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On the Positive Recurrence of Finite Regenerative Stochastic Models

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Abstract: We consider a general approach to establish the positive recurrence (stability) of regenerative stochastic systems. The approach is based on the renewal theory and a characterization of the remaining renewal time of the embedded renewal process generated by regeneration. We discuss how this analysis is simplified for some classes of the stochastic systems. The general approach is then illustrated by the stability analysis of a k-out-of-n repairable system containing n unreliable components with exponential lifetimes. Then we extend the stability analysis to the system with non-exponential lifetimes.

Keywords: regenerative stochastic system; k-out-of-n repairable system; stability; unreliable component

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1. Introduction

In this research, we discuss the stability problem of a wide class of regenerative systems, that is, the systems of the dynamics described by the so-called *stochastic regenerative process*. Then we show how the regenerative stability analysis method can be applied to establish the existence of the stationary distribution of a repairable system containing n components which, as we will show, has the regeneration property. In general, the existence of a stationary regime is in high demand and often a challenging problem, and different sophisticated methods are applied to resolve the problem. An overview of various stability analysis methods can be found, for instance, in Ref. [1].

It is a well-known fact [2] that the finiteness of the mean regeneration period length (or, equivalently, the mean distance between two adjacent regeneration instances) is the main ingredient of the stability analysis of the regenerative processes. This property (that is, the finiteness of the mean) is called *positive recurrence* by an analogy, which is used in the analysis of the conventional Markov chains (for the terminology, see Refs. [3,4]). Indeed, under this condition (and under an additional technical requirement, see below) the regenerative process has the stationary distribution, see for instance Refs. [2,5]. To calculate the stationary performance characteristics of any system it is quite important to first establish the corresponding stability conditions. In the case if the dynamics of the system is described by a regenerative (basic) process then, as we just mentioned, to verify the positive recurrence, as defined, we must verify the finiteness of the mean regeneration period length. As a rule, we cannot claim this finiteness in advance because the basic process is not a predefined (governing) process but is a result of a complex interaction of the given governing processes. For queuing systems, the latter are, for instance, the sequences of the inter-arrival and service times while for the unreliable systems it may be the sequences of the life- and repair times. Moreover, in general the proof of the finiteness of the mean regeneration period length turns out often to be quite a challenging problem. In this regard we refer to the recent book [3] where this problem is considered in detail, and the applications of the regenerative stability analysis to various classes of the regenerative queueing models are presented as well.

The main contribution of this research is two-fold. First, we present in brief a methodology allowing to prove the positive recurrence of a wide class of regenerative stochastic systems. Then we focus on the stability analysis of k-out-of-n repairable system belonging to a wide class of *finite systems*. In each such system a basic process $\{X(t),\ t \geq 0\}$ either has a *finite state space* or is *tight*. The latter property (for one-dimensional X) is formulated as follows: for every $\epsilon > 0$, there exists a constant C such that $\inf_{t\geq 0} \mathbf{P}(|X(t)| \leq C) \geq 1 - \epsilon$. For instance, in a finite queueing system, either the total number of customers is finite or the workload process is tight. The approach we apply in this research is a modification of a general method developed in the monograph [3] for the regenerative queues. This method is based on the renewal theory and a characterization of the limiting property of the remaining regeneration time at instant t as $t \to \infty$.

In this research, we demonstrate how this approach allows to simplify the proof of the positive recurrence of the finite k-out-of-n repairable system studied in Ref. [6] on the base of the so-called *semi-regenerative processes*. The analysis in Ref. [6] allowed us to calculate both stationary and non-stationary performance indexes in such a model, while the problem of the existence of the stationary distribution and hence the stationary performance characteristics received less attention. We note that the basic results of the theory of semi-regenerative processes can be found in the survey paper [7]. In particular, we first show how the regenerative method allows to to prove, relatively simply, both the positive recurrence of the basic regenerative process and the existence of its stationary distribution when lifetimes are exponential and then we extend the stability analysis to the repairable system with *non-exponential lifetimes*, provided the repair times are assumed to be unbounded. In this new setting, instead of the exponentiality, the tightness of the remaining repair and lifetimes turn out to be critically important.

Thus, as we show, the regenerative approach is a powerful and effective alternative method which allows us to establish stability (stationarity) of a wide class of the stochastic systems and in particular the finite systems to which the system considered in this work belongs to.

The paper is organized as follows. In the next Section 2, we give the main necessary results from the theory of regenerative processes and formulate the general approach to verification of the positive recurrence of a renewal process composed of regeneration points. In particular, we here formulate a key result from the renewal theory (Proposition 1) concerning the limiting characterization of the remaining regeneration time, which is a base for the regenerative stability analysis. Then, in Section 3, we give a detailed description of the k-out-of-n system, including basic notations and assumptions, and then prove our main stability result (Theorem 1), stating the positive recurrence of this system. As we show, the latter result is a direct corollary of Proposition 1. Finally, in Section 4, we analyze the stability of the k-out-of-n repairable system with *non-exponential* lifetimes. As we will show, this important generalization can be easily studied by an analogy with the original system with exponential lifetimes. To the best of our knowledge, the latter system is considered for the first time. Some other generalizations are discussed as well.

We assume that the readers are aware of the main results of the theory of regenerative processes; otherwise, we may advise looking at the following basic monographs [2,8].

In what follows, we omit serial index to denote the generic element of a sequence of the independent identically distributed (iid) random variables.

2. Theoretical Background of Regenerative Process

In this section, we first shortly recall a few basic definitions from the theory of regenerative processes, which are used below. Then we present a characterization of the remaining regeneration time, which is the key ingredient of the regenerative stability analysis we use in this research. First, we discuss a general case and then focus on the analysis of a finite stochastic system. Finally, we discuss how the finiteness allows for simplifying the stability analysis. It is useful to note that a detailed introduction to the theory of regenerative processes can be found, for instance, in the following fundamental monographs [2,8,9] and also in the pioneering work of Ref [10]. Before proceeding, we emphasize that the class of stochastic regenerative processes is very wide and includes, in particular, conventional (countable) Markov chains and the so-called Harris Markov chains and processes, see Remark 1 below.

A stochastic process is called *(classical) regenerative* if its path can be split into independent identically distributed (iid) random elements called *regeneration cycles*. The instances which separate these cycles are called *regeneration instants*.

The k-out-of-n system we consider below as the main target model provides an important example in which the basic process describing the dynamics of the system turns out to be classical regenerative. However, right now we mention, as an important example of the regenerative process, the workload (unfinished work) in the $GI/G/1$ queueing system with iid interarrival times and iid service times (for notation see, for example, Ref. [2]). In this system, the regenerations of the workload process (and the queue size) are generated by the newly arriving customers that meet a completely idle system. More formally, we consider a right-continuous stochastic process $\{X(t),\ t \geq 0\}$, with a metric state space. A more detailed description can be found in [2,4,8]. We assume that there are (random) instants $0 = T_0 < T_1 < T_2 < \cdots$ with $T_n \to \infty$ with probability 1 (w.p.1), and define the n-th cycle of the process $\{X(t)\}$ as

$$G_n := \{X(T_n + t) :\ 0 \leq t < T_{n+1} - T_n\},\ \ n \geq 0\,,$$

where $T_{n+1} - T_n$ is the n-th cycle length.

Definition 1 ([8])**.** *The process $\{X(t)\}$ is called zero-delayed classically regenerative if the cycles, $\{T_{n+1} - T_n,\ G_n,\ n \geq 0\}$, are iid. The times $\{T_n\}$ are called regeneration instants.*

Remark 1. *Sometimes the cycle lengths are not included in Definition 1 because they in general are functions of the cycles.*

The regenerative process is called delayed if the initial cycle $\{T_1,\ G_0\}$ has another distribution, see Ref. [2], however we will not consider this case in this research. See Sections 2.3 and 2.5 in Ref. [3] on how to analyze a delayed regenerative queueing process.

Remark 2. *As it follows from Ref. [11], indeed the existence of the first regeneration time instant T_1 implies the existence of an infinite sequence of the regeneration instants. In this regard, we also mention the recent paper [12] in which a similar result is proved for Harris Markov chains and processes. However, the regenerations that can be constructed in Harris Markov chains in general are not classical ones but rather one-dependent regenerations. A detailed discussion related to construction of one-dependent regenerations in the Harris Markov chains in the queueing context is presented in Refs. [13–15] .*

Recall that the regeneration instants are denoted by $\{T_n\}$ and denote by T the generic regeneration period length. It means that T is distributed as any difference $T_{n+1} - T_n$ provided $T_{n+1} < \infty$. Let

$$T(t) = \min_n (T_n - t : \; T_n - t > 0)$$

be the remaining regeneration time at instant t.

As it was mentioned above, our purpose is to find conditions which imply the finiteness of the mean regeneration period, that is

$$\mathbf{E}[T_{n+1} - T_n] \equiv \mathbf{E}[T] < \infty. \tag{1}$$

We shortly explain why this condition is so important in the stability analysis of the regenerative processes. A detailed analysis can be found in Refs. [2,3]. Assume that we would like to calculate a limiting performance measure $f(X(t))$ of a (non-negative) continuous-time regenerative process $\{X(t)\}$ with regeneration points $\{T_n\}$ and the generic length T, where f is a measurable function. Note that then the process $\{f(X(t)\}$ is also regenerative with the same regeneration epochs [2]. Denote the maximum of the regenerative process $\{f(X(t))\}$ over a (generic) regeneration cycle by $M = \sup_{0 \le t < T} |f(X(t)|$. Then, provided $\mathbf{E}[T] < \infty$ (and under additional assumptions $\mathbf{E}[MT] < \infty$, $\mathbf{E}[M] < \infty$, see Chapter 2 in Ref. [8] and Theorem 1.1 in Ref. [3]), the target performance measure is calculated as the (w.p.1) limit

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t f(X(u))du = \frac{\mathbf{E}[\int_0^T f(X(t))dt]}{\mathbf{E}[T]}. \tag{2}$$

The limit (2) is also called the time average almost surely sample-path limit, see Ref. [4]. If, moreover, the cycle length T is non-lattice (or non-arithmetic, that is, not concentrated on a set $\{d, 2d, \ldots\}$ w.p.1, for any $d > 0$) then the right hand side (r.h.s). of (2) is also the stationary distribution of $f(X(t))$ as $t \to \infty$. These results confirm a crucial importance of the finiteness of the mean regeneration period length for the stability of a regenerative process. It is known that if $\mathbf{E}[T] = \infty$ then the remaining regeneration time $T(t) \Rightarrow \infty$ where indicator $\Rightarrow$ denotes convergence in probability, see Formula (4.11) in Chapter XI in Ref. [5]. In other words, in this case, for each $x \ge 0$,

$$\lim_{t \to \infty} \mathbf{P}\{T(t) > x\} = 1. \tag{3}$$

Thus, if (3) does not hold then $\mathbf{E}[T] < \infty$ and, to establish the finiteness of the mean regeneration length, it is enough to show that $T(t) \not\Rightarrow \infty$. This leads to the key condition which we may verify to establish the required finiteness in (1) and which we formulate, for convenience, as the following statement. In this regard, see Ref. [2].

Proposition 1. Consider a (zero-delayed) classically regenerative process $\{f(X(t))\}$ with regeneration cycle length T and the remaining regeneration time $T(t)$ at instant t. If there exist a deterministic sequence of times $z_i \to \infty$, and constants $D < \infty$ and $\varepsilon > 0$ such that

$$\inf_i \mathbf{P}\{T(z_i) \le D\} \ge \varepsilon, \tag{4}$$

then (1) holds and the process $\{f(X(t))\}$ is *positive recurrent*. Moreover, if the random variable T is spread-out (that is, a convolution of the distribution of T with itself has an absolutely continuous component) then also convergence in distribution $f(X(t))$ to the stationary version $\{f(X)\}$, as $t \to \infty$, takes place.

Remark 3. *It is easy to show that assumption $T(t) \not\Rightarrow \infty$ can be written as condition (4) which in turn is the most convenient in practice to verify the positive recurrence of a concrete model. There are numerous stochastic models in Ref. [3] (mainly, queueing systems) in which positive recurrence is established using condition (4). It follows that verification of condition (4) is a key step of the regenerative stability analysis.*

In what follows, for easy notation, we will consider a regenerative process $\{X(t)\}$. In practice, to verify condition (4) we first must show that a similar condition holds true for the basic process itself, that is

$$\inf_i \mathbf{P}\{X(u_i) \le D_1\} \ge \varepsilon_1, \tag{5}$$

for a deterministic sequence $u_i \to \infty$ and some finite positive constants D_1, ε_1. For example, in the conventional $GI/G/1$ queueing system assumption (5) is satisfied under condition $\rho \equiv \mathbf{E}[S]/\mathbf{E}[\tau] < 1$, where S and τ are the generic service time and generic inter-arrival time, respectively, see for instance, Ref. [3]. To pass from (5) to the desired inequality (4), we need one more assumption which guarantees that (assuming for simplicity only that the process $\{X(t)\}$ is one-dimensional), being in the compact $[0, D_1]$ at an instant u_i, the process visits a regeneration state (in the queueing setting, it is typically the state $\{0\}$) with a positive probability within a finite period of time. At that both the lower bound and the length of period do not depend on time, see the right hand side of (4) and (5). However, it is exactly the required inequality (4). We note that in the mentioned classical $GI/G/1$ queueing model, the latter step is achieved due to multiple use of the so-called regeneration condition $\mathbf{P}\{\tau > S\} > 0$, which follows from the condition $\rho < 1$. (In a multiserver system, the regeneration condition must be introduced additionally [3]).

We call the unloading procedure the step leading from condition (5) to condition (4). In the context of this paper, it is worth mentioning that the unloading procedure can be typically applied if an exponential distribution is used to describe the dynamics of the system under consideration.

If the system under consideration is then finite, by definition, the basic process is either finite or tight. For instance, in the queueing setting, such a process might be the number of customers or the unfinished work (accumulated work). In this case, the inequality (5) for a basic process holds automatically. It is intuitive that a tight process may, with a positive probability, fluctuate within a bounded set infinitely long by not visiting a regeneration state, in which case the basic requirement is violated. The simplest confirming example is a single-server closed queueing system in which we circulate $N > 1$ customers and the service time is not exponential. In this system, a customer leaving the system immediately returns for a new service, and the system will never idle. One can show that in such a system classical regeneration can not be constructed. In addition, the corresponding examples related to the closed queueing networks can be found in Section 10.3 of Ref. [3]. However, under very mild assumptions—for instance, if regeneration condition is satisfied or an exponential governing random variable exists—a tight regenerative process turns out to be positive recurrent. We show this below, considering a finite repairable k-out-of-n system.

Remark 4. *Once the finiteness of $\mathbf{E}[T]$ is proven, then the remaining regeneration time process $\{T(t)\}$ is indeed tight, see for instance, Chapter 3 in Ref. [3].*

3. Stability Analysis of a Repairable k-out-of-n System: Exponential Lifetimes

First, we give a detailed description of the model. Namely, we consider a repairable k-out-of-n system, which is functioning as follows. The system consists of n non-reliable identical (stochastically) components that are failed time to time and then are restored by a single repair device. This device repairs the failed components in FIFO order, and after repair they become as good as new. A key feature is that the whole system fails (occurs in the 'fail state') when any (arbitrary) $k \geq 1$ components fail. This property explains the name of this model. There are numerous papers devoted to the analysis of this model see, for instance, the papers mentioned above [6,7], which contain a brief review of the research on this topic. In addition, various aspects of the analysis of this system can be found in Refs. [16–19] (the authors thank a referee who pointed out the latter sources). Indeed, as we mentioned above, the main purpose of this research is to describe the general methodology of stability analysis with focus on the finite systems. In this regard, the analysis of the "k-out-of-n" repairable model presents an illustration of this approach, and we do not pretend to give a detailed review of such models and to discuss their properties and applications.

The specific feature of the system we consider below is that we allow two possible policies of the repair after the failure of the system happens. Namely,

(i) A partial repair: whenever the first failed component (among k failed) becomes available after repair, the system resumes the (standard) functioning, while the device continues to repair the remaining failed components, if there are any. We stress that in this scenario, immediately after the number of the available components exceeds n-k, the system becomes active and resumes the work;
(ii) The full repair, in which case the device repairs the entire system as a whole and, after the repair, the system resumes to work as a 'new one'.

We denote by $\{A_i\}$ the iid lifetimes, which are assumed to be exponential with parameter α. For partial repair scenario, the iid repair times are denoted by $\{B_i\}$ with a predefined distribution $B(t) = \mathbf{P}\{B_i \leq t\}$ and a finite mean, while in the full repair scenario, the iid repair times, denoted by $\{G_i\}$, have in general another general distribution $G(t) = \mathbf{P}\{G_i \leq t\}$ with a finite mean. In accordance with our agreement, in what follows, by A, B, G we denote the generic (exponential) time up to a failure, the generic repair time (for partial repair) and the repair time for the full scenario repair, respectively.

Now we consider the basic process $\{X(t),\ t \geq 0\}$ where $X(t)$ denotes the number of the failed components at the instant t and recall that, by assumption, $X(0) = 0$. It is easy to understand that the process $\{X(t)\}$ is classically regenerative, and its regeneration instants $\{T_i\}$ are recursively defined by the following way:

$$T_0 = 0,\ \ T_{i+1} = \min(t > T_i : X(t) = 0),\ i \geq 1.$$

We note that this regeneration property holds true because each time the process $\{X(t)\}$ visits the state $\{0\}$ the functioning of the system (stochastically) restarts anew, in particular, since the time up to failure are assumed to be exponential. As above (see Section 2), we denote by T the generic regeneration period length.

Our purpose is to construct a positive lower bound of the target probability $\mathbf{P}\{T(t) \leq x\}$ which is independent of time instant t. In other words, this bound is uniform in t. Then it is immediate that $T(t) \not\Rightarrow \infty$ (in probability), that is, (4) holds and the desired result $\mathbf{E}[T] < \infty$ follows.

Now we prove the following stability result.

Theorem 1. *Assume that $X(0) = 0$. Then the regenerative process $\{X(t)\}$ describing the repairable k-out-of-n system with exponential lifetimes and general repair times is a positive recurrent. Moreover, the stationary distribution of the process exists as $t \to \infty$.*

Proof. To prove the theorem, we construct a positive lower bound of the target probability $\mathbf{P}\{T(t) \leq x\}$, which is independent of time instant t. To this end, we define by $\xi_c(t),\ \xi_p(t)$ the remaining repair time at instant t, provided the complete restore or partial restore is performed at instant t, respectively. By definition, $\xi_c(t) = 0$ or $\xi_p(t) = 0$, if no complete repair, respectively, partial repair is performed at the instant t. Denote by $\varphi(t) = \max(\xi_c(t),\ \xi_p(t))$. An important step of the subsequent stability analysis is that, provided $\mathbf{E}[B] < \infty$, $\mathbf{E}[G] < \infty$, the family $\{\varphi(t),\ t \geq 0\}$ is tight, that is, for each $\varepsilon > 0$ there exists a constant $c < \infty$ such that

$$\inf_{t\geq 0} \mathbf{P}\{\varphi(t) \leq c\} \geq 1 - \varepsilon.$$

A detailed proof of the tightness can be found in Chapter 3 from Ref. [3], but we will now outline the proof for easy reading. The basic idea of the proof is adapted from Ref. [20], which then has been developed in Ref. [21]. Consider, for instance, the family $\{\xi_p(t)\}$. To prove its tightness, we first construct a (classic) renewal process generated by the iid partial repair times $\{B_i\}$. It is easy to show that the remaining renewal time at an instant t in the new renewal process, denoted by $\xi_o(t)$, is a tight process [3]. A key difficulty in the further analysis is caused by the fact that the original remaining time $\xi_p(t)$ and the remaining time $\xi_o(t)$ in the new renewal process are not directly comparable. Indeed, $\xi_p(t)$ is a shifted (remaining) renewal time in the new renewal process by the quantity $I(t)$ which is the idle time, within interval $[0,\ t]$, when $\xi_p(t) = 0$, that is when no partial repairs are made (again see Ref. [3]). Thus, we can not directly deduce the tightness of the original process $\{\xi_p(t)\}$ from the (known) tightness of the process $\{\xi_o(t)\}$. Instead, a delicate construction proposed in Ref. [20] is used to show that a 'distance' between $\xi_o(t)$ and the shifted original remaining time $\xi_p(t - I(t))$ is a tight process as well. The remaining details can be found in Section 3.1 of Chapter 3 from Ref. [3]. The tightness of the family $\{\xi_c(t)\}$ is then proved similarly. Thus, the two-dimensional family $\{(\xi_c(t),\ \xi_p(t)),\ t \geq 0\}$ is a tight process, and the tightness of the family $\{\varphi(t)\}$ follows immediately from the tightness of the process $\{(\xi_c(t) + \xi_p(t))\}$.

To construct the mentioned lower bound we write, for arbitrary $x,\ c > 0$, the following inequality:

$$\mathbf{P}\{T(t) \leq x + c\} \geq \sum_{i=0}^{k} \mathbf{P}\Big\{T(t) \leq x + c,\ X(t) = i,\ \ \varphi(t) \leq c\Big\}.$$

It is worth mentioning that, in the event $\{X(t) = i\}$, a regeneration happens in an interval $[t,\ t + x + c]$ if all i failed components are restored and no component (including restored ones) fails in this interval. It then follows from (2) that

$$\begin{aligned}\mathbf{P}\{T(t) \leq x + c\} \quad \geq \quad & \sum_{i=0}^{k} \mathbf{P}\Big\{\varphi(t) + B_1 + \cdots + B_{i-1} \leq x + c; \\ & A_j \geq x + c,\ j = 1, \ldots, n; X(t) = i\Big\},\end{aligned} \tag{6}$$

where B_i, A_i denote stochastic (independent) copies of B and A, respectively (and $B_{-1} = B_0 = 0$ by definition). We recall that $\varphi(t) = 0$ on the event $\{X(t) = 0\}$, that is when all components are working at the instant t. We only make inequality (6) stronger if we replace the sum $B_1 + \cdots + B_{i-1}$ by the sum $B_1 + \cdots + B_k$ with $i \leq k \leq n$. Then (6) leads to

$$\begin{aligned}\mathbf{P}\{T(t) \leq x + c\} \quad \geq \quad & \sum_{i=0}^{k} \mathbf{P}\{\varphi(t) + B_1 + \cdots + B_k \leq x + c; \\ & A_j \geq x + c,\ j = 1, \ldots, n; X(t) = i\} \\ \geq \quad & \mathbf{P}\{\varphi(t) + B_1 + \cdots + B_n \leq x + c;\ A_j \geq x + c,\ j = 1, \ldots, n\} \\ \geq \quad & \mathbf{P}\{\varphi(t) \leq c,\ B_i \leq x/n,\ i = 1, \ldots, n;\ \ A_j \geq x + c,\ j = 1, \ldots, n\} \\ = \quad & \mathbf{P}\{\varphi(t) \leq c\}[B(x/n)]^n e^{-\alpha n(x+c)},\end{aligned} \tag{7}$$

where we use the independence of $\varphi(t)$, the repair times $\{B_i\}$, and the exponential lifetimes times $\{A_j\}$ (with parameter α). Now we take an arbitrary $\varepsilon \in (0, 1)$ and select x_0, c_0 such that

$$B\Big(\frac{x_0}{n}\Big) \geq 1 - \varepsilon, \quad \inf_{t \geq 0} \mathbf{P}\{\varphi(t) \leq c_0\} \geq 1 - \varepsilon,$$

which is possible by the finiteness of $\mathbf{E}[B]$ and the tightness of $\{\varphi(t)\}$, respectively. Then it follows from (7) that, for all $t \geq 0$, $x \geq x_0$, $c \geq c_0$,

$$\mathbf{P}\{T(t) \leq x + c\} \geq (1 - \varepsilon)^{n+1} e^{-\alpha n (x_0 + c_0)} =: \delta > 0,$$

where, we emphasize, that the bound δ is independent of the time instant t, that is

$$\inf_{t \geq 0} \mathbf{P}\{T(t) \leq D\} \geq \delta, \tag{8}$$

for each constant $D \geq x_0 + c_0$. Thus, $T(t) \not\Rightarrow \infty$ (in probability) and it follows from (4) that $\mathbf{E}[T] < \infty$. In other words, the basic process $\{X(t)\}$ is positive recurrent classically regenerative and, because the (generic) regeneration cycle length T contains, with a positive probability, the (absolutely continuous) exponential lifetime then the stationary distribution of the process $\{X(t)\}$, as $t \to \infty$, exists as well [2,3]. $\square$

Remark 5. *There is a similarity between the proof of Theorem 1 and the corresponding proof for a closed queueing network, in which we circulate a finite number of customers. In the latter case, we construct a lower bound of the probability that all customers are collected in a single-server station (it is possible under some extra technical assumptions). This event then leads to a regeneration of the closed network (see Section 10.3 in Ref. [3]).*

The proof of Theorem 1 is short and rather simple and, in our opinion, this result clearly demonstrates that the presented regenerative analysis is a power tool of the stability analysis of complex stochastic systems. Indeed, the approach we apply in this research allows to simplify stability analysis even within the standard regenerative methodology. For instance, consider a finite $M/G/1$ queueing system, with Poisson input with rate λ and generic interarrival time τ, in which an arriving customer (at instant t) is admitted to the system if his service time S and the accumulated work $V(t)$ (workload) are bounded as $S + V(t) \leq C$ where C is a given finite constant (capacity), see Ref. [22]. Because the remaining regeneration time $T(t)$ (up to the system empties) satisfies,

$$\inf_{t \geq 0} \mathbf{P}\{T(t) \leq C\} \geq \mathbf{P}\{\tau \geq C\} = e^{-\lambda C}, \tag{9}$$

we immediately conclude, by previous argument, that the workload process is positive recurrent. Moreover, our analysis is directly extended to the system in which the bound C may be a state-dependent finite (w.p.1) random variable. On the other hand, the establishing bound (9) is only a first step of the regenerative stability analysis presented in Ref. [22].

4. The Repairable System with Non-Exponential Lifetimes

Now we present an extension of the developed above approach to a k-out-of-n repairable system in which the lifetime in general is not exponential. In this model, the only full repair generates classical regeneration of the process $\{X(t)\}$. It is because the remaining lifetimes at instant t, denoted by $A_i(t)$, have unknown distributions at the instants (if they exist)when more than n-k components are in workable states. As a result, the distribution of the basic process at such instances remains unknown as well. It is in a contrast with the case when lifetime A is exponential in which case, using the memoryless property, we replace any variable $A_i(t)$ by an independent stochastic copy of A keeping the distribution of the basic process (cf. Ref. (6)). W denote now by B_t the remaining repair time at instant t

($B_t = 0$ if no component under repair). Additionally. we put $A_i(t) = 0$ if the ith component is being under repair at instant t.

Theorem 2. *Assume that $X(0) = 0$, the repair time B is unbounded and the distribution of the full repair time G is non-arithmetic. Then the statement of Theorem 1 holds true, that the process $\{X(t)\}$ is positive recurrent and has the stationary distribution.*

Proof. It is worth mentioning that in this case we will construct a lower bound of the probability that a full failure happens, unlike the basic model with exponential lifetime A, where we estimate the probability that all components become workable (repaired). Keeping the main previous notation, we define the following event:

$$\mathcal{E}(t) = \{B_t \le c, \max_{1\le i\le n} A_i(t) \le x_0, A \le y_0, B > x_0 + y_0 + c\},$$

where c, x_0, y_0 are arbitrary positive constants. The event $\mathcal{E}(t)$ can be interpreted as follows: the current repair (if any) is finished in the interval $[t, t + c]$, all components stop to work in the interval $[t, t + x_0]$, the lifetime A of the first component starting to work after the instant t is upper bounded by the constant y_0 and, finally, the time B of the first repair starting after instant t exceeds $x_0 + y_0 + c$ that is, this repair finishes beyond the interval $[t, t + x_0 + y_0 + c]$. Because the device is blocked until it finishes a repair, then it is easy to see that, in the event $\mathcal{E}(t)$, there is an instant t_0, within the interval $[t, t + x_0 + y_0 + c]$, when k components are failed first time and thus a full repair begins. Then the next regeneration instant happens when the full repair time G finishes. Note that we can take t_0 as a regeneration instant as well. As a result, we obtain

$$\mathbf{P}(T(t) \le +x_0 + y_0 + c) \ge \mathbf{P}(\mathcal{E}(t)).$$

On the other hand, it follows by the independence that

$$\mathbf{P}(\mathcal{E}(t)) \ge \mathbf{P}(A \le y_0)\mathbf{P}(B > x_0 + y_0 + c)\mathbf{P}(B_t \le c, \max_{1\le i\le n} A_i(t) \le x_0).$$

Now we note that the tightness of $\{B_t\}$ and each remaining time $\{A_i(t)\}$ (and hence the tightness of $\{\max_{1\le i\le n} A_i(t)\}$) is established as in Theorem 1. It allows us to take constants c, x_0 such that the last multiplier in the r.h.s of (4) is positive. Because the random variable A is finite w.p.1, then we select constant y_0 in such a way that $\mathbf{P}(A \le y_0) > 0$. It remains to recall that the repair time is assumed to be unbounded, implying that the second multiplier in the r.h.s. of (4) is positive as well. Because all these multipliers are independent of time instant t, then we obtain a lower bound similar to (8). As to the existence of the limiting distribution of the process $\{X(t)\}$, it remains to remember that the full repair time G, being a part of any regeneration cycle, is non-arithmetic. Thus the regeneration cycle length is non-arithmetic as well, in which case the stationary distribution of the process exists [2], and the proof is completed. □

Remark 6. *We emphasize that in general the system considered in Theorem 1 is not a particular case of what we study in Theorem 2 because, in the former system, the repair time B may be bounded.*

Remark 7. *A detailed analysis of the proof of Theorem 2 shows that its statement stays valid for the non-homogeneous system with the non-identical components, namely with the component-dependent lifetimes and repair times. In this case, in the event $\mathcal{E}(t)$, B_t must be replaced by the maximum of (potential) remaining repair times at instant t, the variable A is replaced by the maximum of the lifetimes and the repair time B is replaced by the minimum of the repair times, over the set of numbers of the components.*

5. Conclusions

In this research, we present a general approach allowing us to establish the positive recurrence (stability) of the regenerative stochastic systems. The approach exploits a limiting property of the remaining renewal time in the embedded process generated by the regeneration points of the basic process describing the dynamics of the model. It allows us to establish the positive recurrence of a wide class of the stochastic models possessing the regeneration property. In this work, we in particular show how this approach simplifies the proof of the positive recurrence of finite systems (in which the basic processes are finite or tight), proving positive recurrence of a k-out-of-n repairable system. First, we analyze the system with exponential lifetimes and then extend the stability analysis to a system with non-exponential lifetimes and unbounded repair times. The latter setting is new and has not been considered earlier.

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Abbreviations

The following abbreviations are used in this manuscript:

iid	independent identically distributed
r.h.s.	right hand side
w.p.	with probability

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