi, t)d\xi = h(x, t), \tag{8}$$

where $\sigma \in (0, \mu)$. Let $y = 0$, if $0 \leq t \leq s$, and $y = \int_s^t (t - \tau)y_{\tau\tau}d\tau$, if $s \leq t \leq T$, where s is any arbitrary fixed number in $[0, T]$. It is obvious that $y \in D_s(\wp) = \{y \in D(\wp)/y = 0$ for $t \leq s\} \subseteq D_0(\wp)$, and has a high order of smoothness. We can infer from above that

$$G(x,t) = \partial/\partial t\left(R(\sigma,t)\int_x^\mu (\xi - x)y_{tt}(\xi, t)d\xi\right). \tag{9}$$

Hence the following lemma which can be proved as in Ref. [11]. □

Lemma 2. *The function G defined by (9) is in $L^2(\mathcal{D})$.*

Upon substitution of (9) into (7) and then carrying out all integrations by parts and using the poincare type inequality

$$\|y\|^2_{L^2(\mathcal{D}_s)} \leq 4T^2 \|y_t\|^2_{L^2(\mathcal{D}_s)},$$

we obtain

$$\begin{aligned} &\left\| \int_x^\mu y_{tt}(\xi,s)d\xi \right\|^2_{L^2(0,\mu)} + \|y_{tt}(.,s)\|^2_{L^2(0,\mu)} + \|y_t(.,T)\|^2_{L^2(0,\mu)} \\ &\leq K^* \left(\left\| \int_x^\mu y_{tt}(\xi,t)d\xi \right\|^2_{L^2(\mathcal{D}_s)} + \|y_{tt}\|^2_{L^2(\mathcal{D}_s)} + \|y_t\|^2_{L^2(\mathcal{D}_s)} \right), \end{aligned} \tag{10}$$

where

$$K^* = 2\frac{\max(\delta R_2, (R_1+R_2), R_0 A^* B^*)}{R_0 \min(1,\delta,R_0)},$$

and

$$\begin{aligned} A^* &= \frac{4R_1R_2^2}{R_0^3} + \frac{R_1^2+R_2^2+R_3^2+4R_1R_2}{R_0} + \frac{4T^2\left[\left(R_2^2+R_4^2+R_5^2\right)\right]}{R_0}, \\ B^* &= \max\{k_1,k_2\}. \end{aligned}$$

Now if we introduce the function $w(x,t) = \int_t^T y_{\tau\tau} d\tau$, then $y_t(x,t) = w(x,s) - w(x,t)$ and $y_t(x,T) = w(x,s)$.

Thus (10) becomes

$$\begin{aligned} &\left\| \int_x^\mu y_{tt}(\xi,s)d\xi \right\|^2_{L^2(0,\mu)} + \|y_{tt}(.,s)\|^2_{L^2(0,\mu)} + \\ &\quad +(1-2K^*(T-s))\|w(.,s)\|^2_{L^2(0,\mu)} \\ &\leq 2K^* \left(\left\| \int_x^\mu y_{tt}(\xi,t)d\xi \right\|^2_{L^2(\mathcal{D}_s)} + \|y_{tt}\|^2_{L^2(\mathcal{D}_s)} + \|w\|^2_{L^2(\mathcal{D}_s)} \right). \end{aligned} \tag{11}$$

If we let $0 \leq (1-2K^*(T-s_0)) \leq \frac{1}{2}$, it follows then from (11) that

$$\begin{aligned} &\left\| \int_x^\mu y_{tt}(\xi,s)d\xi \right\|^2_{L^2(0,\mu)} + \|y_{tt}(.,s)\|^2_{L^2(0,\mu)} + \|w(.,s)\|^2_{L^2(0,\mu)} \\ \leq \; & 4K^* \left(\left\| \int_x^\mu y_{tt}(\xi,t)d\xi \right\|^2_{L^2(\mathcal{D}_s)} + \|y_{tt}\|^2_{L^2(\mathcal{D}_s)} + \|w\|^2_{L^2(\mathcal{D}_s)} \right), \end{aligned} \tag{12}$$

which holds for all $s \in [T-s_0, T]$, where $s_0 = 1/4K^*$.

Application of Gronwall's lemma to (12) leads to

$$\left\| \int_x^\mu y_{tt}(\xi,s)d\xi \right\|^2_{L^2(0,\mu)} + \|y_{tt}(.,s)\|^2_{L^2(0,\mu)} + \|w(.,s)\|^2_{L^2(0,\mu)} \leq 0, \text{ for all } s \in [T-s_0,T],$$

from which it follows that $G = 0$, a.e in $\mathcal{D}_{T-s_0} = (0,\mu) \times [T-s_0, T]$. Proceeding in this way step by step along rectangles of length s_0, we shall exhaust the whole rectangle $\mathcal{D} = (0,\mu) \times (0,T)$, hence, we prove that $G = 0$ almost everywhere in $\mathcal{D}$. To finish the proof of Theorem 2, we let $\mathcal{H} = (g, H_1, H_2) \in H$ to be orthogonal to any element of the range $R(\wp)$ of $\wp$, that is, such that

$$(\wp u, \mathcal{H})_H = (\mathcal{L}u, g)_{L^2(\mathcal{D})} + (\ell_1 u, H_1)_{L^2(0,\mu)} + (\ell_2 u, H_2)_{L^2(0,\mu)} = 0, \tag{13}$$

for all $u \in D(\wp)$. That is, we must show that $\mathcal{H} = (g, H_1, H_2) = 0$. If we put $u \in D_0(\wp)$ into (13), we have

$$(\mathcal{L}u, g)_{L^2(\mathcal{D})} = 0, \; u \in D_0(\wp). \tag{14}$$

Applying Theorem 5 to (14), it follows that $g = 0$. Here, (13) then takes the form

$$(\ell_1 u, H_1)_{L^2(0,\mu)} + (\ell_2 u, H_2)_{L^2(0,\mu)} = 0, \text{for all } u \in D(\wp). \tag{15}$$

However, since the sets $R(\ell_1)$ and $R(\ell_2)$ are dense in the space $L^2(0, \mu)$, then the relation (15) implies that $H_1 = 0$ and $H_2 = 0$. Hence $\mathcal{H} = (g, H_1, H_2) = 0$ and thus $\overline{R(\wp)} = H$. This completes the proof of Theorem 2.

5. Solvability of the Nonlinear Problem

To establish the existence and uniqueness of the weak solution of the nonlinear problem (1), we need to use the above results concerning the associated linear problem. For the nonlinear problem (1), we apply an iterative process based on the obtained results of the linear problem, we establish the existence and uniqueness of its weak solution.

Let us now consider the following auxiliary problem with homogeneous equation

$$\begin{cases} \mathcal{L}Z = Z_{tt} - (R(x,t)Z_x)_x - \delta Z_{ttxx} = 0, \\ \ell_1 Z = Z(x,0) = H_1(x), \quad x \in (0,\mu), \\ \ell_2 Z = Z_t(x,0) = H_2(x), \quad x \in (0,\mu), \\ \quad Z(0,t) = 0, \quad t \in (0,T), \\ \quad \int_0^\mu xZ(x,t)dx = 0, \quad t \in (0,T) \end{cases} \tag{16}$$

If u is a solution of problem (1) and Z is a solution of problem (16), then $\theta = u - Z$ satisfies the problem

$$\begin{cases} \mathcal{L}\theta = \theta_{tt} - (R(x,t)\theta_x)_x - \delta\theta_{ttxx} = F(x,t,\theta), \\ \quad \theta(x,0) = 0, \theta_t(x,0) = 0, \quad x \in (0,\mu), \\ \int_0^\mu x\theta(x,t)dx = 0, \quad \theta(0,t) = 0, \quad t \in (0,T), \end{cases} \tag{17}$$

where $F(x,t,\theta) = g(x,t,\theta + Z)$. The function F satisfies the condition

$$|F(x,t,u_1) - F(x,t,u_2)| \leq d(|u_1 - u_2|), \tag{18}$$

for all $(x,t) \in \mathcal{D}$.

According to Theorem 2, problem (16) has a unique solution that depends continuously on $H_1, H_2 \in L^2(0,\mu)$. We turn back to solve the problem (17). We shall prove that problem (17) has a unique weak solution.

First let

$$\Sigma(\mathcal{D}) = \left\{ v \in C^1(\mathcal{D}) : v_{tx}, v_{txx}, v_{ttx}, v_{ttxx} \in C(\mathcal{D}) \right\}.$$

Assume that $v, \theta \in \Sigma(\mathcal{D}), v(x,T) = 0$, $v(x,0) = 0$, $v_t(x,0) = 0$, $v_t(x,T) = 0$, $v(\mu,t) = 0$, $\theta(x,0) = \theta_t(x,0) = 0$ and $\int_0^\mu xv(x,t)dx = 0$, $\int_0^\mu x\theta(x,t)dx = 0$. For $v \in \Sigma(\mathcal{D})$, we observe that

$$\begin{aligned} -(\mathcal{L}\theta, \mathcal{J}_x(\rho v))_{L^2(\mathcal{D})} &= -(\theta_{tt} - (R(x,t)\theta_x)_x - \delta\theta_{ttxx}, \mathcal{J}_x(\rho v))_{L^2(\mathcal{D})} \\ &= -(\theta_{tt}, \mathcal{J}_x(\rho v))_{L^2(\mathcal{D})} + ((R(x,t)\theta_x)_x, \mathcal{J}_x(\rho v))_{L^2(\mathcal{D})} \\ &\quad +(\delta\theta_{ttxx}, \mathcal{J}_x(\rho v))_{L^2(\mathcal{D})} \\ &= -(F, \mathcal{J}_x(\rho v))_{L^2(\mathcal{D})}, \end{aligned} \tag{19}$$

where $\mathcal{J}_x(\rho z) = \int_x^\mu \xi z(\xi,t)d\xi$.

By using the above conditions on v and θ, a quick computation of each term on the right and left-hand sides of (19), gives

$$\begin{aligned} -(\theta_{tt}, \mathcal{J}_x(\rho v))_{L^2(\mathcal{D})} &= \int_{\mathcal{D}} \theta_t \mathcal{J}_x(\rho v_t) dt dx \\ &= -(v_t, \mathcal{J}_x \theta_t)_{L^2{}_\rho(\mathcal{D})}, \end{aligned} \tag{20}$$

$$\begin{aligned} &((R(x,t)\theta_x)_x, \mathcal{J}_x(\rho v))_{L^2(\mathcal{D})} \\ =\ & (R(x,t)v, \theta_x)_{L^2{}_\rho(\mathcal{D})}, \end{aligned} \tag{21}$$

$$\begin{aligned} (\delta\theta_{ttxx}, \mathcal{J}_x(\rho v))_{L^2(\mathcal{D})} &= \delta \int_0^T \int_0^\mu \theta_{ttx} x v dx dt = -\delta \int_0^\mu \int_0^T \theta_{tx} x v_t dx dt \\ &= \delta \int_0^\mu \theta_{tx}(x,T) x v(x,T) dx - \delta \int_0^\mu \theta_{tx}(x,0) x v(x,0) dx - \delta(v_t, \theta_{tx})_{L^2{}_\rho(\mathcal{D})} \\ &= -\delta(v_t, \theta_{tx})_{L^2{}_\rho(\mathcal{D})}, \end{aligned} \tag{22}$$

$$\begin{aligned} -(F, \mathcal{J}_x(\rho v))_{L^2(\mathcal{D})} &= \int_0^T [(\mathcal{J}_x(F(\xi,t,\theta).\mathcal{J}_x(\rho v)]_0^\mu dt + \int_0^T \int_0^\mu \mathcal{J}_x(F(\xi,t,\theta) x v dx dt \\ &= (\mathcal{J}_x(F), v)_{L^2{}_\rho(\mathcal{D})}. \end{aligned} \tag{23}$$

Insertion of (20)–(23) into (19) yields

$$\begin{aligned} &-(v_t, \mathcal{J}_x \theta_t)_{L^2{}_\rho(\mathcal{D})} + (R(x,t)v, \theta_x)_{L^2{}_\rho(\mathcal{D})} - \delta(v_t, \theta_{tx})_{L^2{}_\rho(\mathcal{D})} \\ =\ & (v, \mathcal{J}_x(F))_{L^2{}_\rho(\mathcal{D})}. \end{aligned} \tag{24}$$

If we make the notation

$$H(v,\theta) = -(v_t, \mathcal{J}_x \theta_t)_{L^2{}_\rho(\mathcal{D})} + (R(x,t)v, \theta_x)_{L^2{}_\rho(\mathcal{D})} - \delta(v_t, \theta_{tx})_{L^2{}_\rho(\mathcal{D})},$$

then we have

$$H(v,\theta) = (v, \mathcal{J}_x(F))_{L^2{}_\rho(\mathcal{D})}. \tag{25}$$

Hence, we have the definition

Definition 1. *A function $\theta \in L^2(\mathcal{D})$ is called a weak solution of problem (17) if (25) holds and $\theta(0,t) = 0$.*

Theorem 4. *Suppose that condition (18) holds, and that $d < \frac{\eta_1}{\sqrt{\eta_2 T}} e^{-\frac{\eta_2}{\eta_1}\frac{T}{2}}$, then problem (17), has a weak solution belonging to $L^2(\mathcal{D})$.*

Proof. We first consider the iterated problems

$$\begin{cases} \theta_{tt}^{(n)} - (R(x,t)\theta_x^{(n)})_x - \delta\theta_{ttxx}^{(n)} = F(x,t,\theta^{(n-1)}), \\ \theta^{(n)}(x,0) = 0,\ \theta_t^{(n)}(x,0) = 0, \\ \theta^{(n)}(0,t) = 0, \quad \int_0^\mu x\theta^{(n)}(x,t)dx = 0, \end{cases} \tag{26}$$

where $\left(\theta^{(n)}\right)_{n\in\mathbb{N}}$ is the constructed iteration sequence starting with first element $\theta^{(0)} = 0$. Given the elements $\theta^{(n-1)}$, for $n = 1, 2, \ldots$, then solve problems (26). Theorem 2 asserts that for fixed n, each problem (26) has a unique solution $\theta^{(n)}(x,t)$. $\square$

Theorem 5. *Now, consider the new problem*

$$\begin{cases} U_{tt}^{(n)} - (R(x,t)U_x^{(n)})_x - \delta U_{ttxx}^{(n)} = \sigma^{(n-1)}(x,t), \\ U^{(n)}(x,0) = 0,\ U_t^{(n)}(x,0) = 0, \\ U^{(n)}(0,t) = 0, \int_0^\mu xU^{(n)}(x,t)dx = 0, \end{cases} \tag{27}$$

with

$$U^{(n)}(x,t) = \theta^{(n+1)}(x,t) - \theta^{(n)}(x,t),$$

and where

$$\sigma^{(n-1)}(x,t) = F(x,t,\theta^{(n)}) - F(x,t,\theta^{(n-1)}).$$

Lemma 3. *Assume that condition (18) holds, then for the linearized problem (27), we have the a priori estimate*

Proof of Lemma 3.

$$\left\|U^{(n)}\right\|_{H^1(0,T;L^2(0,\mu))} \leq \eta \left\|U^{(n-1)}\right\|_{H^1(0,T;L^2(0,\mu))}, \tag{28}$$

where η is a positive constant given by

$$\eta = \frac{d\sqrt{\eta_2 T}}{\eta_1} e^{\frac{\eta_2}{\eta_1}\frac{T}{2}}, \tag{29}$$

with

$$\eta_1 = \min\left\{\frac{R_0}{2}, \frac{\delta}{2}\right\},$$

and

$$\eta_2 = \max\left\{\frac{R_2 + R_3}{2}, \frac{R_3\mu^2}{4} + \frac{5\mu^4}{24}\right\}.$$

Taking the inner product in $L^2(\mathcal{D}_\tau)$, with $0 \leq \tau \leq T$, of the differential equation in (27) and the integro-differential operator

$$\mathcal{N}U = \int_x^\mu (x - \zeta)U_t^{(n)}(\xi,t)d\xi,$$

we have

$$\begin{aligned} & (U_{tt}^{(n)} - (R(x,t)U_x^{(n)})_x - \delta U_{ttxx}^{(n)}, x\mathcal{J}_x U_t^{(n)} - \mathcal{J}_x(\zeta U_t^{(n)}))_{L^2(\mathcal{D}_\tau)} \\ = \; & (\sigma^{(n-1)}(x,t), x\mathcal{J}_x U_t^{(n)} - \mathcal{J}_x(\zeta U_t^{(n)}))_{L^2(\mathcal{D}_\tau)}. \end{aligned}$$

Then

$$\begin{aligned} & (U_{tt}^{(n)}, x\mathcal{J}_x U_t^{(n)})_{L^2(\mathcal{D}_\tau)} - ((R(x,t)U_x^{(n)})_x, x\mathcal{J}_x U_t^{(n)})_{L^2(\mathcal{D}_\tau)} - \delta(U_{ttxx}^{(n)}, x\mathcal{J}_x U_t^{(n)})_{L^2(\mathcal{D}_\tau)} \\ & -(U_{tt}^{(n)}, \mathcal{J}_x(\zeta U_t^{(n)}))_{L^2(\mathcal{D}_\tau)} + ((R(x,t)U_x^{(n)})_x, \mathcal{J}_x(\zeta U_t^{(n)}))_{L^2(\mathcal{D}_\tau)} + \delta(U_{ttxx}^{(n)}, \mathcal{J}_x(\zeta U_t^{(n)}))_{L^2(\mathcal{D}_\tau)} \\ = \; & (\sigma^{(n-1)}(x,t), x\mathcal{J}_x U_t^{(n)})_{L^2(\mathcal{D}_\tau)} - (\sigma^{(n-1)}(x,t), \mathcal{J}_x(\zeta U_t^{(n)}))_{L^2(\mathcal{D}_\tau)}. \end{aligned} \tag{30}$$

In light of the boundary conditions in (27), successive integrations by parts of each term of (30) lead to

$$\begin{aligned} &(U_{tt}^{(n)}, x\mathcal{J}_x U_t^{(n)})_{L^2(\mathcal{D}^\tau)} \\ = &\int_0^\tau \int_0^\mu \mathcal{J}_x\left(U_{tt}^{(n)}\right)\mathcal{J}_x U_t^{(n)} dxdt - \int_0^\tau \int_0^\mu \mathcal{J}_x\left(U_{tt}^{(n)}\right) xU_t^{(n)} dxdt \\ = &\frac{1}{2}\left\|\mathcal{J}_x\left(U_t^{(n)}(x,\tau)\right)\right\|_{L^2(0,\mu)}^2 - (\mathcal{J}_x\left(U_{tt}^{(n)}\right),\ U_t^{(n)})_{L^2{}_\rho(\mathcal{D}_\tau)}, \end{aligned} \tag{31}$$

$$\begin{aligned} &-((R(x,t)U_x^{(n)})_x, x\mathcal{J}_x U_t^{(n)})_{L^2(\mathcal{D}^\tau)} \\ = &\frac{1}{2}\left\|\sqrt{R(.,\tau)}U^{(n)}(.,\tau)\right\|_{L^2(0,\mu)}^2 - \frac{1}{2}\left\|\sqrt{R_t(.,t)}U^{(n)}\right\|_{L^2(\mathcal{D}_\tau)}^2 \\ &-(R_x(x,t)U^{(n)}, \mathcal{J}_x U_t^{(n)})_{L^2(\mathcal{D}_\tau)} - (R(x,t)U_x^{(n)}, U_t^{(n)})_{L^2{}_\rho(\mathcal{D}_\tau)}, \end{aligned} \tag{32}$$

$$-\delta(U_{ttxx}^{(n)}, x\mathcal{J}_x U_t^{(n)})_{L^2(\mathcal{D}_\tau)} = \frac{\delta}{2}\left\|U_t^{(n)}(.,\tau)\right\|_{L^2(0,\mu)}^2 - \delta(U_{ttx}^{(n)}, U_t^{(n)})_{L_\rho{}^2(\mathcal{D}_\tau)}, \tag{33}$$

$$-(U_{tt}^{(n)}, \mathcal{J}_x(\zeta U_t^{(n)}))_{L^2(\mathcal{D}_\tau)} = \left(\mathcal{J}_x U_{tt}^{(n)}, U_t^{(n)}\right)_{L_\rho{}^2(\mathcal{D}_\tau)}, \tag{34}$$

$$((R(x,t)U_x^{(n)})_x, \mathcal{J}_x(\zeta U_t^{(n)}))_{L^2(\mathcal{D}_\tau)} = (R(x,t)U_x^{(n)}, U_t^{(n)})_{L_\rho^2(\mathcal{D}_\tau)}, \tag{35}$$

$$\delta(U_{ttxx}^{(n)}, \mathcal{J}_x(\zeta U_t^{(n)}))_{L^2(\mathcal{D}_\tau)} = \delta(U_{ttx}^{(n)}, U_t^{(n)})_{L_\rho^2(\mathcal{D}_\tau)}, \tag{36}$$

The right-hand side of (30) can be estimated as

$$\begin{aligned} &(\sigma^{(n-1)}(x,t), x\mathcal{J}_x U_t^{(n)})_{L^2(\mathcal{D}_\tau)} \\ = &\int_0^\tau \int_0^\mu \sigma^{(n-1)} x\mathcal{J}_x U_t^{(n)} dxdt \\ \leq &\frac{1}{2}\int_0^\tau \int_0^\mu \left|\sigma^{(n-1)}\right|^2 dxdt + \frac{\mu}{2}\int_0^\tau \int_0^\mu \left|\sqrt{x}\mathcal{J}_x U_t^{(n)}\right|^2 dxdt \\ \leq &\frac{1}{2}\int_0^\tau \int_0^\mu \left|F(x,t,\theta^{(n)}) - F(x,t,\theta^{(n-1)})\right|^2 dxdt + \frac{\mu}{2}\int_0^\tau \left\|\mathcal{J}_x U_t^{(n)}\right\|_{L^2{}_\rho(0,\mu)}^2 dt \\ \leq &\frac{d^2}{2}\int_0^\tau \int_0^\mu \left|\theta^{(n)} - \theta^{(n-1)}\right|^2 dxdt + \frac{\mu}{2}\int_0^\tau \left\|\mathcal{J}_x U_t^{(n)}\right\|_{L^2{}_\rho(0,\mu)}^2 dt \\ \leq &\frac{d^2}{2}\int_0^\tau \left\|U^{(n-1)}\right\|_{L^2(0,\mu)}^2 dt + \frac{\mu^3}{6}\cdot\frac{\mu}{2}\int_0^\tau \left\|U_t^{(n)}\right\|_{L^2(0,\mu)}^2 dt \\ = &\frac{d^2}{2}\left\|U^{(n-1)}\right\|_{L^2(\mathcal{D}_\tau)}^2 + \frac{\mu^4}{12}\left\|U_t^{(n)}\right\|_{L^2(\mathcal{D}_\tau)}^2, \end{aligned} \tag{37}$$

$$\begin{aligned} &-(\sigma^{(n-1)}(x,t), \mathcal{J}_x(\zeta U_t^{(n)}))_{L^2(\mathcal{D}_\tau)} \\ \leq &\left|-(\sigma^{(n-1)}(x,t), \mathcal{J}_x(\zeta U_t^{(n)}))_{L^2(\mathcal{D}_\tau)}\right| \\ \leq &\frac{1}{2}\left\|\sigma^{(n-1)}\right\|_{L^2(\mathcal{D}_\tau)}^2 + \frac{1}{2}\left\|\mathcal{J}_x(\zeta U_t^{(n)})\right\|_{L^2(\mathcal{D}_\tau)}^2 \\ \leq &\frac{d^2}{2}\left\|U^{(n-1)}\right\|_{L^2(\mathcal{D}_\tau)}^2 + \frac{\mu^4}{8}\int_0^\tau \left\|U_t^{(n)}\right\|_{L^2(0,\mu)}^2 dt \\ = &\frac{d^2}{2}\left\|U^{(n-1)}\right\|_{L^2(\mathcal{D}_\tau)}^2 + \frac{\mu^4}{8}\left\|U_t^{(n)}\right\|_{L^2(\mathcal{D}_\tau)}^2. \end{aligned} \tag{38}$$

A combination of (31)–(38) and (30), yields

$$
\begin{aligned}
&\frac{1}{2}\left\|\mathcal{J}_x\left(U_t^{(n)}(x,\tau)\right)\right\|^2_{L^2(0,\mu)}+\frac{1}{2}\left\|\sqrt{R(.,\tau)}U^{(n)}(.,\tau)\right\|^2_{L^2(0,\mu)}\\
&+\frac{\delta}{2}\left\|U_t^{(n)}(.,\tau)\right\|^2_{L^2(0,\mu)}\\
\leq\;&\frac{1}{2}\left\|\sqrt{R_t(.,t)}U^{(n)}\right\|^2_{L^2(\mathcal{D}_\tau)}+(R_x(x,t)U^{(n)},\mathcal{J}_xU_t^{(n)})_{L^2(\mathcal{D}_\tau)}\\
&+d^2\left\|U^{(n-1)}\right\|^2_{L^2(\mathcal{D}_\tau)}+\frac{5\mu^4}{24}\left\|U_t^{(n)}\right\|^2_{L^2(\mathcal{D}_\tau)}.
\end{aligned}
\tag{39}
$$

By using conditions T_1 and T_2, Cauchy-ε inequality and the inequality of Poincare type $\|\mathcal{J}_xH\|^2_{L^2(\mathcal{D}_\tau)}\leq\frac{\mu^2}{2}\|H\|^2_{L^2(\mathcal{D}_\tau)}$ (See Ref. [11]), inequality (39) reduces to

$$
\begin{aligned}
&\frac{1}{2}\left\|\mathcal{J}_x\left(U_t^{(n)}(x,\tau)\right)\right\|^2_{L^2(0,\mu)}+\eta_1\left(\left\|U^{(n)}(.,\tau)\right\|^2_{L^2(0,\mu)}+\left\|U_t^{(n)}(.,\tau)\right\|^2_{L^2(0,\mu)}\right)\\
\leq\;&d^2\left\|U^{(n-1)}\right\|^2_{L^2(\mathcal{D}_\tau)}+\eta_2\left(\left\|U^{(n)}\right\|^2_{L^2(\mathcal{D}_\tau)}+\left\|U_t^{(n)}\right\|^2_{L^2(\mathcal{D}_\tau)}\right),
\end{aligned}
\tag{40}
$$

where

$$
\begin{cases}
\eta_1=\min\{\frac{R_0}{2},\frac{\delta}{2}\}\\
\eta_2=\max\{\frac{R_2+R_3}{2},(\frac{R_3\mu^2}{4}+\frac{5\mu^4}{24})\}.
\end{cases}
\tag{41}
$$

If we discard the first term on the left-hand side of (40), we obtain

$$
\begin{aligned}
&\left\|U^{(n)}(.,\tau)\right\|^2_{L^2(0,\mu)}+\left\|U_t^{(n)}(.,\tau)\right\|^2_{L^2(0,\mu)}\\
\leq\;&\frac{d^2}{\eta_1}\left\|U^{(n-1)}\right\|^2_{L^2(\mathcal{D}_\tau)}+\frac{\eta_2}{\eta_1}\left(\left\|U^{(n)}\right\|^2_{L^2(\mathcal{D}_\tau)}+\left\|U_t^{(n)}\right\|^2_{L^2(\mathcal{D}_\tau)}\right).
\end{aligned}
\tag{42}
$$

Application of Gronwall's lemma (Lemma 3.3 [21]) to (42) gives

$$
\begin{aligned}
&\left\|U^{(n)}(.,\tau)\right\|^2_{L^2(0,\mu)}+\left\|U_t^{(n)}(.,\tau)\right\|^2_{L^2(0,\mu)}\\
\leq\;&\frac{d^2\eta_2}{{\eta_1}^2}\,e^{\frac{\eta_2T}{\eta_1}}\left\|U^{(n-1)}\right\|^2_{L^2(\mathcal{D})}\\
&\frac{d^2\eta_2}{{\eta_1}^2}\,e^{\frac{\eta_2T}{\eta_1}}\left(\left\|U^{(n-1)}\right\|^2_{L^2(\mathcal{D})}+\left\|U^{(n-1)}\right\|^2_{L^2(\mathcal{D})}\right).
\end{aligned}
\tag{43}
$$

Integration of both sides of (43) with respect to τ over the interval $(0,T)$, gives the estimate (28), that is

$$
\left\|U^{(n)}\right\|_{H^1(0,T;L^2(0,\mu))}\leq\eta\left\|U^{(n-1)}\right\|_{H^1(0,T;L^2(0,\mu))},
$$

where the constant η is given by (29). We continue the proof of Theorem 4. From the criteria of convergence of series, we see that the series $\sum_{n=1}^{\infty}U^{(n)}$ converges if $\frac{d\sqrt{\eta_2T}}{\eta_1}\,e^{\frac{\eta_2}{\eta_1}\cdot\frac{T}{2}}<1$, that is if $d<\frac{\eta_1}{\sqrt{\eta_2T}}e^{-\frac{\eta_2}{\eta_1}\cdot\frac{T}{2}}$.

The sequence $(\theta^{(n)})_{n\in N}$ defined by

$$
\begin{aligned}
\theta^{(n)}(x,t)\;&=\;\sum_{k=0}^{n-1}U^{(k)}+\theta^{(0)}(x,t)\\
&=\;\sum_{k=0}^{n-1}\theta^{(k+1)}(x,t)-\theta^{(k)}(x,t)+\theta^{(0)}(x,t),\quad n=1,2,\ldots
\end{aligned}
$$

is convergent to an element $\theta \in H^1(0,T;L^2(0,\mu))$, where $U^{(n)}(x,t) = \theta^{(n+1)}(x,t) - \theta^{(n)}(x,t)$.

Now to prove that the limit function θ is a solution of the problem under consideration (17), we should show that θ satisfies (25) and that $\theta(0,t) = 0$ as mentioned in Definition 1.

For problem (26), we have

$$H(\theta^{(n)},v) = (v,\mathcal{J}_xF(\xi,t,\theta^{(n-1)}))_{L^2_\rho(\mathcal{D})}. \tag{44}$$

From (44), we have

$$\begin{aligned} & H(\theta^{(n)}-\theta,v)+H(\theta,v) \\ = & (v,\mathcal{J}_x(F(\xi,t,\theta^{(n-1)})-F(\xi,t,\theta)))_{L^2_\rho(\mathcal{D})}+(v,\mathcal{J}_xF(\xi,t,\theta))_{L^2_\rho(\mathcal{D})} \\ = & (v,\mathcal{J}_xF(\xi,t,\theta^{(n-1)})-\mathcal{J}_xF(\xi,t,\theta))_{L^2_\rho(\mathcal{D})} \\ & +(v,\mathcal{J}_xF(\xi,t,\theta))_{L^2_\rho(\mathcal{D})}. \end{aligned} \tag{45}$$

From the partial differential equation in (26), we have

$$\begin{aligned} & (v,\frac{\partial^2}{\partial t^2}\mathcal{J}_x(\theta^{(n)}-\theta))_{L^2_\rho(\mathcal{D})}-(v,\mathcal{J}_x(\frac{\partial}{\partial\xi}(R(\xi,t)\frac{\partial}{\partial\xi}(\theta^{(n)}-\theta)))_{L^2_\rho(\mathcal{D})} \\ & -\delta(v,\frac{\partial^2}{\partial t^2}\mathcal{J}_x(\frac{\partial^2}{\partial\xi^2}(\theta^{(n)}-\theta)))_{L^2_\rho(\mathcal{D})} \\ = & H(\theta^{(n)}-\theta,v). \end{aligned} \tag{46}$$

Integration by parts of each term on the left-hand side of (46), and use of conditions on the functions v and θ, gives

$$\begin{aligned} & (v,\frac{\partial^2}{\partial t^2}\mathcal{J}_x(\theta^{(n)}-\theta))_{L^2_\rho(\mathcal{D})} \\ = & -\int_0^T\int_0^\mu \mathcal{J}_x(\xi v)\frac{\partial^2}{\partial t^2}(\theta^{(n)}-\theta)dxdt \\ = & \int_0^T\int_0^\mu \mathcal{J}_x(\xi v_t)\frac{\partial}{\partial t}(\theta^{(n)}-\theta)dxdt \\ = & \int_0^\mu\Big[\mathcal{J}_x(\xi v_t)(\theta^{(n)}-\theta)\Big]_0^T dx-\int_0^T\int_0^\mu \mathcal{J}_x(\xi v_{tt})\,(\theta^{(n)}-\theta)dxdt \\ = & -(\mathcal{J}_x(\xi v_{tt}),\,(\theta^{(n)}-\theta))_{L^2(\mathcal{D})}, \end{aligned} \tag{47}$$

$$\begin{aligned} & -(v,\mathcal{J}_x(\frac{\partial}{\partial\xi}(R(\xi,t)\frac{\partial}{\partial\xi}(\theta^{(n)}-\theta)))_{L^2_\rho(\mathcal{D})} \\ = & \int_0^T\int_0^\mu \mathcal{J}_x(\xi v)(\frac{\partial}{\partial x}(R(x,t)\frac{\partial}{\partial x}(\theta^{(n)}-\theta)))dxdt \\ = & \int_0^T\Big[xvR(x,t)(\theta^{(n)}-\theta))\Big]_{x=0}^{x=\mu}dt-\int_0^T\int_0^\mu(vR(x,t)+xv_xR(x,t)+xvR_x(x,t))(\theta^{(n)}-\theta)dxdt \\ = & -(v,R(x,t))(\theta^{(n)}-\theta))_{L^2(\mathcal{D})}-(xv_x,R(x,t))(\theta^{(n)}-\theta))_{L^2(\mathcal{D})}-(xv,R_x(x,t)(\theta^{(n)}-\theta))_{L^2(\mathcal{D})}, \end{aligned} \tag{48}$$

$$\begin{aligned}
& -\delta(v, \frac{\partial^2}{\partial t^2}\mathcal{J}_x(\frac{\partial^2}{\partial \xi^2}(\theta^{(n)}-\theta)))_{L^2_\rho(\mathcal{D})} \\
= \quad & \delta\int_0^T\int_0^\mu xv_t\frac{\partial}{\partial t}\mathcal{J}_x(\frac{\partial^2}{\partial \xi^2}(\theta^{(n)}-\theta))dxdt \\
= \quad & \delta\int_0^T\int_0^\mu\left(-\int_x^\mu \xi v_{tt}d\xi\right)(-\frac{\partial^2}{\partial x^2}(\theta^{(n)}-\theta))dxdt \\
= \quad & -\delta\int_0^T\int_0^\mu(xv_{tt})(\frac{\partial}{\partial x}(\theta^{(n)}-\theta))dxdt \\
= \quad & \delta\int_0^T\left[xv_{tt}(\theta^{(n)}-\theta)\right]_{x=0}^{x=\mu}dt-\delta\int_0^T\int_0^\mu(v_{tt}+xv_{xtt})(\theta^{(n)}-\theta)dxdt \\
= \quad & -\delta(v_{tt},(\theta^{(n)}-\theta))_{L^2(\mathcal{D})}-\delta(xv_{xtt},(\theta^{(n)}-\theta))_{L^2(\mathcal{D})}. \qquad (49)
\end{aligned}$$

Combination of equalities (46)–(49) yields

$$\begin{aligned}
H(\theta^{(n)}-\theta,v) \quad = \quad & -(\mathcal{J}_x(\xi v_{tt}),\ (\theta^{(n)}-\theta))_{L^2(\mathcal{D})} \\
& -(v,R(x,t))(\theta^{(n)}-\theta))_{L^2(\mathcal{D})}-(xv_x,R(x,t))(\theta^{(n)}-\theta))_{L^2(\mathcal{D})} \\
& -(xv,R_x(x,t)(\theta^{(n)}-\theta))_{L^2(\mathcal{D})} \\
& -\delta(v_{tt},(\theta^{(n)}-\theta))_{L^2(\mathcal{D})}-\delta(xv_{xtt},(\theta^{(n)}-\theta))_{L^2(\mathcal{D})}. \qquad (50)
\end{aligned}$$

We apply Cauchy–Schwarz inequality to the terms on the right-hand side of (50) as follows

$$\begin{aligned}
-(\mathcal{J}_x(\xi v_{tt}),\ (\theta^{(n)}-\theta))_{L^2(\mathcal{D})} \quad \le \quad & \|\mathcal{J}_x(\xi v_{tt})\|_{L^2(\mathcal{D})}\left\|\theta^{(n)}-\theta\right\|_{L^2(\mathcal{D})} \\
\le \quad & \frac{\mu^2}{2}\|v_{tt}\|_{L^2(\mathcal{D})}\left\|\theta^{(n)}-\theta\right\|_{L^2(\mathcal{D})}, \qquad (51)
\end{aligned}$$

$$\begin{aligned}
& -(v,R(x,t))(\theta^{(n)}-\theta))_{L^2(\mathcal{D})}-(xv_x,R(x,t))(\theta^{(n)}-\theta))_{L^2(\mathcal{D})}-(xv,R_x(x,t)(\theta^{(n)}-\theta))_{L^2(\mathcal{D})} \\
\le \quad & (R_1+\mu R_3)\|v\|_{L^2(\mathcal{D})}\left\|\theta^{(n)}-\theta\right\|_{L^2(\mathcal{D})}+\mu R_1\|v_x\|_{L^2(\mathcal{D})}\left\|\theta^{(n)}-\theta\right\|_{L^2(\mathcal{D})}, \qquad (52)
\end{aligned}$$

$$\begin{aligned}
& -\delta(v_{tt},(\theta^{(n)}-\theta))_{L^2(\mathcal{D})}-\delta(xv_{xtt},(\theta^{(n)}-\theta))_{L^2(\mathcal{D})} \\
\le \quad & \delta\|v_{tt}\|_{L^2(\mathcal{D})}\left\|\theta^{(n)}-\theta\right\|_{L^2(\mathcal{D})}+\delta\mu\ \|v_{xtt}\|_{L^2(\mathcal{D})}\left\|\theta^{(n)}-\theta\right\|_{L^2(\mathcal{D})}, \qquad (53)
\end{aligned}$$

Substitution of (51)–(53) into (50) gives the inequality

$$\begin{aligned}
& H(\theta^{(n)}-\theta,v) \\
\le \quad & (R_1+\mu R_3)\|v\|_{L^2(\mathcal{D})}\left\|\theta^{(n)}-\theta\right\|_{L^2(\mathcal{D})} \\
& +\mu R_1\|v_x\|_{L^2(\mathcal{D})}\left\|\theta^{(n)}-\theta\right\|_{L^2(\mathcal{D})} \\
& +(\frac{\mu^2}{2}+\delta)\|v_{tt}\|_{L^2(\mathcal{D})}\left\|\theta^{(n)}-\theta\right\|_{L^2(\mathcal{D})} \\
& +\delta\mu\|v_{xtt}\|_{L^2(\mathcal{D})}\left\|\theta^{(n)}-\theta\right\|_{L^2(\mathcal{D})} \\
\le \quad & c(\|v\|_{L^2(\mathcal{D})}+\|v_x\|_{L^2(\mathcal{D})}+\|v_{tt}\|_{L^2(\mathcal{D})}+\|v_{xtt}\|_{L^2(\mathcal{D})})\left\|\theta^{(n)}-\theta\right\|_{L^2(\mathcal{D})}. \qquad (54)
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
& (v, \mathcal{J}_x F(\xi, t, \theta^{(n-1)}) - \mathcal{J}_x F(\xi, t, \theta))_{L^2_\rho(\mathcal{D})} \\
\leq\ & \mu \, \|v\|_{L^2(\mathcal{D})} \left\| \mathcal{J}_x F(\xi, t, \theta^{(n-1)}) - \mathcal{J}_x F(\xi, t, \theta) \right\|_{L^2(\mathcal{D})} \\
\leq\ & \frac{\mu^2}{\sqrt{2}} \|v\|_{L^2(\mathcal{D})} \left\| F(\xi, t, \theta^{(n-1)}) - F(\xi, t, \theta) \right\|_{L^2(\mathcal{D})} \\
\leq\ & \frac{d\mu^2}{\sqrt{2}} \|v\|_{L^2(\mathcal{D})} \left\| \theta^{(n)} - \theta \right\|_{L^2(\mathcal{D})}.
\end{aligned} \tag{55}$$

Taking into account (54) and (55), and passing to the limit in (45) as $n \to \infty$, we obtain

$$H(\theta, v) = (v, \mathcal{J}_x F)_{L^2_\rho(\mathcal{D})}.$$

Now since $\theta \in H^1(0, T; L^2(0, \mu))$, then $\int_0^t \theta(x, \nu) d\nu \in C(\overline{\mathcal{D}})$, and we conclude that $\theta(0, t) = 0$. □

6. Uniqueness of Solution for the Nonlinear Case

Theorem 6. *If condition (18) is satisfied, then the solution of problem (17) is unique.*

Proof. Suppose that $\theta_1, \theta_2 \in H^1(0, T; L^2(0, \mu))$ are two solutions of (17), then $U = \theta_1 - \theta_2$ satisfies

$$\begin{cases} U_{tt} - (R(x,t)U_x)_x - \delta U_{ttxx} = \sigma(x,t) \\ U(x,0) = 0, U_t(x,0) = 0, \\ U(0,t) = 0, \ \int_0^\mu xU(x,t)dx = 0. \end{cases} \tag{56}$$

where

$$\sigma(x,t) = F(x,t,\theta_1) - F(x,t,\theta_2).$$

Taking the inner product in $L^2(\mathcal{D})$ of the differential operator

$$NU = U_{tt} - (R(x,t)U_x)_x - \delta U_{ttxx}$$

and the integro-differential operator

$$JU = x\mathcal{J}_x U_t - \mathcal{J}_x^*(\zeta U_t),$$

and following the same procedure done in establishing the proof of Lemma 3, we have

$$\|U\|_{H^1(0,T;L^2(0,\mu))} \leq \eta \|U\|_{H^1(0,T;L^2(0,\mu))}, \tag{57}$$

where

$$\eta = \frac{d\sqrt{\eta_2 T}}{\eta_1} e^{\frac{\eta_2}{\eta_1} \cdot \frac{T}{2}},$$

with

$$\eta_1 = \min\{\frac{R_0}{2}, \frac{\delta}{2}\}, \ \eta_2 = \max\{\frac{R_2 + R_3}{2}, (\frac{R_3\mu^2}{4} + \frac{5\mu^4}{24})\}.$$

Since $\eta < 1$, it follows from (57) that

$$(1 - \eta)\|U\|_{H^1(0,T;L^2(0,\mu))} = 0. \tag{58}$$

From the last inequality (58), we deduce that $U = \theta_1 - \theta_2 = 0$, which implies that $\theta_1 = \theta_2 \in H^1(0, T; L^2(0, \mu))$. Hence the uniqueness of the solution of problem (17). □

7. Conclusions

The primary focus of this paper lies in the investigation and solvability of an initial boundary value problem for a semi-linear pseudo-hyperbolic equation. The problem is subject to both a Dirichlet condition and an integral condition. Through rigorous analysis, the existence, uniqueness, and continuous dependence of a strong solution for the specified initial boundary problem have been ascertained. In terms of the associated linear problem, the uniqueness of its generalized solution has been substantiated based on an a priori energy inequality and the application of the Gronwall–Bellman Lemma. Further, it is demonstrated that the operator range generated by the considered problem is dense, thereby confirming the problem's solvability. For the nonlinear counterpart of the problem, an iterative process is employed. This iterative methodology leverages the results previously obtained for the associated linear problem to affirm both the existence and uniqueness of the solution for the nonlinear problem under consideration. Thus, this paper contributes to the mathematical framework for solving complex initial boundary value problems with non-local constraints, offering robust solutions that are both unique and continuously dependent on the initial conditions.

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