# Lateral-Concentration Inhomogeneities in Flows of Suspensions of Rod-like Particles: The Approach of the Theory of Anisotropic Micropolar Fluid 

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#### Abstract

To tackle suspensions of particles of any shape, the thermodynamics of a Cosserat continuum are developed by the method suggested by Landau and Khalatnikov for the mathematical description of the super-fluidity of liquid 2 He . Such an approach allows us to take into account the rotation of particles and their form. The flows of suspensions of neutrally buoyant rod-like particles are considered in detail. These suspensions include linear polymer solutions, FD-virus and worm-like micelles. The anisotropy of the suspensions is determined through the inclusion of the micro-inertia tensor in the rheological constitutive equations. The theory predicts gradient banding, temporal volatility of apparent viscosity and hysteresis of the flux-pressure curve. The transition from the isotropic phase to the nematic phase is also captured. Our mathematical model predicts the formation of flock-like inhomogeneities of concentration jointly with the hindrance effect.


Keywords: suspensions; rod-like particles; micropolar fluid; thermodynamics; hysteresis; concentration inhomogeneity

MSC: 76A05

## 1. Introduction

In many suspensions, particles are not spherical. Some complex fluids can even be treated as suspensions of fibers [1]. Examples include worm-like micelles, FD-virus, liquid crystals and etc. Such fluids enjoy unusual properties like shear banding [2-4], flow instabilities [5], appearance of a nematic phase [6] and formation of flocks [7,8]. In the present paper, thermodynamics and rheology of suspensions of neutrally-buoyant particles with rather general form are developed. To this end, the notion of the Cosserat continuum and the theory of micropolar fluids are applied. This allows to take into account the rotation of particles and their shape using the microinertia tensor. In addition to microrotation and microinertia, such fluids can support the nonsymmetric stress tensor and the couple stress. A new mathematical model is formulated that takes into account concentrated suspensions and their anisotropy. As for suspensions of rod-like particles, shear flows of the Poiseuille type were studied and the indicated properties characteristic of such suspensions were obtained. The practice of transporting oil in pipelines shows that reaching the optimal operating mode depends on the history of pumping [9]. It is proved that hysteresis occurs and it underlies this effect.

In [10], flows were considered in which the heterogeneity of the concentration of non-spherical particles can be neglected. Here, a more general mathematical model has been developed that takes into account concentration inhomogeneities in suspensions of such particles. Particularly the effect of the cluster formation is addressed as far as the rod-like particles are concerned.

Research into flows with rod-shaped particles goes back to Jeffery's work on an ellipsoid carried by a shear fluid flow [11]. According to Jeffrey's theory, the particle periodically rotates in orbits that depend on its initial orientation. This method has been used in a number of kinematic models [12,13]. Applicability of the generalized Jeffery equations was confirmed by a number of experiments [14,15]. This approach has also spawned extensions that take into account rods interactions $[16,17]$ and predict fiber orientation distributions in molded parts [18].

Typically, the rheological equations of fluids are determined in experiments with shear flows [19]. For FD-viruses, the dependence of viscosity on concentration and shear rate was studied in [20]. Using special friction coefficients, a formula for viscosity was derived in [20]. More details on the viscosity formulas are provided in [1,19,21]. Within the framework of the conservation laws of continuum mechanics, the thermodynamics of suspensions of neutral-buoyant particles is developed here. The derived conservation laws fit particles of any form. However, a positive sign of entropy production is guaranteed only for axially symmetric particles.

The thermodynamic arguments of the present paper are based on the Landau-Khalatnikov mathematical modeling method applied to the superfluidity of liquid helium II [22,23]. In the case of reversible processes, this method allows one to unambiguously determine the forces of interaction between particles and fluid by harmonizing the energy conservation law, both with other conservation laws and with the second law of thermodynamics. For dissipative processes, the interaction forces are determined in accordance with the principles of irreversible de Groot-Masur thermodynamics. The Landau-Khalatnikov approach is applicable in the theory of multiphase flows in poroelastic media [24-26]. It turned out that the same method can be applied to suspensions, even if the particles are allowed to rotate $[27,28]$.

In the present article, the Landau-Khalatnikov method is generalized for suspensions with any particle shape and extended to the theory of micropolar fluids [29]. According to this theory, the number of particles is sufficiently large in any infinitesimal volume. Therefore, highly concentrated suspensions fall within the range of applicability of this approach. In particular, within the framework of the theory of micropolar fluid it is shown that the Segre-Silberberg effect is due to the rotation of particles [30]. This effect means that particles accumulate in a concentric annular region during laminar flow in a circular pipe [31]. Centrifugal sedimentation in suspensions is also due to the rotation of particles [32].

In mathematical models of suspensions, the microstructure and rotation of particles are usually neglected, and the difference in normal stresses is used to take into account the effect of anisotropy [33]. As in [29], the Cosserat continuum concept is applied here and the microinertia tensor is placed in the constitutive stress-strain laws to account for anisotropy. The constitutive equations of the theory of micropolar fluids involve several viscosities that determine the contribution of the symmetric and skew-symmetric parts of the velocity and spin gradients to stress and couple stress. There are skew-symmetric and anisotropic viscosities apart from the common shear viscosity. The issue of skew-symmetric viscosity has been studied in [32,34], but there are no publications on anisotropic viscosity yet.

The equations here are solved numerically only in the one-dimensional case, so comparisons with three-dimensional experiments are not given. But these calculations show that the introduced viscosities play a major role in explaining the effects of hysteresis, inhomogeneity of concentration, phase transition, and gradient banding.

In Section 2 the thermodynamics of a micropolar fluid is developed. Section 3 is devoted to the derivation of equations for one-dimensional Poiseuille-like flows from the obtained general conservation laws. The calculation results are provided in Section 4; they explain the phenomena described in the abstract.

## 2. Thermodynamics of Micropolar Fluids

### 2.1. Thermodynamics of Classical Compressible Fluids without Dissipation

We start from the thermodynamics of the Euler fluid. Let $E_{0}, V, M, \theta, p, S$ and $\mu$ stand for internal energy, volume, mass, absolute temperature, pressure, entropy and chemical potential, respectively. The Gibbs identity writes

$$
d E_{0}=\theta d S-p d V+\mu d M
$$

Let us move on to quantities per unit volume:

$$
e=\frac{E_{0}}{V}, \quad \eta=\frac{S}{V}, \quad \rho=\frac{M}{V}
$$

where $e, \eta$ and $\rho$ are specific internal energy, specific entropy and density, respectively. Observe that sometimes it is convenient to use the entropy assigned to the unit mass: $s=S / M=\eta / \rho$. One can formulate the Gibbs identity as follows:

$$
\frac{d V}{V}[-p+\mu \rho+\theta \eta-e]=d e-\theta d \eta-\mu d \rho
$$

The right-hand side of the above equality refers to the unit volume, so $[\cdots]=0$. Hence,

$$
\begin{equation*}
p=-e+\theta \eta+\mu \rho, \quad d e=\theta d \eta+\mu d \rho, \quad d p=\eta d \theta+\rho d \mu . \tag{1}
\end{equation*}
$$

It is assumed that the energy satisfies the state equation $e=e(\rho, \eta)$.
The Euler fluid flow equations are [23]

$$
\begin{equation*}
\dot{\rho}+\rho \operatorname{divv}=0, \quad \rho \dot{\mathbf{v}}=-\nabla p, \quad \dot{s}=0 \tag{2}
\end{equation*}
$$

where $\mathbf{v}$ is the fluid velocity and $\dot{s}$ stands for the material derivative:

$$
\begin{equation*}
\frac{d s}{d t} \equiv \dot{s}=s_{t}+(\mathbf{v} \cdot \nabla) s=s_{t}+v_{j} \frac{\partial s}{\partial x_{j}}, \quad \dot{v}_{i}=v_{i t}+(\mathbf{v} \cdot \nabla) v_{i} \tag{3}
\end{equation*}
$$

The third equation in (2) implies the adiabatic condition, and it is written in terms of the specific entropy $s$ related to mass unity.

By passing to the specific entropy $\eta=\rho s$, one can verify easily that Equation (2) can be written as the following system of conservation laws:

$$
\begin{equation*}
\rho_{t}+\operatorname{div} \mathbf{j}=0, \quad \mathbf{j}_{t}+\operatorname{div} \Pi=0, \quad \eta_{t}+\operatorname{div}(\eta \mathbf{v})=0 \tag{4}
\end{equation*}
$$

where

$$
\begin{equation*}
\mathbf{j}=\rho \mathbf{v}, \quad \Pi=p I+\mathbf{j} \otimes \mathbf{v}, \quad(\mathbf{j} \otimes \mathbf{v})_{k s}=j_{k} v_{s}, \quad(\operatorname{div} \Pi)_{i}=\partial \Pi_{i k} / \partial x_{k} \tag{5}
\end{equation*}
$$

Here, $\mathbf{j} \otimes \mathbf{v}$ is the tensor product of the vectors $\mathbf{j}$ and $\mathbf{v}, \Pi$ is the momentum flux tensor, $I$ is the identical matrix, $I_{i j}=\delta_{i j}$.

It is a consequence of (4) that the total energy $E=e+\mathbf{j} \cdot \mathbf{v} / 2$ satisfies the energy equation

$$
\begin{equation*}
E_{t}+\operatorname{div} \mathbf{Q}=0 \quad \text { with } \quad \mathbf{Q}=\mathbf{v}(p+E) \tag{6}
\end{equation*}
$$

here, the dot stands for the scalar product, $\mathbf{j} \cdot \mathbf{v}=j_{i} v_{i}$.

### 2.2. Thermodynamics of Micropolar Fluids without Dissipation

We consider the granular fluid as a Cosserat continuum [35]. It implies the following. Let $\mathbf{x}=\mathbf{x}(\boldsymbol{\xi}, t)$ denote the position vector of a material point with the Lagrange coordinates $(\boldsymbol{\xi}, t)$. We introduce the orthogonal tensor $\mathrm{G}(\boldsymbol{\xi}, t)$ to determine the orientation of three orthogonal director-vectors $\mathbf{d}_{i}(\boldsymbol{\xi}, t)(i=1,2,3)$ :

$$
\mathbf{d}_{i}(\boldsymbol{\xi}, t)=\mathrm{G}(\boldsymbol{\xi}, t)\left\langle\mathbf{d}_{i}(\boldsymbol{\xi}, 0)\right\rangle, \quad \mathrm{G} \cdot \mathrm{G}^{*}=\mathrm{G}^{*} \cdot \mathrm{G}=I
$$

Here, $\mathrm{G}^{*}$ is the adjoint matrix, such that

$$
\mathbf{a} \cdot \mathrm{G}\langle\mathbf{b}\rangle=\mathbf{b} \cdot \mathrm{G}^{*}\langle\mathbf{a}\rangle, \quad \forall \mathbf{a}, \mathbf{b} \in \mathbb{R}^{3}, \quad\left(\mathrm{G}^{*}\right)_{i j}=\mathrm{G}_{j i} .
$$

So, any material point is considered as a solid body. It is assumed that $\mathbf{x}(\boldsymbol{\xi}, 0)=\boldsymbol{\xi}$, i.e., $\xi$ stands for the initial position of $\mathbf{x}$. Let us denote by $d V$ the infinitesimal volume centered at the vector $\mathbf{x}(\boldsymbol{\xi}, t)$. Any point from $d V$ at the initial moment is characterized by a relative position vector $\xi^{\gamma}$. At each moment $t$, the vector $\xi^{\gamma}$ is defined by the formula

$$
\mathbf{x}^{r}=\mathrm{G}(\boldsymbol{\xi}, t)\left\langle\boldsymbol{\xi}^{r}\right\rangle \equiv \mathbf{x}^{r}\left(t, \boldsymbol{\xi}, \boldsymbol{\xi}^{r}\right), \quad \boldsymbol{\xi}^{r}=\mathrm{G}^{-1}(\boldsymbol{\xi}, t)\left\langle\mathbf{x}^{r}\right\rangle .
$$

The relative velocity and the velocity of the mass center determine the motion of the volume $d V$ :

$$
\begin{gathered}
\mathbf{v}(\boldsymbol{\xi}, t)=\frac{\partial \mathbf{x}(\boldsymbol{\xi}, t)}{\partial t} \equiv \mathbf{v}(\mathbf{x}, t) \\
\mathbf{v}^{r}\left(t, \boldsymbol{\xi}, \boldsymbol{\xi}^{r}\right)=\frac{\partial \mathbf{x}^{r}\left(t, \boldsymbol{\xi}, \boldsymbol{\xi}^{r}\right)}{\partial t}=\mathrm{G}_{t}(\boldsymbol{\xi}, t)\left\langle\boldsymbol{\xi}^{r}\right\rangle=\mathrm{G}_{t}(\boldsymbol{\xi}, t) \cdot \mathrm{G}^{-1}(\boldsymbol{\xi}, t)\left\langle\mathbf{x}^{r}\right\rangle
\end{gathered}
$$

In what follows, $\left\{\mathbf{e}_{i}\right\}_{1}^{3}$ is any orthogonal basis in $\mathbb{R}^{3}$. Let us define the rotation tensor:

$$
\Omega(\mathbf{x}, t)=\dot{\mathrm{G}}(\mathbf{x}, t) \cdot \mathrm{G}^{*}(\mathbf{x}, t) \equiv \mathrm{G}_{t}(t, \boldsymbol{\xi}) \cdot \mathrm{G}^{*}(t, \boldsymbol{\xi})
$$

It is skew-symmetric and satisfies the formula

$$
\Omega\langle\mathbf{h}\rangle=\boldsymbol{\omega} \times \mathbf{h}, \quad \forall \mathbf{h} \in \mathbb{R}^{3}, \quad(\Omega\langle\mathbf{h}\rangle)_{i} \equiv \Omega_{i j} h_{j},
$$

with " $\times$ " denoting the vector product and $\boldsymbol{\omega}(\mathbf{x}, t)$ standing for the angular velocity (or gyration),

$$
2 \omega=\mathbf{e}_{i} \times \Omega\left\langle\mathbf{e}_{i}\right\rangle=\epsilon: \Omega^{*}
$$

Here, $\epsilon$ is the Levi-Civita third order tensor,

$$
\begin{aligned}
\epsilon\langle\mathbf{a}, \mathbf{b}, \mathbf{c}\rangle & =\mathbf{a} \cdot(\mathbf{b} \times \mathbf{c}), \quad \mathbf{e}_{i} \times \mathbf{e}_{j}=\epsilon_{s i j} \mathbf{e}_{s} \\
\epsilon_{s i j} & \equiv \epsilon\left\langle\mathbf{e}_{s}, \mathbf{e}_{i}, \mathbf{e}_{j}\right\rangle, \quad(\epsilon: \Omega)_{i} \equiv \varepsilon_{i j k} \Omega_{j k},
\end{aligned}
$$

where $\epsilon_{i n j}$ is the permutation symbol [29]. It follows from the above definitions that

$$
\Omega=\epsilon: \boldsymbol{\omega}, \quad(\epsilon: \boldsymbol{\omega})_{i j}=\epsilon_{i n j} \omega_{n}
$$

In what follows, we use the scalar product $A: B=A_{i j} B_{i j}$ of two $3 \times 3$-matrices $A$ and $B$. With these notations, we have

$$
\mathbf{u} \cdot \operatorname{div} A=\operatorname{div}\left(A^{*} \mathbf{u}\right)-A: \nabla \mathbf{u}, \quad(\nabla \mathbf{u})_{i j}=\frac{\partial u_{i}}{\partial x_{j}} .
$$

Let $\mathbf{s}$ and $c$ stand for the local internal specific spin and particle concentration. By definition,

$$
\begin{equation*}
\mathbf{s}=J \boldsymbol{\omega}, \quad s_{i}=J_{i j} \omega_{j} \tag{7}
\end{equation*}
$$

where the micro-inertia tensor J obeys the identity [29],

$$
\begin{equation*}
\dot{J}=(\epsilon: \omega) J-J(\epsilon: \omega) \tag{8}
\end{equation*}
$$

For the Cosserat continuum without dissipation, microrotations of two different particles do not interact. Hence,

$$
\begin{equation*}
\dot{\rho}+\rho \operatorname{div} \mathbf{v}=0, \quad \rho \dot{\mathbf{v}}=-\nabla p, \quad \dot{\mathbf{s}}=0, \quad \dot{s}=0, \quad \dot{c}=0 . \tag{9}
\end{equation*}
$$

The third equation in (9) is the internal angular momentum conservation law. The latter equation in (9) implies that the number of particles is constant in any fluid-solid infinitesimal material volume.

One can verify that Equation (9) can be written as the following conservation law system:

$$
\begin{gather*}
\rho_{t}+\operatorname{div} \mathbf{j}=0, \quad \mathbf{j}_{t}+\operatorname{div} \Pi=0, \quad \eta_{t}+\operatorname{div}(\eta \mathbf{v})=0  \tag{10}\\
(\rho c \mathbf{s})_{t}+\operatorname{div} M=0, \quad(\rho c)_{t}+\operatorname{div}(c \mathbf{j})=0 \tag{11}
\end{gather*}
$$

where $M=c \mathbf{s} \otimes \mathbf{j}$.
By the Gibbs identity,

$$
\begin{equation*}
d e=\theta d \eta+\mu d \rho+z d(\rho c) \tag{12}
\end{equation*}
$$

where $z$ is the second chemical potential, we obtain the equalities

$$
\theta=\frac{\partial e}{\partial \eta}, \quad \mu=\frac{\partial e}{\partial \rho}, \quad z=\frac{\partial e}{\partial(\rho c)} .
$$

With concentration being one more thermodynamical variable, pressure can be defined by the same argument as above, and it satisfies the formulas [26]

$$
\begin{equation*}
p=-e+\theta \eta+\mu \rho+z \rho c, \quad d p=\eta d \theta+\rho d \mu+\rho c d z . \tag{13}
\end{equation*}
$$

It is proved in Appendix A (Lemma A3), that energy Equation (6) holds but for the total energy given by the formula

$$
\begin{equation*}
E=e+\frac{\mathbf{j} \cdot \mathbf{v}}{2}+\frac{c \rho \mathbf{s} \cdot \boldsymbol{\omega}}{2} . \tag{14}
\end{equation*}
$$

### 2.3. Thermodynamics of Micropolar Fluids with Dissipation

We extend the model (10)-(13) to equations allowing for dissipation. To this end, we argue as in [23] and write the conservation laws

$$
\begin{gather*}
\rho_{t}+\operatorname{div} \mathbf{j}=0, \quad \mathbf{j}_{t}+\operatorname{div}\left(\Pi+\Pi_{d}\right)=0, \quad \eta_{t}+\operatorname{div}\left(\eta \mathbf{v}+\frac{\mathbf{q}}{\theta}\right)=\frac{R}{\theta}  \tag{15}\\
(\rho c \mathbf{s})_{t}+\operatorname{div}\left(M+M_{d}\right)=\mathbf{f}_{d}, \quad(\rho c)_{t}+\operatorname{div}(c \mathbf{j}+\mathbf{l})=0, \quad \dot{J}=\Omega J-J \Omega  \tag{16}\\
E_{t}+\operatorname{div}\left(\mathbf{Q}+\mathbf{Q}_{d}\right)=0 \tag{17}
\end{gather*}
$$

with dissipative unknowns $\zeta=\left\{\Pi_{d}, M_{d}, \mathbf{1}, \mathbf{q}, \mathbf{f}_{d}, \mathbf{Q}_{d}, R\right\}$.
Observe that Equation (17) should result from the system (15) and (16). Here, $q$ and $R$ are the heat flux and the entropy production, respectively. Conservation laws (15)-(17) should agree with the thermodynamic Equations (12)-(14). The set of unknowns $\zeta$ can be determined due to the fact that Equation (17) follows from (12)-(16).

Using (12) and Lemmas A6 and A7 in Appendix A, we can write the time derivatives $e_{t}$ and $E_{t}$ as follows:

$$
e_{t}=\theta \eta_{t}+\mu \rho_{t}+z \frac{\partial}{\partial t}(\rho c), \quad E_{t}=\frac{\partial}{\partial t}\left(e+\frac{\mathbf{j} \cdot \mathbf{j}}{2 \rho}+\frac{c \rho \mathbf{s} \cdot \boldsymbol{\omega}}{2}\right)=\sum_{1}^{6} D_{i}
$$

where

$$
\begin{gathered}
D_{1}=\rho_{t}\left(\mu-\frac{\mathbf{j} \cdot \mathbf{j}}{2 \rho^{2}}\right), \quad D_{2}=\eta_{t} \theta, \quad D_{3}=(\rho c)_{t}\left(z-\frac{\mathbf{s} \cdot \boldsymbol{w}}{2}\right), \quad D_{4}=\mathbf{j}_{t} \cdot \frac{\mathbf{j}}{\rho} \\
D_{5}=(\rho c \mathbf{s})_{t} \cdot \boldsymbol{\omega}, \quad D_{6}=\rho c\left(\operatorname{div}\left(\mathbf{v} \frac{\mathbf{s} \cdot \boldsymbol{w}}{2}\right)-\frac{\mathbf{s} \cdot \boldsymbol{w}}{2} \operatorname{div} \mathbf{v}-\nabla \boldsymbol{\omega}: \mathbf{s} \otimes \mathbf{v}\right)
\end{gathered}
$$

Eliminating all the time derivatives $\rho_{t}, \eta_{t}$, etc. in $D_{i}$ by the conservation laws, we arrive at the following presentation for $E_{t}$ :

$$
\begin{equation*}
E_{t}=-\operatorname{div}\left(\mathbf{Q}+\mathbf{Q}_{d}\right)+g \tag{18}
\end{equation*}
$$

where

$$
\begin{gathered}
g=R+\frac{\mathbf{q} \cdot \nabla \theta}{\theta}+\Pi_{d}: \nabla \mathbf{v}+M_{d}: \nabla \boldsymbol{\omega}+\boldsymbol{\omega} \cdot \mathbf{f}_{d}+\mathbf{l} \cdot \nabla z_{1} \\
z_{1}=z-\frac{\mathbf{s} \cdot \boldsymbol{\omega}}{2}, \quad \mathbf{Q}=\mathbf{v}(p+E), \quad \mathbf{Q}_{d}=\mathbf{q}+\Pi_{d}^{*} \mathbf{v}+M_{d}^{*} \boldsymbol{\omega}+\mathbf{l} z_{1} .
\end{gathered}
$$

Detailed calculations are performed in Appendix A, Lemma A8.
Clearly, Equation (18) becomes the energy conservation law with $g=0$ if we define the entropy production as follows:

$$
\begin{equation*}
R=-\Pi_{d}: \nabla \mathbf{v}-M_{d}: \nabla \boldsymbol{\omega}-\mathbf{1} \cdot \nabla z_{1}-\frac{\mathbf{q} \cdot \nabla \theta}{\theta}-\boldsymbol{\omega} \cdot \mathbf{f}_{d} \tag{19}
\end{equation*}
$$

Let us choose $\Pi_{d}, M_{d}, \mathbf{q}, 1$ and $\mathbf{f}_{d}$ to provide the restriction $R \geq 0$. We introduce the symmetric and skew-symmetric parts of a matrix $A$ as follows:

$$
A_{s}=\frac{A+A^{*}}{2}, \quad A_{a}=\frac{A-A^{*}}{2}
$$

Clearly, $A=A_{s}+A_{a}$. Any skew-symmetric matrix $M$ obeys the formula $M \mathbf{h}=$ $\mathbf{m} \times \mathbf{h} \quad \forall \mathbf{h}$, where the vector $\mathbf{m}$ is defined uniquely. Observe that

$$
\left(\nabla \mathbf{v}-(\nabla \mathbf{v})^{*}\right) \mathbf{h}=\operatorname{rot} \mathbf{v} \times \mathbf{h} \quad \forall \mathbf{h} \in \mathbb{R}^{3}, \quad \operatorname{rot} \mathbf{v}=-\epsilon: \nabla \mathbf{v}
$$

Particularly, $M_{1}: M_{2}=2 \mathbf{m}_{1} \cdot \mathbf{m}_{2}$, if $M_{i} \mathbf{h}=\mathbf{m}_{i} \times \mathbf{h}$. In the micropolar fluid theory [29] the matrices

$$
B=\nabla \mathbf{v}-\Omega, \quad A=\nabla \boldsymbol{\omega}
$$

are the rate of strain tensors. The equality

$$
B_{a} \mathbf{h}=-\boldsymbol{\omega}_{r} \times \mathbf{h}
$$

holds, where $\omega_{r}=\omega-\operatorname{rot} \mathbf{v} / 2$ is the relative angular velocity.
Let us denote

$$
S=-\Pi_{d}, \quad M_{d}=-N+M_{2} .
$$

Because of the identity

$$
S: \Omega=-(\epsilon: S) \cdot \omega
$$

the formula

$$
\begin{equation*}
-\Pi_{d}: \nabla \mathbf{v}-\boldsymbol{\omega} \cdot \mathbf{f}_{d}=S: B-\boldsymbol{\omega} \cdot\left(\epsilon: S+\mathbf{f}_{d}\right) \tag{20}
\end{equation*}
$$

holds. Given the positive parameters $\alpha_{1}, \alpha_{4}$ and $\varkappa$, we choose $\mathbf{f}_{d}, \mathbf{1}, M_{2}$ and $\mathbf{q}$ by the formulas

$$
\begin{gather*}
\mathbf{f}_{d}=-\epsilon: S, \quad M_{2}=\alpha_{4} \epsilon:\left(\mathbf{l} \times \boldsymbol{\omega}_{r}\right), \quad \mathbf{q}=-\varkappa \nabla \theta,  \tag{21}\\
\nabla z_{1}=-\frac{\mathbf{1}}{\alpha_{1}}-\alpha_{2} \mathbf{l} \times \omega_{r}+\alpha_{4} \operatorname{rot} \omega \times \omega_{r} . \tag{22}
\end{gather*}
$$

As $\mathbf{1} \times \boldsymbol{\omega}_{r}=(\epsilon: \mathbf{l}) \boldsymbol{\omega}_{r}$, Equation (22) is equivalent to the following:

$$
\begin{equation*}
S \mathbf{1}=-\nabla z_{1}+\alpha_{4} \operatorname{rot} \boldsymbol{\omega} \times \boldsymbol{\omega}_{r}, \quad S=\alpha_{1}^{-1} I-\alpha_{2} \epsilon: \omega_{r} \tag{23}
\end{equation*}
$$

Due to the identity

$$
\nabla \omega:\left[\epsilon:\left(\mathbf{l} \times \omega_{r}\right)\right]+\mathbf{l} \cdot\left(\operatorname{rot} \omega \times \omega_{r}\right)=0
$$

we have

$$
\begin{equation*}
-M_{d}: \nabla \omega-1 \cdot \nabla z_{1}=N: A+\frac{|1|^{2}}{\alpha_{1}} . \tag{24}
\end{equation*}
$$

Hence, it follows from (19)-(24) that

$$
\begin{equation*}
R=S: B+N: A+\frac{\varkappa|\nabla \theta|^{2}}{\theta}+\frac{|\mathbf{1}|^{2}}{\alpha_{1}} . \tag{25}
\end{equation*}
$$

As in [10], we apply the constitutive equations

$$
\begin{gather*}
S=2 \mu_{s} B_{s}+2 \mu_{a} B_{a}+2 \mu_{a n} \sigma^{2} J B  \tag{26}\\
N=2 v A+\frac{2 v_{a n}}{\sigma^{2}} A J, \tag{27}
\end{gather*}
$$

with $\mu_{s}, \mu_{a}, \mu_{a n}, v, v_{a n}[\mathrm{~g} /(\mathrm{cm} \cdot \mathrm{s})]$ being the viscosities and with $\sigma\left[\mathrm{cm}^{-1}\right]$ standing for the specific particles surface area. Anisotropy manifests itself through the presence of $J$ in the constitutive laws (26) and (27). We borrow this idea from [29]. Observe that the objectivity of $S$ and $N$ follows from the objectivity of $B, A$ and $J[29]$.

Let $\mathbf{e}_{j}$ and $\lambda_{j}$ be the eigenvectors and the eigenvalues of the symmetric tensor $J$. Then,

$$
J B: B=\sum_{1}^{3} \lambda_{j}\left|B^{*} \mathbf{e}_{j}\right|^{2} .
$$

One can verify that $\lambda_{j} \geq 0$ provided the particles are axisymmetric. For such suspensions the inequalities

$$
\begin{gathered}
S: B=2 \mu_{s} B_{s}: B_{s}+2 \mu_{a} B_{a}: B_{a}+2 \mu_{a n} \sigma^{2} \sum_{1}^{3} \lambda_{j}\left|B^{*} \mathbf{e}_{j}\right|^{2} \geq 0, \\
A J: J=\sum \lambda_{j}\left|A \mathbf{e}_{j}\right|^{2}
\end{gathered}
$$

hold. Thus, $R \geq 0$ for the axisymmetric particles, when $\mu_{s} \geq 0, \mu_{a} \geq 0, \mu_{a n} \geq 0, v \geq 0$, $v_{a n} \geq 0, \alpha_{1} \geq 0$.

Let us comment on the physical sense of $\alpha_{1}$ and $\alpha_{4}$. To this end, we consider a simple isotropic case where $v_{a n}=0$ and $\alpha_{2}=0$. In such a case, Equation (23) is equivalent to the following:

$$
\begin{equation*}
\mathbf{1}=-\alpha_{1} \nabla z_{1}+\alpha_{1} \alpha_{4} \operatorname{rot} \boldsymbol{\omega} \times \boldsymbol{\omega}_{r} . \tag{28}
\end{equation*}
$$

With $z_{1}$ being an effective chemical potential, it is pointed out in [23] that $\alpha_{1}$ coincides with the diffusion coefficient $D$ up to a multiplier. When $v_{a n}=0$, the term $\operatorname{rot} \boldsymbol{\omega} \times \boldsymbol{\omega}_{r}$ becomes

$$
\operatorname{rot} \boldsymbol{\omega} \times \boldsymbol{\omega}_{r}=2 A_{a} \boldsymbol{\omega} \times \boldsymbol{\omega}_{r}=v^{-1} N_{a} \boldsymbol{\omega} \times \boldsymbol{\omega}_{r} .
$$

Hence,

$$
\begin{equation*}
\mathbf{1}=-\alpha_{1} \nabla z_{1}+D_{w} N_{a} \boldsymbol{\omega}_{r}, \quad D_{w}=\alpha_{1} \alpha_{4} v^{-1} . \tag{29}
\end{equation*}
$$

The term $D_{w} N_{a} \boldsymbol{\omega}_{r}$ describes a torque on a rotating particle, with $D_{w}$ standing for the spin diffusion [23]. Thus, $\alpha_{1}$ and $\alpha_{4}$ correlate with the diffusion and spin diffusion coefficients, respectively.

The derived conservation laws can be summarized as follows:

$$
\begin{gather*}
\rho_{t}+\operatorname{div}(\rho \mathbf{v})=0,  \tag{30}\\
(\rho \mathbf{v})_{t}+\operatorname{div}(\rho \mathbf{v} \otimes \mathbf{v})=-\nabla p+\operatorname{div} S,  \tag{31}\\
(\rho c \mathbf{s})_{t}+\operatorname{div}\left(\rho c \mathbf{s} \otimes \mathbf{v}+\alpha_{4} \epsilon:\left(\mathbf{1} \times \boldsymbol{\omega}_{r}\right)\right)=\operatorname{div} N-\epsilon: S, \quad \mathbf{s}=J \omega, \tag{32}
\end{gather*}
$$

$$
\begin{gather*}
(\rho c)_{t}+\operatorname{div}(\rho c \mathbf{v}+\mathbf{1})=0  \tag{33}\\
\eta_{t}+\operatorname{div}\left(\eta \mathbf{v}+\frac{\mathbf{q}}{\theta}\right)=\frac{R}{\theta}  \tag{34}\\
J_{t}+(\mathbf{v} \cdot \nabla) J=(\epsilon: \boldsymbol{\omega}) J-J(\epsilon: \boldsymbol{\omega}) \tag{35}
\end{gather*}
$$

The flux 1 and stress tensors $S, N$ obey the constitutive laws (22), (26) and (27). The entropy production $R$ is defined by Equation (25).

To complete the model (22), (26), (27), (30)-(41) for the variables

$$
\rho, p, c, \eta, \theta, z, \mathbf{v}, \boldsymbol{\omega}, \mathbf{q}, J
$$

one should add thermodynamic relations. Given internal energy $e(\rho, \eta, c)$, we write the Gibs law in the form

$$
\begin{equation*}
d e=\theta d \eta+\mu d \rho+z d(\rho c), \quad \theta=\frac{\partial e}{\partial \eta}, \quad \mu=\frac{\partial e}{\partial \rho}, \quad z=\frac{\partial e}{\partial(\rho c)} . \tag{36}
\end{equation*}
$$

The pressure $p$ is defined by (13). The heat flux $\mathbf{q}$ is defined by the Fourier law $\mathbf{q}=-\varkappa \nabla \theta$. The parameters

$$
\alpha_{1}, \alpha_{2}, \alpha_{4}, \varkappa, \mu_{s}, \mu_{a}, \mu_{a n}, \sigma, v, \nu_{a n}
$$

are assumed to be the known function of the thermodynamic variables $c, p$ and $\theta$.

## 3. Poiseuille Flow

We neglect heat effects and address the pressure-driven flows in the layer $|y|<H$ (Figure 1a) along the horizontal $x$-axis with

$$
\nabla p=\left(p_{x}, 0,0\right), \quad p_{x}(t)<0, \quad v_{2}=v_{3}=0, \quad v_{1} \equiv v
$$

We assume that the rod particles are all identical, rotate around the $z$ axis and lie in the plane $z=0$. This implies that $\omega=(0,0, \omega)$. We look for solutions, such that $v, \omega, \varphi$ and $c$ depend on $y$ and $t$ only.


Figure 1. (a) Particle's orientation in one-dimensional flows. (b) Approximation of the rod-like particle by a cylinder.

The micro-inertia tensor $J$ is described as follows. Let $h$ and $r$ be the length and radius of the cylinder $V_{0}$ stretched along the $y$-axis, Figure 1b. The inertia tensor $J\left(V_{0}\right)$ of $V_{0}$ is defined by the integral

$$
J\left(V_{0}\right)=\int_{V_{0}}|\zeta|^{2} \cdot I-\xi \otimes \xi d \xi \quad \text { or } \quad J_{i j}\left(V_{0}\right)=\int_{V_{0}}\left|\xi_{k}\right|^{2} \delta_{i j}-\xi_{i} \xi_{j} d \xi
$$

where $I$ is the identity matrix and $\mathbf{a} \otimes \mathbf{b}$ denotes the tensor product of two vectors $\mathbf{a}$ and $\mathbf{b}$, $(\mathbf{a} \otimes \mathbf{b})_{i j}=a_{i} b_{j}$. Calculations give the formula

$$
J\left(V_{0}\right)=\frac{h^{2}}{3}\left(\begin{array}{ccc}
3 a^{2} / 4+1 & 0 & 0 \\
0 & 3 a^{2} / 2 & 0 \\
0 & 0 & 3 a^{2} / 4+1
\end{array}\right), \quad a=\frac{r}{h}
$$

In the limit, as the aspect ratio $a$ goes to zero, we arrive at the matrix

$$
J^{0}=\lim _{r \rightarrow 0} J\left(V_{0}\right)=j_{0}\left(\begin{array}{lll}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{array}\right), \quad j_{0}=h^{2} / 3
$$

standing for the inertia tensor of the rod stretched along the $y$-axis. In what follows, we use the "dot" symbol for the material derivative (3) related to the velocity vector $\mathbf{v}$. Let the cylinder $V$ be the result of rotation of the cylinder $V_{0}$ around the axis $z$ by an angle $\varphi$ measured from the axis $y$ counterclockwise (Figure 1a).

The spin $s$ is defined by the formula

$$
\mathbf{s}=J(V) \cdot \boldsymbol{\omega}=\int_{V} \mathbf{x} \times(\boldsymbol{\omega} \times \mathbf{x}) d x=Q_{\varphi} J\left(V_{0}\right) Q_{\varphi}^{*} \cdot \boldsymbol{\omega}
$$

Here,

$$
Q_{\varphi}=\left(\begin{array}{ccc}
\cos \varphi & -\sin \varphi & 0  \tag{37}\\
\sin \varphi & \cos \varphi & 0 \\
0 & 0 & 1
\end{array}\right), \quad \Omega=\left(\begin{array}{ccc}
0 & -\omega & 0 \\
\omega & 0 & 0 \\
0 & 0 & 0
\end{array}\right) .
$$

By definition,

$$
\begin{equation*}
\Omega=\dot{Q}_{\varphi} Q_{\varphi}^{*}, \quad \Omega \cdot \mathbf{h}=\boldsymbol{\omega} \times \mathbf{h} \forall \mathbf{h}, \quad\left(Q^{*}\right)_{i j}=Q_{j i} . \tag{38}
\end{equation*}
$$

It follows from the above formulas that $J(V)=Q_{\varphi} J\left(V_{0}\right) Q_{\varphi}^{*}$. As $r$ tends to zero, we arrive at the following representation of the inertia tensor $J(\varphi)$ :

$$
J(\varphi)=Q_{\varphi} J^{0} Q_{\varphi}^{*}=j_{0}\left(\begin{array}{ccc}
\cos ^{2} \varphi & \sin \varphi \cos \varphi & 0  \tag{39}\\
\sin \varphi \cos \varphi & \sin ^{2} \varphi & 0 \\
0 & 0 & 1
\end{array}\right), \quad \frac{\partial \varphi}{\partial t}=\varphi_{t}=\omega .
$$

Assume that the initial values $\varphi_{0}(y)$ are prescribed. Denoting $J_{0}(y)=J\left(\varphi_{0}(y)\right)$, we arrive at the following initial condition:

$$
\begin{equation*}
\left.J\right|_{t=0}=J_{0} \tag{40}
\end{equation*}
$$

The material derivative $\dot{J}$ coincides with the time derivative $J_{t}$ for one-dimensional flows. Due to (37), Equation (8) reduces to the following system:

$$
\begin{equation*}
\frac{\partial}{\partial t} J_{11}=-2 \omega J_{12}, \quad \frac{\partial}{\partial t} J_{12}=\omega\left(J_{11}-J_{22}\right), \quad \frac{\partial}{\partial t} J_{22}=2 \omega J_{12}, \tag{41}
\end{equation*}
$$

and $J_{i j}=0$ otherwise. By virtue of the equalities

$$
\frac{\partial J_{i j}}{\partial t}=\frac{\partial J_{i j}}{\partial \varphi} \frac{\partial \varphi}{\partial t}=J_{i j}^{\prime} \omega, \quad J_{i j}^{\prime}=\frac{\partial J_{i j}}{\partial \varphi},
$$

the system (41) becomes

$$
\begin{equation*}
J_{11}^{\prime}=-2 J_{12}, \quad J_{12}^{\prime}=J_{11}-J_{22}, \quad J_{22}^{\prime}=2 J_{12},\left.\quad J_{i j}\right|_{\varphi=\varphi_{0}}=J_{0} \tag{42}
\end{equation*}
$$

The matrix $J$ given by the formula (39) is the solution to problem (42).
We calculate the matrices $B, B_{s}$ and $B_{a}$ :

$$
\begin{gather*}
B=\left(\begin{array}{ccc}
0 & v_{y}+\omega & 0 \\
-\omega & 0 & 0 \\
0 & 0 & 0
\end{array}\right), \quad B^{*}=\left(\begin{array}{ccc}
0 & -\omega & 0 \\
v_{y}+\omega & 0 & 0 \\
0 & 0 & 0
\end{array}\right),  \tag{43}\\
B_{s}=\left(\begin{array}{ccc}
0 & v_{y} / 2 & 0 \\
v_{y} / 2 & 0 & 0 \\
0 & 0 & 0
\end{array}\right), \quad B_{a}=\left(\begin{array}{ccc}
0 & v_{y} / 2+\omega & 0 \\
-v_{y} / 2-\omega & 0 & 0 \\
0 & 0 & 0
\end{array}\right), \\
j_{0}^{-1} J B=\left(\begin{array}{ccc}
-\omega \cos \varphi \sin \varphi & \cos ^{2} \varphi\left(v_{y}+\omega\right) & 0 \\
-\omega \sin ^{2} \varphi & \cos \varphi \sin \varphi\left(v_{y}+\omega\right) & 0 \\
0 & 0 & 0
\end{array}\right) .
\end{gather*}
$$

Similarly, we find that

$$
A=\left(\begin{array}{ccc}
0 & 0 & 0  \tag{44}\\
0 & 0 & 0 \\
0 & \omega_{y} & 0
\end{array}\right), \quad j_{0}^{-1} A J=\left(\begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & 0 \\
\omega_{y} \cos \varphi \sin \varphi & \omega_{y} \sin ^{2} \varphi & 0
\end{array}\right)
$$

Let us denote

$$
\varepsilon_{1}=\frac{\mu_{a}}{\mu_{s}}, \quad \varepsilon_{20}=\frac{\mu_{a n} j_{0} \sigma^{2}}{\mu_{s}}, \quad \varepsilon_{30}=\frac{v_{a n} j_{0} \sigma^{2}}{v}, \quad B^{0}=B_{s}+\varepsilon_{1} B_{a}+\varepsilon_{20} J B .
$$

Calculations reveal that matrix $B^{0}$ is given by the formula

$$
B^{0}=\left(\begin{array}{ccc}
-\varepsilon_{20} \omega \cos \varphi \sin \varphi & \frac{v_{y}\left(1+\varepsilon_{1}+2 \varepsilon_{20} \cos ^{2} \varphi\right)}{2}+\omega\left(\varepsilon_{1}+\varepsilon_{20} \cos ^{2} \varphi\right) & 0  \tag{45}\\
\frac{v_{y}\left(1-\varepsilon_{1}\right)}{2}-\omega\left(\varepsilon_{1}+\varepsilon_{20} \sin ^{2} \varphi\right) & \varepsilon_{20} \cos \varphi \sin \varphi\left(v_{y}+\omega\right) & 0 \\
0 & 0 & 0
\end{array}\right)
$$

Under the incompressibility condition $\rho=$ const, the equation $\operatorname{div} \mathbf{v}=0$ is satisfied. Other conservation laws (31)-(33) become

$$
\begin{gather*}
\frac{\partial \varphi}{\partial t}=\omega, \quad \rho \frac{\partial v}{\partial t}=-p_{x}+\frac{\partial S_{12}}{\partial y},  \tag{46}\\
\frac{\partial\left(\rho j_{0} c \omega\right)}{\partial t}+\alpha_{4}\left(\frac{\partial\left(l_{1} \omega_{r}\right)}{\partial x}+\frac{\partial\left(l_{2} \omega_{r}\right)}{\partial y}\right)=\frac{\partial N_{32}}{\partial y}+S_{21}-S_{12}, \omega_{r}=\omega+\frac{v_{y}}{2},  \tag{47}\\
\rho j_{0} \frac{\partial c}{\partial t}+\frac{\partial l_{1}}{\partial x}+\frac{\partial l_{2}}{\partial y}=0 . \tag{48}
\end{gather*}
$$

The constitutive law (22) can be written as the linear system for $l_{i}$ :

$$
\begin{equation*}
\alpha_{1}^{-1} l_{1}+\alpha_{2} \omega_{r} l_{2}=-z_{1 x}, \quad-\alpha_{2} \omega_{r} l_{1}+\alpha_{1}^{-1} l_{2}=-z_{1 y}-\alpha_{4} \omega_{r} \omega_{y} \tag{49}
\end{equation*}
$$

with

$$
\begin{equation*}
z_{1}=z-j_{0} \omega^{2} / 2 \tag{50}
\end{equation*}
$$

Generally, the chemical potential $z$ is a given function of the thermodynamic variables $c, p$ and $\theta$, and so

$$
\begin{equation*}
z_{x}=k c_{x}+k_{p} p_{x}+k_{\theta} \theta_{x}, \quad z_{y}=k c_{y}+k_{p} p_{y}+k_{\theta} \theta_{y} . \tag{51}
\end{equation*}
$$

Here, $k, k_{p}$ and $k_{\theta}$ are diffusion, barodiffusion and thermodiffusion, respectively [23]. Due to our hypothesis, we set $k_{\theta}=0$ and assume that $k$ and $k_{p}$ are known constants.

As the pressure gradient $p_{x}$ does not depend on the space variables, the fluxes $l_{1}$ and $l_{2}$ defined by the system (49) do not depend on the variable $x$. Thus, the $x$-derivatives in the Equations (47) and (48) disappear. Projecting the Equation (31) on the $y$-axis, we obtain

$$
\begin{equation*}
0=\frac{\partial}{\partial y}\left(-p+S_{22}\right) \tag{52}
\end{equation*}
$$

Hence,

$$
\begin{equation*}
z_{1 x}=k_{p} p_{x}, \quad z_{1 y}=k \frac{\partial c}{\partial y}+k_{p} \frac{\partial}{\partial y}\left(S_{22}-j_{0} \frac{\omega^{2}}{2}\right) \tag{53}
\end{equation*}
$$

For one-dimensional flows, the constitutive laws (26) and (27) reduce to

$$
\begin{equation*}
S_{i j}=2 \mu_{s} B_{i j}^{0}, \quad N_{32}=2 \frac{v}{\sigma^{2}} A_{32}+2 v_{a n}(A J)_{32} . \tag{54}
\end{equation*}
$$

Observe that

$$
S_{21}-S_{12}=2 \mu_{s}\left(B_{21}^{0}-B_{12}^{0}\right), B_{21}^{0}-B_{12}^{0}=-v_{y}\left(\varepsilon_{1}+\varepsilon_{20} \cos ^{2} \varphi\right)-\omega\left(2 \varepsilon_{1}+\varepsilon_{20}\right) .
$$

We formulate boundary conditions at $|y|=H$ as follows:

$$
\begin{equation*}
\mathbf{v}=0, \quad \boldsymbol{\omega}=\frac{\Lambda}{2} \nabla \times \mathbf{v}, \quad \mathbf{l} \cdot \mathbf{n}=0, \quad 0 \leq \Lambda \leq 1 . \tag{55}
\end{equation*}
$$

The first equation in (55) states that velocity obeys the no-slip condition. The second condition in (55) has the meaning that the micro-rotation $\omega$ depends linearly on the macrorotation $\nabla \times \mathbf{v} / 2$ [32]. The third condition concerns the concentration flux, with $\mathbf{n}$ being the outward normal vector.

Let us move on to dimensionless variables, assuming that reference values $V$ and $T$ for speed and time are fixed. We denote $\Omega=V / H$ and choose $T=1 / \Omega$. The dimensionless variables are defined as follows:

$$
\begin{gathered}
S^{\prime}=\frac{1}{2 \mu_{s} \Omega} S, N^{\prime}=\frac{N}{2 \mu_{s} H \Omega}, B^{\prime 0}=\frac{B^{0}}{\Omega}, \quad l_{i}^{\prime}=\frac{l_{i}}{\rho V}, y^{\prime}=\frac{y}{H}, t^{\prime}=\frac{t}{T} \\
v^{\prime}=\frac{v}{V}, \omega^{\prime}=\frac{\omega}{\Omega}, z^{\prime}=\frac{z}{H^{2} \Omega^{2}}, z_{1}^{\prime}=\frac{z_{1}}{H^{2} \Omega^{2}} .
\end{gathered}
$$

In what follows, we use the dimensionless parameters

$$
\begin{gathered}
\operatorname{Re}=\frac{H^{2} \rho \Omega}{\mu_{s}}, \Pi=\frac{\left|p_{x}\right| H^{2}}{2 V \mu_{s}}, \Gamma=\frac{v}{H^{2} \mu_{s} \sigma^{2}}, M_{s}=\frac{\mu_{s}}{\rho H V}, \Lambda_{1}=\frac{\alpha_{1} \Omega}{\rho}, \Lambda_{2}=\alpha_{2} \rho, \\
\Lambda_{4}=\frac{\alpha_{4} \rho \Omega}{2 \mu_{s}}, \quad K=\frac{k}{H^{2} \Omega}, \quad K_{p}=\frac{k_{p} \mu_{s}}{H^{2} \Omega}, \quad \bar{j}_{0}=\frac{j_{0}}{H^{2}} .
\end{gathered}
$$

In new variables, Equations (46)-(48) become

$$
\begin{gather*}
\frac{\partial \varphi}{\partial t^{\prime}}=\omega^{\prime}, \quad \frac{\operatorname{Re}}{2} \frac{\partial v^{\prime}}{\partial t^{\prime}}=\Pi+\frac{\partial S_{12}^{\prime}}{\partial y^{\prime}}, \quad \operatorname{Re} \frac{\partial c}{\partial t^{\prime}}+\frac{1}{M_{s}} \frac{\partial l_{2}^{\prime}}{\partial y^{\prime}}=0  \tag{56}\\
\frac{\bar{j}_{0} \operatorname{Re}}{2} \frac{\partial\left(c \omega^{\prime}\right)}{\partial t^{\prime}}+\Lambda_{4} \frac{\partial\left(l_{2}^{\prime} \omega_{r}^{\prime}\right)}{\partial y^{\prime}}=\frac{\partial N_{32}^{\prime}}{\partial y^{\prime}}+S_{21}^{\prime}-S_{12}^{\prime} \tag{57}
\end{gather*}
$$

In what follows we consider quasi-steady slow flows. Neglecting the terms with small Reynolds number in (56) and (57), we arrive at the equations

$$
\begin{gather*}
\varphi_{t^{\prime}}=\omega^{\prime}, \quad 0=\Pi+\frac{\partial S_{12}^{\prime}}{\partial y^{\prime}}, \quad l_{2}^{\prime}=0  \tag{58}\\
\Lambda_{4} \frac{\partial\left(l_{2}^{\prime} \omega_{r}^{\prime}\right)}{\partial y^{\prime}}=\frac{\partial N_{32}^{\prime}}{\partial y^{\prime}}+S_{21}^{\prime}-S_{12}^{\prime} \tag{59}
\end{gather*}
$$

where

$$
\begin{gather*}
S_{12}^{\prime}=\frac{\left(1+\varepsilon_{1}+2 \varepsilon_{20} \cos ^{2} \varphi\right)}{2} \frac{\partial v^{\prime}}{\partial y^{\prime}}+\omega^{\prime}\left(\varepsilon_{1}+\varepsilon_{20} \cos ^{2} \varphi\right)  \tag{60}\\
l_{2}^{\prime}=\frac{2 K_{p} \Pi \omega_{r}^{\prime}-\Lambda_{1}^{-1} z_{1 y^{\prime}}^{\prime}}{\Lambda_{1}^{-2}+\Lambda_{2}^{2} \omega_{r}^{\prime}}, \quad z_{1 y^{\prime}}^{\prime}=\frac{\partial}{\partial y^{\prime}}\left(2 K_{p} S_{22}^{\prime}-\frac{\bar{j}_{0} \omega^{\prime 2}}{2}+K c\right),  \tag{61}\\
S_{22}^{\prime}=\varepsilon_{20} \cos \varphi \sin \varphi\left(\omega^{\prime}+\frac{\partial v^{\prime}}{\partial y^{\prime}}\right), \quad N_{32}^{\prime}=\Gamma\left(1+\varepsilon_{30} \sin ^{2} \varphi\right) \frac{\partial \omega^{\prime}}{\partial y^{\prime}}  \tag{62}\\
S_{21}^{\prime}-S_{12}^{\prime}=-\left(\varepsilon_{1}+\varepsilon_{20} \cos ^{2} \varphi\right) \frac{\partial v^{\prime}}{\partial y^{\prime}}-\left(2 \varepsilon_{1}+\varepsilon_{20}\right) \omega^{\prime}, \quad \overline{j_{0}}=\frac{j_{0}}{H^{2}} \tag{63}
\end{gather*}
$$

The third equation in (58) follows from the condition $\partial l_{2}^{\prime} / \partial y^{\prime}=0$ and the boundary condition $l_{2}^{\prime}=0$ at $y^{\prime}= \pm 1$.

Due to symmetry, we will limit ourselves to the layer $0<y^{\prime}<1$. The boundary and initial conditions become

$$
\begin{equation*}
\left.\varphi\right|_{t^{\prime}=0}=\varphi_{0}\left(y^{\prime}\right), v^{\prime}(1)=0, v_{y^{\prime}}^{\prime}(0)=0, \omega^{\prime}(1)=-0.5 \Lambda v_{y^{\prime}}^{\prime}(1), \omega^{\prime}(0)=0 . \tag{64}
\end{equation*}
$$

As for the function $c$, we set the condition

$$
\begin{equation*}
\int_{0}^{1} c d y^{\prime}=c_{0}, \quad 0<c_{0}<1 \tag{65}
\end{equation*}
$$

which prescribes the total mass. To solve the above problem (58)-(65) numerically, one should fix the dimensionless parameters $\Pi, \varepsilon_{1}, \varepsilon_{20}, \varepsilon_{30}, \Gamma, \Lambda, \Lambda_{1}, K, K_{p}, \bar{j}_{0}$.

## 4. Results of Calculations

For handling the initial boundary-value problem, (58)-(65) we apply the Wolfram Mathematica solver.

In many complex fluids, there is a shear-banding effect [4,36,37]. It consists in the coexistence of bands with different shear rates. For suspensions of rod-like particles, two types of shear banding are distinguished: gradient banding and vortex banding [2-4].

If the flow consists of stripes with different shear rates along the direction of the velocity gradient, then gradient banding occurs. With reference to the coordinate system of Figure 1a, the flow is directed along the $x$-axis with the velocity vector $\mathbf{v}=(v, 0,0)$, and the $y$-axis is the gradient direction along which the flow has non-zero derivative $\partial v / \partial y$. The macro-vorticity vector $\nabla \times \mathbf{v}$ has the vorticity direction and coincides with the $z$-axis. As the one-dimensional flow under consideration does not depend on the $z$-variable, the system (58)-(63) is not applicable to describe vorticity bands. However, we show that the system (58)-(63) really predicts the gradient banding.

Figures 2 and 3 show the appearance of the gradient banding as the time grows. Such an effect manifests itself through the loss of the convexity of the velocity profile in the Poiseuille-like flow. The performed calculations are based on the following data:

$$
\begin{gather*}
\Pi=2.85, \Lambda=0.3, \Gamma=0.1, \varepsilon_{20}=0.3, \varepsilon_{30}=0.4, \bar{j}_{0}=0.09  \tag{66}\\
K_{p}:=0.0099, K=0.1, \Lambda_{1}=0.001, c_{0}=0.2, \varphi_{0}=0
\end{gather*}
$$

As for the reduced viscosity $\varepsilon_{1}$, it is assumed to be constant and equal to 0.2 in Figure 2, whereas it is the linear function of concentration $\varepsilon_{1}=E c$ in Figure 3. It is proved in [32]
that $E=2.5$, in agreement with the Einstein theory for viscosity of suspensions. Though this theory and the results in [32] concern spherical particles, we apply the closure $\varepsilon_{1}=E c$ for fibers, assuming that it holds also for dilute suspensions of fibers with low aspect ratio and low concentration $c_{0}$.

Many shear banding systems display oscillations and instabilities. A good example is worm-like micelles [38]. In laboratory experiments with fibers, the velocity fluctuations were observed in $[39,40]$ by adapting the particle-tracking technique used in the sedimentation of spheres. Our model predicts chaotic velocity behavior even if a constant pressure gradient is applied (Figure 4).

(a)

(b)

Figure 2. Shear banding at different times in the case of constant reduced viscosity $\varepsilon_{1}$. From top to bottom: profiles of velocity $v^{\prime}$ (blue), micro-spin $\omega^{\prime}$ (brown) and concentration $c$ (green) for $t^{\prime}=1$ (a) and $t=5$ (b). The data for calculation are given by the list (66).

(a)

(b)

Figure 3. Shear-banding development with time in the upper half-layer $0<y^{\prime}<1$ for the Einsteinlike law for the reduced viscosity $\varepsilon_{1}=2.5 c$. The blue, brown and green lines correspond to the profiles of velocity $v^{\prime}$ and its first and second derivatives with respect to $y^{\prime}$ at the times $t^{\prime}=0.5$ (a) and $t^{\prime}=8(\mathbf{b})$. The data for calculation are given by the list (66).


Figure 4. Velocity variation with time in the center of the channel at a constant pressure gradient for $\varphi_{0}=0(\mathbf{a})$ and $\varphi_{0}=\pi / 2(\mathbf{b})$.

Figure 5 depicts the phase transition from the isotropic phase to nematic phases where particles rotate with the same angular velocity, as in a quasi-rigid domain, and the fibers have the same orientation. The following data are used:

$$
\begin{gather*}
\Pi=2.85, \Lambda=0.3, \Gamma=0.1, \varepsilon_{1}=0.2, \varepsilon_{20}=0.3, \varepsilon_{30}=0.4, \bar{j}_{0}=0.09  \tag{67}\\
K_{p}=0.0001, K=0.1, \Lambda_{1}=0.001, c_{0}=0.2, \varphi_{0}=0
\end{gather*}
$$

The nematic phase transitions are seen in experiments with fibers in semidilute regimes [41].


Figure 5. Appearance of nematic phases. The blue, brown and green lines correspond to profiles of velocity (blue), spin (brawn) and concentration (green) at the times $t^{\prime}=0(\mathbf{a})$ and $t^{\prime}=5$ (b). The data for calculation are given by the list (67).

In the survey on sedimentation [41], some effects were reported concerning the behaviour of fiber concentration. Particularly, the formation of flock-like inhomogeneities occurs jointly with hindrance in settling. Although in our considerations the driving force is not gravity but the pressure gradient, similar concentration effects are also observed. The calculation results in Figures 6 and 7 imply that the flock-like inhomogeneities develop over time. The following data are involved:

$$
\begin{gather*}
\Pi=2.85, \Lambda=0.5, \Gamma=10.5, \varepsilon_{1}=0.4, \varepsilon_{20}=0.03, \varepsilon_{30}=0.1, \bar{j}_{0}=0.01  \tag{68}\\
K_{p}=0.01, K=0.01, \Lambda_{1}=0.1, c_{0}=0.2, \varphi_{0}=0 .
\end{gather*}
$$

Observe that the list of data (68) is not the only one that leads to the development of concentration inhomogeneities and instability.


Figure 6. Development of concentration (green) inhomogeneities: $t^{\prime}=0$ (a), $t^{\prime}=5.5$ (b). The blue and brown curves stand for velocity and spin respectively.


Figure 7. Development of concentration (green) inhomogeneities: $t^{\prime}=15.5$ (a), $t^{\prime}=20.5$ (b). The blue and brown curves stand for velocity and spin respectively.

As for hindering, Figure 8 shows that the more total concentration $c_{0}$, the lower the velocity. To perform calculations, we use the data

$$
\begin{gather*}
\Pi=2.05, \Lambda=0.3, \Gamma=1.3, \varepsilon_{20}=4, \varepsilon_{30}=1.5, \bar{j}_{0}=0.3  \tag{69}\\
K_{p}=0.99, K=1, \Lambda_{1}=0.00001, t=1.09, \varphi_{0}=0 .
\end{gather*}
$$

As for $\varepsilon_{1}$, the closure

$$
\varepsilon_{1}=\left(1-\frac{c}{c_{*}}\right)^{-2.5 / c_{*}}-1
$$

is applied with $c_{*}=1$. Though its proof is given in [32] for a high concentration of spherical particles, one can treat it as an approximation for the fiber aspect ratio $a$ close to 1 .


Figure 8. Hindering effect. The more total concentration $c_{0}$, the lower the velocity (blue): $c_{0}=0.2$ (a), $c_{0}=0.6(\mathbf{b})$. In cases ( $\mathbf{a}, \mathbf{b}$ ), the calculations refer to the same time instant $t^{\prime}=1.09$. The brown and green curves stand for spin and concentration respectively.

Next, we consider the issue related to the transportation of oil through pipelines. Polymer additives change the microstructure of oil and improve its pumping. Then, the friction factor depends on the flow history [42]. In practice, a desired flow rate is achieved either by reducing or increasing it. It turns out that the smallest friction losses are obtained in the first case [9]. We claim that the developed mathematical model explains such an effect.

First, we show that there is no simple correlation between the pressure gradient $\Pi$ and the total volumetric flux $Q=2 \int_{0}^{1} v d y$ when solving the system (58)-(65). We calculate the total flux $Q(t)$, starting from the time-dependent pressure gradient $\Pi(t)=\Pi_{0}(1+\sin \pi t)$. The calculations data are

$$
\begin{gather*}
\Pi_{0}=2.85, \Lambda=0.3, \Gamma=1, \varepsilon_{1}=0.2, \varepsilon_{20}=0.3, \varepsilon_{30}=0.4, \bar{j}_{0}=0.09  \tag{70}\\
K_{p}=0.0099, K=0.1, \Lambda_{1}=0.001, c_{0}=0.2, \varphi_{0}=0
\end{gather*}
$$

The resulting parametric curve

$$
\begin{equation*}
\Pi=\Pi_{0}(1+\sin \pi t), \quad Q=Q(t), \quad 0<t<1 \tag{71}
\end{equation*}
$$

on the ( $\Pi, Q$ )-plane is shown in Figure 9a. The first (lower) part of this curve, along which both $\Pi$ and $Q$ increase, corresponds to the time interval $0<t<1 / 2$ (see Figure 10a). Along this curve, $\Pi$ increases from $\Pi_{0}$ to $2 \Pi_{0}$. The second (upper) part of this curve corresponds to the time interval $0<t<1 / 2$. Along this curve, $\Pi$ decreases from $2 \Pi_{0}$ to $\Pi_{0}$ (see Figure 10b). In nonlinear viscous fluids, $\Pi$ is uniquely calculated from $Q$; as a consequence, the first and second curves coincide. As we see, this property is violated due to anisotropy.

(a)

(b)

Figure 9. (a) The hysteresis loop on the ( $\Pi, Q$ )-plane. The total volumetric flux $Q$ and the pressure gradient $\Pi$ are increased by factors 14 and 10, respectively. (b) The hysteresis loop on the ( $\Pi, M$ )plane. The total particles mass flux $M$ and the pressure gradient $\Pi$ are increased by factors 50 and 2 , respectively.

How can the flow $Q$ be determined from $\Pi_{*}$ under the constraint $\Pi_{0}<\Pi_{*}<2 \Pi_{0}$ ? From Figure 9a it follows that it is possible to select two values of $Q$ corresponding to $\Pi_{*}$. Indeed, let the curve intersect in the vertical line $\Pi=\Pi_{*}$ at points $A$ and $B$ :

$$
A=\left(\Pi_{*}, Q_{A}\right), \quad B=\left(\Pi_{*}, Q_{B}\right) .
$$

One can choose $t_{A}$ and $t_{B}$, such that

$$
0<t_{A}<1 / 2<t_{B}<1, \Pi\left(t_{A}\right)=\Pi\left(t_{B}\right)=\Pi_{*}, Q_{A}<Q_{B}, Q_{i}=Q\left(t_{i}\right)
$$

with $i=A, B$. Let us choose points $C=\left(\Pi_{C}, Q_{C}\right)$ and $D=\left(\Pi_{D}, Q_{D}\right)$, so that

$$
\Pi_{C}<\Pi_{*}<\Pi_{D}
$$

As the value of $\Pi$ increases from $\Pi_{C}$ to $\Pi_{*}$, the value of $Q$ increases from $Q_{C}$ to

$$
Q_{A}=\lim _{\Pi \nearrow \Pi_{*}} Q(\Pi)
$$


(a)
(b)

Figure 10. (a) The first part of the hysteresis loop in Figure 9a along which both $\Pi$ and $Q$ increase $(0<t<1 / 2)$. (b) The second part of the hysteresis loop in Figure 9a $(1 / 2<t<1)$.

When the value of $\Pi$ decreases from $\Pi_{D}$ to $\Pi_{*}$, the value of $Q$ decreases from $Q_{D}$ to

$$
Q_{B}=\lim _{\Pi \searrow \Pi_{*}} Q(\Pi) .
$$

Thus, the evolutionary history of the pressure gradient influences the overall flow.
Similarly, we calculate $\Pi$ from $Q$. Again, the background to $Q$ 's changes also turns out to be important. Let the flow $Q_{*}$ be between $\left.Q\right|_{t=0}$ and $\left.Q\right|_{t=1 / 2}$. Considering the intersection of the horizontal line $Q=Q_{*}$ with the entire loop, we come to the points

$$
N=\left(\Pi_{N}, Q_{*}\right), \quad M=\left(\Pi_{M}, Q_{*}\right)
$$

There exist $t_{N}$ and $t_{M}$, such that

$$
0<t_{N}<1 / 2<t_{M}<1, Q\left(t_{N}\right)=Q\left(t_{M}\right)=Q_{*}, \Pi_{M}<\Pi_{N}, \Pi_{i}=\Pi\left(t_{i}\right), \quad i=N, M
$$

Let us choose points $R=\left(\Pi_{R}, Q_{R}\right)$ and $S=\left(\Pi_{S}, Q_{S}\right)$ in such a way that $Q_{R}<Q_{*}<$ $Q_{S}$. When $Q$ increases from $Q_{R}$ to $Q_{*}$, the value of $\Pi$ changes from $\Pi_{R}$ to

$$
\Pi_{N}=\lim _{Q \nearrow Q_{*}} \Pi(Q)
$$

If $Q$ decreases from $Q_{S}$ to $Q_{*}$, the value of $\Pi$ changes from $\Pi_{S}$ to

$$
\Pi_{M}=\lim _{Q \searrow Q_{*}} \Pi(Q)
$$

Thus, it is the prehistory of the evolution of the total flux that influences the pressure gradient. In relation to oil pipelines, the operating flow rate can be achieved either by switching from fast or slow flow. According to our model, the pressure drop to ensure the operating mode in the first case is less.

Now let us look at the friction losses that occur when pumping oil through pipelines. In dimension variables, the average speed $u$ and the friction coefficient $\lambda$ (dimensionless) are given by the formulas

$$
u=\frac{1}{2 H} \int_{-H}^{H} v(y) d y, \quad\left|p_{x}\right|=\frac{\lambda}{2 H} \frac{\rho u^{2}}{2} .
$$

In dimensionless variables, $u$ and $\lambda$ become

$$
U=\int_{0}^{1} v^{\prime}(y) d y=\frac{Q}{2} \quad \text { and } \quad \Lambda \equiv \frac{\operatorname{Re} \cdot \lambda}{8}=\frac{\Pi}{U^{2}}
$$

where $\Lambda$ is the reduced friction factor.
Let the pressure gradient change with time according to the formula

$$
\Pi(t)=\Pi_{0}(1+\sin \pi t)
$$

Calculations show that the curve

$$
U=U(t), \quad \Lambda(t)=\frac{\Pi(t)}{U^{2}(t)}, \quad 0<t<1
$$

behaves as shown in Figure 11. The time interval $0<t<1 / 2$ corresponds to the top part of this curve (see Figure 12a). Both $U$ and $\Pi$ grow along this segment, whereas $\Lambda$ decreases and $\Pi_{0}<\Pi<2 \Pi_{0}$. The time interval $1 / 2<t<1$ fits the curve in Figure 11, (see Figure 12b). The pressure gradient $\Pi$ decreases along this segment.


Figure 11. Hysteresis curve on the $(U, \Lambda)$ plane, with $U$ and $\Lambda$ standing for the mean velocity and the friction factor respectively. The data are the same as in Figure 9.

(a)

(b)

Figure 12. (a) The first part of the hysteresis loop in Figure 11, along which both $\Pi$ and $U$ increase $(0<t<1 / 2)$. (b) The second part of the hysteresis loop in Figure $11(1 / 2<t<1)$.

Let us discuss a mode for determining the friction factor $\Lambda$ corresponding to a mean velocity $U^{*}$. In practice, one need to increase or decrease $U$ to arrive at the value $U_{*}$. Given $U_{*}$ such that

$$
U_{\min }=\left.U\right|_{t=0}<U_{*}<U_{\max }=\left.U\right|_{t=1 / 2}
$$

one can determine $t_{1}$ and $t_{2}$ obeying the restrictions

$$
0<t_{1}<1 / 2<t_{2}<1, \quad U\left(t_{1}\right)=U\left(t_{2}\right)=U_{*}
$$

Denoting $\Lambda_{i}=\Lambda\left(t_{i}\right)$, we derive from Figure 11 that $\Lambda_{1}>\Lambda_{2}$ despite the fact that both $\Lambda_{1}$ and $\Lambda_{2}$ fit the same $U_{*}$. Hence,

$$
\Lambda_{1}=\lim _{U \nearrow U_{*}} \Lambda(U)=\left.\Lambda\right|_{U_{*}-}>\left.\Lambda\right|_{U_{*}+}=\lim _{U \searrow U_{*}} \Lambda(U)=\Lambda_{2}
$$

Let us return to the issue of oil pumping. In practice, the operating flow rate is carried out either by decreasing or increasing the flow. In the first case, the friction losses are less, in accordance with the known field data [9].

Let us introduce the total mass flux of particles $m$ :

$$
m=\int_{-H}^{H} \rho c v d y, \quad M=\int_{0}^{1} c v^{\prime} d y^{\prime}, \quad m=2 \rho H V M
$$

with $M$ standing for the dimensionless total mass flux of particles. We perform calculations starting from the data (70). The resulting parametric curve

$$
\begin{equation*}
\Pi=\Pi_{0}(1+\sin \pi t), \quad M=M(t), \quad 0<t<1 \tag{72}
\end{equation*}
$$

exhibits the hysteresis effect on the $(\Pi, M)$-plane, as is shown in Figure 9b. The form of this hysteresis loop is due to the fact that both the velocity and concentration vary in time and space.

The correlation of stress with velocity gradient is an important rheological issue. It follows from the dimensional steady-state Equation (58) that the shear stress $S_{12}$ obeys the equality

$$
S_{12}=p_{x} y, \quad \tilde{\tau} \equiv-\left.S_{12}\right|_{y=H}=-p_{x} H,
$$

with $\tilde{\tau}$ standing for the stress at the upper plane $y=H$. As a result of calculation, we arrive at the curve $\tilde{\tau}=\tilde{\tau}\left(\dot{\gamma}_{1}\right)$ where $\dot{\gamma}_{1}=-\left.v_{y}\right|_{y=H}$. The transition to dimensionless parameters leads to the formulas

$$
\tau=\frac{\tilde{\tau}}{2 \mu_{s} \Omega}=\Pi, \quad \dot{\gamma}=-\left.v_{y^{\prime}}^{\prime}\right|_{y^{\prime}=1^{\prime}} \quad \dot{\gamma}_{1}=\Omega \dot{\gamma}
$$

The study of the parametric curve

$$
\begin{equation*}
\Pi=\Pi_{0}(1+\sin \pi t), \quad \dot{\gamma}=\dot{\gamma}(t), \quad 0<t<1 \tag{73}
\end{equation*}
$$

results at the hysteresis loop $\Pi=\tau(\dot{\gamma})$, which is shown in Figure 13. The data for calculation were

$$
\begin{gather*}
\Pi_{0}=2.3, \Lambda=0.3, \Gamma=10.5, \varepsilon_{1}=0.4, \varepsilon_{20}=0.2, \varepsilon_{30}=0.2, \bar{j}_{0}=0.09  \tag{74}\\
K_{p}=0.1, K=0.1, \Lambda_{1}=0.001, c_{0}=0.2, \varphi_{0}=0 .
\end{gather*}
$$

Thus, shear stress cannot be unambiguously calculated from the velocity gradient. This effect was observed in worm-like micellar solutions [43].


Figure 13. The stress-strain hysteresis loop $\Pi=\Pi(\dot{\gamma})$.
We define the apparent viscosity $\eta_{a}=\tilde{\tau} / \dot{\gamma}_{1}$, with $\eta=\tau / \dot{\gamma}$ being its dimensionless replica. Performing calculations for the case $\Pi=$ const and the data (74), we arrive at Figure 14 , showing how $\eta$ varies with time.

Sustained temporal oscillations of macroscopic viscosity are observed in [44] for the rod-like suspension.

As far as the function $\eta=\eta(\dot{\gamma})$ is concerned, the shear thinning nature of suspensions of rod-like particles is clearly depicted in Figure 15, in agreement with the observations in [1] (the calculations are based on the data in (74)).


Figure 14. Dynamics of apparent viscosity when $\Pi=\Pi_{0}=$ const.


Figure 15. Correlation between the apparent viscosity $\eta$ and the velocity gradient $\dot{\gamma}$ when $\Pi=\Pi_{0}=$ const.

## 5. Conclusions

Within the framework of the theory of micropolar fluids, conservation laws consistent with thermodynamics are derived for suspensions of non-spherical particles. Constitutive equations are formulated in such a way that they also agree with the thermodynamic laws. In addition, these equations satisfy the principle of objectivity and support anisotropy, as they describe the response of stress not only to the strain rate, but also to the micro-inertia tensor. Particle rotation and the law of conservation of angular momentum underlie the theory of micropolar fluid. The Cauchy stress tensor becomes asymmetric and it is necessary to involve the couple stress tensor. Rheological constitutive equations are formulated for these tensors with the use of two rates of strain tensors. The influence of deformation rates on the local stress state is manifested through viscosity effects. To this end, we introduced skew-symmetric and anisotropic viscosities in addition to shear viscosity. The role of these viscosities was studied in detail for flows of suspensions of rod-like particles. Calculations carried out for pressure-driven flows explained a number of effects, including hysteresis, gradient banding and development of the nematic phase. In addition, these calculations predicted the formation of concentration inhomogeneities. In relation to the transportation of oil through pipelines, it was shown that the flow rate is determined by the history of the pumps.

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## Appendix A

Lemma A1. It follows from the inertia tensor Equation (8) that $\boldsymbol{\omega} \cdot \dot{\mathbf{s}}=\mathbf{s} \cdot \dot{\boldsymbol{\omega}}$ and $\frac{d}{d t}(\boldsymbol{\omega} \cdot \mathbf{s})=0$ provided $\dot{\mathbf{s}}=0$.

Proof. Observe that $\Omega^{*}=-\Omega, \Omega \omega=0$ and $J^{*}=J$. Now, the proof follows from the following equalities:

$$
\omega \cdot \dot{\mathbf{s}}=\omega \cdot[(\Omega J-J \Omega) \omega]+\omega \cdot(J \dot{\omega})=\left(\Omega^{*} \omega\right) \cdot(J \omega)-(J \omega) \cdot(\Omega \omega)+\mathbf{s} \cdot \dot{\omega}
$$

Lemma A2. The internal energy e for the non-dissipative flows of micropolar fluids obeys the equation

$$
\dot{e}=\frac{\dot{\rho}(p+e)}{\rho} .
$$

Proof. We start from the hydrodynamic Equations (9) and the thermodynamic relationships (12) and (13). Clearly,

$$
\dot{e}=\theta \dot{\eta}+\mu \dot{\rho}+z \frac{d}{d t}(\rho c) .
$$

On the other hand, the equations

$$
\frac{d}{d t}\left(\frac{\eta}{\rho}\right)=0, \quad \dot{c}=0
$$

imply that

$$
\dot{\eta}=\rho^{-1} \eta \dot{\rho}, \quad \frac{d}{d t}(\rho c)=c \dot{\rho}
$$

Hence,

$$
\dot{e}=\dot{\rho}(\theta \eta+\rho \mu+z c \rho) \rho^{-1} .
$$

Now, the proof follows from the definition (13) of pressure.
Lemma A3. The total energy E defined by formula (14) for the non-dissipative flows of micropolar fluids obeys the conservation law (6).

Proof. Using the above claims and the equalities $\dot{c}=0$ and $d(\mathbf{s} \cdot \boldsymbol{\omega}) / d t=0$, we calculate that

$$
\dot{E}=\dot{\rho} \rho^{-1}(p+E)+\rho \mathbf{v} \cdot \dot{\mathbf{v}}
$$

Eliminating the material derivatives $\dot{\rho}$ and $\dot{\mathbf{v}}$ by Equation (9), we obtain that

$$
\dot{E}=-\operatorname{div}((p+E) \mathbf{v})+\mathbf{v} \cdot \nabla E
$$

Thus, the claim is proved.
Lemma A4. The quality

$$
\boldsymbol{\omega} \cdot\left(J_{t} \boldsymbol{\omega}\right)=(\mathbf{s} \cdot \boldsymbol{\omega}) \operatorname{div} \mathbf{v}-\operatorname{div}((\mathbf{s} \cdot \boldsymbol{\omega}) \mathbf{v})+2 \nabla \boldsymbol{\omega}: \mathbf{s} \otimes \mathbf{v}
$$

holds.
Proof. Due to Equation (8) and the equalities $\Omega \boldsymbol{\omega}=0, J^{*}=J$, we can write

$$
\boldsymbol{\omega} \cdot\left(J_{t} \boldsymbol{\omega}\right)=\omega_{i} \omega_{j}\left(J_{i j}\right)_{t}=\omega_{i} \omega_{j}\left(\Omega_{i k} J_{k j}-J_{i k} \Omega_{k j}\right)+s_{i} \omega_{i} \operatorname{div} \mathbf{v}-
$$

$$
\begin{gathered}
\frac{\partial}{\partial x_{n}}\left(J_{i j} \omega_{i} \omega_{j} v_{n}\right)+J_{i j} v_{n}\left(\omega_{i} \frac{\partial \omega_{j}}{\partial x_{n}}+\omega_{j} \frac{\partial \omega_{i}}{\partial x_{n}}\right)= \\
(\mathbf{s} \cdot \boldsymbol{\omega}) \operatorname{div} \mathbf{v}-\operatorname{div}((\mathbf{s} \cdot \boldsymbol{\omega}) \mathbf{v})+2 \nabla \boldsymbol{\omega}: \mathbf{s} \otimes \mathbf{v} .
\end{gathered}
$$

Hence, the claim is proved.
Lemma A5. The quality

$$
(\mathbf{s} \cdot \boldsymbol{\omega})_{t}=2 \boldsymbol{\omega} \cdot \mathbf{s}_{t}-\boldsymbol{\omega} \cdot\left(J_{t} \boldsymbol{\omega}\right)
$$

holds.

Proof. Due to symmetry of $J$, we have

$$
(\mathbf{s} \cdot \boldsymbol{\omega})_{t}=\mathbf{s}_{t} \cdot \boldsymbol{\omega}+\mathbf{s} \cdot \boldsymbol{\omega}_{t}=\left(J_{t} \boldsymbol{\omega}\right) \cdot \boldsymbol{\omega}+\boldsymbol{\omega} \cdot\left(J \boldsymbol{\omega}_{t}\right)+\mathbf{s} \cdot \omega_{t}=\left(J_{t} \boldsymbol{\omega}\right) \cdot \boldsymbol{\omega}+2 \mathbf{s} \cdot \boldsymbol{\omega}_{t}
$$

On the other hand, it follows from

$$
\omega \cdot \mathbf{s}_{t}=\omega \cdot(J \omega)_{t}=\omega \cdot\left(J_{t} \omega+J \omega_{t}\right)=\mathbf{s} \cdot \omega_{t}+\omega \cdot\left(J_{t} \omega\right)
$$

that

$$
\mathbf{s} \cdot \boldsymbol{\omega}_{t}=\boldsymbol{\omega} \cdot \mathbf{s}_{t}-\boldsymbol{\omega} \cdot\left(J_{t} \boldsymbol{\omega}\right)
$$

Hence,

$$
(\mathbf{s} \cdot \boldsymbol{\omega})_{t}=2\left(\boldsymbol{\omega} \cdot \mathbf{s}_{t}-\boldsymbol{\omega} \cdot\left(J_{t} \boldsymbol{\omega}\right)\right)+\boldsymbol{\omega} \cdot\left(J_{t} \boldsymbol{\omega}\right)
$$

and the claim is proved.
It follows from the above claims that

$$
\begin{equation*}
(\mathbf{s} \cdot \boldsymbol{\omega})_{t}=2 \boldsymbol{\omega} \cdot \mathbf{s}_{t}-[(\mathbf{s} \cdot \boldsymbol{\omega}) \operatorname{div} \mathbf{v}-\operatorname{div}((\mathbf{s} \cdot \boldsymbol{\omega}) \mathbf{v})+2 \nabla \boldsymbol{\omega}: \mathbf{s} \otimes \mathbf{v}] \tag{A1}
\end{equation*}
$$

Lemma A6. The quality

$$
\begin{aligned}
& \frac{\partial}{\partial t}\left(\frac{\rho c(\mathbf{s} \cdot \boldsymbol{\omega})}{2}\right)=\boldsymbol{\omega} \cdot(\rho c \mathbf{s})_{t}-\frac{\mathbf{s} \cdot \boldsymbol{\omega}}{2}(\rho c)_{t}+ \\
& \rho c\left(\operatorname{div}\left(\frac{\mathbf{s} \cdot \boldsymbol{\omega}}{2} \mathbf{v}\right)-\frac{\mathbf{s} \cdot \boldsymbol{\omega}}{2} \operatorname{div} \mathbf{v}-\nabla \boldsymbol{\omega}: \mathbf{s} \otimes \mathbf{v}\right)
\end{aligned}
$$

holds.

Proof. We have

$$
\frac{\partial}{\partial t}\left(\frac{\rho c(\mathbf{s} \cdot \boldsymbol{\omega})}{2}\right)=\frac{\mathbf{s} \cdot \boldsymbol{\omega}}{2}(\rho c)_{t}+\frac{\rho c}{2}(\mathbf{s} \cdot \boldsymbol{\omega})_{t}, \quad \rho c \boldsymbol{\omega} \cdot \mathbf{s}_{t}=\boldsymbol{\omega} \cdot\left[(\rho c \mathbf{s})_{t}-\mathbf{s}(\rho c)_{t}\right] .
$$

Now, the proof follows from (A1).
Lemma A7. The quality

$$
\frac{\partial}{\partial t}\left(\frac{\rho \mathbf{v} \cdot \mathbf{v}}{2}\right)=\frac{\mathbf{j} \cdot \mathbf{j}_{t}}{\rho}-\frac{\mathbf{j} \cdot \mathbf{j} \rho_{t}}{2 \rho^{2}}
$$

holds.
The proof follows from the representation formula $\rho \mathbf{v} \cdot \mathbf{v}=\mathbf{j} \cdot \mathbf{j} / \rho$.
Lemma A8. Equation (18) holds.
Proof. Using the conservation laws and the formulas $\Pi=p I+\mathbf{j} \otimes \mathbf{v}$ and $M=c \rho \mathbf{s} \otimes \mathbf{v}$, we calculate that

$$
\begin{gathered}
D_{1}=-\operatorname{div}\left(\mathbf{v} \rho \mu-\mathbf{v} \rho v^{2} / 2\right)+\rho \mathbf{v} \cdot \nabla \mu-\rho \nabla \mathbf{v}: \mathbf{v} \otimes \mathbf{v} \\
D_{2}=-\operatorname{div}(\mathbf{v} \theta \eta+\mathbf{q})+R+\eta \mathbf{v} \cdot \nabla \theta+\frac{\mathbf{q} \cdot \nabla \theta}{\theta}, \\
D_{3}=-\operatorname{div}[(c \mathbf{j}+\mathbf{l})(z-\mathbf{s} \cdot \boldsymbol{\omega} / 2)]+(c \mathbf{j}+\mathbf{l}) \cdot \nabla(z-\mathbf{s} \cdot \boldsymbol{\omega} / 2), \\
D_{4}=-\operatorname{div}\left(p \mathbf{v}+\mathbf{v} \rho v^{2}+\Pi_{d}^{*} \mathbf{v}\right)+p \operatorname{div} \mathbf{v}+\nabla \mathbf{v}:\left(\mathbf{j} \otimes \mathbf{v}+\Pi_{d}\right), \\
D_{5}=-\operatorname{div}\left((\mathbf{s} \cdot \boldsymbol{\omega}) c \rho \mathbf{v}+M_{d}^{*} \boldsymbol{\omega}\right)+c \rho \nabla \boldsymbol{\omega}: \mathbf{s} \otimes \mathbf{v}+\boldsymbol{\omega} \cdot \mathbf{f}_{d}+\nabla \boldsymbol{\omega}: M_{d} .
\end{gathered}
$$

Applying the equality $\nabla p=\rho \nabla \mu+\eta \nabla \theta+\rho c \nabla z$, which follows from (13), and looking for $z$ in the form $z=z_{1}+\mathbf{s} \cdot \boldsymbol{\omega} / 2$, we find that

$$
\sum_{1}^{5} D_{i}=-\operatorname{div} \mathbf{Q}_{1}+L
$$

with

$$
\begin{gathered}
\mathbf{Q}_{1}=p \mathbf{v}+(\rho \mu+\theta \eta+c \rho z) \mathbf{v}+\mathbf{v}\left(\frac{\rho v^{2}}{2}+\frac{c \rho(\mathbf{s} \cdot \boldsymbol{\omega})}{2}\right)+\mathbf{q}+1 z_{1}+\Pi_{d}^{*} \mathbf{v}+M_{d}^{*} \boldsymbol{\omega}, \\
L=\mathbf{v} \cdot \nabla p-\rho c \mathbf{v} \cdot \frac{\nabla(\mathbf{s} \cdot \boldsymbol{\omega})}{2}+p \operatorname{div} \mathbf{v}+\Pi_{d}: \nabla \mathbf{v}+c \rho \nabla \boldsymbol{\omega}: \mathbf{s} \otimes \mathbf{v}+\nabla \boldsymbol{\omega}: M_{d}+ \\
R+\frac{\mathbf{q} \cdot \nabla \theta}{\theta}+\mathbf{l} \cdot \nabla z_{1}+\boldsymbol{\omega} \cdot \mathbf{f}_{d} .
\end{gathered}
$$

Now, we use the formula $p=-e+\rho \mu+\theta \eta+c \rho z$ and add $D_{6}$. As a result, we obtain

$$
\sum_{1}^{6} D_{i}=-\operatorname{div}\left(\mathbf{Q}+\mathbf{Q}_{d}\right)+g
$$

with the notations in (18). The claim is proved.

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