mathematics

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A Generalized Log Gamma Approach: Theoretical Contributions and an Application to Companies' Life Expectancy

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Abstract: The survival of a company has been a topic of growing interest in the scientific community. Measuring the life expectancy of Portuguese telecommunications companies using generalized log-gamma (GLG) distribution is a new research endeavor. Regarding the new theoretical contributions, original expressions for the moments and mode of the GLG distribution are presented. In this empirical study, data on the entrepreneurial fabric in the Information and Communication sector from 2004 to 2018, when some companies were born or died, were used. In addition to the GLG, three other statistical distributions with two parameters are analyzed: gamma, Weibull, and log-normal. Maximum likelihood parameters and confidence intervals for survival probabilities are estimated and compared. The Akaike information criterion is used to compare the performance of the four estimated models. The results show that GLG distribution is a promising solution to assess the resilience and longevity of a firm.

Keywords: probability distributions; parameter estimation; parametric models; generalized log-gamma; companies' life expectancy

MSC: 62E10; 62E15; 62P20; 91B82

Citation: Dias Gonçalves, J.H.; Ferreira Gomes, J.J.; Rubio, L.; Ramos, F.R. A Generalized Log Gamma Approach: Theoretical Contributions and an Application to Companies' Life Expectancy. *Mathematics* **2023**, *11*, 4792. https://doi.org/10.3390/math11234792

Academic Editors: Carmen Patino-Alonso and Steve Drekic

Received: 20 July 2023
Revised: 25 August 2023
Accepted: 3 September 2023
Published: 27 November 2023

1. Introduction

In an increasingly global and competitive economic environment, the creation of a company faces major challenges. Their resilience and survival, especially in the early stages, are under constant scrutiny. Many do not hold up and fail. Analyzing the specific reasons companies fail, general trends, or even the macroeconomic conditions affecting business failure have been long-standing topics in the scientific literature (e.g., [1–4]). The attempt to model and understand real phenomena, such as the one presented, is a timeless and undeniably useful topic that involves articulating statistical, mathematical, and computational methods and procedures. Some studies reported in the scientific literature related to this topic are presented below.

In the Portuguese context, some previous works have focused on studying the survival of new firms [5–7]. On the other hand, the survival of new firms based on certain conditions of entry and permanence in the market is addressed by [6] and developed by the same authors in a later work [7]. In addition, [8] proposes an empirical model to understand the impact that the founding conditions of a firm have on its life expectancy; [9] suggests a mortality table for domestic firms. On an international scale, [10] analyzes over 25,000 North American firms from 1950 to 2009 and concludes that the life expectancy of a firm does not depend on its age. The search for company longevity is approached in [11] from a business strategy perspective, in which key indicators that promote the growth of a company's longevity are presented. According to the S&P 500 index, the average life of companies in 2020 was just over 21 years, whereas in 1965, it was about 32 years [12].

Since this index includes the 500 largest companies and represents almost 80% of the available market capitalization, there is a natural concern about the life span of smaller companies. It is reasonable to assume that fewer companies will survive than their market-leading counterparts. In essence, most studies establish a positive correlation between company size and life expectancy and emphasize industry size as a factor in the health of the organizations within it. Thus, considering the impacts it may have on the economic life of the business fabric or even of a country, it is necessary to understand the lifetime of a set of companies. However, it is important to identify robust and accurate models (e.g., fitted probability distributions) that assist in the modeling process, providing support for compression and forecasting. When studying firm survival, it is crucial to obtain density and survival functions for the whole lifetime of a company. Decision makers can use these tools to assess sectoral resilience and make comparisons before, for example, making a long-term investment decision that may affect their portfolio.

From the pioneering works of [13–15] and the analysis of other theoretical works on generalized log-gamma (GLG) [16–18], some features are easy to notice. Moreover, approximate conditional inference techniques on the distributions of location and scale parameters, based on normal approximations to the signed square root distributions of likelihood ratio statistics, are examined by [19]. By leveraging these methods, it becomes possible to derive approximate inferences for both the quantiles and the scale parameter of the generalized log-gamma distribution. However, [20] proposed robust estimators for the generalized log-gamma distribution using a (weighted) Q_τ estimator, which minimizes a scale τ that measures differences between the empirical and theoretical quantiles. Unfortunately, this estimator does not follow a normal distribution asymptotically but is a practical starting point for a one-step weighted likelihood estimator. As mentioned in [21], the GLG and other particular distributions mentioned in the scientific literature seem applicable in analyzing lifetime and reliability data. This applicability becomes even more relevant when considering the sectors of activity with a business fabric characterized by a strong heterogeneity regarding service life. More specifically, a small number of companies have a very long history of activity (dominating the sector) and a constant emergence of small and medium-sized companies not surviving in the market, failing soon after starting their activity.

Many applications of the generalized log-gamma distribution can be found in the literature, for example, the Coale–McNeil nuptiality model studied by [22]. For example, the author proposes an approach to developing country-specific standard schedules and illustrates its usefulness in a regression analysis. An application to fertility projection is proposed by modeling the fertility calendar by birth order. Moreover, [23] modified the generalized log-gamma regression model to consider the presence of long-term survivors. The authors of [24] analyzed a real data set from the medical domain using a four-parameter extension of the generalized gamma distribution capable of modeling a bathtub-shaped hazard rate function and examined other commonly used models such as log-exponential, log-Weibull, and log-normal regression with cure rates, determining that a generalized log-gamma regression model that incorporates a cure rate is best. The beauty and importance of this distribution lie in its ability to model monotonic and non-monotonic failure rate functions, which are quite common in lifetime and reliability data analysis. The authors of [25] used the generalized Gamma distribution to model traumatic brain injury data, considering different stages of hospitalization related to the survival rates of a limited number of patients after a traffic accident. We propose different estimation methods to obtain improved estimators of the model parameters, which can be recommended for use in small samples. The authors of [26] analyzed the potential of the inverse generalized gamma (IGG) distribution for modeling reliability data with an upside-down bathtub hazard rate function using real environmental data. In this study, Bayesian inferences are presented for the parameters of the IGG distribution using non-informative priors, such as the Jeffreys prior and the reference prior. The authors of [27] analyzed the estimation of the generalized gamma distribution parameters from left-truncated and right-censored data. A stochastic version of the expectation maximization (EM) algorithm is proposed as an alternative method for computing approximate maximum likelihood estimations (MLE). The proposed stochastic EM algorithm is demonstrated to be a useful alternative estimation method for model fitting of the generalized gamma distribution. The authors of [28] studied cell phone ages, considering that they follow a gamma distribution under progressive first-fault (PFF) censoring. The data set of radio transceivers was analyzed as an application. For this distribution, these researchers estimated all the unknown parameters and the Shannon and Rényi entropies using the maximum likelihood (ML) method. The asymptotic confidence intervals of the ML estimators of the target parameters were produced using the normal approximation to ML and log-transformed ML. The authors of [29] proposed the asymmetric LGN distribution as a flexible option for the error term in linear regression models. The maximum likelihood method was implemented to estimate the parameters of the LGN model, and the Fisher information matrix was derived, showing that it is non-singular. Illustrations with real data show that the proposed model can be an effective alternative to existing models, such as normal, normal-power, normal Tobit, and normal-power Tobit.

Furthermore, in the context of the distributions of interest to this research, [30] proposed a four-parameter extended generalized gamma model for modeling lifetime data. According to the authors, the proposed heteroscedastic regression model can be used more effectively in the analysis of survival data since it includes several widely known regression models as special models. Various simulations are performed for different parameter settings, sample sizes, and censoring percentages. The authors of [31] defined and studied a four-parameter model called the generalized odd log-logistic flexible Weibull distribution, providing an extensive study of the quantile function. With an application to real engineering data sets, these authors used the maximum likelihood method for estimating the model parameters and performed various simulations for different parameter settings, sample sizes, and censoring percentages.

Following a methodology based on probability distributions, in [32], a family of generalized gamma distributions, T-gamma family, has been proposed using the $T - R\{Y\}$ framework. The family of distributions is generated using the quantile functions of uniform, exponential, log-logistic, logistic, and extreme value distributions. Based on the theoretical developments proposed by the authors, four data sets with various shapes are fitted using members of the T-gamma family of distributions. Also, an interesting methodological approach is proposed using probability distributions [33]. In this study, a new lifetime distribution is proposed to model the lifetime of the series system, which incorporates most of the composite lifetime distribution. Several mathematical properties obtained for this distribution included the moment-generating function, moments, and order statistics. According to the authors, the proposed distribution has a flexible density function, and the proposed family offers a better fit based on tests on a real data set (the unknown parameters of the proposed generalized family were estimated using the MLE technique) from statistical analysis. Using the distribution employed by [34] via a lifetime application, [35] proposed a new model called the odd log-logistic generalized half-normal distribution for describing fatigue lifetime data, discussing the method of maximum likelihood to fit the model parameters. For different parameter settings and sample sizes, some simulation studies compare the performance of the new lifetime model. The authors of [36] introduced a new generator of probability distribution, the adjusted log-logistic generalized (ALLoG) distribution, and a new extension of the standard one-parameter exponential distribution called the adjusted log-logistic generalized exponential (ALLoGExp) distribution. Using the MLE method to estimate the model parameters, the importance and flexibility of the ALLoGExp distribution were demonstrated with a real and uncensored lifetime data set, and its fit was compared with five other exponential-related distributions.

On this basis, it is considered beneficial to develop research focusing on GLG distribution and its applications. Thus, according to the state of the art described in the previous paragraphs, this work proposes maximum likelihood estimates for the generalized log gamma model parameters and the confidence intervals for survival probabilities. In addition, expressions for the moments and the GLG distribution mode are presented. As an original application, the survival problem of telecommunications companies is analyzed, which, at the time of presenting this paper, had not been addressed using the GLG model and the techniques proposed in this research. From the set of data obtained (despite the limitations that will be discussed at the end of this study), modeling the life expectancy of companies (following a methodology based on probability distributions supported by the scientific literature) constitutes the objective of the empirical part of this research.

Having described multiple applications of the generalized log-gamma distribution, the parameter estimation techniques found in the literature, and the main objective of this present work, each section will be divided as follows: Section 1 provides an overview of the literature related to the application problem proposed in this work, as well as the objectives, research questions, and main contributions of this investigation. Section 2 reviews the GLG distribution: (i) a brief original study on the random variable $X \sim GLG(\mu, \sigma, q)$ is presented; (ii) GLG is presented as a general case of some other known statistical distributions; (iii) some original theoretical contributions on GLG are presented. Next, Section 3 presents the data to be used in this study and some considerations on the methodological procedures, where the estimation methodology focused on the maximum likelihood method in the presence of censored data are discussed. Section 4 presents the results of applying the proposed models to a real-world dataset. Finally, Section 5 concludes with a discussion of the results, conclusions, and references to the limitations and future work.

2. Generalized Log-Gamma Distribution: Theoretical Framework and Complements

2.1. Generalized Log-Gamma Density Function

According to [15], the GLG density function is introduced in Definition 1, with parameters μ, σ, and q. The gamma function (denoted as Γ) is used with the argument q^{-2}, where μ and σ represent the mean and standard deviation, respectively.

Definition 1. *For all $x > 0$, $\sigma > 0$, $\mu \in \mathbb{R}$ and $w = x - \mu)/\sigma$, let GLG be the density function with parametersμ, σ and q, defined by*

$$f(x|\mu,\sigma,q) = \begin{cases} \frac{|q|\left(q^{-2}\right)^{q^{-2}}}{x\sigma\ \Gamma\left(q^{-2}\right)} \exp\left\{q^{-2}[qw - \exp(qw)]\right\}; q \neq 0 \\ \frac{\left(\frac{1}{2}\right)^{\frac{1}{2}}}{x\sigma\sqrt{\pi}} \exp\left[-\frac{(\ln x - \mu)^2}{2\sigma^2}\right]; \qquad q = 0 \end{cases} \tag{1}$$

Note that if a random variable U has a Gamma distribution with shape parameter q^{-2} ($q \neq 0$) and scale parameter 1, $U \sim Gamma\left(q^{-2}, 1\right)$, then,

$$X = \exp\left[\mu + \sigma\frac{\ln\left(q^2 U\right)}{q}\right]$$

follows a GLG distribution, with parameters μ, σ, q, and the X support Ω_X, for $X \sim GLG(\mu, \sigma, q)$, is giving by $\Omega_X = \mathbb{R}^+$. Indeed, suppose that the variable U assumes a particular value $u_a > 0$. In this case, X will take a particular value $x_a = \exp\left\{\mu + \sigma\left[\ln\left(q^2 u_a\right)/q\right]\right\}$, and it can be observed that

$$\lim_{u_a \to 0^+} x_a \quad \text{and} \quad \lim_{u_a \to +\infty} x_a$$

depend on whether the parameter q is positive or negative. Due to the above, for $q \neq 0$, it follows that

$$\lim_{u_a \to 0^+} x_a = \lim_{u_a \to 0^+} \exp\left[\mu + \sigma \frac{\ln(q^2 u_a)}{q}\right] = \begin{cases} 0^+ & (q > 0) \\ \\ +\infty & (q < 0) \end{cases} \tag{2}$$

using properties of logarithms and limits,

$$\lim_{u_a \to +\infty} x_a = \exp\left(\mu + \frac{2\sigma \ln|q|}{q}\right) \lim_{u_a \to +\infty} \exp\left(\frac{\sigma \ln u_a}{q}\right) = \begin{cases} +\infty & (q > 0) \\ \\ 0^+ & (q < 0) \end{cases} \tag{3}$$

In addition, the function $x_a = \exp\{\mu + \sigma[\ln(q^2 u_a)/q]\}$ is continuous for all $u_a > 0$.

- Case 1: $q > 0$, x_a is increasing with u_a, and assume all values from $\lim_{u_a \to 0^+} x_a = 0^+$ to $\lim_{u_a \to +\infty} x_a = +\infty$
- Case 2: $q < 0$, x_a decreases with u_a, and assume all values from $\lim_{u_a \to 0^+} x_a = +\infty$ to $\lim_{u_a \to +\infty} x_a = 0^+$

Both results allow us to conclude that x_a takes any positive value, so the support of X is given by $\Omega_X = \mathbb{R}^+$.

2.2. Particular Cases of Generalized Log Gamma Distribution

In this section, four special cases of the GLG distribution are discussed. First, it is shown that the generalized gamma distribution corresponds to a reparameterization of GLG for positive values of q [15]. Then, two brief subsections are presented, where gamma and Weibull are written according to GLG. Finally, a log-normal distribution is referred to as a GLG special case when $q \to 0$.

2.2.1. Generalized Gamma Distribution

The generalized gamma distribution was introduced by [13] and can be seen as the half-part of GLG for $q > 0$. When $q < 0$, it is not possible to establish a parametric correspondence between the two distributions. Thus, it can be said that GLG is a generalized gamma extension.

A random variable X follows a generalized gamma distribution, with a positive scale parameter β and two positive shape parameters d and n, if the correspondent density function is given via (4) as follows:

$$f(x|\beta, d, n) = \frac{\left(\frac{n}{\beta^d}\right) x^{d-1} \exp\left[-\left(\frac{x}{\beta}\right)^n\right]}{\Gamma\left(\frac{d}{n}\right)} \tag{4}$$

for positive values of x. With $\gamma = d/n$, the density function in (4) can be rewritten as

$$f(x|\beta, \gamma, n) = \frac{\left(\frac{n}{\beta^{n\gamma}}\right) x^{n\gamma-1} \exp\left[-\left(\frac{x}{\beta}\right)^n\right]}{\Gamma(\gamma)} \tag{5}$$

for positive values of x, β, γ and n.

Proposition 1. *Consider $X \sim GLG(\mu, \sigma, q)$. If $q > 0$, then X also follows a generalized gamma distribution with scale parameter β and shape parameters γ and n. The two sets of parameters are related using*

$$\begin{cases} \mu = ln\beta + ln\gamma/n \\ q = 1/\sqrt{\gamma} \\ \sigma = 1/n\sqrt{\gamma} \end{cases} \tag{6}$$

and

$$\begin{cases} \beta = exp[\mu + 2(\sigma/q)lnq] \\ \gamma = 1/q^2 \\ n = q/\sigma \end{cases} . \tag{7}$$

Proof. Note that the generalized gamma density function can be rewritten as Equation (1). Hence, GLG and generalized gamma sets of parameters can match. Rewriting the density function described in (7) yields the following:

$$f(x|\beta,\gamma,n) = \frac{\left(\frac{n}{\beta^{n\gamma}}\right)x^{n\gamma-1}\exp\left[-\left(\frac{x}{\beta}\right)^n\right]}{\Gamma(\gamma)} = \frac{n\exp\{n\gamma s - \exp[ns]\}}{x\,\Gamma(\gamma)}$$

It is known that $s = \ln x - \ln \beta$, which means the natural logarithm of the scale parameter β is already isolated, allowing to achieve an equivalency with GLG density. The next step consists of adding and subtracting $\ln \gamma/n$ to the expression of s. The objective is to wrap the two shape parameters γ and n with the scale parameter β. This is related to the introduction of the GLG density, which incorporates three parameters as arguments for an exponential function. Hence, considering $s = \ln x - \ln \beta$ and $t = \ln x - \left(\ln \beta + \frac{\ln \gamma}{n}\right)$,

$$\begin{aligned} f(x|\beta,\gamma,n) &= \frac{n\ \exp\left\{n\gamma\left(\ln x - \ln \beta - \frac{\ln\gamma}{n} + \frac{\ln\gamma}{n}\right) - \exp\left[n\left(\ln x - \ln\beta - \frac{\ln\gamma}{n} + \frac{\ln\gamma}{n}\right)\right]\right\}}{x\ \Gamma(\gamma)} \\ &= \frac{n\ \exp(\ln\gamma^{\gamma})\exp[n\gamma t - \gamma\ \exp(nt)]}{x\ \Gamma(\gamma)} \end{aligned}$$

Finally, it follows that:

$$f(x|\beta,\gamma,n) = \frac{n\gamma^{\gamma}\exp\{\gamma[nt - \exp(nt)]\}}{x\,\Gamma(\gamma)} = \frac{n\gamma^{\gamma}}{x\,\Gamma(\gamma)}\exp\{\gamma[nt - \exp(nt)]\}$$

Considering $\gamma = q^{-2}$ and

$$qw = q(\ln x - \mu)/\sigma = \underbrace{n}_{q/\sigma}\left[\ln x - \underbrace{(\ln\beta + \ln\gamma/n)}_{\mu}\right]$$

We can conclude that the two sets of parameters are related according to a system of equations solvable for the parameters of GLG:

$$\begin{cases} \mu = \ln\beta + \ln\gamma/n \\ q^{-2} = \gamma \\ q/\sigma = n \\ \frac{|q|\left(q^{-2}\right)^{q^{-2}}}{\sigma\,\Gamma(q^{-2})} = \frac{n\gamma^{\gamma}}{\Gamma(\gamma)} \end{cases} \Leftrightarrow \begin{cases} \mu = \ln\beta + \ln\gamma/n \\ q = 1/\sqrt{\gamma} \\ \sigma = q/n \\ \frac{|q|n\left(q^{-2}\right)^{q^{-2}}}{q\,\Gamma(q^{-2})} = \frac{n\left(q^{-2}\right)^{q^{-2}}}{\Gamma(q^{-2})} \end{cases} \Leftrightarrow \begin{cases} \mu = \ln\beta + \ln\gamma/n \\ q = 1/\sqrt{\gamma} \\ \sigma = 1/n\sqrt{\gamma} \\ \frac{|q|}{q} = 1 \end{cases}$$

Note that $n = q/\sigma$ is a sufficient condition to prove $q > 0$, due to the parameters n and σ being positive. Consequently, $q^2 = 1/\gamma$ is equivalent to $q = 1/\sqrt{\gamma}$ and $|q| = q$ and the relationship indicated in the equation system (6) is obtained. To achieve the result described in (7), it is sufficient to solve the equation system (6) for the generalized gamma parameters β, γ and n

$$\begin{cases} \mu = \ln\beta + \ln\gamma/n \\ q = 1/\sqrt{\gamma} \\ \sigma = 1/n\sqrt{\gamma} \end{cases} \Leftrightarrow \begin{cases} \ln\beta = \mu - \ln\gamma/n \\ \gamma = q^{-2} \\ n\sqrt{\gamma} = 1/\sigma \end{cases} \Leftrightarrow \begin{cases} \beta = exp[\mu + 2(\sigma/q)\ln q] \\ \gamma = 1/q^2 \\ n = q/\sigma \end{cases}$$

Therefore, the generalized gamma is a GLG special case when $q > 0$. □

Proposition 1 allows us to frame the gamma and Weibull distributions as two GLG special cases with a positive q.

2.2.2. Gamma Distribution

As mentioned in [37], the gamma distribution was first used by Laplace in 1836 in Laplace's theory of errors. Two years later, Bienaymé introduced a Bayesian study of linear shapes in multinomial variables. In 1844, Ellis obtained the gamma distribution as a sum of exponential random variables. The authors of [38] mentioned that the gamma distribution naturally appeared in the theory associated with normal distributed variables. It is known that the gamma distribution is a generalized gamma special case with a unitary shape parameter n. Then, replacing n by 1 in (5), the correspondent density function is obtained as follows:

$$f(x|\beta,\gamma)) = \frac{\left(\frac{1}{\beta}\right)^{\gamma} x^{\gamma-1} \exp\left(-\frac{x}{\beta}\right)}{\Gamma(\gamma)} \tag{8}$$

where $x > 0$. Scale parameter β and shape parameter γ are both positive. Considering that the gamma distribution is a special case of the generalized gamma with a unitary shape parameter n, after replacing this parameter in the system (6), we can obtain the following:

$$\begin{cases} \mu = \ln\beta + \ln\gamma \\ q = 1/\sqrt{\gamma} \\ \sigma = 1/\sqrt{\gamma} \end{cases} \tag{9}$$

2.2.3. Weibull Distribution

The Weibull distribution was introduced by the Physicist Waloddi Weibull in 1939 for modeling material resistance. At the beginning of the 1950s, this parametric model was directed at a wide range of applications. According to [38], Weibull is notable for the attention he has received in the scientific community since the early 1970s. It can be observed that the Weibull distribution is a special case of generalized gamma with a unit shape parameter γ. Replacing this parameter by 1 in Equation (5), we can obtain the density function as follows:

$$f(x|\beta,n) = \frac{n}{\beta}\left(\frac{x}{\beta}\right)^{n-1} \exp\left[-\left(\frac{x}{\beta}\right)^{n}\right] \tag{10}$$

where $x > 0$, β is a positive scale parameter, and n is a positive shape parameter.

Considering that the Weibull distribution is a generalized gamma special case with a unitary shape parameter γ, by replacing this parameter with 1 in the system (6), we can obtain the following:

$$\begin{cases} \mu = \ln\beta + \ln 1/n \\ q = 1/\sqrt{1} \\ \sigma = 1/n\sqrt{1} \end{cases} \Leftrightarrow \begin{cases} \mu = \ln\beta \\ q = 1 \\ \sigma = 1/n \end{cases} \tag{11}$$

2.2.4. Log-Normal Distribution

The pioneers in log-normal distribution development are referred to in [38]: Galton, e McAlister, Kapteyen, Van Uven, and Gibrat. In the economic sciences, this distribution is sometimes called the Cobb–Douglas distribution and has a particular interest in production function representations.

Log-normal can be introduced as a generalized log-gamma special case. In the study developed on the GLG creation, [15] started to refer to the generalized gamma density function described in Equation (5) and converged to the log-normal density function when the shape parameter $\gamma \to +\infty$. Starting from this premise, the author proved that the log-normal is a GLG special case when $q \to 0$. The theoretical contributions of other authors [39–41] were also relevant to achieving this result. So, the density function (1) can be obtained, where x and σ assume only positive values and $\mu \in R$. When it is necessary to model data with a very asymmetrical distribution on the right, the log-normal distribution is, within two-parameter models, one of the most adequate approaches.

2.3. Theoretical Contributions to the Study of the Generalized Log Gamma Distribution

A review of the literature developed over almost five decades on GLG shows that there is still room for theoretical developments on this parametric model. The statistical distribution proposed by [42] is the starting point for defining a wide family of distributions. However, its four parameters made it difficult to handle and apply in practice.

The generalized gamma distribution was introduced by [13] and consists of eliminating the location parameter defined by [42] while keeping the two shape parameters and the scale parameter. Three years later, ref. [14] advanced the methodology for estimating the parameters of the generalized gamma. Later, ref. [15] suggested the GLG, which consists of reparameterization and an extension of the result proposed by Stacy [13,14]. In fact, GLG allows negative values for the shape parameter q, proving to be more flexible than Stacy's proposal. The main advantage of Prentice's contribution is the ease of estimation using the maximum likelihood method.

Since [15], theoretical advances have been made, accompanied by relevant practical applications. In the late 1970s, ref. [43] showed that the extension of generalized gamma is linked to a family of parametric models and described likelihood-based methods to evaluate some special cases. In addition, they presented applications to various data sets, notably in the field of medicine. Inference methods directed at GLG made remarkable advances in 1987 [19]. In a short chapter on models with additional shape parameters, ref. [44] discusses GLG in his book on statistical models and methods for lifetime data. More recently, ref. [20] introduced robust GLG estimators.

Thus, building on pioneering works and other later theoretical contributions (such as some of the papers mentioned above), new theoretical contributions are presented in this research. Original expressions for the moments and mode of the distribution are proposed. In addition, some forms of GLG density are analyzed via graphical representations. The aim is to graphically highlight the mode and moments, as well as to show some cases where the absence of these measures appears. Finally, the expected value, variance, and mode are analyzed considering the value of q.

2.3.1. Theoretical Contributions

To obtain the general expression for the GLG distribution moments, see Lemma 1.

Lemma 1 (GLG Distribution Moments)**.** *Considering a positive integer number m, if $X \sim GLG(\mu, \sigma, q)$, then*

$$E(X^m) = \exp\left(m\mu + 2m\sigma\frac{ln|q|}{q}\right)\frac{\Gamma\left(q^{-2} + \frac{m\sigma}{q}\right)}{\Gamma(q^{-2})} \tag{12}$$

only if $q > -\frac{1}{m\sigma} \wedge q \neq 0$. When $m \to +\infty$, $E(X^m)$ only exists for $q > 0$.

Proof. Denoting $f_U(u|q)$ the density function of U,

$$f_U(u|q) = \frac{u^{q^{-2}-1}\exp(-u)}{\Gamma(q^{-2})} \qquad u > 0,\ \ q \neq 0,$$

where $\Gamma(a) = \int_0^{+\infty} u^{a-1} \exp(-u)du$ is the gamma function for a positive number a. To isolate the random variable U, one can rewrite the expression for X in the following form:

$$X = \exp\left[\mu + \sigma\frac{\ln\left(q^2U\right)}{q}\right] = \exp\left(\mu + \sigma\frac{\ln q^2}{q} + \frac{\sigma}{q}\ln U\right) = \\ = \exp\left(\mu + \sigma\frac{\ln q^2}{q}\right)\exp\left(\frac{\sigma}{q}\ln U\right) = \exp\left(\mu + \sigma\frac{\ln q^2}{q}\right)U^{\frac{\sigma}{q}}$$

So, GLG m moment is given via:

$$E(X^m) = E\left\{\left[\exp\left(\mu + \sigma\frac{\ln q^2}{q}\right)U^{\frac{\sigma}{q}}\right]^m\right\} = \exp\left(m\mu + 2m\sigma\frac{\ln|q|}{q}\right)E\left(U^{m\frac{\sigma}{q}}\right)$$

The expected value can now be calculated as follows:

$$E(X^m) = \underbrace{\exp\left(m\mu + 2m\sigma\frac{\ln|q|}{q}\right)}_{C}\int_0^{+\infty} u^{\frac{m\sigma}{q}} f_U(u|q)d \\ = C\int_0^{+\infty} u^{\frac{m\sigma}{q}}\,\frac{u^{q^{-2}-1}\exp(-u)}{\Gamma(q^{-2})}du = \\ = \frac{C}{\Gamma(q^{-2})}\int_0^{+\infty} u^{\left(q^{-2}+\frac{m\sigma}{q}\right)-1}\exp(-u)du = \frac{C}{\Gamma(q^{-2})}\Gamma\left(q^{-2} + \frac{m\sigma}{q}\right)$$

Finally,

$$E(X^m) = \exp\left(m\mu + 2m\sigma\frac{\ln|q|}{q}\right)\frac{\Gamma\left(q^{-2} + \frac{m\sigma}{q}\right)}{\Gamma(q^{-2})},$$

only if $q^{-2} + \frac{m\sigma}{q} > 0 \ \wedge\ q \neq 0$.

Otherwise, this moment is equal to $+\infty$, which implies the inexistence of $E(X^m)$. Simplifying the condition referred to above, with $q \neq 0$, yields

$$q^{-2} + \frac{m\sigma}{q} > 0 \ \Leftrightarrow \frac{1}{q^2} + \frac{m\sigma q}{q^2} > 0 \ \Leftrightarrow\ \frac{1+m\sigma q}{q^2} > 0 \\ \underbrace{\Leftrightarrow}_{q^2>0} 1 + m\sigma q > 0 \ \Leftrightarrow m\sigma q > -1 \\ \underbrace{\Leftrightarrow}_{\sigma>0} q > -\frac{1}{m\sigma} \wedge q \neq 0$$

Since $\lim\limits_{m\to+\infty} -\frac{1}{m\sigma} = 0$, when m is a sufficiently large number, the correspondent moment only exists for $q > 0$. □

Regarding the existence of the mode in the GLG distribution and how it provides useful statistical information (especially when the first movement does not exist), see Lemma 2.

Lemma 2 (GLG Distribution Mode). *Considering* $X \sim LGG(\mu,\sigma,q)$*. If* $q < \frac{1}{\sigma}$*, with* $q \neq 0$*, then the mode is given* via

$$Mo = exp\left[\mu + \frac{\sigma ln(1-\sigma q)}{q}\right]. \tag{13}$$

Otherwise, GLG density is a decreasing function.

Proof. To $q \neq 0$, rewriting the first branch of Equation (1) yields,

$$\begin{aligned} f(x|\mu,\sigma,q) &= \frac{|q|\left(q^{-2}\right)^{q^{-2}}}{x\sigma\,\Gamma(q^{-2})}\exp\left\{q^{-2}[qw - \exp(qw)]\right\} \\ &= \frac{|q|\left(q^{-2}\right)^{q^{-2}}}{x\sigma\,\Gamma(q^{-2})}\exp\left[q^{-1}w - q^{-2}\exp(qw)\right] \end{aligned}$$

Obtaining the first derivative of f respect to x,

$$\begin{aligned} \frac{\partial f}{\partial x} &= \left[\frac{|q|\left(q^{-2}\right)^{q^{-2}}}{x\sigma\,\Gamma(q^{-2})}\right]' \exp\left[q^{-1}w - q^{-2}\exp(qw)\right] \\ &+ \frac{|q|\left(q^{-2}\right)^{q^{-2}}}{x\sigma\,\Gamma(q^{-2})}\left\{\exp\left[q^{-1}w - q^{-2}\exp(qw)\right]\right\} \\ &= -\frac{|q|\left(q^{-2}\right)^{q^{-2}}}{x^2\sigma\,\Gamma(q^{-2})}\exp\left[q^{-1}w - q^{-2}\exp(qw)\right] \\ &+ \frac{|q|\left(q^{-2}\right)^{q^{-2}}}{x\sigma\,\Gamma(q^{-2})}\exp\left[q^{-1}w - q^{-2}\exp(qw)\right]\left[q^{-1}\frac{\partial w}{\partial x} - q^{-2}\exp(qw)q\frac{\partial w}{\partial x}\right] \end{aligned}$$

Changing the terms' order in the previous expression and considering $w = (\ln x - \mu)/\sigma$,

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{x|q|\left(q^{-2}\right)^{q^{-2}}}{x^2\;\sigma\;\Gamma(q^{-2})}\exp\left[q^{-1}w - q^{-2}\exp(qw)\right]\left[q^{-1}\frac{1}{\sigma x} - q^{-2}\exp(qw)q\frac{1}{\sigma x}\right] \\ &\quad - \frac{|q|\left(q^{-2}\right)^{q^{-2}}}{x^2\;\sigma\Gamma(q^{-2})}\exp\left[q^{-1}w - q^{-2}\exp(qw)\right] \\ &= \underbrace{\frac{|q|\left(q^{-2}\right)^{q^{-2}}}{x^2\;\sigma\Gamma(q^{-2})}}_{>0}\underbrace{\exp\left[q^{-1}w - q^{-2}\exp(qw)\right]}_{>0}\underbrace{\left[\frac{q^{-1}}{\sigma} - \frac{q^{-1}}{\sigma}\exp(qw) - 1\right]}_{D} \end{aligned}$$

Since $q \neq 0$, this derivative is only null when $D = 0$. This equation can be solved using the following:

$$D = 0 \Longleftrightarrow -\frac{q^{-1}}{\sigma}\exp\left(q\frac{\ln x - \mu}{\sigma}\right) = 1 - \frac{q^{-1}}{\sigma} \Longleftrightarrow \exp\left(q\frac{\ln x - \mu}{\sigma}\right) = 1 - \sigma q$$

This equation only has a solution when $1 - \sigma q > 0$, which is equivalent to $q < \frac{1}{\sigma}$. In this case,

$$q\frac{\ln x - \mu}{\sigma} = \ln(1 - \sigma q\Big) \Longleftrightarrow \ln x = \mu + \frac{\sigma\ln(1 - \sigma q)}{q}.$$

Finally, we can obtain,

$$x = \exp\left[\mu + \frac{\sigma\ln(1 - \sigma q)}{q}\right],\; q < \frac{1}{\sigma}.$$

Let us now solve the in equation $\partial f/\partial x > 0$, to find the values of x where GLG density is an increasing function.

This relationship is only observed when $D > 0$, so

$$\underbrace{\frac{q^{-1}}{\sigma} - \frac{q^{-1}}{\sigma}\exp(qw) - 1}_{D} > 0 \Longleftrightarrow -\frac{q^{-1}}{\sigma}\exp(qw) > 1 - \frac{q^{-1}}{\sigma} \Longleftrightarrow q^{-1}\exp\,(qw) < q^{-1} - \sigma \tag{14}$$

The resolution of in Equation (14) depends on q.

- Case 1: $q > 0$

If $q > 0$, the previous expression (14) is equivalent to

$$\exp(qw) < 1 - \sigma q. \tag{15}$$

For $1 - \sigma q \leq 0 \Leftrightarrow q \geq \frac{1}{\sigma}$, the in Equation (15) is impossible, so $\partial f/\partial x \leq 0$.

It was already proved that for the same values of q, $\partial f/\partial x \neq 0$, then $\partial f/\partial x < 0$. This means GLG density is a decreasing function when $q \geq \frac{1}{\sigma}$.

When $0 < q < \frac{1}{\sigma}$, in Equation (15) is equivalent to

$$qw < \ln(1 - \sigma q) \Longleftrightarrow q\underbrace{\frac{\ln x - \mu}{\sigma}}_{w} < \ln(1 - \sigma q) \Longleftrightarrow \ln x < \mu + \frac{\sigma \ln(1-\sigma q)}{q} \Longleftrightarrow$$
$$\Longleftrightarrow x < \exp\left[\mu + \frac{\sigma \ln(1-\sigma q)}{q}\right].$$

So, for $0 < q < \frac{1}{\sigma}$, GLG density increases when $x \leq \exp\left[\mu + \frac{\sigma \ln(1-\sigma q)}{q}\right]$ and decreases after that value, confirming that $x = \exp\left[\mu + \frac{\sigma \ln(1-\sigma q)}{q}\right]$ is the distribution mode.

- Case 2: $q < 0$

When $q < 0$, it can be concluded that $1 - \sigma q > 1$. So, in this case, the previous expression (14) is equivalent to

$$\exp(qw) > \underbrace{1 - \sigma q}_{>1} \Longleftrightarrow qw > \ln(1 - \sigma q) \Longleftrightarrow q\underbrace{\frac{\ln x - \mu}{\sigma}}_{w} > \ln(1 - \sigma q) \Longleftrightarrow$$
$$\Longleftrightarrow q(\ln x - \mu) > \sigma \ln(1 - \sigma q) \underbrace{\Longleftrightarrow}_{q<0} \ln x < \mu + \frac{\sigma \ln(1-\sigma q)}{q} \Longleftrightarrow$$
$$\Longleftrightarrow x < \exp\left[\mu + \frac{\sigma \ln(1-\sigma q)}{q}\right].$$

So, the same result for the $0 < q < \frac{1}{\sigma}$ case is obtained. The proof is now concluded. □

2.3.2. Density function: Graphical Representations

This section presents graphical representations of the GLG density function, considering three different situations.

- *First situation:* $-\frac{1}{\sigma} < q < \frac{1}{\sigma}$

The first case, when $-\frac{1}{\sigma} < q < \frac{1}{\sigma}$, corresponds to the case when expected values and modes exist. The two graphical representations in Figure 1 represent two GLG density functions, both considering $\mu = \sigma = 1$, where in Figure 1A $q = 0.5$ and Figure 1B refers to $q = -0.5$. Both distributions have a heavy tail on the right (characteristic of the GLG distribution), giving rise to a $Mo \approx 0.6796$ and $E(X) \approx 3.3979$ in the first case, and $Mo \approx 1.2081$ and $E(X) \approx 7.2488$ in the second case. The mode is higher when $q = -1/2$ and the density maximum value decreases. Furthermore, a negative q is associated with longer and thicker tails, corresponding to higher expected values.

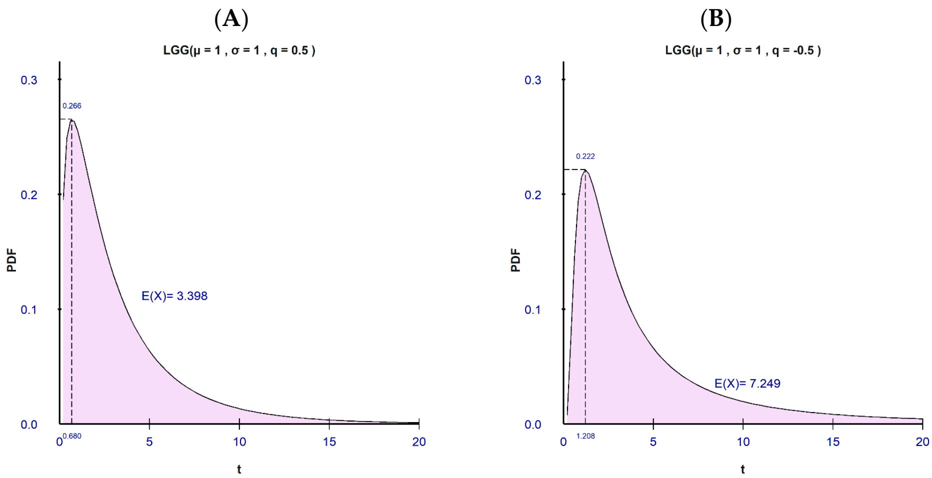

Figure 1. GLG density function (expected value and mode both exist): (**A**) q positive; (**B**) q negative.

- *Second situation:* $q \geq \frac{1}{\sigma}$

In this case, the GLG density is a decreasing function of x, and the expected value is a finite number. This situation occurs when $q \geq \frac{1}{\sigma}$. Figure 2 represents two GLG density functions, both considering $\mu = \sigma = 1$. Case Figure 2A considers $q = 1$, which originates $E(X) \approx 2.7183$. Case Figure 2B considers $q = 2$ and results in $E(X) \approx 1.8375$. From Figure 2, we can see that (considering the same values of μ and σ), when q increases and its difference from $\frac{1}{\sigma}$ becomes larger, the expected value tends to decrease.

- *Third situation:* $q \leq -\frac{1}{\sigma}$

Finally, $q \leq -\frac{1}{\sigma}$ is considered, which corresponds to the case without the expected value and mode equal to $\exp\left[\mu + \frac{\sigma \ln(1-\sigma q)}{q}\right]$. The two graphical representations in Figure 3 represent two GLG density functions, both considering $\mu = \sigma = 1$, where in Figure 3A $q = -1$, and Figure 3B refers to $q = -2$.

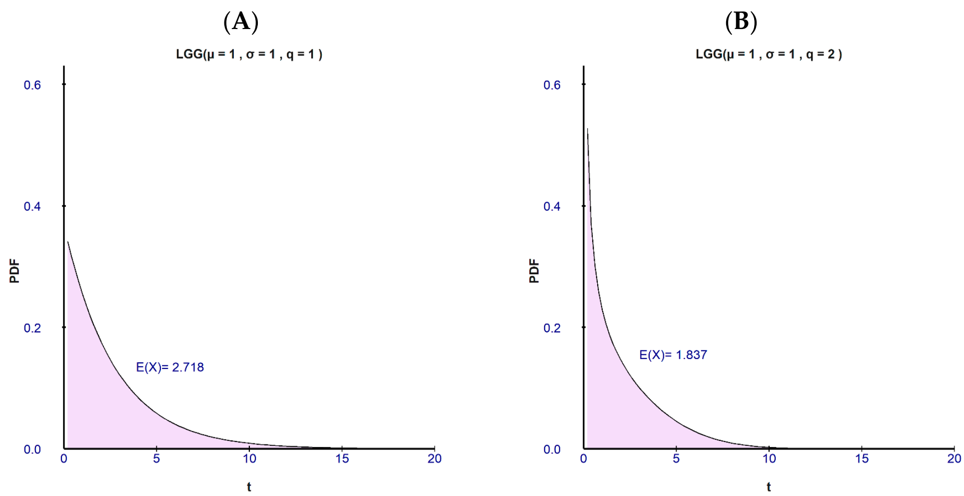

Figure 2. GLG density function (with expected value and without mode): (**A**) $q = 1$; (**B**) $q = 2$.

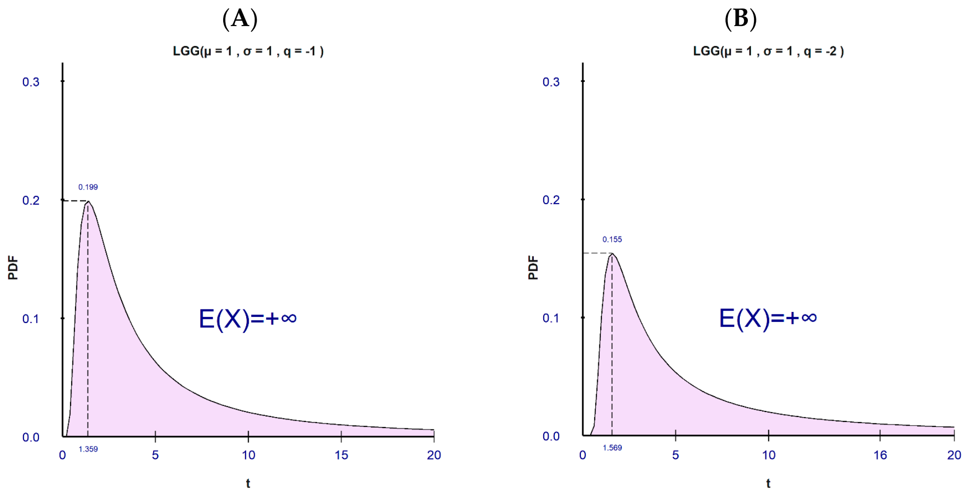

Figure 3. GLG density function (without expected value and with mode): (**A**) $q = -1$; (**B**) $q = -2$.

Both distributions (without expected value) have longer and thicker tails on the right, giving rise to a $Mo \approx 1.3591$ in the first case and $Mo \approx 1.5694$ in the second case.

2.3.3. Expected Value, Variance, and Mode as Functions of the Shape Parameter

To access some parameters of the GLG distribution (expected value, variance, and mode) as a function of the shape parameter q, three graphical representations are considered in Figure 4. All the measures consider $\mu = \sigma = 1$.

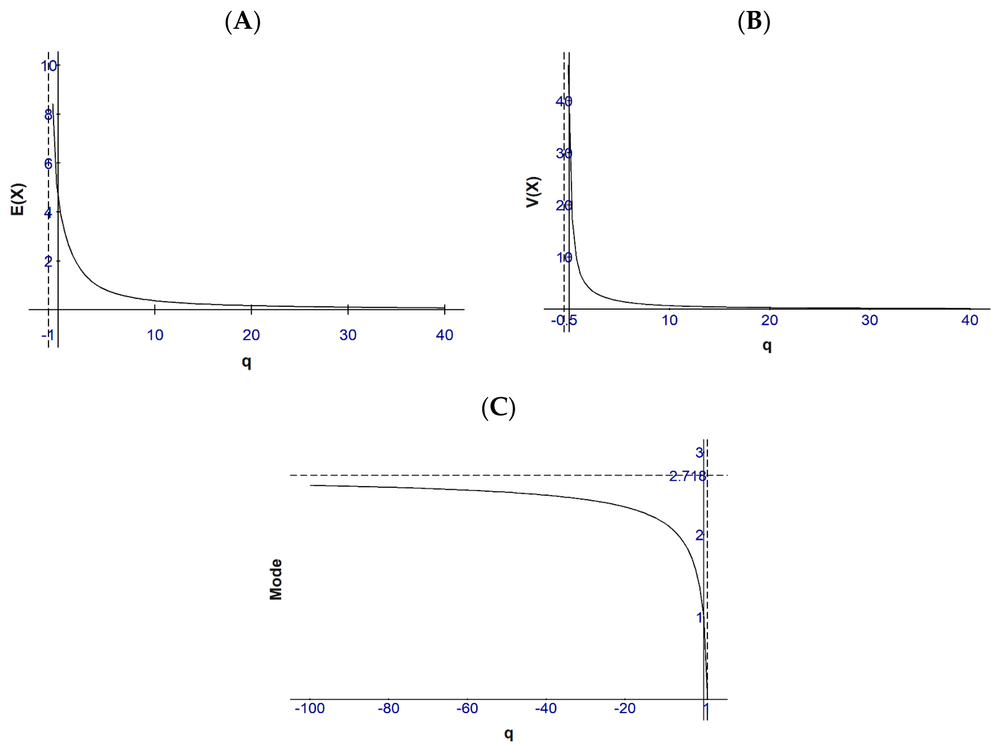

Figure 4. Parameters of the GLG distribution as a function of the shape parameter q: (**A**) Expected value; (**B**) Variance; (**C**) Mode.

Case Figure 4A and Case Figure 4B show that for unitary values of μ and σ, considering the result in Lemma 1, the expected value only exists for $q > -1$ and, since variance is equal to $E(X^2) - E(X)$, it can only be obtained for $q > -1/2$. It is possible to observe that the expected value and variance are lower for the highest values of q. Case Figure 4C represents the mode evolution as q varies. Considering the result in Lemma 2, a mode does not exist for $q \geq 1$. It can be observed that the lowest values of q are associated with higher mode values. When $q \to -\infty$, the mode approaches a finite value. Retaking the result in Lemma 2, the following limit can be computed:

$$\lim_{q \to -\infty} Mo = \lim_{q \to -\infty} \exp\left[\mu + \frac{\sigma \ln(1-\sigma q)}{q}\right] = \exp(\mu)\exp\left\{\overbrace{\lim_{q \to -\infty}\left[\frac{\sigma \ln(1-\sigma q)}{q}\right]}^{+\infty/-\infty}\right\} =$$

$$\underbrace{=}_{Cauchy's\ Rule} \exp(\mu)\exp\left[\lim_{q \to -\infty}\left(\frac{\sigma \frac{-\sigma}{1-\sigma q}}{1}\right)\right] = \exp(\mu)\exp\left(-\frac{\sigma^2}{+\infty}\right) = \exp(\mu)$$

So, when $q \to -\infty$, the mode approaches the location parameter exponentially. In Case Figure 4C, the horizontal asymptote is given by the equation $y = e$.

3. Data and Methods

This section begins by describing the data used in the empirical part of this paper. Next, reference is made to some methodological procedures, where the estimation methodology centered on the maximum likelihood method in the presence of censored data are discussed. As for the theoretical models involved in this research (probability distributions), the considerations and theoretical framework were presented in the previous section. As for the computational implementation, this was performed in the R programming language. The 4.3.1. R software packages used in this study are `ggplot2` [45], `survival` [46], and `flexsurv` [47]. Briefly, the survival package is used to incorporate the indication of whether it is an observed or censored lifetime into the firm's uptime. The `flexsurv` package is used to estimate the parametric models and obtain the point estimates and confidence intervals of the survival probabilities using the methods proposed by [48].

3.1. Data

To address the objective of applying the theoretical models discussed, life expectancy data from companies were used.

The dataset was provided by the Instituto Nacional de Estatística (INE) via the Integrated Business Accounting Systems—SCIE INE [49]. The developed analysis is based on the groups of Section J—Information and Communication; according to NACE Rev. 2 (European Classification of Economic Activities). The sectoral group on which this research focuses is G61—Telecommunications.

This study is based on the period 2004–2018. The fact that some enterprises were born and died during this period allows us to know and assess the survival time of a substantial number of enterprises. However, for the remaining ones, it is not possible to accurately determine their time on the market. Therefore, the presence of censored data are considered.

In Figure 5, four types of censoring are distinguished: (i) case 1 presents companies for which the period of activity is known; (ii) case 2 aggregates companies that emerged in the period 2004–2018 and remain in the market after that period; (iii) in case 3, organizations created before 2004 and activity ceased before 2018 were considered; (iv) case 4 corresponds to companies with activities started before 2004 and remained after 2018. This generic graphical approach was based on [27].

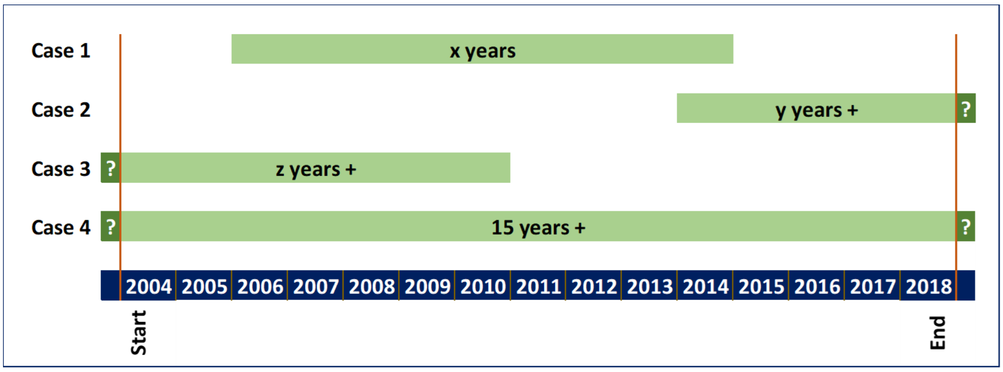

Figure 5. A graphic representation of the censorship typologies present in this study.

Table 1 presents the number of companies with observed and censored life expectancies (disaggregated by case) among the 1922 companies analyzed. The presented data indicate a censorship rate of around 47.45%. Considering the different types of censorship, most companies belong to Case 2; it is guaranteed that they will remain active after 2018. Companies belonging to Case 4 represent a minority, including the most consolidated companies in the economic sector.

Table 1. Number of companies with observed and censored life expectations.

Designation	Number
Companies with observed life expectations	1010
Companies with censored life expectations	912
Censored life expectations—Case 2	(714)
Censored life expectations—Case 3	(150)
Censored life expectations—Case 4	(48)

For more detailed information on non-censored firms (only firms with known times of activity are considered, corresponding to Case 1), some descriptive statistics on the number of years of activity are presented in Table 2.

Table 2. Number of years of activity of non-censored enterprises: descriptive statistics.

Count	Mean	Q1	Median	Q3	Max.
1010	3.3	2	3	4	14

For all individual enterprises considered, these enterprises stay in business between three and four years on average. Half of the companies do not reach 4 years (median = 3), which shows that these sector groups are relatively fragile and not very resilient. In fact, none of these companies fully covered the period 2004–2018. Table 3 specifies the duration of activity of the enterprises that were not fully observed, i.e., the enterprises included in Cases 2, 3, and 4 of Figure 5.

Table 3. Number of companies by censored times of activity.

Times of Activity (Censored)	Number		
	Case 2	Case 3	Case 4
1+	126	26	0
2+	107	22	0
3+	88	7	0
4+	76	24	0
5+	56	17	0
6+	55	8	0
7+	41	3	0
8+	49	8	0
9+	24	12	0
10+	17	7	0
11+	33	8	0
12+	10	4	0
13+	10	3	0
14+	14	1	0
15+	8	0	48

From the data presented, it is possible to notice a high number of enterprises that have only been observed for one or two years. This presents an additional challenge to the estimation process, given the scarcity of information available in these situations.

3.2. Methods: Estimation Methodology with Censored Data

Having presented the data under study in the present section, some considerations are made about the methodological methods and procedures followed in this research. Regarding the models used in this study, four parametric models are applied to a firm's life expectancy: GLG, gamma, Weibull, and log-normal.

Given the characteristics of the data under study (an asymmetric distribution with a heavy right tail), it is believed that the mentioned family of distributions could be a good option for modeling purposes (more details in Section 4). Before presenting the results, a more detailed description of the estimation methodology with censored data are necessary.

The estimation process only starts after determining the values of the variable c_i, which represents the number of activity years for the company i in the period 2004–2018. In Case 1 of Figure 5, c_i represents the total lifetime of a ceased enterprise during the analyzed period. In Case 2, the variable c_i may be considered the age of the company in 2018, that is, at the end of the period. In Cases 3 and 4, c_i is a minorant to companies' life expectancy. Hence,

$$c_i = Final\_year_i - Starting\_year_i + 1 \tag{16}$$

where $Starting_{year_i}$ and $Final\_year_i$ refer, respectively, to the first and last years of the activity of the company i, during the period 2004–2018. Knowing that $X_i = c_i + l_i$, with $l_i \in [0, +\infty]$, it can define a status variable δ_i, which assumes a value of 1 if X_i is an observed value and 0 otherwise. The most suitable estimation methodologies applied to the censored data consider the maximum likelihood method because it adjusts to the occurrence of censorship in each observation. However, the maximum likelihood method does not allow for the consideration of overly complex censorship mechanisms [50].

In the present analysis, the simplification focuses on conceptualizing Cases 2 to 4 of Figure 5 as right-censored observations. Some companies will only be observed until the censorship time c_i, and it can be guaranteed that $X_i > c_i$. It can observe the concretization of the random variables $(T_1, T_2, \ldots, T_n)$, where

$$T_i = \begin{cases} X_i, & if\ X_i \leq c_i \\ c_i, & if\ X_i > c_i \end{cases}. \tag{17}$$

Considering n companies, with censorship times $(c_1, c_2, \ldots, c_n)$ and observed activity times $(x_1, x_2, \ldots, x_n)$, each business individual is associated to the pair (x_i, c_i), and the lifetime expectation is only observed if $x_i \leq c_i$. Assuming the variables X_i are independent and identically distributed, with a probability density function $f(x)$ and a survival function $S(x)$, following the definition suggested by [51], data can be represented by n pairs of random variables (T_i, Δ_i), where

$$T_i = min(X_i, c_i) \text{ and } \Delta_i = \begin{cases} 1, & \text{if } X_i \leq c_i \\ \\ 0, & \text{if } X_i > c_i \end{cases}. \quad (18)$$

Considering a sample of (t_i, δ_i) obtained from the pair of random variables (T_i, Δ_i), we can verify that δ_i matches the binary variable status defined previously, that is, δ_i assumes the value 1, if the company lifetime expectancy is known, and zero, if that activity time is censored. Consequently, $T_i = X_i$ for the unitary status value and $T_i = c_i$ otherwise. On the basis that X_i are identically distributed and independent of the censorship, the random vector (T_i, Δ_i) is also identically distributed. Hence, it can be proved that the joint probability density function of (T, Δ) is given via

$$f_{T,\Delta}(t, \delta) = f\,(x)^{\delta} S(c)^{1-\delta} \quad (19)$$

where f and S are, respectively, the density and the survival functions of the random variable X [51]. If the pairs of vectors (T_i, Δ_i) are independent, the likelihood function can be written using

$$L = \prod_{i=1}^{n} f(x_i)^{\delta_i} S(c_i)^{1-\delta_i} \quad (20)$$

The parameter values that maximize L or $\ln L$ are the same. Hence,

$$\ln L = \sum_{i=1}^{n} \delta_i \ln f(x_i) + \sum_{i=1}^{n} (1 - \delta_i) \ln S(c_i) \quad (21)$$

Assuming the random variable X distribution is made up of p parameters $\left(\theta_1, \theta_2, \ldots, \theta_p\right)$, maximum likelihood estimators $\left(\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_p\right)$ are the solutions of the first order conditions, described on the system of Equation (22).

$$\begin{cases} \frac{\partial M}{\partial \theta_1} = 0 \\ \frac{\partial M}{\partial \theta_2} = 0 \\ \quad\vdots \\ \quad\vdots \\ \quad\vdots \\ \frac{\partial M}{\partial \theta_p} = 0 \end{cases}. \quad (22)$$

It should also be noted that for those optimal solutions, the second-order conditions must be satisfied, that is, the Hessian Matrix H (23) must be negative definite.

$$H = \begin{bmatrix} \frac{\partial^2 M}{\partial {\theta_1}^2} & \frac{\partial^2 M}{\partial \theta_1 \partial \theta_2} & \cdots & \frac{\partial^2 M}{\partial \theta_1 \partial \theta_p} \\ \frac{\partial^2 M}{\partial \theta_2 \partial \theta_1} & \frac{\partial^2 M}{\partial {\theta_2}^2} & \cdots & \frac{\partial^2 M}{\partial \theta_2 \partial \theta_p} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial^2 M}{\partial \theta_p \partial \theta_1} & \frac{\partial^2 M}{\partial \theta_p \partial \theta_2} & \cdots & \frac{\partial^2 M}{\partial {\theta_p}^2} \end{bmatrix} \quad (23)$$

4. Results

After the description of the data made in Section 3.1, this section is dedicated to the presentation of the results of the four considered models. To access the four parametric models, the respective parameters are estimated. The results are presented in Table 4.

Table 4. Estimation of model parameters with 95% confidence intervals (LCL and UCL) and moments.

Generalized Log Gamma				Gamma				Weibull				Log Normal			
PAR	EST	LCL	UCL	PAR	EST	LCL	UCL	PAR	EST	LCL	UCL	PAR	EST	LCL	UCL
μ	0.92	0.78	1.05	β	5.16	4.63	5.74	β	7.87	7.45	8.30	meanlog	1.65	1.59	1.70
σ	0.93	0.86	1.00	γ	1.41	1.31	1.52	n	1.18	1.12	1.23	sdlog	1.02	0.98	1.07
q	−1.69	−2.02	−1.36	-	-	-	-	-	-	-	-	-	-	-	-

Legend: PAR—Parameters; EST—Estimative; LCL/UCL—Lower/Upper confidence limits.

Figure 6 includes the probability density functions (PDF) resulting from the estimation process via the maximum likelihood method. The variable t refers to the activity time in years.

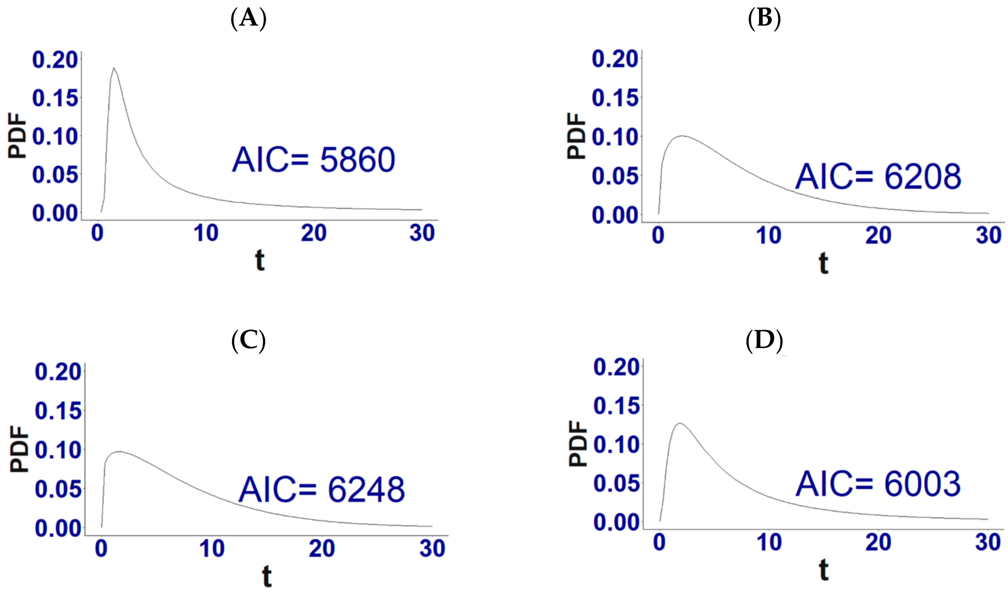

Figure 6. Density functions for the sectoral group G61: (**A**) Generalized log gamma; (**B**) Gamma; (**C**) Weibull; (**D**) Log normal.

As shown in Figure 6, the value of the Akaike Information Criterion (AIC) [52] is presented in each graphical representation. This criterion allows us to compare the model's quality in terms of goodness of fit, and the associated coefficient is defined as

$$AIC = -2L(\hat{\theta}_1,\ \hat{\theta}_2, \ldots,\ \hat{\theta}_k | X_1,\ X_2, \ldots, X_k) + 2k \tag{24}$$

where, $L(\theta_1,\ \theta_2, \ldots,\ \theta_k | X_1,\ X_2, \ldots, X_k)$ is the likelihood function of the model with k parameters based on a sample of size n and $\hat{\theta}_1,\ \hat{\theta}_2, \ldots,\ \hat{\theta}_k$ refer to the parameter estimators using the MLE method. In this work, the Akaike criterion has been used to evaluate the goodness of fit for each studied model. Regarding the calculated AIC values, the best model for the G61 sector group density function is the generalized Log Gamma. Nevertheless, the data set studied here presents a large amount of censored data (see Table 1). Therefore, in future work, we propose to study, validate, and implement new metrics for goodness of fit when building models with a large amount of censored data.

In addition, for an easier visualization of model contrasts, the PDF and survival function, $S(t)$, are overlappingly displayed in Figure 7.

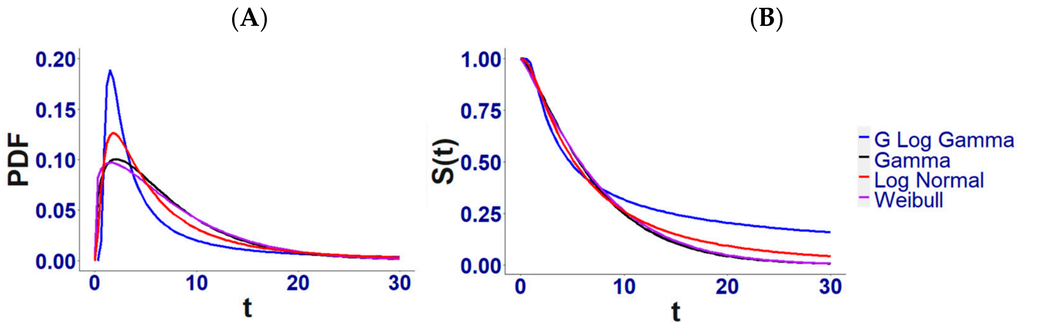

Figure 7. Graphical representations for the sectoral group G61: (**A**) Density function; (**B**) Survival function.

The GLG is the model that best fits our data. Considering the distributions of the two parameters, log-normal stands out relative to gamma and Weibull. As shown in Figure 7, it can be noted that GLG considers a higher failure probability in the first years of activity, best covering the most fragile parts of the companies. Regarding the long-term survival probabilities (higher than 30 years), GLG admits a more elevated life expectancy than the remaining models, more effectively incorporating the life expectancy of the most resilient companies. Comparing the four models and considering the AIC information, it is possible to order the four models from the best fit to the least fit: GLG, log-normal, gamma, and Weibull. After a detailed examination of gamma and Weibull parameterizations as particular cases of GLG, it can be observed that for these two cases, the parameter q assumes a positive value, while for the GLG estimated model, this parameter is negative. Then, it can be deduced that the condition $q > 0$ is a central reason for the worst results of the gamma and Weibull models. The most favorable result associated with the log-normal results corresponds to the closest proximity of a negative value for the parameter q, in this case, $q = 0$ (see Section 2.2).

Once the parametric models have been estimated and their quality assessed, one main application of this study is estimating the survival probabilities of companies. Table 5 contains the estimations and the confidence intervals (95%) of survival probabilities considering the four chosen parametric models.

Table 5. Survival probabilities estimates for the sectoral group G61 and 95% confidence intervals.

Time	Generalized Log Gamma			Gamma			Weibull			Log Normal		
	LCL	EST	UCL	LCL	EST	UCL	LCL	EST	UCL	LCL	EST	UCL
1	0.96	0.97	0.97	0.92	0.93	0.94	0.91	0.92	0.92	0.94	0.95	0.95
2	0.77	0.79	0.81	0.82	0.83	0.85	0.80	0.82	0.83	0.81	0.82	0.84
3	0.63	0.65	0.67	0.72	0.73	0.75	0.71	0.72	0.74	0.69	0.70	0.72
4	0.53	0.55	0.58	0.62	0.64	0.66	0.62	0.64	0.66	0.58	0.60	0.62
5	0.47	0.49	0.51	0.53	0.55	0.57	0.54	0.56	0.58	0.49	0.52	0.54
6	0.41	0.44	0.46	0.45	0.47	0.50	0.46	0.48	0.50	0.42	0.44	0.47
7	0.37	0.40	0.42	0.38	0.41	0.43	0.40	0.42	0.44	0.36	0.39	0.41
8	0.34	0.37	0.39	0.32	0.35	0.37	0.34	0.36	0.38	0.31	0.34	0.36
9	0.31	0.34	0.36	0.27	0.29	0.32	0.29	0.31	0.33	0.27	0.30	0.32
10	0.29	0.32	0.34	0.23	0.25	0.27	0.24	0.27	0.29	0.24	0.26	0.28
11	0.27	0.30	0.33	0.19	0.21	0.24	0.20	0.23	0.25	0.21	0.23	0.25
12	0.26	0.28	0.31	0.16	0.18	0.20	0.17	0.19	0.22	0.19	0.21	0.23
13	0.24	0.27	0.30	0.13	0.15	0.17	0.14	0.16	0.19	0.17	0.19	0.21
14	0.23	0.26	0.28	0.11	0.13	0.15	0.12	0.14	0.16	0.15	0.17	0.19
15	0.22	0.25	0.27	0.09	0.11	0.13	0.10	0.12	0.14	0.13	0.15	0.17

Legend: EST—Estimative; LCL/UCL—Lower/Upper confidence limits.

From the table, we can observe that the survival probability is lower than 50%, starting in the fifth/sixth year. The two parameter models (gamma, Weibull, and log-normal) recorded less optimistic survival probability estimations, which are more pronounced in gamma and Weibull. Furthermore, note that after the eighth year, it can be observed that the estimated probabilities for the GLG distribution are always higher than those of the other distributions (gamma, Weibull, and log-normal), providing more optimal survival probabilities associated, for example, with the strongest companies in the market.

5. Discussion and Conclusions

Business competitiveness is nothing new. However, because of an increasingly global environment, it is becoming more and more evident. This situation, combined with other constraints, contributes to many companies not being able to resist and prematurely closing their operations. The business fabric of the Information and Communication sector is no exception. To better understand this phenomenon (from an analysis and prediction perspective), it is important to identify the mathematical/statistical models that fit this type of data behavior. In light of the above information, when assessing the life span of enterprises, particularly in this business area, the literature points to a marked and unbalanced heterogeneity in terms of proportion.

A considerable number of firms do not exceed four or five years of activity, while a small part remains active for many years. This leads to a strongly skewed probabilistic distribution of lifetime, with a heavy tail on the right. In the scientific literature, several probability distributions with these characteristics can be identified. Among these possibilities is the GLG probability distribution, for which the literature identifies other distributions as possible cases (gamma, Weibull, and log-normal distributions). We believe that by using GLG, we have modeled the resilience of the telecommunications sector in Portugal in a way that has never been seen before. This study employs the Akaike criterion to assess the goodness of fit of each model. However, the dataset under investigation contains a substantial volume of censored data (see Table 1). Therefore, in future work, we suggest exploring, validating, and incorporating new metrics to evaluate the goodness of fit when working with large censored datasets. To apply GLG to a dataset referring to the life span of companies, which is not yet reported in the literature, a theoretical study of this parametric distribution (and consequent family of distributions) was necessary. This work identified some gaps in the literature. In the theoretical developments, expressions for the moments and mode of the GLG distribution are presented. From this study, it is possible to conclude that the existence of moments centered at the origin rarely arises for a negative q. Regarding the mode, it exists whenever q is less than the scale parameter inverse. It was inferred that the interval $-1/\sigma < q < 1/\sigma$ enables the existence of a mode and an expected value. When the parameters μ and σ are fixed, it can be observed that the GLG expected value, variance, and mode decrease when the shape parameter q is higher. Furthermore, negative values of q are associated with longer and thicker density function tails. This approach using GLG, which allows the q parameter to be less than zero, will give a new impetus to modeling company lifetimes.

To achieve the objectives related to the empirical side of this research, four parametric models were applied to a dataset on the lifetime of firms in the groups of Section J—Information and Communication. For a more robust analysis, given some difficulties in the dataset, a censoring methodology is presented in this research. Based on the state of the art, the originality of the application of GLG models to the life expectancy of companies can be reported. The developed study points out GLG models as a promising solution to model the longevity of firms in the sector. Given that there is a strong presence of censored data, the maximum likelihood method reveals the best estimation alternative. In addition, it should be noted that the model's complexity, combined with the data treatment exigencies, implies the use of sophisticated optimization algorithms. In the project development, the `flexsurvreg` function from the library `flexsurv` of the R statistical software made the estimation process possible using the BFGS (Broyden, Fletcher, Goldfarb, and Shanno) optimization algorithms, which are characterized by being a Quasi-Newton method that numerically calculates the second derivatives of the Hessian Matrix. It can be said that there are no convergency problems in the estimation methodology developed. All the models show probabilities of surviving more than 15 years, far below the censorship rate. This can be assumed to be a positive indicator regarding the estimation process because the censorship rates represent a major factor in the probability of surviving after 15 years. In this case, the analysis period has a fifteen-year duration. However, in the future, this horizon may be extended by including more data, which will represent a valuable addition to the conclusions reported in this study.

For this sectorial group, the estimated parameter q assumed a negative value, which justifies the non-use of generalized gamma, restricted to a positive value of q. Concerning modeling the life expectancy of companies, it is possible to see that GLG is more suitable than its three special cases with two parameters. The complexity and behavior of a lifetime require a three-parameter model that can characterize the high probability of failure in the first years of activity for the most fragile companies and, simultaneously, highlight the long-life expectancy of the most resilient ones. Although the presence of censored data has created new challenges in the estimation process, it was possible to provide a set of strong results, which made those conclusions possible. Considering the GLG model, the probability of a company surviving ten or more years is 31.9%. The same model shows that nearly 50% of companies survive for more than five years. Based on GLG estimates, the probability of surviving fifteen years or more is almost 25%. Considering the above, it is worth mentioning that this study is a starting point for an in-depth understanding of the success and failure records of companies and can be applied to any sectoral group in domestic or non-domestic economies. The consolidation of the conclusions referred to in this study depends on data collection, and it is worth noting the ambition to extend the analysis to a broader period in the future. However, in addition to the theoretical contributions highlighted, the practical parts of this research highlight the following contributions: (i) it allows us to compare the resilience between different sectors of the economy (for example, we could compare the resilience of the telecommunications sector with the press sector); (ii) it allows us to compare the resilience of the same sector in two different time periods: for example, comparing 2004–2018 with 2008–2022 to see if there is a statistically significant difference between these periods.

Thus, trusting that this work will shed light on the subject, some limitations in this present research are recognized: (i) a reductionist view of reality, modeling the expected life of firms without considering factors that may be determinant for their survival (e.g., initial investment, capital stock, number of employees, impact on the economy); (ii) the information used in this study has some limitations, as the database does not include information on possible mergers and reorganizations of companies, which may lead to one-off biases; (iii) making inferences about the suitability of these models since they have only been tested on a specific dataset. In addition, future work should explore improvements in modeling this phenomenon, namely (i) evaluating the possibility of using other models that better fit the data analyzed, (ii) testing these models on another data set, (iii) improving the dataset by including information on firms that can be used as additional variables in studying the life expectancy of firms, and (iv) considering regression models that allow for a more robust evaluation and more accurate forecasting.

In summary, the contributions of this research are acknowledged, although some limitations are noted, some of which are beyond the scope of this study. In addition to the application made to a real data set, the theoretical contributions made provide contributions to the subject and to the scientific debate. If we use this distribution with less censored data and some associated explanatory variables, we may have found the right path to the ultimate goal: a malleable probability distribution with tails that can adapt to life expectancies close to infinity and, at the same time, adapt to very high early death rates.

Author Contributions: All authors (J.H.D.G., J.J.F.G., L.R. and F.R.R.) contributed to the study's conception. Individual contributions: conceptualization, J.H.D.G. and F.R.R.; literature review, J.H.D.G., L.R. and F.R.R.; theoretical/mathematical frameworks, J.H.D.G., J.J.F.G. and F.R.R.; methodology, J.H.D.G. and J.J.F.G.; formal analysis, L.R. and F.R.R.; computational implementations, J.H.D.G. and J.J.F.G.; data analysis, J.H.D.G., J.J.F.G., L.R. and F.R.R.; writing—original draft preparation, J.H.D.G.

and F.R.R.; writing—review and editing, F.R.R.; final validation, F.R.R. All authors have read and agreed to the published version of the manuscript.

Funding: L.R. is supported by the Universidad del Norte. J.J.F.G. and F.R.R. are financed by national funds through FCT—Fundação para a Ciência e a Tecnologia under the project UIDB/00006/2020.

Data Availability Statement: The data supporting the results of this study are available from the National Institute of Statistics (INE, https://www.ine.pt). However, there are restrictions on the availability of these data, which were used under license for this study and are not publicly available. Nevertheless, the data are available to the authors upon reasonable request and with authorization from INE.

Acknowledgments: This work is partially financed by national funds through FCT—Fundação para a Ciência e a Tecnologia under the project UIDB/00006/2020. J.H.D.G dedicates this working paper to her mother, Maria Madalena. The same author would like to thank the employer institution—the Portuguese Regulatory Authority for the Media (ERC)—for their timetable flexibility, which was essential to this scientific project development. The authors would like to thank Miguel Graça Balbina and Beatriz Nogueira Sequeira for their suggestions. The authors express gratitude to Statistics Portugal for making data available from the Integrated Business Accounting Systems (SCIE).

Conflicts of Interest: The authors declare no conflict of interest. The funders had no role in the design of this study, in the collection, analysis, or interpretation of data, in the writing of this manuscript, or in the decision to publish these results.

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