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Global Regular Axially Symmetric Solutions to the Navier–Stokes Equations: Part 1

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Abstract: The axially symmetric solutions to the Navier–Stokes equations are considered in a bounded cylinder $\Omega \subset \mathbb{R}^3$ with the axis of symmetry. S_1 is the boundary of the cylinder parallel to the axis of symmetry and S_2 is perpendicular to it. We have two parts of S_2. For simplicity, we assume the periodic boundary conditions on S_2. On S_1, we impose the vanishing of the normal component of velocity, the angular component of velocity, and the angular component of vorticity. We prove the existence of global regular solutions. To prove this, it is necessary that the coordinate of velocity along the axis of symmetry vanishes on it. We have to emphasize that the technique of weighted spaces applied to the stream function plays a crucial role in the proof of global regular axially symmetric solutions. The weighted spaces used are such that the stream function divided by the radius must vanish on the axis of symmetry. Currently, we do not know how to relax this restriction. In part 2 of this topic, the periodic boundary conditions on S_2 are replaced by the conditions that both the normal component of velocity and the angular component of vorticity must vanish. Moreover, it is assumed that the normal derivative of the angular component of velocity also vanishes on S_2. A transformation from part 1 to part 2 is not trivial because it needs new boundary value problems, so new estimates must be derived.

Keywords: Navier–Stokes equations; axially symmetric solutions; cylindrical domain; existence of global regular solutions

MSC: 35A01; 35B01; 35B65; 35Q30; 76D03; 76D05

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1. Introduction

The regularity problem for axially symmetric solutions to the Navier–Stokes equations has a long history. However, there are only two results where the global regular axially symmetric solutions are proved, assuming the vanishing of the angular component of velocity (see papers [1] by O.A. Ladyzhenskaya and [2] by M.R. Ukhovskii and V.I. Yudovich).

Other results (see the papers cited in [3–7]) describe the existence of global regular axially symmetric solutions imposing different Serrin-type conditions. The conditions are such that certain coordinates, either of velocity or of derivatives of velocity or vorticity, belong to $L_q(0T; L_p(\Omega))$ spaces for appropriately chosen parameters p and q.

This paper closely aligns with the results presented by O.A. Ladyzhenskaya, M.R. Ukhovskii, and V.I. Yudovich, as the vanishing of the stream function divided by the radius implies the existence of global regular axially symmetric solutions. The aim of this paper is to provide a proof of the global estimate (24).

The estimate can imply any global regularity of solutions to problem (6), assuming appropriate regularity of data.

We must emphasize that the methods and proofs presented in this paper are completely new. The proofs and results in Sections 3, 5, and 6 are original.

Before the formal introduction starts, we outline the main steps of the proof of Theorem 1. The main difficulty in the regularity theory of the Navier–Stokes equations lies in handling the nonlinear terms. We need to transform them in such a way that they can be absorbed by the main linear terms. In this paper, we consider problems (17)–(20) for functions Φ and Γ defined by (16). Applying the energy method, we derive inequality (111) with a strongly nonlinear term denoted by I_3.

The main task of this paper is to estimate I_3 by quantities that can be absorbed by the terms from the l.h.s. of (111).

I_3 is estimated in (122). Using notation (132), we derive from (111) and (122) the inequality (see (134))

$$(*) \qquad X^2 \le \phi_1 X^{2-\delta} + \phi(\text{data}),$$

where ϕ_1 depends on $|v_\varphi|_{d,\infty,\Omega^t}$, $|v_\varphi|_{\infty,\Omega^t}$ and $d > 3$, $\delta > 0$.

For $\delta > 0$, the Young inequality can be applied, so (133) holds. The existence of such a positive δ follows from inequality (173), which can be written in the following form:

$$(**) \qquad |\Phi|^2_{2,\Omega^t} \le \phi_2 \|\Gamma\|_{1,2,\Omega^t} + \phi(\text{data}),$$

where ϕ_2 depends on $|v_\varphi|_{\infty,\Omega^t}$.

Inequality $(**)$ implies the existence of the positive δ. For $\delta = 0$, we were not able to apply the Young inequality in $(*)$, so we could not prove Theorem 1.

Hence, $(**)$ is probably the most important inequality in this paper. It is a totally new result.

In the next step, we eliminate $|v_\varphi|_{d,\infty,\Omega^t}$, $d = 12$. To show this, we need to delve into the proof of Lemma 13. To derive (141) from (140), we need the estimate

$$(***) \qquad \int_\Omega \frac{\psi_1^2}{r^{6'}} dx \le c\|\Gamma\|^2_{1,\Omega}.$$

The Hardy inequality implies that $(***)$ does not hold for 6 but holds for any number less than 6. It is denoted by $6'$. Inequality $(***)$ follows from (202).

Then, we derive (145). Using (133) in (145) yields the following inequality:

$$(****) \qquad |v_\varphi|^{6'}_{12,\infty,\Omega^t} \le c|v_\varphi|^{\frac{4\varepsilon}{\theta}}_{12,\infty,\Omega^t} + \phi(\text{data}).$$

To apply the Young inequality in $(****)$, we require that $6' > \frac{4\varepsilon}{\theta}$. It is shown in Remark 4 that the inequality holds. We need $6'$ to be close to 6. Then, $(****)$ implies (137).

Moreover, to prove (137), we need the existence of such solutions to problem (6), where v_φ is not small. The existence of such solutions is proven in Appendix A. Hence, for such local solutions, we prove global estimate (24). Once we have (24), we can extend the local solution incrementally over time.

Finally, we can easily derive estimate (152) because $|v_\varphi|_{\infty,,\Omega^t}$ appears in (137) with arbitrarily small power.

Using estimates (137) and (152) in (133) implies (24) and proves Theorem 1.

In this paper, we prove the existence of global regular axially symmetric solutions to the Navier–Stokes equations in a cylindrical domain $\Omega \subset \mathbb{R}^3$:

$$\Omega = \{x \in \mathbb{R}^3 \colon x_1^2 + x_2^2 < R^2, |x_3| < a\},$$

where a and R are the given positive numbers. We denote by $x = (x_1, x_2, x_3)$ the Cartesian coordinates. It is assumed that the x_3-axis is the axis of symmetry of Ω.

Moreover,

$$
\begin{aligned}
&S_1 = \{x \in \mathbb{R}^3 \colon \sqrt{x_1^2 + x_2^2} = R, x_3 \in (-a, a)\},\\
&S_2 = S_2(-a) \cup S_2(a) \quad \text{and}\\
&S_2(a_0) = \{x \in \mathbb{R}^3 \colon \sqrt{x_1^2 + x_2^2} < R, x_3 = a_0 \in \{-a, a\}\},
\end{aligned}
$$

where S_1 is parallel to the axis of symmetry and $S_2(a_0)$ is perpendicular to it. $S_2(a_0)$ meets the axis of symmetry at a_0.

To describe the considered problem, we introduce cylindrical coordinates r, φ, and z by the relations

$$
x_1 = r\cos\varphi, \quad x_2 = r\sin\varphi, \quad x_3 = z. \tag{1}
$$

The following orthonormal system

$$
\bar{e}_r = (\cos\varphi, \sin\varphi, 0), \;\; \bar{e}_\varphi = (-\sin\varphi, \cos\varphi, 0), \;\; \bar{e}_z = (0, 0, 1) \tag{2}
$$

is connected to the cylindrical coordinates.

Any vector, u, for the axially symmetric motions can be decomposed as follows:

$$
u = u_r(r,z,t)\bar{e}_r + u_\varphi(r,z,t)\bar{e}_\varphi + u_z(r,z,t)\bar{e}_z, \tag{3}
$$

where u_r, u_φ, and u_z are cylindrical coordinates of u.

Therefore, velocity v and vorticity $\omega = \operatorname{rot} v$ are decomposed in the form

$$
v = v_r(r,z,t)\bar{e}_r + v_\varphi(r,z,t)\bar{e}_\varphi + v_z(r,z,t)\bar{e}_z \tag{4}
$$

and

$$
\omega = \omega_r(r,z,t)\bar{e}_r + \omega_\varphi(r,z,t)\bar{e}_\varphi + \omega_z(r,z,t)\bar{e}_z. \tag{5}
$$

The paper is devoted to the proof of global regular axially symmetric solutions to the problem

$$
\begin{array}{ll}
v_{,t} + v\cdot\nabla v - \nu\Delta v + \nabla p = f & \text{in } \Omega^T = \Omega\times(0,T),\\
\operatorname{div} v = 0 & \text{in } \Omega^T,\\
v \text{ satisfies periodic boundary conditions} & \text{on } S_2^T = S_2\times(0,T),\\
v\cdot\bar{n}|_{S_1} = 0, \;\; \omega_\varphi|_{S_1} = 0, \;\; v_\varphi|_{S_1} = 0 & \text{on } S_1^T = S_1\times(0,T),\\
v|_{t=0} = v(0) & \text{in } \Omega,
\end{array} \tag{6}
$$

where $v = (v_1(x,t), v_2(x,t), v_3(x,t)) \in \mathbb{R}^3$ is the velocity of the fluid, $p = p(x,t) \in \mathbb{R}$ is the pressure, $f = (f_1(x,t), f_2(x,t), f_3(x,t)) \in \mathbb{R}^3$ is the external force field, and $\nu > 0$ is the constant viscosity coefficient.

Expressing problem (6) in the cylindrical coordinates of velocity yields

$$
\begin{aligned}
&v_{r,t} + v\cdot\nabla v_r - \frac{v_\varphi^2}{r} - \nu\Delta v_r + \nu\frac{v_r}{r^2} = -p_{,r} + f_r,\\
&v_{\varphi,t} + v\cdot\nabla v_\varphi + \frac{v_r}{r}v_\varphi - \nu\Delta v_\varphi + \nu\frac{v_\varphi}{r^2} = f_\varphi,\\
&v_{z,t} + v\cdot\nabla v_z - \nu\Delta v_z = -p_{,z} + f_z,\\
&(rv_r)_{,r} + (rv_z)_{,z} = 0\\
&v_r|_{S_1} = 0, \;\; v_\varphi|_{S_1} = 0, \;\; v_{r,z} - v_{z,r}|_{S_1} = 0,\\
&v_r|_{t=0} = v_r(0), \;\; v_\varphi|_{t=0} = v_\varphi(0), \;\; v_z|_{t=0} = v_z(0),
\end{aligned} \tag{7}
$$

where we have the periodic boundary conditions on S_2 and

$$\begin{aligned} &v\cdot\nabla=(v_r\bar{e}_r+v_z\bar{e}_z)\cdot\nabla=v_r\partial_r+v_z\partial_z,\\ &\Delta u=\frac{1}{r}(ru_{,r})_{,r}+u_{,zz}. \end{aligned} \tag{8}$$

Formulating problem (6) in terms of the cylindrical coordinates of vorticity implies

$$\begin{aligned} &\omega_{r,t}+v\cdot\nabla\omega_r-\nu\Delta\omega_r+\nu\frac{\omega_r}{r^2}=\omega_rv_{r,r}+\omega_zv_{r,z}+F_r,\\ &\omega_{\varphi,t}+v\cdot\nabla\omega_\varphi-\frac{v_r}{r}\omega_\varphi-\nu\Delta\omega_\varphi+\nu\frac{\omega_\varphi}{r^2}=\frac{2}{r}v_\varphi v_{\varphi,z}+F_\varphi,\\ &\omega_{z,t}+v\cdot\nabla\omega_z-\nu\Delta\omega_z=\omega_rv_{z,r}+\omega_zv_{z,z}+F_z,\\ &\omega_r|_{t=0}=\omega_r(0),\ \ \omega_\varphi|_{t=0}=\omega_\varphi(0),\ \ \omega_z|_{t=0}=\omega(0) \end{aligned} \tag{9}$$

and we have boundary conditions $(7)_5$ on S_1 and the periodic boundary conditions on S_2, where $F=\operatorname{rot} f$ and

$$F=F_r(r,z,t)\bar{e}_r+F_\varphi(r,z,t)\bar{e}_\varphi+F_z(r,z,t)\bar{e}_z. \tag{10}$$

The function

$$u=rv_\varphi \tag{11}$$

is called swirl. It is a solution to the problem

$$\begin{aligned} &u_{,t}+v\cdot\nabla u-\nu\Delta u+\frac{2\nu}{r}u_{,r}=rf_\varphi\equiv f_0,\\ &u|_{S_1}=0 \text{ and } u \text{ satisfies periodic boundary conditions on } S_2,\\ &u|_{t=0}=u(0). \end{aligned} \tag{12}$$

The cylindrical components of vorticity can be described in terms of the cylindrical components of the velocity and swirl in the following form

$$\begin{aligned} &\omega_r=-v_{\varphi,z}=-\frac{1}{r}u_{,z},\\ &\omega_\varphi=v_{r,z}-v_{z,r},\\ &\omega_z=\frac{1}{r}(rv_\varphi)_{,r}=v_{\varphi,r}+\frac{v_\varphi}{r}=\frac{1}{r}u_{,r}. \end{aligned} \tag{13}$$

Equation $(7)_4$ implies the existence of the stream function ψ, which is a solution to the problem

$$\begin{aligned} &-\Delta\psi+\frac{\psi}{r^2}=\omega_\varphi,\\ &\psi|_{S_1}=0,\\ &\psi \text{ satisfies periodic boundary conditions on } S_2. \end{aligned} \tag{14}$$

Moreover, cylindrical components of velocity can be expressed in terms of the stream function in the following way

$$\begin{aligned} &v_r=-\psi_{,z},\ \ v_z=\frac{1}{r}(r\psi)_{,r}=\psi_{,r}+\frac{\psi}{r},\\ &v_{r,r}=-\psi_{,zr},\ \ v_{r,z}=-\psi_{,zz},\\ &v_{z,z}=\psi_{,rz}+\frac{\psi_{,z}}{r},\ \ v_{z,r}=\psi_{,rr}+\frac{1}{r}\psi_{,r}-\frac{\psi}{r^2}. \end{aligned} \tag{15}$$

Introduce the pair

$$(\Phi,\Gamma)=(\omega_r/r,\omega_\varphi/r). \tag{16}$$

Formula (6) from [8] implies that quantities (16) satisfy the following equations

$$\Phi_{,t} + v\cdot\nabla\Phi - \nu\left(\Delta + \frac{2}{r}\partial_r\right)\Phi - (\omega_r\partial_r + \omega_z\partial_z)\frac{v_r}{r} = F_r/r \equiv \bar{F}_r \tag{17}$$

and

$$\Gamma_{,t} + v\cdot\nabla\Gamma - \nu\left(\Delta + \frac{2}{r}\partial_r\right)\Gamma + 2\frac{v_\varphi}{r}\Phi = F_\varphi/r \equiv \bar{F}_\varphi. \tag{18}$$

We add the following initial and boundary conditions to solutions of (17) and (18)

$$\begin{aligned}&\Phi|_{S_1} = 0,\ \ \Gamma|_{S_1} = 0,\ \Phi,\ \Gamma \text{ satisfy the periodic}\\ &\text{boundary conditions on } S_2,\end{aligned} \tag{19}$$

$$\Phi|_{t=0} = \Phi(0),\ \ \Gamma|_{t=0} = \Gamma(0). \tag{20}$$

Next, we express cylindrical coordinates of velocity in terms of $\psi_1 = \psi/r$

$$\begin{aligned}&v_r = -r\psi_{1,z}, && v_z = (r\psi_1)_{,r} + \psi_1 = r\psi_{1,r} + 2\psi_1,\\ &v_{r,r} = -\psi_{1,z} - r\psi_{1,rz}, && v_{r,z} = -r\psi_{1,zz},\\ &v_{z,z} = r\psi_{1,rz} + 2\psi_{1,z}, && v_{z,r} = 3\psi_{1,r} + r\psi_{1,rr}.\end{aligned} \tag{21}$$

The aim of this paper is to prove the existence of global regular axially symmetric solutions to problem (6). For this purpose, we have to find a global estimate that guarantees the existence of global regular solutions.

Function ψ_1 is a solution to the problem

$$\begin{aligned}&-\Delta\psi_1 - \frac{2}{r}\psi_{1,r} = \omega_1 \quad \text{in } \Omega = (0,R)\times(-a,a),\\ &\psi_1|_{r=R} = 0,\\ &\psi_1 \text{ satisfies the periodic boundary conditions on } S_2,\end{aligned} \tag{22}$$

where

$$\omega_1 = \omega_\varphi/r. \tag{23}$$

We have $\omega_1 = \Gamma$.

To state the main result, we first introduce the necessary assumptions.

Assumption 1. *Assume that the following quantities are finite:*

$$\begin{aligned}&D_1 = \|f\|_{L_2(\Omega^t)} + \|v(0)\|_{L_2(\Omega)},\\ &D_2 = \|f_0\|_{L_{\infty,1}(\Omega^t)} + \|u(0)\|_{L_\infty(\Omega)},\\ &f_0 = rf_\varphi,\ \ u = rv_\varphi,\\ &D_3^2 = D_1^2D_2^2 + \|u_{,z}(0)\|^2_{L_2(\Omega)} + \|f_0\|^2_{L_2(\Omega^t)},\\ &D_4^2 = D_1^2(1+D_2) + \|u_{,r}(0)\|^2_{L_2(\Omega)} + \|f_0\|^2_{L_2(\Omega^t)} + \|f_0\|_{L_2(0,t;L_{4/3}(S_1))},\end{aligned}$$

where D_1 and D_2 are introduced in (46) and (52), respectively, and D_3 and D_4 are introduced in (159) and (160), respectively. Let

$$\begin{aligned}&D_5 = D_2(D_1 + D_2 + D_3),\\ &D_6 = D_2^{1-\varepsilon_0}D_3,\end{aligned}$$

where ε_0 is an arbitrarily small positive number. Moreover,

$$D_7 = \|F_r\|^2_{L_2(0,t;L_{6/5}(\Omega))} + \|F_z\|^2_{L_2(0,t;L_{6/5}(\Omega))} + \|\omega_r(0)\|^2_{L_2(\Omega)} + \|\omega_z(0)\|^2_{L_2(\Omega)}$$

is defined in Lemma 16.

Next,

$$D_8 = \phi(D_2)(\|\bar{F}_r\|^2_{L_2(0,t;L_{6/5}(\Omega))} + \|\bar{F}_\varphi\|^2_{L_2(0,t;L_{6/5}(\Omega))}) + \|\Phi(0)\|^2_{L_2(\Omega)} + \|\Gamma(0)\|^2_{L_2(\Omega)},$$

where $\bar{F}_r = F_r/r$, $\bar{F}_\varphi = F_\varphi/r$, $\Phi = \frac{\omega_r}{r}$, $\Gamma = \frac{\omega_\varphi}{r}$ and D_8 appear in (111).

In Lemma 13, the following quantity is defined

$$D_9(12) = 12\|f_\varphi\|_{L_{12}(0,t;L_{36/25}(\Omega))} + \|v_\varphi(0)\|_{L_{12}(\Omega)}.$$

Finally, in Lemma 14, we introduce the quantity

$$D_{10} = \|f_\varphi/r\|_{L_1(0,t;L_\infty(\Omega))} + \|v_\varphi(0)\|_{L_\infty(\Omega)}.$$

The main result is as follows:

Theorem 1. *Assume that Assumption 1 holds. Then, there exists an increasing positive function ϕ, such that*

$$\|\Phi\|_{V(\Omega^t)} + \|\Gamma\|_{V(\Omega^t)} \le \phi(D_1, \cdots, D_{10}). \tag{24}$$

Remark 1. *Estimate (24) implies any regularity of solutions to problem (6), assuming sufficient regularity of data.*

To prove (24), we require that ψ_1 and v_z vanish on the axis of symmetry.

Proof of Theorem 1. Inequality (113) in the form

$$\frac{d}{dt}|\Phi|^2_{2,\Omega} + |\nabla\Phi|^2_{2,\Omega} \le I + \int_\Omega \bar{F}_r \Phi dx \tag{25}$$

is the first step of the proof of (24), where $\Phi = -\frac{v_{\varphi,z}}{r}$, $\bar{F}_r = \frac{F_r}{r}$ and

$$I \le \int_\Omega \left| v_\varphi \partial_r \frac{v_r}{r} \Phi_{,z} \right| dx + \int_\Omega \left| v_\varphi \partial_z \frac{v_r}{r} \Phi_{,r} \right| dx \equiv I_1 + I_2.$$

Our aim is to estimate I_1 and I_2 by a product of norms $\|\Phi\|_{V(\Omega^t)}$, $\|\Gamma\|_{V(\Omega^t)}$.

Since the L_∞-estimate of swirl rv_φ is bounded by D_2 (see Lemma 2) and $\frac{v_r}{r} = -\psi_{1,z}$, we obtain the estimates

$$\begin{aligned} I_1 &\le D_2 \left| \frac{\psi_{1,rz}}{r} \right|_{2,\Omega} |\Phi_{,z}|_{2,\Omega}, \\ I_2 &\le D_2 \left| \frac{\psi_{1,zz}}{r} \right|_{2,\Omega} |\Phi_{,r}|_{2,\Omega}. \end{aligned} \tag{26}$$

To examine estimate (26), we recall that ψ_1 is a solution to problem (22).

In Lemma 4, we prove the existence of weak solutions to problem (22) and derive the estimate (56)

$$\|\psi_1\|_{1,\Omega} \le c|w_1|_{6/5,\Omega}. \tag{27}$$

In Section 3, we increase the regularity of weak solutions by deriving estimates for higher derivatives.

From (82), we have

$$\left|\frac{\psi_{1,rz}}{r}\right|_{2,\Omega} \le c|\Gamma_{,z}|_{2,\Omega}. \tag{28}$$

The estimate holds for the weak solutions to problem (22) because [9] yields the expansion of ψ_1 near the axis of symmetry

$$\psi_1 = a_1(z,t) + a_2(z,t)r^2 + a_3(z,t)r^4 + \cdots \tag{29}$$

Hence, $\psi_{1,r} = 2a_2(z,t)r$ and the norm $|\frac{\psi_{1,rz}}{r}|_{2,\Omega}$ can be finite.

To estimate I_2, we need

$$\left|\frac{\psi_{1,zz}}{r}\right|_{2,\Omega} \le c|\Gamma_{,z}|_{2,\Omega}. \tag{30}$$

The estimate holds for such a class of regularized weak solutions to problem (22), where

$$\psi_1|_{r=0} = 0. \tag{31}$$

This means that in expansion (29), we have $a_1(z,t) = 0$.

The existence of solutions to problem (22) (also see (61)), satisfying restriction (31) and estimate (30), follows from the theory developed by Kondratiev (see [10]) for elliptic boundary value problems in domains with cones in weighted Sobolev spaces.

In this paper, the existence is proven in Lemmas 8 and 17. From [10], it also follows that we can prove the existence of different solutions to problem (22) belonging to different weighted Sobolev spaces.

The difference between such two distinct solutions is equal to the expression derived from the Cauchy theorem for complex functions related to contour integration. This is described in more detail in [11].

Restriction (31) means that we have to work with a very restricted class of weak solutions to (22). This also means that v_z must vanish on the axis of symmetry.

Using estimates (28) and (30) in (25) yields

$$\frac{d}{dt}|\Phi|_{2,\Omega}^2 + |\nabla\Phi|_{2,\Omega}^2 \le cD_2|\Gamma_{,z}|_{2,\Omega}|\nabla\Phi|_{2,\Omega} + \int\limits_\Omega \bar{F}_r\Phi dx. \tag{32}$$

We have to emphasize that we are not able to prove estimate (24) without restriction (31).

Now, we integrate (120) with respect to time. Then, we obtain

$$\begin{aligned} |\Gamma|_{2,\Omega}^2 + \|\Gamma\|_{1,2,\Omega^t}^2 \le 2\left|\int\limits_{\Omega^t} \frac{v_\varphi}{r}\Phi\Gamma dxdt'\right| \\ + c|\bar{F}_\varphi|_{6/5,2,\Omega^t}^2 + c|\Gamma(0)|_{2,\Omega}^2. \end{aligned} \tag{33}$$

Integrating (32) with respect to time and adding to (33) yields

$$\begin{aligned} \|\Phi\|_{V(\Omega^t)}^2 + \|\Gamma\|_{V(\Omega^t)}^2 \le c(D_2)\left|\int\limits_{\Omega^t} \frac{v_\varphi}{r}\Phi\Gamma dxdt'\right| \\ + c(D_2)(|\bar{F}_r|_{6/5,2,\Omega^t}^2 + |\bar{F}_\varphi|_{6/5,2,\Omega^t}^2) + c(D_2)(|\Phi(0)|_{2,\Omega}^2 + |\Gamma(0)|_{2,\Omega}^2). \end{aligned} \tag{34}$$

Now, we have to estimate the first term on the r.h.s. of (34).

Introducing the quantity (see (132))

$$X(t) = \|\Phi\|_{V(\Omega^t)} + \|\Gamma\|_{V(\Omega^t)} \tag{35}$$

and recalling that constant D_8 is introduced in Assumption 1, inequality (34) takes the form

$$X^2(t) \leq c(D_2)\left|\int\limits_{\Omega^t} \frac{v_\varphi}{r}\Phi\Gamma dx dt'\right| + cD_8^2, \tag{36}$$

where the first integral is called I_3.

Using estimate (123) and estimate of L_1^2 in the proof of Lemma 11, we obtain from (36) the inequality

$$X^2(t) = c(D_2)|v_\varphi|^{\varepsilon}_{d,\infty,\Omega^t}|\Phi|^{\theta}_{2,\Omega^t}|\nabla\Phi|^{1-\theta}_{2,\Omega^t}|\nabla\Gamma|_{2,\Omega^t} + cD_8^2, \tag{37}$$

where $\theta = \left(1-\frac{3}{d}\right)\varepsilon_1 - \frac{3}{d}\varepsilon_2$, $d > 3$, $\varepsilon = \varepsilon_1 + \varepsilon_2 < 1$.

To derive any estimate from (37), we use (173) in the form

$$|\Phi|^2_{2,\Omega} \leq c(D_5 + D_6|v_\varphi|^{\varepsilon_0}_{\infty,\Omega^t})\|\Gamma\|_{1,2,\Omega^t} + cD_7, \tag{38}$$

where ε_0 can be assumed to be an arbitrarily small positive number and D_5, D_6, and D_7 are defined in Assumption 1. This is a very important estimate because the square of $|\Phi|_{2,\Omega^t}$ depends linearly on $\|\Gamma\|_{1,2,\Omega^t}$.

Using (38) in (37) yields the following (the estimate of I_3 is described in (122)):

$$X^2(t) \leq c|v_\varphi|^{\varepsilon}_{d,\infty,\Omega^t}[c_1(1 + |v_\varphi|^{\frac{1}{2}\theta\varepsilon_0}_{\infty,\Omega^t})X^{\frac{1}{2}\theta} + c_2]X^{2-\theta} + cD_8^2, \tag{39}$$

where c_1 and c_2 depend on D_5, D_6, and D_7.

Since $2 - \frac{1}{2}\theta$, $2-\theta$ are less than 2 Lemma 12 yields the inequality

$$X^2 \leq c_0|v_\varphi|^{\frac{4\varepsilon}{\theta}}_{d,\infty,\Omega^t}(1 + |v_\varphi|^{2\varepsilon_0}_{\infty,\Omega^t}) + c_0|v_\varphi|^{\frac{2\varepsilon}{\theta}}_{d,\infty,\Omega^t} + cD_8^2, \tag{40}$$

where $c_0 = \phi(D_2, D_5, D_6, D_7)$.

Setting $d = 12$ and assuming that v_φ is not small, we derive (137) in the form

$$|v_\varphi|^{6'}_{12,\infty,\Omega^t} \leq c|v_\varphi|^{b_0\varepsilon_0}_{\infty,\Omega^t} + \phi(D_2, D_5, D_6, D_7, D_8, D_9), \tag{41}$$

where b_0 is a positive number.

The smallness of v_φ, which must be excluded in the proof of (41), is described in Appendix A.

To prove (41), we have to pass from (140) to (141). Therefore, we need the following estimate:

$$\int\limits_{\Omega^t} \frac{\psi_1^2}{r^{6'}} dx dt' \leq c\|\Gamma\|^2_{1,2,\Omega^t}, \tag{42}$$

where $6' < 6$ and $6'$ are very close to 6. Moreover, $6'$ is such a number where (42) holds (also see Remark 8).

Estimate (42) is crucial to the proof of (141), which is very important in deriving (151). Inequalities (151) and (152) imply the main result of this paper: estimate (24).

Replacing $6'$ with 6 estimate (42) takes the form

$$\int\limits_0^t\int\limits_\Omega \frac{\psi_1^2}{r^6} dx dt' \leq c\int\limits_0^t \|\Gamma\|^2_{H_0^1(\Omega)} dt', \tag{43}$$

where the r.h.s. cannot be estimated by $\|\Gamma\|_{V(\Omega^t)}$.

Estimate (42) follows from Lemma 18 and imposes the following additional restrictions on ψ_1:

$$\psi_1|_{r=0} = 0, \quad \psi_{1,r}|_{r=0} = 0. \tag{44}$$

However, the theory developed in [9] implies that $\psi_{1,r}|_{r=0} = 0$.

Exploiting (41) in (40) yields

$$X \le c(1 + |v_\varphi|^{d_1\varepsilon_0}_{\infty,\Omega^t})|v_\varphi|^{d_2\varepsilon_0}_{\infty,\Omega^t} + \phi(D_2, D_5, D_6, D_7, D_8, D_9), \tag{45}$$

where d_1 and d_2 are positive finite numbers.

Finally, we find the estimate for $|v_\varphi|_{\infty,\Omega^t}$ (see (152)). Using (152) in (45) yields (24). This ends the proof of Theorem 1. □

The problem of regularity of axially symmetric solutions to the Navier–Stokes equations has a long history. The first regularity results in the case of vanishing swirl were independently derived by O. A. Ladyzhenskaya and Ukhovskii-Yudovich, as referenced in [1,2]. Many references concerning the nonvanishing swirl case can be found in [3].

We have to emphasize that we were able to prove Theorem 1 because the theory of weighted Sobolev spaces developed in [11] was used.

2. Notation and Auxiliary Results

First, we introduce some notations

Definition 1. *We use the following notation for Lebesgue and Sobolev spaces*

$$\|u\|_{L_p(\Omega)} = |u|_{p,\Omega}, \quad \|u\|_{L_p(\Omega^t)} = |u|_{p,\Omega^t},$$
$$\|u\|_{L_{p,q}(\Omega^t)} = \|u\|_{L_q(0,t;L_p(\Omega))} = |u|_{p,q,\Omega^t},$$

where $p, q \in [1, \infty]$. Next,

$$\|u\|_{H^s(\Omega)} = \|u\|_{s,\Omega}, \quad \|u\|_{W^s_p(\Omega)} = \|u\|_{s,p,\Omega},$$
$$\|u\|_{L_q(0,t;W^k_p(\Omega))} = \|u\|_{k,p,q,\Omega^t}, \quad \|u\|_{k,p,p,\Omega^t} = \|u\|_{k,p,\Omega^t},$$

where $s, k \in \mathbb{N} \cup \{0\}$, $H^s(\Omega) = W^s_2(\Omega)$.

We need energy-type space $V(\Omega^t)$, which is appropriate for the description of weak solutions to the Navier–Stokes equations

$$\|u\|_{V(\Omega^t)} = |u|_{2,\infty,\Omega^t} + |\nabla u|_{2,\Omega^t}.$$

We recall weighted Sobolev spaces, defined by

$$\|f\|_{H^k_\mu(\mathbb{R}_+)} = \bigg(\int_{\mathbb{R}_+} \sum_{j=0}^k |\partial^j_r f|^2 r^{2(\mu+j-k)} r dr\bigg)^{1/2}$$

and

$$\|f\|_{H^k_\mu(\Omega)} = \bigg(\int_\Omega \sum_{|\alpha|=0}^k |D^\alpha_{r,z} f|^2 r^{2(\mu+|\alpha|-k)} r dr dz\bigg)^{1/2},$$

where Ω contains the axis of symmetry, $D^\alpha_{r,z} = \partial^{\alpha_1}_r \partial^{\alpha_2}_z$, $|\alpha| = \alpha_1 + \alpha_2$, $\alpha_i \in \mathbb{N} \cup \{0\}$, $i = 1, 2$, $k \in \mathbb{N} \cup \{0\}$, and $\mu \in \mathbb{R}_+$. Moreover, we have

$$H^0_0(\Omega) = L_{2,0}(\Omega) = L_2(\Omega),$$
$$H^0_\mu(\Omega) = L_{2,\mu}(\Omega)$$

and

$$\|f\|_{L_{2,\mu}(\Omega)} = |f|_{2,\mu,\Omega}.$$

Lemma 1. *Let $f \in L_{2,1}(\Omega^t)$, $v(0) \in L_2(\Omega)$. Then, solutions to (7) satisfy the estimate*

$$\begin{aligned}&\|v(t)\|^2_{L_2(\Omega)} + \nu \int\limits_{\Omega^t} (|\nabla v_r|^2 + |\nabla v_\varphi|^2 + |\nabla v_z|^2) dxdt' \\ &+ \nu \int\limits_{\Omega^t} \left(\frac{v_r^2}{r^2} + \frac{v_\varphi^2}{r^2}\right) dxdt' \le 3\|f\|^2_{L_{2,1}(\Omega^t)} + 2\|v(0)\|^2_{L_2(\Omega)} \equiv D_1^2.\end{aligned} \tag{46}$$

Proof. Multiplying $(7)_1$ by v_r, $(7)_2$ by v_φ, $(7)_3$ by v_z, adding the results, and integrating over Ω yield

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int\limits_\Omega (v_r^2 + v_\varphi^2 + v_z^2) dx + \nu \int\limits_\Omega (|\nabla v_r|^2 + |\nabla v_\varphi|^2 + |\nabla v_z|^2) dx \\ &+ \nu \int\limits_\Omega \left(\frac{v_r^2}{r^2} + \frac{v_\varphi^2}{r^2}\right) dx + \int\limits_\Omega (p_{,r} v_r + p_{,z} v_z) dx \\ &= \int\limits_\Omega (f_r v_r + f_\varphi v_\varphi + f_z v_z) dx.\end{aligned} \tag{47}$$

The last term on the l.h.s. of (47) vanishes in virtue of the equation of continuity $(7)_4$ and the boundary conditions.

Using $v^2 = v_r^2 + v_\varphi^2 + v_z^2$, (47) takes the form

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|v\|^2_{L_2(\Omega)} + \nu \int\limits_\Omega (|\nabla v_r|^2 + |\nabla v_\varphi|^2 + |\nabla v_z|^2) dx \\ &+ \nu \int\limits_\Omega \left(\frac{v_r^2}{r^2} + \frac{v_\varphi^2}{r^2}\right) dx = \int\limits_\Omega (f_r v_r + f_\varphi v_\varphi + f_z v_z) dx.\end{aligned} \tag{48}$$

Applying the Hölder inequality to the r.h.s. of (48) yields

$$\frac{d}{dt}\|v\|_{L_2(\Omega)} \le \|f\|_{L_2(\Omega)}, \tag{49}$$

where $f^2 = f_r^2 + f_\varphi^2 + f_z^2$.

Integrating (49) with respect to time gives

$$\|v\|_{L_2(\Omega)} \le \|f\|_{L_{2,1}(\Omega^t)} + \|v(0)\|_{L_2(\Omega)}. \tag{50}$$

Integrating (48) with respect to time, using the Hölder inequality on the r.h.s. of (48), and exploiting (50), we obtain

$$\begin{aligned}&\frac{1}{2}\|v(t)\|^2_{L_2(\Omega)} + \nu \int\limits_{\Omega^t} (|\nabla v_r|^2 + |\nabla v_\varphi|^2 + |\nabla v_z|^2) dxdt' \\ &+ \nu \int\limits_{\Omega^t} \left(\frac{v_r^2}{r^2} + \frac{v_\varphi^2}{r^2}\right) dxdt' \le \|f\|_{L_{2,1}(\Omega^t)} (\|f\|_{L_{2,1}(\Omega^t)} \\ &+ \|v(0)\|_{L_2(\Omega)}) + \frac{1}{2}\|v(0)\|^2_{L_2(\Omega)}.\end{aligned} \tag{51}$$

The above inequality implies (46). This concludes the proof. □

Lemma 2. *Consider problem (12). Assume that $f_0 \in L_{\infty,1}(\Omega^t)$ and $u(0) \in L_\infty(\Omega)$. Then,*

$$\|u(t)\|_{L_\infty(\Omega)} \le \|f_0\|_{L_{\infty,1}(\Omega^t)} + \|u(0)\|_{L_\infty(\Omega)} \equiv D_2. \tag{52}$$

Proof. Multiplying $(12)_1$ by $u|u|^{s-2}$, $s > 2$, integrating over Ω and by parts, we obtain

$$\begin{aligned}&\frac{1}{s}\frac{d}{dt}\|u\|^s_{L_s(\Omega)} + \frac{4\nu(s-1)}{s^2}\|\nabla|u|^{s/2}\|^2_{L_2(\Omega)} + \frac{\nu}{s}\int\limits_\Omega(|u|^s)_{,r}drdz\\&=\int\limits_\Omega f_0u|u|^{s-2}dx.\end{aligned} \tag{53}$$

From [9], it follows that $u|_{r=0} = 0$. Moreover, using boundary conditions, (53) implies

$$\frac{d}{dt}\|u\|_{L_s(\Omega)} \le \|f_0\|_{L_s(\Omega)}. \tag{54}$$

Integrating (54) with respect to time and passing with $s \to \infty$, we derive (52). This ends the proof. □

Lemma 3. *Let estimates (46) and (52) hold. Then,*

$$\|v_\varphi\|_{L_4(\Omega^t)} \le D_1^{1/2}D_2^{1/2}. \tag{55}$$

Proof. We have

$$\int\limits_{\Omega^t}|v_\varphi|^4dxdt' = \int\limits_{\Omega^t}r^2v_\varphi^2\frac{v_\varphi^2}{r^2}dxdt' \le \|rv_\varphi\|^2_{L_\infty(\Omega^t)}\int\limits_{(\Omega^t)}\frac{v_\varphi^2}{r^2}dxdt' \le D_2^2D_1^2.$$

This implies (55) and concludes the proof. □

Lemma 4. *Consider problem (22). Assume that $\omega_1 \in L_{6/5}(\Omega)$, where $\Omega = (0,R)\times(-a,a)$. Then, there exists a weak solution to problem (22), such that $\psi_1 \in H^1(\Omega)$ and the estimate*

$$\|\psi_1\|_{1,\Omega} \le c|\omega_1|_{6/5,\Omega} \tag{56}$$

holds.

Proof. Multiplying $(22)_1$ by ψ_1 and using the boundary conditions, we obtain

$$\|\psi_1\|^2_{1,\Omega} + \int\limits_{-a}^{a}\psi_1^2|_{r=0}dz = \int\limits_\Omega\omega_1\psi_1dx.$$

Applying the Hölder and Young inequalities to the r.h.s. implies (56). The Fredholm theorem gives existence. This ends the proof. □

Remark 2. *We have to emphasize that the weak solution ψ_1 of (22) does not vanish on the axis of symmetry. It also follows from [9].*

From Lemma 2.4 in [8], we also have

Lemma 5. *Let $f \in C^\infty((0,R)\times(-a,a))$, $f|_{r\ge R} = 0$. Let $1 < r \le 3$, $0 \le s \le r$, $s \le 2$, $q \in \left[r, \frac{r(3-s)}{3-r}\right]$. Then, there exists a positive constant $c = c(s,r)$, such that*

$$\left(\int\limits_\Omega\frac{|f|^q}{r^s}dx\right)^{1/q} \le c|f|_{r,\Omega}^{\frac{3-s}{q}-\frac{3}{r}+1}|\nabla f|_{r,\Omega}^{\frac{3}{r}-\frac{3-s}{q}}, \tag{57}$$

where f does not depend on φ.

Notation 1 (see [11]). *First, we introduce the Fourier transform. Let $f \in S(\mathbb{R})$, where $S(\mathbb{R})$ is the Schwartz space of all complex-valued rapidly decreasing and infinitely differentiable functions on $\mathbb{R}$. Then, the Fourier transform of f and its inverse are defined by*

$$\hat{f}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\lambda\tau} f(\tau) d\tau, \quad \check{\hat{f}}(\tau) = f(\tau) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\lambda\tau} \hat{f}(\lambda) d\lambda \tag{58}$$

and $\hat{\check{f}} = \check{\hat{f}} = f$.

By $H^k_\mu(\mathbb{R}_+)$, we denote a weighted space with the norm

$$\|u\|^2_{H^k_\mu(\mathbb{R}_+)} = \sum_{i=0}^k \int_{\mathbb{R}_+} |\partial_r^i u|^2 r^{2(\mu-k+i)} r dr.$$

In view of transformation $\tau = -\ln r$, $r = e^{-\tau}$, $dr = -e^{-\tau} d\tau$, we have the equivalence

$$\sum_{i=0}^k \int_{\mathbb{R}_+} |\partial_r^i u|^2 r^{2(\mu-k+i)} r dr \sim \sum_{i=0}^k \int_{\mathbb{R}} |\partial_\tau^i u'|^2 e^{2h\tau} d\tau \tag{59}$$

which holds for $u'(\tau) = u'(-\ln r) = u(r)$, $h = k - 1 - \mu$.

In view of the Fourier transform (58) and the Parseval identity, we have

$$\int_{-\infty+ih}^{+\infty+ih} \sum_{j=0}^k |\lambda|^{2j} |\hat{u}(\lambda)|^2 d\lambda = \int_{\mathbb{R}} \sum_{j=0}^k |\partial_\tau^j u|^2 e^{2h\tau} d\tau. \tag{60}$$

3. Estimates for the Stream Function ψ_1

In this section, we derive many estimates for $\psi_1 = \psi/r$, where ψ is the stream function, in terms of $\|\Gamma\|_{1,2,\Omega^t} + |\Gamma|_{2,\infty,\Omega^t}$ (recall that $\Gamma = \omega_1$). Function ψ_1 was introduced by Thomas Hou in [12]. Lemma 6 is proved by applying the energy-type method. Inequalities (85) and (93) are proved by applying the technique of Kondratiev (see [10]) to problem (61). We need the inequalities to prove inequality (173), so they are very important.

To prove (85) and (93), we require that $\psi_1|_{r=0} = 0$; however, the theory developed by [9] does not imply the restriction.

Recall that ψ_1 is a solution to the problem

$$\begin{aligned} &-\psi_{1,rr} - \psi_{1,zz} - \frac{3}{r}\psi_{1,r} = \omega_1 && \text{in } \Omega = (0,R) \times (-a,a), \\ &\psi_1|_{r=R} = 0, \\ &\psi_1 \text{ satisfies the periodic boundary conditions} && \text{on } S_2. \end{aligned} \tag{61}$$

Lemma 6. *For sufficiently regular solutions to (61), the following estimates hold*

$$\begin{aligned} &\int_\Omega (\psi^2_{1,rr} + \psi^2_{1,rz} + \psi^2_{1,zz}) dx + \int_\Omega \frac{1}{r^2} \psi^2_{1,r} dx + \int_{-a}^a \psi^2_{1,z}|_{r=0} dz \\ &+ \int_{-a}^a \psi^2_{1,r}|_{r=R} dz \le c|\omega_1|^2_{2,\Omega} \end{aligned} \tag{62}$$

and

$$\int_\Omega (\psi^2_{1,zzr} + \psi^2_{1,zzz}) dx + \int_{-a}^a \psi^2_{1,zz}|_{r=0} dz \le c|\omega_{1,z}|^2_{2,\Omega} \tag{63}$$

and

$$\begin{aligned}&\int_\Omega (\psi_{1,rrz}^2+\psi_{1,rzz}^2+\psi_{1,zzz}^2)dx+\int_{-a}^a \psi_{1,zz}^2|_{r=0}dz\\&+\int_{-a}^a \psi_{1,rz}^2|_{r=R}dz\le c|\omega_{1,z}|_{2,\Omega}^2.\end{aligned}\tag{64}$$

Proof. First, we prove (62). Multiplying $(61)_1$ by $\psi_{1,zz}$ and integrating over Ω yields

$$-\int_\Omega \psi_{1,rr}\psi_{1,zz}dx-\int_\Omega \psi_{1,zz}^2dx-3\int_\Omega \frac{1}{r}\psi_{1,r}\psi_{1,zz}dx=\int_\Omega \omega_1\psi_{1,zz}dx.\tag{65}$$

Integrating by parts with respect to r in the first term implies

$$\begin{aligned}&-\int_\Omega (\psi_{1,r}\psi_{1,zz}r)_{,r}drdz+\int_\Omega \psi_{1,r}\psi_{1,zzr}dx+\int_\Omega \psi_{1,r}\psi_{1,zz}drdz\\&-\int_\Omega \psi_{1,zz}^2dx-3\int_\Omega \psi_{1,r}\psi_{1,zz}drdz=\int_\Omega \omega_1\psi_{1,zz}dx.\end{aligned}$$

Continuing, we have

$$\begin{aligned}&-\int_{-a}^a \psi_{1,r}\psi_{1,zz}r\Big|_{r=0}^{r=R}dz+\int_\Omega \psi_{1,r}\psi_{1,zzr}dx-\int_\Omega \psi_{1,zz}^2dx\\&-2\int_\Omega \psi_{1,r}\psi_{1,zz}drdz=\int_\Omega \omega_1\psi_{1,zz}dx.\end{aligned}\tag{66}$$

The first integral in (66) vanishes because $\psi_{1,r}r|_{r=0}=0$, $\psi_{1,zz}|_{r=R}=0$. Integrating by parts with respect to z in the last term on the l.h.s. of (66), and using the periodic boundary conditions on S_2, we obtain

$$\int_\Omega \psi_{1,r}\psi_{1,zzr}dx-\int_\Omega \psi_{1,zz}^2dx+2\int_\Omega \psi_{1,rz}\psi_{1,z}drdz=\int_\Omega \omega_1\psi_{1,zz}dx.\tag{67}$$

Integrating by parts with respect to z in the first term in (67) and using the boundary conditions on S_2, we have

$$\int_\Omega (\psi_{1,zr}^2+\psi_{1,zz}^2)dx-\int_\Omega (\psi_{1,z}^2)_{,r}drdz=-\int_\Omega \omega_1\psi_{1,zz}dx,\tag{68}$$

where the last term on the l.h.s. equals

$$-\int_{-a}^a \psi_{1,z}^2\Big|_{r=0}^{r=R}dz=\int_{-a}^a \psi_{1,z}^2\Big|_{r=0}dz$$

because $\psi_{1,z}|_{r=R}=0$. Using this in (68) and applying the Hölder and Young inequalities to the r.h.s. of (68) yield

$$\int_\Omega (\psi_{1,rz}^2+\psi_{1,zz}^2)dx+\int_{-a}^a \psi_{1,z}^2|_{r=0}dz\le c|\omega_1|_{2,\Omega}^2.\tag{69}$$

We multiply $(61)_1$ by $\frac{1}{r}\psi_{1,r}$ and integrate over Ω. Then, we have

$$3\int_\Omega \left|\frac{1}{r}\psi_{1,r}\right|^2 dx = -\int_\Omega \psi_{1,rr}\frac{1}{r}\psi_{1,r}dx - \int_\Omega \psi_{1,zz}\frac{1}{r}\psi_{1,r}dx - \int_\Omega \omega_1\frac{1}{r}\psi_{1,r}dx. \tag{70}$$

The first term on the r.h.s. of (70) equals

$$-\frac{1}{2}\int_\Omega \partial_r\psi_{1,r}^2 drdz = -\frac{1}{2}\int_{-a}^{a}\psi_{1,r}^2\bigg|_{r=0}^{r=R}dz = -\frac{1}{2}\int_{-a}^{a}\psi_{1,r}^2|_{r=R}dz,$$

because $\psi_{1,r}|_{r=0} = 0$ (see [9]). Applying the Hölder and Young inequalities to the last two terms on the r.h.s. of (70) implies

$$\int_\Omega \left|\frac{1}{r}\psi_{1,r}\right|^2 dx + \frac{1}{2}\int_{-a}^{a}\psi_{1,r}^2\bigg|_{r=R}dz \le c(|\psi_{1,zz}|_{2,\Omega}^2 + |\omega_1|_{2,\Omega}^2). \tag{71}$$

Inequalities (69) and (71) imply the estimate

$$\begin{aligned}&\int_\Omega(\psi_{1,rz}^2+\psi_{1,zz}^2)dx + \int_\Omega\left|\frac{1}{r}\psi_{1,r}\right|^2dx + \int_{-a}^{a}\psi_{1,z}^2\bigg|_{r=0}dz\\&+\int_{-a}^{a}\psi_{1,r}^2\bigg|_{r=R}dz \le c|\omega_1|_{1,\Omega}^2.\end{aligned} \tag{72}$$

From $(61)_1$, we have

$$|\psi_{1,rr}|_{2,\Omega}^2 \le |\psi_{1,zz}|_{2,\Omega}^2 + 3\left|\frac{1}{r}\psi_{1,r}\right|_{2,\Omega}^2 + |\omega_1|_{2,\Omega}^2. \tag{73}$$

Inequalities (72) and (73) imply (62).

Now, we show (63). We differentiate $(61)_1$ with respect to z, multiply by $-\psi_{1,zzz}$, and integrate over Ω. Then, we obtain

$$\int_\Omega \psi_{1,rrz}\psi_{1,zzz}dx + \int_\Omega \psi_{1,zzz}^2dx + 3\int_\Omega \frac{1}{r}\psi_{1,rz}\psi_{1,zzz}dx = -\int_\Omega \omega_{1,z}\psi_{1,zzz}dx. \tag{74}$$

Integrating by parts with respect to z yields

$$\int_\Omega \psi_{1,rrz}\psi_{1,zzz}dx = \int_\Omega(\psi_{1,rrz}\psi_{1,zz})_{,z}dx - \int_\Omega \psi_{1,rrzz}\psi_{1,zz}dx, \tag{75}$$

where the first integral vanishes in view of periodic boundary conditions on S_2. Integrating by parts with respect to r in the second integral in (75) gives

$$-\int_\Omega(\psi_{1,rzz}\psi_{1,zz}r)_{,r}drdz + \int_\Omega \psi_{1,rzz}^2dx + \int_\Omega \psi_{1,rzz}\psi_{1,zz}drdz,$$

where the first integral vanishes because

$$\psi_{1,rzz}r|_{r=0} = 0, \quad \psi_{1,zz}|_{r=R} = 0.$$

In view of the above considerations, (74) takes the form

$$\begin{aligned}&\int_\Omega (\psi_{1,rzz}^2+\psi_{1,zzz}^2)dx+\int_\Omega \psi_{1,rzz}\psi_{1,zz}drdz\\&+3\int_\Omega \psi_{1,rz}\psi_{1,zzz}drdz=-\int_\Omega \omega_{1,z}\psi_{1,zzz}dx.\end{aligned} \tag{76}$$

Integrating by parts with respect to z in the last term on the l.h.s. of (76) and using the periodic boundary conditions on S_2, we have

$$\begin{aligned}&\int_\Omega (\psi_{1,rzz}^2+\psi_{1,zzz}^2)dx-\int_\Omega \partial_r\psi_{1,zz}^2drdz\\&=-\int_\Omega \omega_{1,z}\psi_{1,zzz}dx.\end{aligned} \tag{77}$$

Applying the Hölder and Young inequalities to the r.h.s. of (77) yields

$$\int_\Omega (\psi_{1,rzz}^2+\psi_{1,zzz}^2)dx+\int_{-a}^a \psi_{1,zz}^2\Big|_{r=0}dz\le c|\omega_{1,z}|_{2,\Omega}^2,$$

where we used that $\psi_{1,zz}|_{r=R}=0$.

The above inequality implies (63).

Finally, we show (64). We differentiate $(61)_1$ with respect to z, multiply by $\psi_{1,rrz}$, and integrate over Ω. Then, we have

$$\begin{aligned}&-\int_\Omega \psi_{1,rrz}^2dx-\int_\Omega \psi_{1,zzz}\psi_{1,rrz}dx-3\int_\Omega \frac{1}{r}\psi_{1,rz}\psi_{1,rrz}dx\\&=\int_\Omega \omega_{1,z}\psi_{1,rrz}dx.\end{aligned} \tag{78}$$

Integrating by parts with respect to z in the second term in (78) implies

$$\begin{aligned}&-\int_\Omega \psi_{1,zzz}\psi_{1,rrz}dx=\int_\Omega \psi_{1,zz}\psi_{1,rrzz}dx=\int_\Omega (\psi_{1,zz}\psi_{1,rzz}r)_rdrdz\\&-\int_\Omega \psi_{1,rzz}^2dx-\int_\Omega \psi_{1,zz}\psi_{1,rzz}drdz,\end{aligned}$$

where the first term vanishes because

$$\psi_{1,rzz}r|_{r=0}=0,\quad \psi_{1,zz}|_{r=R}=0.$$

Then, (78) takes the form

$$\begin{aligned}&\int_\Omega (\psi_{1,rrz}^2+\psi_{1,rzz}^2)dx+\int_\Omega \psi_{1,zz}\psi_{1,rzz}drdz\\&+3\int_\Omega \psi_{1,rz}\psi_{1,rrz}drdz=-\int_\Omega \omega_{1,z}\psi_{1,rrz}dx.\end{aligned} \tag{79}$$

The second term in (79) equals

$$\frac{1}{2}\int_{-a}^a \psi_{1,zz}^2\Big|_{r=0}^{r=R}dz=-\frac{1}{2}\int_{-a}^a \psi_{1,zz}^2\Big|_{r=0}dz$$

because $\psi_{1,zz}|_{r=R} = 0$, and the last term on the l.h.s. of (79) has the form

$$\frac{3}{2}\int_{\Omega} \partial_r \psi_{1,rz}^2 drdz = \frac{3}{2}\int_{-a}^{a} \psi_{1,rz}^2\Big|_{r=0}^{r=R} dz = \frac{3}{2}\int_{-a}^{a} \psi_{1,rz}^2\Big|_{r=R} dz$$

because $\psi_{1,rz}|_{r=0} = 0$.

Using the above expressions in (79) implies the equality

$$\begin{aligned}&\int_{\Omega} (\psi_{1,rrz}^2 + \psi_{1,rzz}^2)dx - \frac{1}{2}\int_{-a}^{a} \psi_{1,zz}^2\Big|_{r=0} dz + \frac{3}{2}\int_{-a}^{a} \psi_{1,rz}^2\Big|_{r=R} dz \\ &= -\int_{\Omega} \omega_{1,z}\psi_{1,rrz} dx.\end{aligned} \tag{80}$$

Applying the Hölder and Young inequalities on the r.h.s. of (80) gives

$$\begin{aligned}&\int_{\Omega} (\psi_{1,rrz}^2 + \psi_{1,rzz}^2)dx - \frac{1}{2}\int_{-a}^{a} \psi_{1,zz}^2\Big|_{r=0} dz \\ &+ \frac{3}{2}\int_{-a}^{a} \psi_{1,rz}^2\Big|_{r=R} dz \le c|\omega_{1,z}|_{2,\Omega}^2.\end{aligned} \tag{81}$$

Inequalities (81) and (63) imply (64). This ends the proof. □

Lemma 7. *For sufficiently regular solutions to (61), the following inequality*

$$\left|\frac{1}{r}\psi_{1,rz}\right|_{2,\Omega} \le c|\omega_{1,z}|_{2,\Omega} \tag{82}$$

holds.

Proof. Differentiating (61) with respect to z implies

$$-\psi_{1,rrz} - \psi_{1,zzz} - \frac{3}{r}\psi_{1,rz} = \omega_{1,z}. \tag{83}$$

From (83), we have

$$\left|\frac{1}{r}\psi_{1,rz}\right|_{2,\Omega} \le c(|\psi_{1,rrz}|_{2,\Omega} + |\psi_{1,zzz}|_{2,\Omega} + |\omega_{1,z}|_{2,\Omega}). \tag{84}$$

Using (64) in (84) yields (82). This concludes the proof. □

Now, we estimate $|\frac{\psi_{1,zz}}{r}|_{2,\Omega}$.

Lemma 8. *Let ψ_1 be a weak solution to problem (61), which vanishes on the axis of symmetry. Then, such sufficiently regular solutions to problem (61) satisfy the following estimate:*

$$\int_{\Omega} \frac{\psi_{1,zz}^2}{r^2} dx + \int_{\Omega} \left(\psi_{1,zrr}^2 + \frac{\psi_{1,zr}^2}{r^2} + \frac{\psi_{1,z}^2}{r^4}\right) dx \le c|\omega_{1,z}|_{2,\Omega}^2. \tag{85}$$

Proof. Differentiating (61) with respect to z yields

$$\begin{aligned} & -\Delta\psi_{1,z} - \frac{2}{r}\psi_{1,zr} = \omega_{1,z}, \\ & \psi_{1,z}|_{S_1} = 0, \\ & \psi_{1,z} \text{ satisfies periodic boundary conditions on } S_2. \end{aligned} \tag{86}$$

Applying Lemma 17 (also see Lemma 3.1 from [11]) to problem (86) gives

$$\int_\Omega \left(\psi_{1,zrr}^2 + \frac{\psi_{1,zr}^2}{r^2} + \frac{\psi_{1,z}^2}{r^4}\right)dx \le c(|\omega_{1,z}|_{2,\Omega}^2 + |\psi_{1,zzz}|_{2,\Omega}^2) \le c|\omega_{1,z}|_{2,\Omega}^2, \tag{87}$$

where (63) is used in the last inequality.

To examine solutions to (86) we use the notation

$$u = \psi_{1,z}. \tag{88}$$

Then, (86) takes the form

$$\begin{aligned} & -\Delta u - \frac{2}{r}u_{,r} = \omega_{1,z}, \\ & u|_{S_1} = 0, \\ & u \text{ satisfies periodic boundary conditions on } S_2. \end{aligned} \tag{89}$$

We multiply $(88)_1$ by ur^{-2}, integrate over Ω, and express the Laplacian operator in cylindrical coordinates. Then, we have

$$-\int_\Omega \left(u_{,rr} + \frac{1}{r}u_{,r} + u_{,zz}\right)ur^{-2}dx - 2\int_\Omega \frac{1}{r}u_{,r}ur^{-2}dx = \int_\Omega \omega_{1,z}ur^{-2}dx. \tag{90}$$

Integrating by parts with respect to z in the third term under the first integral, we obtain

$$\int_\Omega \frac{u_{,z}^2}{r^2}dx = \int_\Omega \left(u_{,rr} + \frac{3}{r}u_{,r}\right)ur^{-2}dx + \int_\Omega \omega_{1,z}ur^{-2}dx. \tag{91}$$

Applying the Hölder and Young inequalities to the r.h.s. integrals, using $u = \psi_{1,z}$ and (87), we derive

$$\int_\Omega \frac{\psi_{1,zz}^2}{r^2}dx \le c\int_\Omega \left(\psi_{1,zrr}^2 + \frac{\psi_{1,zr}^2}{r^2} + \frac{\psi_{1,z}^2}{r^4}\right)dx + c|\omega_{1,z}|_{2,\Omega}^2. \tag{92}$$

Using (87) in (92) implies (85). This concludes the proof. □

Remark 3. *Lemma 8 is necessary in the proof of global regular axially symmetric solutions to problem (6). However, it imposes strong restrictions on the solutions to (6) because the condition $\psi_1|_{r=0} = 0$ implies $v_z|_{r=0} = 0$. We do not know how to omit this restriction in the proof presented in this paper.*

Lemma 9. *Let $\mu > 0$ and $\omega_1 \in H_\mu^1(\Omega)$. Then, for sufficiently smooth solutions to (61), the following estimate is valid*

$$\int_\Omega \left(\psi_{1,rrr}^2 + \frac{\psi_{1,rr}^2}{r^2} + \frac{\psi_{1,r}^2}{r^4}\right)r^{2\mu}dx \le cR^{2\mu}\|\omega_1\|_{1,\Omega}^2. \tag{93}$$

Proof. To prove the lemma, we introduce a partition of unity $\{\zeta^{(i)}(r)\}_{i=1,2}$, such that

$$\sum_{i=1}^{2}\zeta^{(i)}(r)=1$$

and

$$\zeta^{(1)}(r)=\begin{cases}1 & r\le r_0,\\ 0 & r\ge r_0+\lambda,\end{cases}$$
$$\zeta^{(2)}(r)=\begin{cases}0 & r\le r_0,\\ 1 & r\ge r_0+\lambda,\end{cases}$$

where $r_0<R$ and $\zeta^{(i)}(r)$, $i=1,2$, are smooth functions.

We introduce the notation

$$\psi_1^{(i)}=\psi_1\zeta^{(i)},\quad \omega_1^{(i)}=\omega_1\zeta^{(i)},\ \ i=1,2. \tag{94}$$

Then, function (94) satisfies the equations

$$\begin{aligned}&-\psi_{1,rr}^{(i)}-\psi_{1,zz}^{(i)}-\frac{3}{r}\psi_{1,r}^{(i)}=-2\psi_{1,r}\dot{\zeta}^{(i)}-\psi_1\ddot{\zeta}^{(i)}-\frac{3}{r}\psi_1\dot{\zeta}^{(i)}\\ &+\omega_1^{(i)}\equiv g^{(i)},\ \ i=1,2,\end{aligned} \tag{95}$$

where the dot denotes the derivative with respect to r.

First, we consider the case $i=1$. Differentiating (95) for $i=1$ with respect to r yields

$$-\psi_{1,rrr}^{(1)}-\psi_{1,rzz}^{(1)}-\frac{3}{r}\psi_{1,rr}^{(1)}+\frac{3}{r^2}\psi_{1,r}^{(1)}=g_{,r}^{(1)}. \tag{96}$$

We introduce the notation

$$v=\psi_{1,r}^{(1)},\quad f=g_{,r}^{(1)}. \tag{97}$$

Then, (96) takes the form

$$\begin{aligned}&-v_{,rr}-v_{,zz}-\frac{3}{r}v_{,r}+\frac{3}{r}v=f\quad\text{in }\ \Omega_{r_0},\\ &v|_{r=r_0}=0,\\ &v|_{S_2}\text{ satisfies periodic boundary conditions,}\end{aligned} \tag{98}$$

where $\Omega_{r_0}=\{x\in\Omega\colon r\in(0,r_0),z\in(-a,a)\}$ and $r_0<R$.

Multiplying $(98)_1$ by r^2 yields

$$-r^2v_{,rr}-3rv_{,r}+3v=r^2(f+v_{,zz})\equiv g(r,z)$$

or equivalently

$$-r\partial_r(r\partial_r v)-2r\partial_r v+3v=g(r,z). \tag{99}$$

We introduce the new variable

$$\tau=-\ln r,\quad r=e^{-\tau}.$$

Since $r\partial_r=-\partial_\tau$, Equation (99) takes the form

$$-\partial_\tau^2 v+2\partial_\tau v+3v=g(e^{-\tau},z)\equiv g'(\tau,z). \tag{100}$$

Applying the Fourier transform (58) to (100) gives

$$\lambda^2\hat{v}+2i\lambda\hat{v}+3\hat{v}=\hat{g}'. \tag{101}$$

Looking for solutions to the algebraic equation

$$\lambda^2 + 2i\lambda + 3 = 0$$

we see that it has two solutions

$$\lambda_1 = -3i, \;\; \lambda_2 = i.$$

For $\lambda \notin \{-3i, i\}$, we can write solutions to (101) in the form

$$\hat{v} = \frac{1}{\lambda^2 + 2i\lambda + 3}\hat{g}' \equiv R(\lambda)\hat{g}'. \tag{102}$$

Since $R(\lambda)$ does not have poles on the line $\operatorname{Im}\lambda = 1 - \mu = h$, $\mu \in (0,1)$, we can use Lemma 3.1 from [11]. Then, we obtain

$$\begin{aligned} &\int\limits_{-\infty+ih}^{\infty+ih} \sum_{j=0}^{2} |\lambda|^{2(2-j)} |\hat{v}|^2 d\lambda \le c \int\limits_{-\infty+ih}^{+\infty+ih} \sum_{j=0}^{2} |\lambda|^{2(2-j)} |R(\lambda)\hat{g}'|^2 d\lambda \\ &\le c \int\limits_{-\infty+ih}^{+\infty+ih} |\hat{g}'|^2 d\lambda. \end{aligned} \tag{103}$$

By applying the Parseval identity, inequality (103) becomes

$$\int\limits_{\mathbb{R}} \sum_{j=0}^{2} |\partial_\tau^j v|^2 e^{2h\tau} d\tau \le c \int\limits_{\mathbb{R}} |g'|^2 e^{2h\tau} d\tau.$$

Passing to variable r yields

$$\sum_{j=0}^{2} \int\limits_{\mathbb{R}_+} |\partial_r^j v|^2 r^{2(\mu+j-2)} r dr \le c \int\limits_{\mathbb{R}_+} |g|^2 r^{2(\mu-2)} r dr.$$

Using that $g = r^2(f + v_{,zz})$, we have

$$\sum_{j=0}^{2} \int\limits_{\mathbb{R}_+} |\partial_r^j v|^2 r^{2(\mu+j-2)} r dr \le c \int\limits_{\mathbb{R}_+} |f + v_{,zz}|^2 r^{2\mu} r dr. \tag{104}$$

Recalling notation (97), we derive from (104) the inequality

$$\begin{aligned} \sum_{j=0}^{2} \int\limits_{\Omega} |\partial_r^j \psi_{1,r}^{(1)}|^2 r^{2(\mu+j-2)} dx &\le c \int\limits_{\Omega} |g_{,r}^{(1)}|^2 r^{2\mu} dx \\ &+ c \int\limits_{\Omega} |\psi_{1,rzz}|^2 r^{2\mu} dx. \end{aligned} \tag{105}$$

In view of (63),

$$|\psi_{1,rzz}|_{2,\Omega} \le c|\omega_{1,z}|_{2,\Omega}. \tag{106}$$

The first term on the r.h.s. of (105) can be estimated by

$$|g_{,r}^{(1)}|_{2,\mu,\Omega} \le c(|\psi_{1,rr}|_{2,\Omega} + |\psi_{1,r}|_{2,\Omega} + |\psi_1|_{2,\Omega} + |\omega_{1,r}|_{2,\Omega} + |\omega_1|_{2,\Omega}). \tag{107}$$

Lemma 6 and inequalities (105), (106), and (107) imply

$$\begin{aligned}&\int\limits_{\Omega}\left(|\psi_{1,rrr}^{(1)}|^2+\frac{|\psi_{,rr}^{(1)}|^2}{r^2}+\frac{|\psi_{,r}^{(1)}|^2}{r^4}\right)r^{2\mu}rdrdz\\&+\int\limits_{\Omega}|\psi_{1,rzz}|^2dx\le c(|\omega_{1,r}|_{2,\Omega}^2+|\omega_1|_{2,\Omega}^2).\end{aligned}\tag{108}$$

Function $\psi_1^{(2)}$ is a solution to the problem

$$\begin{aligned}&-\Delta\psi_1^{(2)}=-2\psi_{1,r}\dot\zeta^{(2)}-\psi_1\ddot\zeta^{(2)}+\frac{2}{r}\psi_{1,r}^{(2)}\\&\quad-\frac{3}{r}\psi_1\dot\zeta^{(2)}+\omega_1^{(2)} && \text{in } \bar\Omega_{r_0},\\&\psi_1^{(2)}|_{r=R}=0,\ \ \psi^{(2)}=0 && \text{for } r\le r_0,\\&\psi^{(2)} \text{ satisfies periodic boundary conditions} && \text{on } S_2,\end{aligned}\tag{109}$$

where $\bar\Omega_{r_0}=\{x\in\mathbb{R}^3\colon r_0\le r\le R, z\in(-a,a)\}$ and the dot denotes the derivative with respect to r.

For solutions to (109), the following estimate holds

$$\|\psi_1^{(2)}\|_{3,\Omega}\le c(\|\psi_{1,r}\|_{1,\Omega}+\|\psi_1\|_{1,\Omega}+\|\omega_1^{(2)}\|_{1,\Omega})\le c\|\omega_1\|_{1,\Omega}.\tag{110}$$

From (56), (108), and (110), inequality (93) follows. This ends the proof. □

4. Estimates for Φ and Γ

Let $\Omega=\{(r,z)\colon r\in(0,R), z\in(-a,a)\}$. Let $\Phi=\omega_r/r$, $\Gamma=\omega_\varphi/r$ and Φ, Γ are solutions to problems (17)–(20).

Lemma 10. *Assume that $\Phi(0),\Gamma(0)\in L_2(\Omega)$, $\bar F_r,\bar F_\varphi\in L_2(0,t;L_{6/5}(\Omega))$. Let D_2 be defined by (52) and let*

$$I_3=\int\limits_{\Omega^t}\left|\frac{v_\varphi}{r}\Phi\Gamma\right|dxdt'<\infty.$$

Then

$$\begin{aligned}&|\Phi(t)|_{2,\Omega}^2+|\Gamma(t)|_{2,\Omega}^2+\nu(\|\Phi\|_{1,2,\Omega^t}^2+\|\Gamma\|_{1,2,\Omega^t}^2)\\&\le\phi(D_2)\left|\int\limits_{\Omega^t}\frac{v_\varphi}{r}\Phi\Gamma dxdt'\right|+\phi(D_2)(|\bar F_r|_{6/5,2,\Omega^t}^2\\&\quad+|\bar F_\varphi|_{6/5,2,\Omega^t}^2)+|\Phi(0)|_{2,\Omega}^2+|\Gamma(0)|_{2,\Omega}^2\\&\equiv\phi(D_2)I_3+D_8.\end{aligned}\tag{111}$$

Proof. Multiplying (17) by Φ and integrating over Ω yields

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}|\Phi|_{2,\Omega}^2+|\nabla\Phi|_{2,\Omega}^2-\int\limits_{-a}^{a}\Phi\Big|_{r=0}^{r=R}dz\\&=\int\limits_{\Omega}(\omega_r\partial_r+\omega_z\partial_z)\frac{v_r}{r}\Phi dx+\int\limits_{\Omega}\bar F_r\Phi dx.\end{aligned}\tag{112}$$

To derive the second term on the l.h.s. of (112), we consider (17) in

$$\bar\Omega=\{x\in\mathbb{R}^3\colon r<R, z\in(-a,a),\varphi\in(0,2\pi)\}.$$

Then, by applying the Green theorem and the boundary conditions, we obtain the second term on the l.h.s. of (112) on $\bar{\Omega}$. Considering that all quantities in (112) do not depend on φ, we can omit the integration with respect to φ and obtain (112).

Using $\Phi|_{r=R} = 0$ and (13), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|\Phi|^2_{2,\Omega} + |\nabla\Phi|^2_{2,\Omega} \le \int_\Omega (\omega_r\partial_r + \omega_z\partial_z)\frac{v_r}{r}\Phi dx + \int_\Omega \bar{F}_r\Phi dx \\
&\le \int_\Omega \bigg(-v_{\varphi,z}\partial_r\frac{v_r}{r} + \frac{\partial_r(rv_\varphi)}{r}\partial_z\frac{v_r}{r}\bigg)\Phi r dr dz + \int_\Omega \bar{F}_r\Phi dx \\
&= \int_\Omega v_\varphi\bigg(\bigg(\partial_z\partial_r\frac{v_r}{r}\bigg)\Phi + \partial_r\frac{v_r}{r}\partial_z\Phi\bigg)dx \\
&\quad + \int_\Omega \partial_r\bigg(rv_\varphi\partial_z\frac{v_r}{r}\Phi\bigg)drdz - \int_\Omega v_\varphi\bigg(\bigg(\partial_z\partial_r\frac{v_r}{r}\bigg)\Phi + \partial_z\frac{v_r}{r}\partial_r\Phi\bigg)dx \\
&\quad + \int_\Omega \bar{F}_r\Phi dx = \int_{-a}^{a} rv_\varphi\partial_z\frac{v_r}{r}\Phi\bigg|_{r=0}^{r=R} dz + \int_\Omega v_\varphi\bigg(\partial_r\frac{v_r}{r}\partial_z\Phi - \partial_z\frac{v_r}{r}\partial_r\Phi\bigg)dx \\
&\quad + \int_\Omega \bar{F}_r\Phi dx \equiv \int_{-a}^{a} rv_\varphi\partial_z\frac{v_r}{r}\Phi\bigg|_{r=0}^{r=R} dz + I + \int_\Omega \bar{F}_r\Phi dx,
\end{aligned} \tag{113}$$

Using the periodic boundary conditions on S_2, the boundary term vanishes because $v_\varphi|_{r=R} = 0$, $v_r|_{r=R} = 0$, $\Phi|_{r=R} = 0$, and

$$\int_{-a}^{a} rv_\varphi\partial_z\frac{v_r}{r}\Phi\bigg|_{r=0} dz = 0$$

because [9] implies the following expansions near the axis of symmetry

$$\begin{aligned}
v_\varphi &= a_1(z,t)r + a_2(z,t)r^3 + \cdots, \\
v_r &= \bar{a}_1(z,t)r + \bar{a}_2(z,t)r^3 + \cdots
\end{aligned}$$

and $\Phi = -\frac{v_{\varphi,z}}{r}$.

Finally, $I \le I_1 + I_2$, where

$$\begin{aligned}
I_1 &\le \int_\Omega \bigg|v_\varphi\partial_r\frac{v_r}{r}\Phi_{,z}\bigg|dx, \\
I_2 &\le \int_\Omega \bigg|v_\varphi\partial_z\frac{v_r}{r}\Phi_{,r}\bigg|dx.
\end{aligned} \tag{114}$$

Now, we estimate I_1 and I_2. Recall that $\frac{v_r}{r} = -\psi_{1,z}$. Then,

$$\begin{aligned}
I_1 &\le \int_\Omega |v_\varphi\psi_{1,rz}\Phi_{,z}|dx = \int_\Omega \bigg|rv_\varphi\frac{\psi_{1,rz}}{r}\Phi_{,z}\bigg|dx \\
&\le |rv_\varphi|_{\infty,\Omega}\bigg|\frac{\psi_{1,rz}}{r}\bigg|_{2,\Omega}|\Phi_{,z}|_{2,\Omega} \equiv I_1^1.
\end{aligned}$$

From (52) and (82), we have (recall that $\Gamma = \omega_1$):

$$I_1^1 \le cD_2|\Gamma_{,z}|_{2,\Omega}|\Phi_{,z}|_{2,\Omega}. \tag{115}$$

Similarly, we calculate

$$\begin{aligned} I_2 &\le \int\limits_\Omega |v_\varphi \psi_{1,zz}\Phi_{,r}|dx \le |rv_\varphi|_{\infty,\Omega}\left|\frac{\psi_{1,zz}}{r}\right|_{2,\Omega}|\Phi_{,r}|_{2,\Omega} \\ &\le cD_2|\Gamma_{,z}|_{2,\Omega}|\Phi_{,r}|_{2,\Omega}, \end{aligned} \tag{116}$$

where (85) is used.

Finally, the last term on the r.h.s. of (113) is bounded by

$$\varepsilon|\Phi|^2_{6,\Omega} + c(1/\varepsilon)|\bar F_r|^2_{6/5,\Omega}. \tag{117}$$

Using estimates (115)–(117) in (113), assuming that ε is sufficiently small and applying the Poincaré inequality, we obtain

$$\frac{d}{dt}|\Phi|^2_{2,\Omega} + \|\Phi\|^2_{1,\Omega} \le cD_2|\Gamma_{,z}|_{2,\Omega}|\nabla\Phi|_{2,\Omega} + c|\bar F_r|^2_{6/5,\Omega}. \tag{118}$$

Multiplying (18) by Γ, integrating over Ω, using the boundary conditions and explanations about applying the Green theorem appeared below (112), we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}|\Gamma|^2_{2,\Omega} + |\nabla\Gamma|^2_{2,\Omega} - \int\limits_{-a}^{a}\Gamma^2\Big|_{r=0}^{r=R}dz \\ &\le 2\left|\int\limits_\Omega \frac{v_\varphi}{r}\Phi\Gamma dx\right| + \int\limits_\Omega \bar F_\varphi\Gamma dx. \end{aligned} \tag{119}$$

Using $\Gamma|_{r=R} = 0$, applying the Hölder and Young inequalities to the last term on the r.h.s. of (119) and using the Poincaré inequality, we derive

$$\frac{d}{dt}|\Gamma|^2_{2,\Omega} + \|\Gamma\|^2_{1,\Omega} \le 2\int\limits_\Omega \frac{v_\varphi}{r}\Phi\Gamma dx + c|\bar F_\varphi|^2_{6/5,\Omega}. \tag{120}$$

From (118) and (120), we have

$$\begin{aligned} &\frac{d}{dt}(|\Phi|^2_{2,\Omega} + |\Gamma|^2_{2,\Omega}) + \|\Phi\|^2_{1,\Omega} + \|\Gamma\|^2_{1,\Omega} \le \phi(D_2)\left|\int\limits_\Omega \frac{v_\varphi}{r}\Phi\Gamma dxdt'\right| \\ &+ \phi(D_2)(|\bar F_r|^2_{6/5,\Omega} + |\bar F_\varphi|^2_{6/5,\Omega}), \end{aligned} \tag{121}$$

where ϕ is an increasing positive function. Integrating (121) with respect to time yields (111). This ends the proof. □

Lemma 11. *Let the assumptions of Lemma 16 hold.*

Let $v_\varphi \in L_\infty(0,t;L_d(\Omega))$, $d > 3$. Let $\theta = \left(1 - \frac{3}{d}\right)\varepsilon_1 - \frac{3}{d}\varepsilon_2 > 0$, $\varepsilon = \varepsilon_1 + \varepsilon_2$. Let $\varepsilon_0 > 0$ be arbitrarily small.

Then,

$$\begin{aligned} I_3 \le\ & c|v_\varphi|^{\varepsilon}_{d,\infty,\Omega^t}[c_1(1 + |v_\varphi|^{\frac{1}{2}\theta\varepsilon_0}_{\infty,\Omega^t})\|\Gamma\|^{\frac{1}{2}\theta}_{1,2,\Omega^t} \\ &+ c_2]|\nabla\Phi|^{1-\theta}_{2,\Omega^t}|\nabla\Gamma|_{2,\Omega^t}, \end{aligned} \tag{122}$$

where c_1 and c_2, depending on D_5, D_6, and D_7, are introduced in L_1^4 below.

Proof. We examine

$$I_3 = \int\limits_{\Omega^t} \left| r v_\varphi \frac{\Phi}{r} \frac{\Gamma}{r} \right| dx dt' \\ \leq \int\limits_{\Omega^t} |r v_\varphi|^{1-\varepsilon} |v_\varphi|^\varepsilon \left| \frac{\Phi}{r^{1-\varepsilon_1}} \right| \left| \frac{\Gamma}{r^{1-\varepsilon_2}} \right| dx dt' = I_3^1,$$

where $\varepsilon = \varepsilon_1 + \varepsilon_2$ and ε_i, for $i = 1, 2$, are positive numbers.

Using (52) and applying the Hölder inequality in I_3^1 yields

$$I_3^1 \leq D_2^{1-\varepsilon} \bigg(\int\limits_{\Omega^t} |v_\varphi|^{2\varepsilon} \left| \frac{\Phi}{r^{1-\varepsilon_1}} \right|^2 dx dt' \bigg)^{1/2} \left| \frac{\Gamma}{r^{1-\varepsilon_2}} \right|_{2,\Omega^t} \\ \equiv D_2^{1-\varepsilon} L |\Gamma / r^{1-\varepsilon_2}|_{2,\Omega^t} \equiv I_3^2.$$

By the Hardy inequality, we obtain

$$\left\| \frac{\Gamma}{r} \right\|_{L_{2,\varepsilon_2}(\Omega^t)} \leq c \|\nabla \Gamma\|_{L_{2,\varepsilon_2}(\Omega^t)} \leq c R^{\varepsilon_2} |\nabla \Gamma|_{2,\Omega^t}. \tag{123}$$

Now, we estimate L,

$$L = \bigg(\int\limits_0^t \int\limits_\Omega |v_\varphi|^{2\varepsilon} \left| \frac{\Phi}{r^{1-\varepsilon_1}} \right|^2 dx dt' \bigg)^{1/2} \\ \leq \bigg[\int\limits_0^t |v_\varphi|_{2\varepsilon\sigma,\Omega}^{2\varepsilon} \bigg(\int\limits_\Omega \left| \frac{\Phi}{r^{1-\varepsilon_1}} \right|^q dx \bigg)^{2/q} dt' \bigg]^{1/2} \equiv L_1,$$

where $1/\sigma + 1/\sigma' = 1$, $q = 2\sigma'$. Let $d = 2\varepsilon\sigma$. Then,

$$\sigma' = \frac{d}{d - 2\varepsilon} \quad \text{so} \quad q = \frac{2d}{d - 2\varepsilon}.$$

Continuing,

$$L_1 \leq \sup_t |v_\varphi|_{d,\Omega}^\varepsilon \bigg(\int\limits_0^t \left| \frac{\Phi}{r^{1-\varepsilon_1}} \right|_{q,\Omega}^2 dt' \bigg)^{1/2} \equiv L_1^1 L_1^2.$$

Now, we estimate the second factor L_1^2.

For this purpose, we use Lemma 5 for $r = 2$. Let $\frac{s}{q} = 1 - \varepsilon_1$. Then, $q \in [2, 2(3 - s)]$. Since $s = (1 - \varepsilon_1)q$ we have the restriction $2 \leq q \leq 6 - 2s = 6 - 2(1 - \varepsilon_1)q$. Then,

$$2 \leq q \leq \frac{6}{3 - 2\varepsilon_1} \tag{124}$$

and $\frac{6}{3-2\varepsilon_1} > 2$ for any $\varepsilon_1 \in (0, 1)$.

Hence, Lemma 5 implies

$$
\begin{aligned}
L_1^2 &= \bigg(\int_0^t \bigg| \frac{\Phi}{r^{1-\varepsilon_1}} \bigg|_{q,\Omega}^2 dt' \bigg)^{1/2} \\
&\le c \bigg(\int_0^t |\Phi|_{2,\Omega}^{2(\frac{3-s}{q}-\frac{1}{2})} |\nabla\Phi|_{2,\Omega}^{2(\frac{3}{2}-\frac{3-s}{q})} dt' \bigg)^{1/2} \\
&\le c|\Phi|_{2,\Omega^t}^{\frac{3-s}{q}-\frac{1}{2}} |\nabla\Phi|_{2,\Omega^t}^{\frac{3}{2}-\frac{3-s}{q}} \equiv L_1^3,
\end{aligned}
$$

where we used that for $\theta = \frac{3-s}{q} - \frac{1}{2}$, $1-\theta = \frac{3}{2} - \frac{3-s}{q}$ so the Hölder inequality can be applied.

Using (173) in L_1^3, we have

$$
\begin{aligned}
L_1^3 &\le c(D_5^{\frac{1}{2}\theta} |\nabla\Gamma|_{2,\Omega^t}^{\frac{1}{2}\theta} + D_6^{\frac{1}{2}\theta} |v_\varphi|_{\infty,\Omega^t}^{\frac{1}{2}\theta\varepsilon_0} \|\Gamma\|_{1,2,\Omega^t}^{\frac{1}{2}\theta} + D_7^{\frac{1}{2}\theta}) \cdot |\nabla\Phi|_{2,\Omega^t}^{1-\theta} \\
&\equiv [c_1(1+|v_\varphi|_{\infty,\Omega^t}^{\frac{1}{2}\theta\varepsilon_0}) \|\Gamma\|_{1,2,\Omega^t}^{\frac{1}{2}\theta} + c_2] |\nabla\Phi|_{2,\Omega^t}^{1-\theta} \equiv L_1^4,
\end{aligned}
$$

where c_1 and c_2 depend on D_5, D_6, and D_7.

To justify the above inequality, we have to know that the following inequalities hold:

$$
\theta = \frac{3-s}{q} - \frac{1}{2} > 0 \tag{125}
$$

and

$$
1-\theta = \frac{3}{2} - \frac{3-s}{q} > 0. \tag{126}
$$

Consider (125). Using the form of q and $\frac{s}{q}$, we have

$$
\frac{3}{q} - \frac{s}{q} - \frac{1}{2} > 0 \quad \text{so} \quad \frac{3(d-2\varepsilon)}{2d} - (1-\varepsilon_1) - \frac{1}{2} > 0.
$$

Hence

$$
\frac{3}{2} - \frac{3}{d}\varepsilon - 1 + \varepsilon_1 - \frac{1}{2} > 0 \quad \text{so} \quad \varepsilon_1 - \frac{3}{d}(\varepsilon_1+\varepsilon_2) > 0.
$$

Therefore, the following inequality

$$
\left(1-\frac{3}{d}\right)\varepsilon_1 - \frac{3}{d}\varepsilon_2 > 0 \tag{127}
$$

holds for $d > 3$, and ε_2 is sufficiently small. Moreover, (127) implies

$$
\varepsilon_1 > \frac{3}{d}\frac{d}{d-3}\varepsilon_2 = \frac{3}{d-3}\varepsilon_2. \tag{128}
$$

To examine (126), we calculate

$$
\frac{3}{2} - \frac{3(d-2\varepsilon)}{2d} + 1 - \varepsilon_1 = 1 + \frac{3}{d}\varepsilon - \varepsilon_1 = 1 - \left(1-\frac{3}{d}\right)\varepsilon_1 + \frac{3}{d}\varepsilon_2. \tag{129}
$$

Since (129) must be positive, we have the restriction

$$
1 + \frac{3}{d}\varepsilon_2 > \left(1-\frac{3}{d}\right)\varepsilon_1 \tag{130}
$$

Using (128) in (130) implies

$$1+\frac{3}{d}\varepsilon_2>\frac{3}{d}\varepsilon_2$$

so there is no contradiction.

Hence, we have

$$\begin{aligned}\theta&=\left(1-\frac{3}{d}\right)\varepsilon_1-\frac{3}{d}\varepsilon_2,\\ 1-\theta&=1-\left(1-\frac{3}{d}\right)\varepsilon_1+\frac{3}{d}\varepsilon_2,\end{aligned}\tag{131}$$

where $d>3$.

Finally,

$$I_3\le c|v_\varphi|^{\varepsilon}_{d,\infty,\Omega^t}[c_1(1+|v_\varphi|^{\frac{1}{2}\theta\varepsilon_0}_{\infty,\Omega^t})\|\Gamma\|^{\frac{1}{2}\theta}_{1,2,\Omega^t}+c_2]|\nabla\Phi|^{1-\theta}_{2,\Omega^t}\cdot|\nabla\Gamma|_{2,\Omega^t}.$$

This implies (122) and ends the proof. □

We introduce the quantity

$$X(t)=\|\Phi\|_{V(\Omega^t)}+\|\Gamma\|_{V(\Omega^t)}.\tag{132}$$

Lemma 12. *Let the assumptions of Lemmas 10 and 11 hold. Let* $\theta=\left(1-\frac{3}{d}\right)\varepsilon_1-\frac{3}{d}\varepsilon_2$, $\varepsilon=\varepsilon_1+\varepsilon_2$. *Then*

$$X^2\le c_0|v_\varphi|^{\frac{4\varepsilon}{\theta}}_{d,\infty,\Omega^t}(1+|v_\varphi|^{2\varepsilon_0}_{\infty,\Omega^t})+c_0|v_\varphi|^{\frac{2\varepsilon}{\theta}}_{d,\infty,\Omega^t}+D_8^2,\tag{133}$$

where $c_0=\phi(D_5,D_6,D_7)$.

Proof. In view of notation (132), inequalities (111) and (122) imply

$$\begin{aligned}X^2\le{}&c|v_\varphi|^{\varepsilon}_{d,\infty,\Omega^t}[c_1(1+|v_\varphi|^{\frac{1}{2}\theta\varepsilon_0}_{\infty,\Omega^t})X^{1-\frac{1}{2}\theta}\\&+c_2X^{1-\theta}]X+D_8\equiv\alpha_1X^{2-\frac{1}{2}\theta}+\alpha_2X^{2-\theta}+D_8^2.\end{aligned}\tag{134}$$

Applying the Young inequality in (134) implies

$$X^2\le c\alpha_1^{\frac{4}{\theta}}+c\alpha_2^{\frac{2}{\theta}}+D_8^2.$$

This yields (133) and concludes the proof. □

Remark 4. *Consider exponents in (133). Then,*

$$\delta=\frac{4\varepsilon}{\theta}=\frac{4\varepsilon}{(1-\frac{3}{d})\varepsilon_1-\frac{3}{d}\varepsilon_2},\quad\delta_0=\frac{2\varepsilon}{(1-\frac{3}{d})\varepsilon_1-\frac{3}{d}\varepsilon_2}.\tag{135}$$

For ε_2 *small, we have*

$$\delta=\frac{4}{1-\frac{3}{d}}+\varepsilon_*,\quad\delta_0=\frac{2}{1-\frac{3}{d}}+\varepsilon_{0*},$$

where ε_* *and* ε_{0*} *are positive numbers that can be chosen to be very small.*

For $d=12$, *it follows that*

$$\delta=\frac{16}{3}+\varepsilon_*,\quad\delta_0=\frac{8}{3}+\varepsilon_{0*}.\tag{136}$$

This ends the remark.

Lemma 13. *Assume that* $\varepsilon_1 > a\varepsilon_2$, $s > 1$, $a = \frac{16+6'}{3\cdot 6'-16}$, $b = \frac{2\cdot 6'(3\varepsilon_1-\varepsilon_2)}{(6'\cdot 3-16)(\varepsilon_1-a\varepsilon_2)}$, *and we choose* $6'$ *to be arbitrarily close to 6, and*

$$D_9^s(s) = s^2|f_\varphi|^s_{\frac{3s}{2s+1},s,\Omega^t} + |v_\varphi(0)|^s_{s,\Omega} < \infty.$$

Then, excluding cases where either $v_\varphi = 0$ *or* v_φ *is small, we have*

$$|v_\varphi|^{6'}_{12,\infty,\Omega^t} \le c|v_\varphi|^{b\varepsilon_0}_{\infty,\Omega^t} + \phi(D_5, D_6, D_7) + c(D_8 + D_9^{12}) \tag{137}$$

Proof. We multiply $(7)_2$ by $v_\varphi|v_\varphi|^{s-2}$, integrate over Ω, and exploit the relation $\frac{v_r}{r} = -\psi_{1,z}$. Then, we obtain

$$\begin{aligned} &\frac{1}{s}\frac{d}{dt}|v_\varphi|^s_{s,\Omega} + \frac{4\nu(s-1)}{s^2}|\nabla|v_\varphi|^{s/2}|^2_{2,\Omega} = \int_\Omega \psi_{1,z}|v_\varphi|^s dx \\ &+ \int_\Omega f_\varphi v_\varphi |v_\varphi|^{s-2} dx. \end{aligned} \tag{138}$$

Integrating by parts in the first term on the r.h.s. of (138) and applying the Hölder and Young inequalities yield

$$\left|\int_\Omega \psi_{1,z}|v_\varphi|^s dx\right| \le \varepsilon|\partial_z|v_\varphi|^{s/2}|^2_{2,\Omega} + c(1/\varepsilon)\int_\Omega \psi_1^2|v_\varphi|^s dx.$$

By the Poincaré inequality,

$$|\nabla|v_\varphi|^{s/2}|^2_{2,\Omega} \ge c|v_\varphi|^s_{3s,\Omega}$$

we can estimate the second term on the r.h.s. of (138) by

$$|f_\varphi|_{\frac{3s}{2s+1},\Omega}|v_\varphi|^{s-1}_{3s,\Omega} \le \varepsilon_1|v_\varphi|^s_{3s,\Omega} + c(1/\varepsilon_1)|f_\varphi|^s_{\frac{3s}{2s+1},\Omega}.$$

Using the above estimates with sufficiently small ε, ε_1 in (138) we derive the inequality

$$\begin{aligned} &\frac{1}{s}\frac{d}{dt}|v_\varphi|^s_{s,\Omega} + \frac{1}{s}|\nabla|v_\varphi|^{s/2}|^2_{2,\Omega} + \frac{1}{s}|v_\varphi|^s_{3s,\Omega} \\ &\le cs\int_\Omega \psi_1^2|v_\varphi|^s dx + cs|f_\varphi|^s_{\frac{3s}{2s+1},\Omega}. \end{aligned} \tag{139}$$

In view of Lemma 2, the first term on the r.h.s. of (139) is bounded by

$$cs|u|^{6'}_{\infty,\Omega^t}\int_\Omega \frac{\psi_1^2}{r^{6'}}|v_\varphi|^{s-6'}dx \le csD_2^{6'}|v_\varphi|^{s-6'}_{\infty,\Omega}\int_\Omega \frac{\psi_1^2}{r^{6'}}dx,$$

where $6' < 6$ but $6'$ may be assumed arbitrarily close to 6.

Using the estimate in (139) yields

$$\frac{1}{s}\frac{d}{dt}|v_\varphi|^s_{s,\Omega} \le csD_2^{6'}|v_\varphi|^{s-6'}_{\infty,\Omega}\int_\Omega \frac{\psi_1^2}{r^{6'}}dx + cs|f_\varphi|^s_{\frac{3s}{2s+1},\Omega}. \tag{140}$$

Integrating (140) with respect to time and using Lemma 18, we obtain

$$\begin{aligned} |v_\varphi|^s_{s,\Omega} &\le c_1 s^2 D_2^{6'}|v_\varphi|^{s-6'}_{\infty,\Omega^t}\|\Gamma\|^2_{1,2,\Omega^t} \\ &\quad + cs^2|f_\varphi|^s_{\frac{3s}{2s+1},s,\Omega^t} + |v_\varphi(0)|^s_{s,\Omega} \\ &\equiv c_1 s^2 D_2^{6'}|v_\varphi|^{s-6'}_{\infty,\Omega^t}\|\Gamma\|^2_{1,2,\Omega^t} + cD_9^s(s), \end{aligned} \tag{141}$$

$c_1 = cR^{2\mu}\left(1 + \frac{4}{(6-6')^2}\right)$.

Dividing (141) by $|v_\varphi|^{s-6'}_{\infty,\Omega^t}$ implies

$$\left|\frac{|v_\varphi|_{s,\infty,\Omega^t}}{|v_\varphi|_{\infty,\Omega^t}}\right|^{s-6'} |v_\varphi|^{6'}_{s,\Omega} \le c_1 s^2 D_2^{6'} \|\Gamma\|^2_{1,2,\Omega^t} + \frac{c}{|v_\varphi|^{s-6'}_{\infty,\Omega^t}} D_9^s(s). \tag{142}$$

The dividing by $|v_\varphi|_{\infty,\Omega^t}$ is justified because the following two cases are excluded from this paper:

Case 1: In the case, $v_\varphi = 0$ the existence of global regular solutions to problem (6) is proven in [1,2,13].

Case 2: The existence of global regular solutions to problem (6) for v_φ sufficiently small is proven in Appendix A .

Since Cases 1 and 2 are not considered in this paper, we can show the existence of positive constants c_0 and c_1, such that

$$\left|\frac{|v_\varphi|_{s,\infty,\Omega^t}}{|v_\varphi|_{\infty,\Omega^t}}\right|^{s-6'} \ge \bar{c}_0 \tag{143}$$

and

$$\frac{1}{|v_\varphi|^{s-6'}_{\infty,\Omega^t}} \le \bar{c}_1. \tag{144}$$

In view of (143) and (144), inequality (142) takes the form

$$\bar{c}_0 |v_\varphi|^{6'}_{s,\infty,\Omega^t} \le c_1 s^2 \|\Gamma\|^2_{1,2,\Omega^t} + c\bar{c}_1 D_9^s(s). \tag{145}$$

Let $d = 12$. Then, $\theta = \frac{1}{4}(3\varepsilon_1 - \varepsilon_2)$ and (133) for $d = 12$ takes the form

$$X^2 \le c_0 |v_\varphi|^{\frac{16\varepsilon}{3\varepsilon_1-\varepsilon_2}}_{12,\infty,\Omega^t} (1 + |v_\varphi|^{2\varepsilon_0}_{\infty,\Omega^t}) + c_0 |v_\varphi|^{\frac{8\varepsilon}{3\varepsilon_1-\varepsilon_2}}_{12,\infty,\Omega^t} + D_8. \tag{146}$$

Taking (145) for $s = 12$ and using (146) yield

$$\begin{aligned} |v_\varphi|^{6'}_{12,\infty,\Omega^t} \le\ & c_2 |v_\varphi|^{\frac{16\varepsilon}{3\varepsilon_1-\varepsilon_2}}_{12,\infty,\Omega^t} (1 + |v_\varphi|^{2\varepsilon_0}_{\infty,\Omega^t}) \\ & + c_2 |v_\varphi|^{\frac{8\varepsilon}{3\varepsilon_1-\varepsilon_2}}_{12,\infty,\Omega^t} + cD_8 + cD_9^{12}, \end{aligned} \tag{147}$$

where $C_2 = \frac{144 c_1 c_0}{\bar{c}_0}$.

To derive any estimate from (147), we need

$$\frac{16\varepsilon}{3\varepsilon_1 - \varepsilon_2} < 6' \tag{148}$$

We see that (148) holds for

$$\varepsilon_1 > \frac{16 + 6'}{3 \cdot 6' - 16} \varepsilon_2 \equiv a\varepsilon_2, \tag{149}$$

where $a > 11$.

In view of the Young inequality, (147) implies

$$|v_\varphi|^{6'}_{12,\infty,\Omega^t} \le c |v_\varphi|^{b\varepsilon_0}_{\infty,\Omega^t} + c + c(D_8 + D_9^{12}), \tag{150}$$

where $b = \frac{2 \cdot 6'(3\varepsilon_1 - \varepsilon_2)}{(6' \cdot 3 - 16)(\varepsilon_1 - a\varepsilon_2)}$. The above inequality implies (137) and concludes the proof. □

Remark 5. *Exploiting (150) in (146) implies the inequality*

$$X^2 \le c(1 + |v_\varphi|^{2\varepsilon_0}_{\infty,\Omega^t})|v_\varphi|^{d\varepsilon_0}_{\infty,\Omega^t} + \phi(D_5, D_7, D_8, D_9), \tag{151}$$

where $d = \frac{16b\varepsilon}{3\varepsilon_1 - \varepsilon_2}$ *and X is introduced in (132).*

To prove Theorem 1, we need an estimate for $|v_\varphi|_{\infty,\Omega^t}$. For this purpose, we need the result

Lemma 14. *Assume that quantities* D_2, D_5, D_7, D_8, *and* D_9 *are bounded. Assume that* $f_\varphi/r \in L_1(0,t;L_\infty(\Omega))$, $v_\varphi(0) \in L_\infty(\Omega)$.

Then, there exists an increasing positive function ϕ, *such that*

$$|v_\varphi|_{\infty,\Omega^t} \le \phi(D_2, D_5, D_7, D_8, D_9, \|f_\varphi/r\|_{L_1(0,t;L_\infty(\Omega))}, |v_\varphi(0)|_{\infty,\Omega}). \tag{152}$$

Proof. Recall Equation $(7)_2$ for v_φ

$$v_{\varphi,t} + v\cdot\nabla v_\varphi - \nu\left(\Delta v_\varphi - \frac{1}{r^2}v_\varphi\right) = \psi_{1,z}v_\varphi + f_\varphi, \tag{153}$$

where $\frac{v_r}{r} = -\psi_{1,z}$.

Multiplying (153) by $v_\varphi|v_\varphi|^{s-2}$ and integrating over Ω yields

$$\begin{aligned} &\frac{1}{s}\frac{d}{dt}|v_\varphi|^s_{s,\Omega} + \frac{4\nu(s-1)}{s^2}|\nabla|v_\varphi|^{s/2}|^2_{2,\Omega} + \nu\int_\Omega \frac{|v_\varphi|^s}{r^2}dx \\ &= \int_\Omega \psi_{1,z}v_\varphi^2|v_\varphi|^{s-2}dx + \int_\Omega f_\varphi v_\varphi|v_\varphi|^{s-2}dx. \end{aligned} \tag{154}$$

The first term on the r.h.s. of (154) is bounded by

$$\int_\Omega |\psi_{,z}|\,|v_\varphi|^{s/2}\frac{|v_\varphi|^{s/2}}{r}dx \le \varepsilon\int_\Omega \frac{|v_\varphi|^s}{r^2}dx + c(1/\varepsilon)\int_\Omega \psi_{,z}^2|v_\varphi|^s dx,$$

where the second integral is bounded by

$$|rv_\varphi|^2_{\infty,\Omega}\int_\Omega |\psi_{1,z}|^2|v_\varphi|^{s-2}dx \le D_2^2|\psi_{1,z}|^2_{s,\Omega}|v_\varphi|^{s-2}_{s,\Omega}.$$

The second term on the r.h.s. of (154) is estimated by

$$\begin{aligned} &\int_\Omega |f_\varphi|\,|v_\varphi|^{s-1}dx = \int_\Omega \left|\frac{f_\varphi}{r}\right| r|v_\varphi|^{s-1}dx \\ &\le |rv_\varphi|_{\infty,\Omega}\int_\Omega \left|\frac{f_\varphi}{r}\right| |v_\varphi|^{s-2}dx \le D_2\left|\frac{f_\varphi}{r}\right|_{s/2,\Omega}|v_\varphi|^{s-2}_{s,\Omega}. \end{aligned}$$

Using the above estimates in (154) and assuming that ε is sufficiently small, we obtain the inequality

$$\frac{1}{s}\frac{d}{dt}|v_\varphi|^s_{s,\Omega} \le D_2^2\left(|\psi_{1,z}|^2_{s,\Omega}|v_\varphi|^{s-2}_{s,\Omega} + \left|\frac{f_\varphi}{r}\right|_{s/2,\Omega}|v_\varphi|^{s-2}_{s,\Omega}\right).$$

Simplifying, we have

$$\frac{1}{2}\frac{d}{dt}|v_\varphi|^2_{s,\Omega} \le D_2^2\left(|\psi_{1,z}|^2_{s,\Omega} + \left|\frac{f_\varphi}{r}\right|_{s/2,\Omega}\right).$$

Integrating with respect to time and passing with $s \to \infty$, we derive

$$|v_\varphi(t)|^2_{\infty,\Omega} \le D_2^2\bigg(\int\limits_0^t |\psi_{1,z}|^2_{\infty,\Omega}dt' + \int\limits_0^t \left|\frac{f_\varphi}{r}\right|_{\infty,\Omega}dt'\bigg) + |v_\varphi(0)|^2_{\infty,\Omega}. \tag{155}$$

Since $\int_0^t |\psi_{1,z}|^2_{\infty,\Omega}dt' \le X^2$, we can apply (151). Then (155) takes the form

$$\begin{aligned} |v_\varphi|^2_{\infty,\Omega^t} &\le D_2^2(1+|v_\varphi|^{2\varepsilon_0}_{\infty,\Omega^t})|v_\varphi|^{\frac{96\varepsilon}{\varepsilon_1-11\varepsilon_2}\varepsilon_0}_{\infty,\Omega^t} + D_2\phi(D_5,D_7,D_8,D_9) \\ &+ D_2^2\int\limits_0^t \left|\frac{f_\varphi}{r}\right|_{\infty,\Omega}dt' + |v_\varphi(0)|^2_{\infty,\Omega}. \end{aligned} \tag{156}$$

Hence, for ε_0 sufficiently small, we derive (152). This ends the proof. □

Remark 6. *Inequalities (151) and (152) imply*

$$X \le \phi(D_2,D_5,D_7,D_8,D_9,|f_\varphi/r|_{\infty,1,\Omega^t},|v_\varphi(0)|_{\infty,\Omega}) \tag{157}$$

The above inequality proves Theorem 1.

5. Estimates for the Swirl

Applying the energy method and using the estimate for the weak solution (see Lemma 1) and L_∞-estimate for swirl (see Lemma 2), we derive the estimate

$$\|u\|_{L_\infty(0,t;H^1(\Omega))} + \|u\|_{L_2(0,T;H^2(\Omega))} \le \phi(\text{data}).$$

This is a new result. It is necessary in the proof of (173).

In this section, we find estimates for solutions to the problem

$$\begin{aligned} &u_{,t} + v\cdot\nabla u - \nu\Delta u + 2\nu\frac{u_{,r}}{r} = rf_\varphi \equiv f_0 && \text{in } \Omega^t, \\ &u|_{S_1} = 0, && \text{in } \Omega^t, \\ &u|_{S_2} - \text{ periodic boundary conditions}, \\ &u|_{t=0} = u(0) && \text{in } \Omega. \end{aligned} \tag{158}$$

Lemma 15. *Assume that D_1 and D_2 are described by (46) and (52), respectively. Let $u_{,z}(0), u_{,r}(0) \in L_2(\Omega)$, $f_0 \in L_2(\Omega^t)$.*

Then, solutions to (158) satisfy the estimates

$$|u_{,z}(t)|^2_{2,\Omega} + \nu|\nabla u_{,z}|^2_{2,\Omega^t} \le c(D_1^2D_2^2 + |u_{,z}(0)|^2_{2,\Omega} + |f_0|^2_{2,\Omega^t}) \equiv cD_3^2, \tag{159}$$

$$\begin{aligned} &|u_{,r}(t)|^2_{2,\Omega} + \nu(|u_{,rr}|^2_{2,\Omega^t} + |u_{,rz}|^2_{2,\Omega^t}) \le cD_1^2(1+D_2^2) \\ &+ |u_{,r}(0)|^2_{2,\Omega} + |f_0|^2_{2,\Omega^t} + |f_0|^2_{4/3,2,S_1^t} \equiv cD_4^2. \end{aligned} \tag{160}$$

Proof. We differentiate (158) with respect to z, multiply by $u_{,z}$, and integrate over Ω. To apply the Green theorem, we have to consider problem (158) in domain $\bar\Omega = \{x \in \mathbb{R}^3 : r < R, z \in (-a,a), \varphi \in (0,2\pi)\}$. Then, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}|u_{,z}|^2_{2,\bar\Omega} - \nu\int\limits_{\bar\Omega}\operatorname{div}(\nabla u_{,z}u_{,z})d\bar x + \nu\int\limits_{\bar\Omega}|\nabla u_{,z}|^2d\bar x \\ &= -\int\limits_{\bar\Omega} v_{,z}\cdot\nabla u\cdot u_{,z}d\bar x + \int\limits_{\bar\Omega} f_{0,z}u_{,z}d\bar x, \end{aligned} \tag{161}$$

where $d\bar{x} = dxd\varphi$. The second term on the l.h.s. implies a boundary term, which vanishes due to boundary conditions. Since all functions in (161) do not depend on φ, any integral with respect to φ can be omitted.

Integrating by parts with respect to z in the term from the r.h.s. of (161) and using the boundary conditions on S_2, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}|u_{,z}|_{2,\Omega}^2 + \nu|\nabla u_{,z}|_{2,\Omega}^2 - \frac{\nu}{2}\int_{-a}^{a} u_{,z}^2\Big|_{r=0}^{r=R} dz \\ &= \int_\Omega v_{,zz}\cdot\nabla u u dx + \int_\Omega v_{,z}\cdot\nabla u_{,z}\cdot u dx - \int_\Omega f_0 u_{,zz} dx.\end{aligned} \tag{162}$$

The last term on the l.h.s. of (162) vanishes and the first term on the r.h.s. equals

$$\frac{1}{2}\int_\Omega v_{,zz}\cdot\nabla u^2 dx = \frac{1}{2}\int_{S_1} v_{,zz}\cdot\bar{n}u^2 dS_1 = 0.$$

Applying the Hölder and Young inequalities to the other terms from the r.h.s. of (162) yields

$$\frac{d}{dt}|u_{,z}|_{2,\Omega}^2 + \nu|\nabla u_{,z}|_{2,\Omega}^2 \le c|u|_{\infty,\Omega}^2|v_{,z}|_{2,\Omega}^2 + c|f_0|_{2,\Omega}^2. \tag{163}$$

Integrating (163) with respect to time gives

$$\begin{aligned}&|u_{,z}(t)|_{2,\Omega}^2 + \nu|\nabla u_{,z}|_{2,\Omega}^2 \le c|u|_{\infty,\Omega^t}^2|v_{,z}|_{2,\Omega^t}^2 + |u_{,z}(0)|_{2,\Omega}^2 \\ &\quad + c|f_0|_{2,\Omega^t}^2 \le cD_1^2D_2^2 + |u_{,z}(0)|_{2,\Omega}^2 + c|f_0|_{2,\Omega^t}^2.\end{aligned} \tag{164}$$

The above inequality implies (159).

Differentiating (158) with respect to r gives

$$u_{,rt} + v\cdot\nabla u_{,r} + v_{,r}\cdot\nabla u - \nu(\Delta u)_{,r} + \frac{2\nu}{r}u_{,rr} - \frac{2\nu}{r^2}u_{,r} = f_{0,r}. \tag{165}$$

Multiplying (165) by $u_{,r}$ and integrating over Ω yields

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}|u_{,r}|_{2,\Omega}^2 + \int_\Omega v_{,r}\cdot\nabla u u_{,r} dx - \nu\int_\Omega (\Delta u)_{,r}u_{,r} dx \\ &+ 2\nu\int_\Omega \frac{1}{r}u_{,rr}u_{,r} dx - 2\nu\int_\Omega \frac{u_{,r}^2}{r^2} dx = \int_\Omega f_{0,r}u_{,r} dx.\end{aligned} \tag{166}$$

Now, we examine the particular terms in (166). The second term equals

$$\begin{aligned}&\int_\Omega v_{,r}\cdot\nabla u u_{,r} r dr dz = \int_\Omega (v_{r,r}\partial_r u + v_{z,r}\partial_z u)u_{,r} r dr dz \\ &= \int_\Omega (rv_{r,r}u_{,r} + rv_{z,r}u_{,z})u_{,r} dr dz \\ &= \int_\Omega (rv_{r,r}u_{,r}u_{,r} + rv_{z,r}u_{,r}u_{,z}) dr dz \\ &= -\int_\Omega [(rv_{r,r}u_{,r})_{,r} + (rv_{z,r}u_{,r})_{,z}] u dr dz \equiv I,\end{aligned}$$

where we use $rv_{r,r}u_{,r}u|_{r=0} = 0$ (see [9]). Continuing, we write I in the following form:

$$\begin{aligned} I = &-\int\limits_\Omega [(rv_{r,r})_{,r} + (rv_{z,r})_{,z}]u_{,r}udrdz \\ &-\int\limits_\Omega [rv_{r,r}u_{,rr} + rv_{z,r}u_{,rz}]udrdz \equiv I_1 + I_2. \end{aligned}$$

To estimate I_1, we calculate

$$I_1^1 = (rv_{r,r})_{,r} + (rv_{z,r})_{,z} = rv_{r,rr} + v_{r,r} + rv_{z,rz}.$$

Since $v = v_r\bar{e}_r + v_z\bar{e}_z$ is divergence free, we have

$$v_{r,r} + v_{z,z} + \frac{v_r}{r} = 0. \tag{167}$$

Since Equation (167) is satisfied identically in Ω, we can differentiate (167) with respect to r. Then, we have

$$v_{r,rr} + v_{z,zr} + \frac{v_{r,r}}{r} - \frac{v_r}{r^2} = 0.$$

Hence,

$$I_1^1 = \frac{v_r}{r}.$$

Then, I_1 equals

$$I_1 = -\int\limits_\Omega \frac{v_r}{r}u_{,r}udrdz.$$

Therefore,

$$\left|\int\limits_0^t I_1 dt'\right| \le \left|\frac{v_r}{r}\right|_{2,\Omega^t}\left|\frac{u_{,r}}{r}\right|_{2,\Omega^t}|u|_{\infty,\Omega^t}. \tag{168}$$

Next,

$$|I_2| \le \varepsilon(|u_{,rr}|^2_{2,\Omega} + |u_{,rz}|^2_{2,\Omega}) + c(1/\varepsilon)|u|^2_{\infty,\Omega}(|v_{r,r}|^2_{2,\Omega} + |v_{z,r}|^2_{2,\Omega}).$$

The third integral in (166) equals

$$\begin{aligned} J &= -\nu\int\limits_\Omega (\Delta u)_{,r}u_{,r}dx = -\nu\int\limits_\Omega \left(u_{,rrr} + \left(\frac{1}{r}u_{,r}\right)_{,r} + u_{,rzz}\right)u_{,r}rdrdz \\ &= -\nu\int\limits_\Omega \left[\left(u_{,rr} + \frac{1}{r}u_{,r}\right)u_{,r}r\right]_{,r} drdz + \nu\int\limits_\Omega u_{,rr}(u_{,r}r)_{,r}drdz \\ &\quad + \nu\int\limits_\Omega \frac{1}{r}u_{,r}(u_{,r}r)_{,r}drdz + \int\limits_\Omega u^2_{,rz}dx = -\nu\int\limits_{-a}^a \left(u_{,rr} + \frac{1}{r}u_{,r}\right)u_{,r}r\Big|_{r=0}^{r=R} dz \\ &\quad + \nu\int\limits_\Omega (u^2_{,rr} + u^2_{,rz})dx + \nu\int\limits_\Omega \frac{u^2_{,r}}{r^2}dx + 2\nu\int\limits_\Omega u_{,rr}u_{,r}drdz, \end{aligned}$$

where the last term equals

$$\nu\int\limits_\Omega (u^2_{,r})_{,r}drdz = \nu\int\limits_{-a}^a u^2_{,r}\Big|_{r=0}^{r=R} dz = \nu\int\limits_{-a}^a u^2_{,r}\Big|_{r=R} dz \tag{169}$$

because $u_{,r}|_{r=0} = (v_\varphi + v_{\varphi,r}r)|_{r=0} = 0$.

To examine the boundary term in J, we recall the expansion of v_φ near the axis of symmetry (see [9])

$$v_\varphi = a_1(z,t)r + a_2(z,t)r^3 + \cdots,$$

so

$$u = a_1(z,t)r^2 + a_2(z,t)r^4 + \cdots$$

Then, $\left(u_{,rr} + \frac{1}{r}u_{,r}\right)u_{,r}r|_{r=0} = 0$, and we have to emphasize that all calculations in this paper are performed for sufficiently regular solutions.

Therefore, the boundary term in J equals

$$J_1 = -\nu \int_{-a}^{a} \left(u_{,rr} + \frac{1}{r}u_{,r}\right)u_{,r}r\Big|_{r=R} dz.$$

Projecting (158)$_1$ on S_1 yields

$$-\nu\left(u_{,rr} + \frac{1}{r}u_{,r}\right) + 2\nu\frac{u_{,r}}{r} = f_0 \quad \text{on} \quad S_1.$$

Hence,

$$u_{,rr}|_{S_1} = \left(\frac{u_{,r}}{r} - \frac{1}{\nu}f_0\right)\Big|_{S_1}.$$

Using the expression in J_1 gives

$$J_1 = -2\nu \int_{-a}^{a} u_{,r}^2\Big|_{r=R} dz + \int_{-a}^{a} f_0 u_{,r}r\Big|_{r=R} dz.$$

The fourth term in (166) equals (169).

Using the above estimates and expressions in (166) yields

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}|u_{,r}|_{2,\Omega}^2 + \nu\int_\Omega (u_{,rr}^2 + u_{,rz}^2)dx + \nu\int_\Omega \frac{u_{,r}^2}{r^2}dx \\ &- 2\nu\int_\Omega \frac{u_{,r}^2}{r^2}dx \le \int_\Omega \left|\frac{v_r}{r}u_{,r}u\right| drdz \\ &+ \varepsilon(|u_{,rr}|_{2,\Omega}^2 + |u_{,rz}|_{2,\Omega}^2) + c(1/\varepsilon)|u|_{\infty,\Omega}^2(|v_{r,r}|_{2,\Omega}^2 + |v_{z,r}|_{2,\Omega}^2) \\ &+ c(1/\varepsilon)|f_0|_{2,\Omega}^2 + \left|\int_{-a}^{a} f_0 u_{,r}r\Big|_{r=R} dz\right|. \end{aligned} \tag{170}$$

Integrating (170) with respect to time and assuming that ε is sufficiently small, we obtain

$$\begin{aligned} &|u_{,r}(t)|_{2,\Omega}^2 + \nu(|u_{,rr}|_{2,\Omega^t}^2 + |u_{,rz}|_{2,\Omega^t}^2) \le \nu\left|\frac{u_{,r}}{r}\right|_{2,\Omega^t}^2 \\ &+ c\left|\frac{v_r}{r}\right|_{2,\Omega^t}\left|\frac{u_{,r}}{r}\right|_{2,\Omega^t}|u|_{\infty,\Omega^t} + c|u|_{\infty,\Omega^t}^2(|v_{r,r}|_{2,\Omega^t}^2 + |v_{z,r}|_{2,\Omega^t}^2) \\ &+ c|f_0|_{2,\Omega^t}^2 + |u_{,r}(0)|_{2,\Omega}^2 + \nu\int_0^t\int_{-a}^{a} u_{,r}^2\Big|_{r=R} dxdt' \\ &+ \left|\int_0^t\int_{-a}^{a} f_0 u_{,r}r\Big|_{r=R} dxdt'\right|. \end{aligned} \tag{171}$$

Using

$$\int_{\Omega^t} \left|\frac{u_{,r}}{r}\right|^2 dxdt' \le \int_{\Omega^t} \left(|v_{\varphi,r}|^2 + \frac{v_\varphi^2}{r^2}\right) dxdt' \le cD_1^2$$

and

$$\int_0^t \int_{-a}^a u_{,r}^2\Big|_{r=R} dxdt' \le \varepsilon |\nabla u_{,r}|^2_{2,\Omega^t} + c(1/\varepsilon)|u_{,r}|^2_{2,\Omega^t},$$

$$\begin{aligned}\Big|\int_0^t \int_{-a}^a f_0 u_{,r}\Big|_{r=R} dxdt' &\le \varepsilon_1 |u_{,r}|^2_{4,2,S_1^t} + c(1/\varepsilon_1)|f_0|^2_{4/3,2,S_1^t} \\ &\le \varepsilon_1(|u_{,rr}|^2_{2,\Omega^t} + |u_{,rz}|^2_{2,\Omega^t}) + c(1/\varepsilon_1)|f_0|^2_{4/3,2,S_1^t}\end{aligned}$$

and Lemmas 1, 2, we have

$$\begin{aligned}|u_{,r}(t)|^2_{2,\Omega} + \nu(|u_{,rr}|^2_{2,\Omega^t} + |u_{,rz}|^2_{2,\Omega^t}) &\le c(D_1^2 + D_1^2 D_2 + D_1^2 D_2^2) \\ &+ c|f_0|^2_{2,\Omega^t} + c|f_0|^2_{4/3,2,S_1^t} + |u_{,r}(0)|^2_{2,\Omega}.\end{aligned} \tag{172}$$

This inequality implies (160) and concludes the proof. □

6. Estimates for ω_r, ω_z

Inequality (173) is the most important inequality in this paper. To prove it, we need results from Sections 3 and 5 as well as Lemma 2. By the energy method, we derive (174), where the first term on the r.h.s. is nonlinear. The aim of the proof of Lemma 16 is to show that

$$(*) \qquad J \le c|u|_{\infty,\Omega^t}(D_1 + \|u\|_{L_2(0,t;H^2(\Omega))})\|\Gamma\|_{L_2(0,t;H^1(\Omega))}$$

To show $(*)$, we replace ω_r, ω_z in J by derivatives of u described by (13) and express components of velocities v_r and v_z by derivatives of ψ using (15). We perform such calculations in J, where we are able to extract the norm $|u|_{\infty,\Omega^t}$. Then, J becomes bilinear. Then, estimates (46), (159), and (160) imply $(*)$ by the Hölder inequality. Hence, J is bounded by the quantity, which is linear with respect to the norm of Γ. This implies that the main Theorem 1 can be proved.

Lemma 16. *Assume that $D_5 = D_2(D_1 + D_3 + D_4)$ and $D_6 = D_2^{1-\varepsilon_0} D_3$, where D_1, and D_2 are introduced in (46) and (52), and D_3 and D_4 are introduced in (159) and (160), respectively. Let*

$$\begin{aligned}D_7 = |F_r|^2_{6/5,2,\Omega^t} + |F_z|^2_{6/5,2,\Omega^t} + |\omega_r(0)|^2_{2,\Omega} + |\omega_z(0)|^2_{2,\Omega} \\ + |f_\varphi|_{2,S_1^t}(D_3 + D_4) < \infty.\end{aligned}$$

Let ε_0 be an arbitrarily small positive number and let $v_\varphi \in L_\infty(\Omega^t)$.

Let $\Gamma \in L_2(0,t;H^1(\Omega))$.

Then,

$$\begin{aligned}\|\omega_r\|^2_{V(\Omega^t)} + \|\omega_z\|^2_{V(\Omega^t)} + |\Phi|^2_{2,\Omega^t} &\le cD_5|\Gamma_{,z}|_{2,\Omega^t} \\ &+ cD_6|v_\varphi|^{\varepsilon_0}_{\infty,\Omega^t}\|\Gamma\|_{1,2,\Omega^t} + cD_7.\end{aligned} \tag{173}$$

Proof. Multiplying (9)$_1$ by ω_r, (9)$_3$ by ω_z, integrating over Ω^t, and adding yield

$$\begin{aligned}
&\frac{1}{2}(|\omega_r(t)|_{2,\Omega}^2+|\omega_z(t)|_{2,\Omega}^2)+\nu(|\nabla\omega_r|_{2,\Omega^t}^2+|\nabla\omega_z|_{2,\Omega^t}^2)\\
&\quad+\nu\left|\frac{\omega_r}{r}\right|_{2,\Omega^t}^2-\nu\int_{S_1^t}\bar{n}\cdot\nabla\omega_r\omega_r dS_1dt'-\nu\int_{S_1^t}\bar{n}\cdot\nabla\omega_z\omega_z dS_1dt'\\
&=\int_{\Omega^t}[v_{r,r}\omega_r^2+v_{z,z}\omega_z^2+(v_{r,z}+v_{z,r})\omega_r\omega_z]dxdt'\\
&\quad+\int_{\Omega^t}(F_r\omega_r+F_z\omega_z)dxdt'+\frac{1}{2}(|\omega_r(0)|_{2,\Omega}^2+|\omega_z(0)|_{2,\Omega}^2)\\
&\equiv J+\int_{\Omega^t}(F_r\omega_r+F_z\omega_z)dxdt'+\frac{1}{2}(|\omega_r(0)|_{2,\Omega}^2+|\omega_z(0)|_{2,\Omega}^2).
\end{aligned}\tag{174}$$

Since $\omega_r=-v_{\varphi,z}$ and $v_\varphi|_{r=R}=0$, we obtain

$$-\int_{S_1}\bar{n}\cdot\nabla\omega_r\omega_r dS_1=0.$$

Using $\omega_z=v_{\varphi,r}+\frac{v_\varphi}{r}$, we derive

$$-\nu\int_{S_1^t}\bar{n}\cdot\nabla\omega_z\omega_z dS_1dt'=-\nu\int_0^t\int_{-a}^a\partial_r\left(v_{\varphi,r}+\frac{v_\varphi}{r}\right)\left(v_{\varphi,r}+\frac{v_\varphi}{r}\right)\bigg|_{r=R}Rdzdt'\equiv I_1.$$

Since $v_\varphi|_{r=R}=0$ I_1 takes the form

$$I_1=-\nu\int_0^t\int_{-a}^a\left(v_{\varphi,rr}+\frac{v_{\varphi,r}}{r}\right)v_{\varphi,r}\bigg|_{r=R}Rdzdt'.$$

Projecting (7)$_2$ on S_1 yields

$$-\nu\left(v_{\varphi,rr}+\frac{1}{r}v_{\varphi,r}\right)=f_\varphi\quad\text{on}\ \ S_1.$$

Hence,

$$I_1=R\int_0^t\int_{-a}^a f_\varphi v_{\varphi,r}\bigg|_{r=R}dzdt'=\int_0^t\int_{-a}^a f_\varphi\left(u_{,r}-\frac{1}{R}u\right)\bigg|_{r=R}dzdt'.\tag{175}$$

Using (13) and (21) in J implies

$$\begin{aligned}
J=\int_{\Omega^t}\bigg[&-\frac{1}{r^2}u_{,z}^2(\psi_{1,z}+r\psi_{1,rz})+\left(\frac{1}{r}u_{,r}\right)^2(r\psi_{1,zr}+2\psi_{1,z})\\
&-\frac{1}{r^2}u_{,r}u_{,z}(-r\psi_{1,zz}+3\psi_{1,r}+r\psi_{1,rr})\bigg]dxdt'\equiv J_1+J_2+J_3.
\end{aligned}$$

We integrate by parts in J_1 and use the boundary conditions on S_2. Then, we have

$$J_1=\int_{\Omega^t}\frac{1}{r^2}uu_{,zz}(\psi_{1,z}+r\psi_{1,rz})dxdt'+\int_{\Omega^t}\frac{1}{r^2}uu_{,z}(\psi_{1,zz}+r\psi_{1,rzz})dxdt'.$$

Now, we estimate the particular terms in J_1,

$$J_{11} = \left|\int\limits_{\Omega^t} uu_{,zz}\frac{1}{r}\psi_{1,rz}dxdt'\right| \le |u|_{\infty,\Omega^t}|u_{,zz}|_{2,\Omega^t}\left|\frac{1}{r}\psi_{1,rz}\right|_{2,\Omega^t},$$

$$J_{12} = \left|\int\limits_{\Omega^t} u\frac{u_{,z}}{r}\psi_{1,rzz}dxdt'\right| \le |u|_{\infty,\Omega^t}\left|\frac{u_{,z}}{r}\right|_{2,\Omega^t}|\psi_{1,rzz}|_{2,\Omega^t},$$

$$J_{13} = \left|\int\limits_{\Omega^t} u\frac{u_{,z}}{r}\frac{\psi_{1,zz}}{r}dxdt'\right| \le |u|_{\infty,\Omega^t}\left|\frac{u_{,z}}{r}\right|_{2,\Omega^t}\left|\frac{\psi_{1,zz}}{r}\right|_{2,\Omega^t},$$

$$J_{14} = \left|\int\limits_{\Omega^t} \frac{1}{r^2}uu_{,zz}\psi_{1,z}dxdt'\right| = \left|\int\limits_{\Omega^t} uu_{,zz}\frac{\psi_{1,z}}{r^2}dxdt'\right|$$
$$\le |u|_{\infty,\Omega^t}|u_{,zz}|_{2,\Omega^t}\left|\frac{\psi_{1,z}}{r^2}\right|_{2,\Omega^t},$$

where integration by parts can be performed in view of periodic boundary conditions on S_2.

Next, we consider J_2,

$$J_2 = \int\limits_{\Omega^t} \frac{1}{r^2}u_{,r}^2(r\psi_{1,zr}+2\psi_{1,z})rdrdzdt' = \int\limits_{\Omega^t} \frac{1}{r}u_{,r}^2(r\psi_{1,zr}+2\psi_{1,z})drdzdt'$$
$$= \int\limits_0^t\int\limits_{-a}^a \left[\frac{1}{r}uu_{,r}(r\psi_{1,zr}+2\psi_{1,z})\right]\Bigg|_{r=0}^{r=R} dzdt'$$
$$- \int\limits_{\Omega^t} uu_{,rr}\left(\frac{1}{r}\psi_{1,zr}+\frac{2}{r^2}\psi_{1,z}\right)dxdt'$$
$$- \int\limits_{\Omega^t} uu_{,r}\left(\psi_{1,zrr}-\frac{2}{r^2}\psi_{1,z}+\frac{2}{r}\psi_{1,zr}\right)drdzdt',$$

where the boundary term for $r = R$ vanishes because $u|_{r=R} = 0$. To examine the boundary term at $r = 0$, we recall from [9] the expressions near the axis of symmetry

$$u = a_1(z,t)r^2 + a_2(z,t)r^4 + \cdots,$$

so

$$u_{,r} = 2a_1(z,t)r + 4a_2(z,t)r^3 + \cdots$$

Then,

$$\frac{1}{r}uu_{,r}(r\psi_{1,zr}+2\psi_{1,z}) \sim cr^2(r\psi_{1,zr}+2\psi_{1,z}).$$

The above expression vanishes for $r = 0$ because $\psi_{1,z}$ is bounded near the axis of symmetry.

Now, we estimate the particular terms in J_2,

$$J_{21} = \left| \int\limits_{\Omega^t} u u_{,rr} \frac{1}{r} \psi_{1,rz} dx dt' \right| \leq |u|_{\infty,\Omega^t} |u_{,rr}|_{2,\Omega^t} \left| \frac{1}{r} \psi_{1,zr} \right|_{2,\Omega^t},$$

$$J_{22} = \left| \int\limits_{\Omega^t} u u_{,rr} \frac{1}{r^2} \psi_{1,z} dx dt' \right| \leq |u|_{\infty,\Omega^t} |u_{,rr}|_{2,\Omega^t} \left| \frac{1}{r^2} \psi_{1,z} \right|_{2,\Omega^t},$$

$$J_{23} = \left| \int\limits_{\Omega^t} u \frac{u_{,r}}{r} \psi_{1,zrr} dx dt' \right| \leq |u|_{\infty,\Omega^t} \left| \frac{u_{,r}}{r} \right|_{2,\Omega^t} |\psi_{1,zrr}|_{2,\Omega^t},$$

$$J_{24} = \left| \int\limits_{\Omega^t} u \frac{u_{,r}}{r} \frac{1}{r^2} \psi_{1,z} dx dt' \right| \leq |u|_{\infty,\Omega^t} \left| \frac{u_{,r}}{r} \right|_{2,\Omega^t} \left| \frac{1}{r^2} \psi_{1,z} \right|_{2,\Omega^t},$$

$$J_{25} = \left| \int\limits_{\Omega^t} u \frac{u_{,r}}{r} \frac{1}{r} \psi_{1,zr} dx dt' \right| \leq |u|_{\infty,\Omega^t} \left| \frac{u_{,r}}{r} \right|_{2,\Omega^t} \left| \frac{1}{r} \psi_{1,zr} \right|_{2,\Omega^t}.$$

Finally, we examine J_3. Integrating by parts with respect to z, and using the periodic boundary conditions on S_2, we have

$$\begin{aligned} J_3 = &\int\limits_{\Omega^t} u \frac{1}{r^2} u_{,rz} (-r\psi_{1,zz} + 3\psi_{1,r} + r\psi_{1,rr}) dx dt' \\ &+ \int\limits_{\Omega^t} u \frac{1}{r^2} u_{,r} (-r\psi_{1,zzz} + 3\psi_{1,rz} + r\psi_{1,rrz}) dx dt'. \end{aligned}$$

Now, we estimate the particular terms in J_3,

$$J_{31} = \left| \int\limits_{\Omega^t} u u_{,rz} \frac{1}{r} \psi_{1,zz} dx dt' \right| \leq |u|_{\infty,\Omega^t} |u_{,rz}|_{2,\Omega^t} \left| \frac{\psi_{1,zz}}{r} \right|_{2,\Omega^t},$$

$$\begin{aligned} J_{32} &= \left| \int\limits_{\Omega^t} u \frac{1}{r^2} u_{,rz} \psi_{1,r} dx dt' \right| = \left| \int\limits_{\Omega^t} \frac{u}{r^{\varepsilon_0}} u_{,rz} \frac{\psi_{1,r}}{r^{2-\varepsilon_0}} dx dt' \right| \\ &\leq |u|^{1-\varepsilon_0}_{\infty,\Omega^t} |v_\varphi|^{\varepsilon_0}_{\infty,\Omega^t} |u_{,rz}|_{2,\Omega^t} \left| \frac{\psi_{1,r}}{r^2} \right|_{L_2(0,t;L_{2,\varepsilon_0}(\Omega))}, \end{aligned}$$

where $\varepsilon_0 > 0$ can be as small as we want. Thus,

$$\begin{aligned} J_{33} &= \left| \int\limits_{\Omega^t} \frac{u}{r^{\varepsilon_0}} u_{,rz} \frac{1}{r^{1-\varepsilon_0}} \psi_{1,rr} dx dt' \right| \\ &\leq |u|^{1-\varepsilon_0}_{\infty,\Omega^t} |v_\varphi|^{\varepsilon_0}_{\infty,\Omega^t} |u_{,rz}|_{2,\Omega^t} \left| \frac{\psi_{1,rr}}{r^{1-\varepsilon_0}} \right|_{2,\Omega^t}, \end{aligned}$$

$$J_{34} = \left| \int\limits_{\Omega^t} u \frac{u_{,r}}{r} \psi_{1,zzz} dx dt' \right| \leq |u|_{\infty,\Omega^t} \left| \frac{u_{,r}}{r} \right|_{2,\Omega^t} |\psi_{1,zzz}|_{2,\Omega^t},$$

$$J_{35} = \left| \int\limits_{\Omega^t} u \frac{u_{,r}}{r} \frac{1}{r} \psi_{1,rz} dx dt' \right| \leq |u|_{\infty,\Omega^t} \left| \frac{u_{,r}}{r} \right|_{2,\Omega^t} \left| \frac{\psi_{1,rz}}{r} \right|_{2,\Omega^t},$$

$$J_{36} = \left| \int\limits_{\Omega^t} u \frac{u_{,r}}{r} \psi_{1,rrz} dx dt' \right| \leq |u|_{\infty,\Omega^t} \left| \frac{u_{,r}}{r} \right|_{2,\Omega^t} |\psi_{1,rrz}|_{2,\Omega^t}.$$

Summarizing the above estimates, we obtain

$$
\begin{aligned}
|J| \le c|u|_{\infty,\Omega^t}\bigg[&(|u_{,zz}|_{2,\Omega^t}+|u_{,zr}|_{2,\Omega^t}+|u_{,rr}|_{2,\Omega^t})\cdot\\
&\cdot\bigg(\bigg|\frac{1}{r}\psi_{1,rz}\bigg|_{2,\Omega^t}+\bigg|\frac{1}{r}\psi_{1,zz}\bigg|_{2,\Omega^t}+\bigg|\frac{1}{r^2}\psi_{1,z}\bigg|_{2,\Omega^t}\bigg)\\
&+\bigg(\bigg|\frac{u_{,r}}{r}\bigg|_{2,\Omega^t}+\bigg|\frac{u_{,z}}{r}\bigg|_{2,\Omega^t}\bigg)\bigg(|\psi_{1,rzz}|_{2,\Omega^t}+|\psi_{1,zrr}|_{2,\Omega^t}\\
&+|\psi_{1,zzz}|_{2,\Omega^t}+\bigg|\frac{1}{r}\psi_{1,zz}\bigg|_{2,\Omega^t}+\bigg|\frac{1}{r}\psi_{1,zr}\bigg|_{2,\Omega^t}+\bigg|\frac{1}{r^2}\psi_{1,z}\bigg|_{2,\Omega^t}\bigg)\bigg]\\
&+c|u|^{1-\varepsilon_0}_{\infty,\Omega^t}|v_\varphi|^{\varepsilon_0}_{\infty,\Omega^t}|u_{,rz}|_{2,\Omega^t}\bigg(\bigg|\frac{\psi_{1,rr}}{r}\bigg|_{L_2(0,t;L_{2,\varepsilon_0}(\Omega))}+\bigg|\frac{\psi_{1,r}}{r^2}\bigg|_{L_2(0,t;L_{2,\varepsilon_0}(\Omega))}\bigg).
\end{aligned}
$$

Using (52), (159), (160), and the estimates from (46)

$$
\begin{aligned}
&\bigg|\frac{u_{,r}}{r}\bigg|_{2,\Omega^t}\le\bigg|\frac{v_\varphi}{r}\bigg|_{2,\Omega^t}+|v_{\varphi,r}|_{2,\Omega^t}\le cD_1,\\
&\bigg|\frac{u_{,z}}{r}\bigg|_{2,\Omega^t}\le|v_{\varphi,z}|_{2,\Omega^t}\le cD_1
\end{aligned}
$$

we obtain the following estimate for J,

$$
\begin{aligned}
|J|\le c[D_2(D_3+D_4)+D_1D_2]\bigg(&|\psi_{1,rrz}|_{2,\Omega^t}+|\psi_{1,rzz}|_{2,\Omega^t}\\
&+|\psi_{1,zzz}|_{2,\Omega^t}+\bigg|\frac{1}{r}\psi_{1,rz}\bigg|_{2,\Omega^t}+\bigg|\frac{1}{r}\psi_{1,zz}\bigg|_{2,\Omega^t}+\bigg|\frac{1}{r^2}\psi_{1,z}\bigg|_{2,\Omega^t}\bigg)\\
&+cD_2^{1-\varepsilon_0}D_3|v_\varphi|^{\varepsilon_0}_{\infty,\Omega^t}\bigg(\bigg|\frac{1}{r}\psi_{1,rr}\bigg|_{L_2(0,t;L_{2,\varepsilon_0}(\Omega))}+\bigg|\frac{1}{r^2}\psi_{1,r}\bigg|_{L_2(0,t;L_{2,\varepsilon_0}(\Omega))}\bigg)\equiv J'.
\end{aligned}
$$

From (64), we have the following (recall that $\omega_1=\Gamma$):

$$
|\psi_{1,rrz}|_{2,\Omega^t}+|\psi_{1,rzz}|_{2,\Omega^t}+|\psi_{1,zzz}|_{2,\Omega^t}\le c|\Gamma_{,z}|_{2,\Omega^t}. \tag{176}
$$

estimates (82) and (85) imply

$$
\bigg|\frac{\psi_{1,rz}}{r}\bigg|_{2,\Omega^t}+\bigg|\frac{\psi_{1,zz}}{r}\bigg|_{2,\Omega^t}+\bigg|\frac{\psi_{1,z}}{r^2}\bigg|_{2,\Omega^t}\le c|\Gamma_{,z}|_{2,\Omega^t}. \tag{177}
$$

Finally, (93) yields

$$
\bigg|\frac{1}{r}\psi_{1,rr}\bigg|_{L_2(0,t;L_{2,\varepsilon_0}(\Omega))}+\bigg|\frac{1}{r^2}\psi_{1,r}\bigg|_{L_2(0,t;L_{2,\varepsilon_0}(\Omega))}\le cR^{\varepsilon_0}\|\Gamma\|_{1,2,\Omega^t} \tag{178}
$$

Recall that (177) is valid for $\psi_1|_{r=0}=0$.

This restriction implies that $v_z|_{r=0}=0$, so it is a strong restriction on solutions proved in this paper.

Using (176)–(178) in J' yields

$$
J'\le cD_2(D_1+D_3+D_4)|\Gamma_{,z}|_{2,\Omega^t}+cD_2^{1-\varepsilon_0}D_3|v_\varphi|^{\varepsilon_0}_{\infty,\Omega^t}\|\Gamma\|_{1,2,\Omega^t}.
$$

In view of Lemma 15, the term I_1 introduced in (175) is bounded by

$$
I\le c|f_\varphi|_{2,S_1^t}\|u\|_{2,2,\Omega^t}\le c|f_\varphi|_{2,S_1^t}(D_3+D_4).
$$

Using the estimates in (174), we obtain

$$\begin{aligned}
&\|\omega_r\|^2_{V(\Omega^t)} + \|\omega_z\|^2_{V(\Omega^t)} + |\Phi|^2_{2,\Omega^t} \\
&\le cD_2(D_1 + D_3 + D_4)|\Gamma_{,z}|_{2,\Omega^t} + cD_2^{1-\varepsilon_0} D_3 |v_\varphi|^{\varepsilon_0}_{\infty,\Omega^t} \|\Gamma\|_{1,2,\Omega^t} \\
&\quad + c(|F_r|^2_{6/5,2,\Omega^t} + |F_z|^2_{6/5,2,\Omega^t}) + c(|\omega_r(0)|^2_{2,\Omega} \\
&\quad + |\omega_z(0)|^2_{2,\Omega}) + c|f_\varphi|_{2,S_1^t}(D_3 + D_4),
\end{aligned} \tag{179}$$

where we used

$$\begin{aligned}
\left| \int_\Omega (F_r\omega_r + F_z\omega_z) dx dt' \right| &\le \varepsilon(|\omega_r|^2_{6,\Omega} + |\omega_z|^2_{6,\Omega}) \\
&\quad + c(1/\varepsilon)(|F_r|^2_{6/5,\Omega} + |F_z|^2_{6/5,\Omega}).
\end{aligned}$$

Hence, (179) implies (173) and concludes the proof. □

7. Estimates for the Stream Function in Weighted Sobolev Spaces

Recall that the stream function ψ_1 is a solution to problem (22). To increase the regularity of weak solutions, we need appropriate estimates for ψ_1, assuming sufficient regularity of the vorticity ω_1.

Remark 7. *In Lemma 4, the existence of weak solutions to problem (22) satisfying estimate (56) is proven. Inequality (62) implies that the weak solution belongs to $H^2(\Omega)$ and the estimate holds*

$$\|\psi_1\|_{2,\Omega} \le c|\omega_1|_{2,\Omega}. \tag{180}$$

Assuming that $\omega_{1,z} \in L_2(\Omega)$, estimates (63) and (64) increase the regularity of ψ_1, such that $\psi_{1,z} \in H^2(\Omega)$ and the estimate holds

$$\|\psi_{1,z}\|_{H^2(\Omega)} \le c(|\omega_{1,z}|_{2,\Omega} + |\omega_1|_{2,\Omega}). \tag{181}$$

Estimate (181) is derived using the technique of the energy method. However, this method is not sufficiently robust to derive an estimate for $|\psi_{1,rrr}|_{2,\Omega}$.

Moreover, estimate (181) is not sufficient to prove estimate (24) of Theorem 1. To prove Theorem 1 we need estimates (85) and (93). To prove the estimates, we need the theory of weighted Sobolev spaces developed by Kondratiev [10], which is used to examine elliptic boundary value problems in domains with cones.

Unfortunately, estimates (85) and (93) hold for weak solutions where ψ_1 vanishes on the axis of symmetry. This implies that the v_z coordinate of velocity must also vanish on the axis of symmetry. Therefore, Theorem 1 is applicable to a smaller class than the class of weak solutions. This indicates that the regularity problem for axially symmetric solutions to the Navier–Stokes equations is only partially solved.

Now, we show the existence of solutions to problem (22) in weighted Sobolev spaces.

Lemma 17. *Assume that ψ_1 is a solution to (61). Assume that $\omega_{1,z}, \omega_1 \in L_2(\Omega)$. Then,*

$$\begin{aligned}
&\int_\Omega \left(\psi^2_{1,zrr} + \frac{1}{r^2}\psi^2_{1,zr} + \frac{1}{r^4}\psi^2_{1,z} \right) dx + \int_\Omega \psi^2_{1,zzz} dx \\
&\le c \int_\Omega (|\omega_{1,z}|^2 + |\omega_1|^2) dx.
\end{aligned} \tag{182}$$

Proof. To prove the lemma, we need weighted Sobolev spaces defined by Fourier transform (58) and introduced in (59) and (60). Therefore, to examine problem (22) in weighted Sobolev spaces, we have to derive estimates with respect to r and z, separately. To derive an estimate with respect to r, we have to examine solutions to (22) independently, as well in a neighborhood of the axis of symmetry, such as in a neighborhood located in a positive distance from it. To perform such considerations, we treat z as a parameter and introduce a partition of unity $\{\zeta^{(1)}(r), \zeta^{(2)}(r)\}$, such that

$$\sum_{i=1}^{2} \zeta^{(i)}(r) = 1$$

and

$$\zeta^{(1)}(r) = \begin{cases} 1 & \text{for } r \le r_0 \\ 0 & \text{for } r \ge 2r_0 \end{cases}, \quad \zeta^{2)}(r) = \begin{cases} 0 & \text{for } r \le r_0 \\ 1 & \text{for } r \ge 2r_0, \end{cases}$$

where $0 < r_0$ is fixed in such a way that $2r_0 < R$.

Let $\psi_1^{(i)} = \psi_1 \zeta^{(i)}$, $\omega_1^{(i)} = \omega_1 \zeta^{(i)}$ and $\dot{\zeta}^{(i)} = \frac{d}{dr}\zeta^{(i)}$, $\ddot{\zeta}^{(i)} = \frac{d^2}{dr^2}\zeta^{(i)}$, $i = 1, 2$. Moreover, functions $\zeta^{(1)}$ and $\zeta^{(2)}$ are smooth.

Then, we obtain from (22) the following two problems

$$\begin{cases} -\Delta\psi_1^{(1)} - \dfrac{2}{r}\psi_1^{(1)} = \omega_1^{(1)} - 2\psi_{1,r}\dot{\zeta}^{(1)} - \psi_1\ddot{\zeta}^{(1)} & \\ \quad -\dfrac{2}{r}\psi_1\dot{\zeta}^{(1)} & \text{in } \Omega^{(1)}, \\ \psi_1^{(1)} \text{ satisfies periodic boundary conditions} & \text{on } S_2^{(1)}, \end{cases} \tag{183}$$

where

$$\Omega^{(1)} = \{(r,z) \colon r > 0, z \in (-a,a)\},$$
$$S_2^{(1)} = \{(r,z) \colon r > 0, z \in \{-a,a\}\}$$

and

$$\begin{cases} -\Delta\psi_1^{(2)} - \dfrac{2}{r}\psi_1^{(2)} = \omega_1^{(2)} - 2\psi_{1,r}\dot{\zeta}^{(2)} - \psi_1\ddot{\zeta}^{(2)} & \\ \quad -\dfrac{2}{r}\psi_1\dot{\zeta}^{(2)} & \text{in } \Omega^{(2)} \\ \psi_1^{(2)} = 0 & \text{on } S_1, \\ \psi_1^{(2)} \text{ satisfies periodic boundary conditions} & \text{on } S_2^{(2)}, \end{cases} \tag{184}$$

where

$$\Omega^{(2)} = \{(r,z) \colon r_0 < r < R, z \in (-a,a)\},$$
$$S_2^{(2)} = \{(r,z) \colon r_0 < r < R, z \in \{-a,a\}\}.$$

We temporarily simplify the notation using

$$\begin{aligned} &u = \psi_1^{(1)}, \ \ w = \psi_1^{(2)}, \\ &f = \omega_1^{(1)} - 2\psi_{1,r}\dot{\zeta}^{(1)} - \psi_1\ddot{\zeta}^{(1)} - \frac{2}{r}\psi_1\dot{\zeta}^{(1)}, \\ &b = \omega_1^{(2)} - 2\psi_{1,r}\dot{\zeta}^{(2)} - \psi_1\ddot{\zeta}^{(2)} - \frac{2}{r}\psi_1\dot{\zeta}^{(2)}. \end{aligned} \tag{185}$$

Then, (183) and (184) become

$$\begin{aligned} &-\Delta u - \frac{2}{r}u_{,r} = f && \text{in } \Omega^{(1)}, \\ &u \ - \text{ satisfies periodic boundary conditions} && \text{on } S_2^{(1)} \end{aligned} \tag{186}$$

and

$$\begin{aligned} & -\Delta w - \frac{2}{r} w_{,r} = b && \text{in } \Omega^{(2)}, \\ & w = 0 && \text{on } S_1, \\ & w \text{ satisfies periodic boundary conditions} && \text{on } S_2^{(2)}. \end{aligned} \tag{187}$$

Next, we use the simplified notation

$$pbc - \text{periodic boundary conditions.} \tag{188}$$

First, we consider problem (186). We rewrite it in the form

$$\begin{aligned} & -u_{,rr} - \frac{3}{r} u_{,r} = f + u_{,zz} && \text{in } \Omega^{(1)}, \\ & u \text{ satisfies } pbc && \text{on } S_2^1. \end{aligned} \tag{189}$$

For a fixed $z \in (-a, a)$, we treat (189) as

$$-u_{,rr} - \frac{3}{r} u_{,r} = f + u_{,zz} \quad \text{in } \mathbb{R}_+. \tag{190}$$

Multiplying (190) by r^2 yields

$$-r^2 u_{,rr} - 3r u_{,r} = r^2(f + u_{,zz}) \equiv g(r,z)$$

or equivalently

$$-r\partial_r(r\partial_r u) - 2r\partial_r u = g(r,z). \tag{191}$$

Introduce the new variable

$$\tau = -\ln r, \quad r = e^{-\tau}. \tag{192}$$

Since $r\partial_r = -\partial_\tau$, we see that (191) takes the form

$$-\partial_\tau^2 u + 2\partial_\tau u = g(e^{-\tau}, z) = g'(\tau, z). \tag{193}$$

Utilizing the Fourier transform (58) to (193), we have

$$\lambda^2 \hat{u} + 2i\lambda\hat{u} = \hat{g}'.$$

For $\lambda \notin \{0, -2i\}$ we have

$$\hat{u} = \frac{1}{\lambda(\lambda + 2i)} \hat{g}' \equiv R(\lambda)\hat{g}'. \tag{194}$$

We introduce the quantity

$$h(k, \mu) = 1 + k - \mu. \tag{195}$$

Consider the case where $k = 0$ and $\mu = 0$. Then, $h(0,0) = 1$. Theorem 1.1 from Section 1 in [10] (also see Lemma 3.1 from [11]) yields the following:

Let $f + u_{,zz} \in L_2(\mathbb{R}_+)$, and $R(\lambda)$ does not have poles on the line $Im\ \lambda = 1$. Then, we have

$$\int_{-\infty+ih(0,0)}^{+\infty+ih(0,0)} \sum_{j=0}^{2} |\lambda|^{2(2-j)} |\hat{u}|^2 d\lambda \le c \int_{-\infty+ih(0,0)}^{+\infty+ih(0,0)} |\hat{g}'|^2 d\lambda. \tag{196}$$

Using (60) and $h(0,0) = 1$, we obtain

$$\int_{\mathbb{R}} \sum_{j=0}^{2} |\partial_\tau^j u|^2 e^{2\tau} d\tau \le c \int_{\mathbb{R}} |g'|^2 e^{2\tau} d\tau.$$

Passing to variable r yields

$$\int\limits_{\mathbb{R}_+} \left(|u_{,rr}|^2 + \frac{1}{r^2}|u_{,r}|^2 + \frac{1}{r^4}|u|^2\right) rdr \leq c \int\limits_{\mathbb{R}_+} |f + u_{,zz}|^2 rdr. \tag{197}$$

Using notation (185) and the estimate for the weak solutions, we obtain from (197) the inequality

$$\begin{aligned} &\int\limits_{\mathbb{R}_+\cap \operatorname{supp}\zeta^{(1)}} \left(\psi_{1,rr}^2 + \frac{1}{r^2}\psi_{1,r}^2 + \frac{1}{r^4}\psi_1^2\right) rdr \\ &\leq c \int\limits_{\mathbb{R}_+\cap \operatorname{supp}\zeta^{(1)}} |\omega_1|^2 rdr + c \int\limits_{\mathbb{R}_+\cap \operatorname{supp}\zeta^{(1)}} (|\psi_{1,r}|^2 + |\psi_1|^2) rdr \\ &\quad + c \int\limits_{\mathbb{R}_+\cap \operatorname{supp}\zeta^{(1)}} |\psi_{1,zz}|^2 rdr. \end{aligned} \tag{198}$$

For solutions to (187), we have the estimate

$$\begin{aligned} \|w\|_{H^2((0,R)\cap \operatorname{supp}\zeta^{(2)})} &\leq c\|b\|_{L_2((0,R)\cap \operatorname{supp}\zeta^{(2)})} \\ &\quad + c\|w_{,r}\|_{L_2((0,R)\cap \operatorname{supp}\zeta^{(2)})}. \end{aligned} \tag{199}$$

In view of notation (185) we obtain

$$\begin{aligned} &\int\limits_{(0,R)\cap \operatorname{supp}\zeta^{(2)}} (|\psi_{1,rr}|^2 + |\psi_{1,r}|^2 + |\psi_1|^2) rdr \\ &\leq c \int\limits_{(0,R)\operatorname{supp}\zeta^{(2)}} (|\omega_1|^2 + |\psi_{1,zz}|^2 + |\psi_{1,r}|^2 + |\psi_1|^2) rdr. \end{aligned} \tag{200}$$

Adding (198) and (200), integrating the result with respect to z, and using (56) yield

$$\begin{aligned} &\int\limits_{\Omega} \left(\psi_{1,rr}^2 + \frac{1}{r^2}\psi_{1,r}^2 + \frac{1}{r^4}\psi_1^2\right) dx \\ &\leq c \int\limits_{\Omega} (|\omega_1|^2 + |\psi_{1,zz}|^2) dx. \end{aligned} \tag{201}$$

Replacing ψ_1 with $\psi_{1,z}$, and ω_1 with $\omega_{1,z}$, we obtain estimate (182) from (201) and (63). This ends the proof. □

Lemma 18. *Assume that ψ_1 is a solution to (61). Assume that $\mu \in (0,1)$, $\omega_1 \in H^1(\Omega)$, $\Omega = (0,R) \times (-a,a)$.*

Then,

$$\begin{aligned} &\int\limits_{\Omega} \left(\psi_{1,rrr}^2 + \frac{1}{r^2}\psi_{1,rr}^2 + \frac{1}{r^4}\psi_{1,r}^2 + \frac{1}{r^6}\psi_1^2\right) r^{2\mu} dx + \|\psi_1\|_{H^2(\Omega)}^2 \\ &\quad + \int\limits_{\Omega} (\psi_{1,zrr}^2 + \psi_{1,zzr}^2 + \psi_{1,zzz}^2) dx \leq c\left(1 + \frac{1}{\mu^2}\right) \|\omega_1\|_{H^1(\Omega)}^2. \end{aligned} \tag{202}$$

Proof. Recall the partition of unity introduced in the proof of Lemma 17. Also recall the local problems (183) and (184), as well as notation (185). Then, we can examine problems (186) and (187). First, we examine problem (186).

Applying the Mellin transform, any solution to (190) can be expressed in the form (194).

In this case, we introduce the following quantity:

$$h(1,\mu) = 2-\mu. \tag{203}$$

Since operator $R(\lambda)$ does not have poles on the line $\operatorname{Im}\lambda = h(1,\mu)$, we have (see Theorem 1.1 from Section 1 in [10])

$$\begin{aligned} &\int\limits_{-\infty+ih(1,\mu)}^{+\infty+ih(1,\mu)} \sum_{j=0}^{3} |\lambda|^{2(3-j)}|\hat{u}|^2 d\lambda \\ &\leq c \int\limits_{-\infty+ih(1,\mu)}^{+\infty+ih(1,\mu)} \sum_{j=0}^{1} |\lambda|^{2(1-j)}|\hat{g}'|^2 d\lambda. \end{aligned} \tag{204}$$

Using (60) for $h(1,\mu) = 2-\mu$, we obtain

$$\begin{aligned} &\int\limits_{\mathbb{R}} \sum_{j=0}^{3} |\partial_\tau^j u|^2 e^{2(3-j)\tau} d\tau \\ &\leq c \int\limits_{\mathbb{R}} \sum_{j=0}^{1} |\partial_\tau^j g'|^2 e^{2(1-j)\tau} d\tau. \end{aligned} \tag{205}$$

In view of equivalence (59), inequality (205) takes the form

$$\begin{aligned} &\int\limits_{\mathbb{R}_+} \left(|u_{rrr}|^2 + \frac{1}{r^2}|u_{rr}|^2 + \frac{1}{r^4}|u_r|^2 + \frac{1}{r^6}|u|^2\right) r^{2\mu} r dr \\ &\leq c \int\limits_{\mathbb{R}_+} |(f+u_{zz})_{,r}|^2 r^{2\mu} r dr + c \int\limits_{\mathbb{R}_+} |f+u_{zz}|^2 r^{2\mu-2} r dr, \end{aligned} \tag{206}$$

where $z \in (-a,a)$ and $\mu \in (0,1)$.

Integrating (206) with respect to z, and exploiting notation (185), yield

$$\begin{aligned} &\int\limits_{-a}^{a} dz \int\limits_{\mathbb{R}_+\cap \operatorname{supp}\zeta^{(1)}} \left(\psi_{1,rrr}^2 + \frac{1}{r^2}\psi_{1,rr}^2 + \frac{1}{r^4}\psi_{1,r}^2 + \frac{1}{r^6}\psi_1^2\right) r^{2\mu} r dr \\ &\leq c \int\limits_{-a}^{a} dz \int\limits_{\mathbb{R}_+\cap \operatorname{supp}\zeta^{(1)}} (|\partial_r(\omega_1+\psi_{1,zz})|^2 + |\omega_1+\psi_{1,zz}|r^{-2}) r^{2\mu} r dr. \end{aligned} \tag{207}$$

For solutions to problem (187) and notation (185), we obtain

$$\begin{aligned} &\int\limits_{-a}^{a} dz \|\psi_1\|^2_{H^3_\mu(\mathbb{R}_+\cap \operatorname{supp}\zeta^{(2)}} \\ &\leq c \int\limits_{-a}^{a} dz (\|\omega_1\|^2_{H^1(\mathbb{R}_+\cap \operatorname{supp}\zeta^{(2)})} + \|\psi_{1,zz}\|^2_{H^1(\mathbb{R}_+\cap \operatorname{supp}\zeta^{(2)})}). \end{aligned} \tag{208}$$

From (207) and (208), as well as the Hardy inequality (see [14] (Chapter 1, Section 2.16)):

$$\int\limits_{\mathbb{R}_+} |\omega_1+\psi_{1,zz}|^2 r^{2\mu-2} r dr \leq \frac{1}{\mu^2} \int\limits_{\mathbb{R}_+} |(\omega_1+\psi_{1,zz})_{,r}|^2 r^{2\mu} r dr \tag{209}$$

we obtain

$$
\begin{aligned}
&\int_\Omega \left(\psi_{1,rrr}^2 + \frac{1}{r^2}\psi_{1,rr}^2 + \frac{1}{r^4}\psi_{1,r}^2 + \frac{1}{r^6}\psi_1^2\right) r^{2\mu} dx \\
&\le c\left(1 + \frac{1}{\mu^2}\right)\left[\|\omega_1\|_{H^1(\Omega)}^2 + \int_\Omega (\psi_{1,zzr}^2 + \psi_{1,zzz}^2) dx\right].
\end{aligned} \tag{210}
$$

Using estimates (56), (62), (63) in (210) implies (202) and ends the proof. □

Remark 8. *Since* $\mu > 0$, *the Hardy inequality (209) does not need* $\omega_1 + \psi_{1,zz}|_{r=0} = 0$.

8. Conclusions

The main result of this paper is the proof of (24). Since $\Gamma = \omega_\varphi / r$, we obtain from (24) the estimate

$$
\|\omega_\varphi\|_{V(\Omega^t)} \le \phi(\text{data}), \tag{211}
$$

where we used the fact that $r < R$ and R is finite. This means that (211) does not hold for the Cauchy problem.

Using problem (14) and relation (15), we obtain

$$
\begin{aligned}
\|v'\|_{L_\infty(0,t;L_6(\Omega))} &\le c\|\psi\|_{L_\infty(0,t;H^2(\Omega))} \\
&\le c\|\omega_\varphi\|_{L_\infty(0,t;L_2(\Omega))} \le \phi(\text{data}),
\end{aligned} \tag{212}
$$

where $v' = (v_r, v_z)$.

Consider the Stokes problem, which follows from (6)

$$
\begin{aligned}
& v_t - \nu\Delta v + \nabla p = -v' \cdot \nabla v + f && \text{in } \Omega^T, \\
& \operatorname{div} v = 0 && \text{in } \Omega^T, \\
& v \cdot \bar{n}|_{S_1} = 0, \ \omega_\varphi|_{S_1} = 0, \ v_\varphi|_{S_1} = 0 && \text{on } S_1^T, \\
& v \ - \ \text{satisfies periodic boundary conditions} && \text{on } S_2^T, \\
& v|_{t=0} = v(0) && \text{in } \Omega.
\end{aligned} \tag{213}
$$

Using (212) and the energy estimate (46), we have

$$
\|v' \cdot \nabla v\|_{L_2(0,t;L_{3/2}(\Omega))} \le \phi(\text{data}). \tag{214}
$$

Assuming more regularity on data than in the proof of Theorem 1, and using the result from [15], we obtain the following estimate for solutions to (213)

$$
\|v\|_{W_{3/2,2}^{2,1}(\Omega^t)} \le \phi(\text{data}). \tag{215}
$$

Anisotropic weighted spaces can be found in [16].

The above inequality implies

$$
\|\nabla v\|_{L_{5/2}(\Omega^t)} \le \phi(\text{data}) \tag{216}
$$

thus, the increase in regularity holds because in (46) we have $\|\nabla v\|_{L_2(\Omega^t)} \le \phi(\text{data})$.

Continuing the considerations, we derive the estimate

$$
\|v\|_{W_2^{2,1}(\Omega^t)} + \|\nabla p\|_{L_2(\Omega^t)} \le \phi(\text{data}). \tag{217}
$$

The existence of solutions to problem (6) follows from appropriately choosing a fixed point theorem.

$W^{2,1}_{p,q}(\Omega^t)$ is the Sobolev space with mixed norm. We have

$$\|u\|_{W^{2,1}_{p,q}(\Omega^t)} = \|u\|_{L_q(0,t;L_p(\Omega))} + \|\partial_x^2 u\|_{L_q(0,t;L_p(\Omega))} + \|\partial_t u\|_{L_q(0,t;L_p(\Omega))}.$$

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Appendix A. Existence of Regular Solutions to (1) for Small Data

Recall the quantities

$$u_1 = \frac{v_\varphi}{r},\ \ \omega_1 = \frac{\omega_\varphi}{r},\ \ \psi_1 = \frac{\psi}{r},\ \ f_1 = \frac{f_\varphi}{r},\ \ F_1 = \frac{F_\varphi}{r}. \tag{A1}$$

In view of [12] system (1) is equivalent to the following one

$$\begin{aligned}
&u_{1,t} + v\cdot\nabla u_1 - \nu\left(\Delta u_1 + \frac{2}{r}u_{1,r}\right) = 2u_1\psi_{1,z} + f_1,\\
&\omega_{1,t} + v\cdot\nabla\omega_1 - \nu\left(\Delta\omega_1 + \frac{2}{r}\omega_{1,r}\right) = 2u_1u_{1,z} + F_1,\\
&-\Delta\psi_1 - \frac{2}{r}\psi_{1,r} = \omega_1,\\
&\text{periodic boundary conditions on } S_2,\\
&u_1|_{t=0} = u_1(0),\\
&\omega_1|_{t=0} = \omega_1(0).
\end{aligned} \tag{A2}$$

Functions u_1, ω_1, ψ_1 have compact support with respect to variable r.

We multiply $(A2)_1$ by $u_1|u_1|^2$, integrate over Ω, and use boundary conditions to yield the following:

$$\frac{d}{dt}|u_1|^4_{4,\Omega} + \nu|u_1|^4_{4,\Omega} \le c|\omega_1|^2_{2,\Omega}|u_1|^4_{4,\Omega} + c|f_1|^4_{4,\Omega}. \tag{A3}$$

We multiply $(A2)_2$ by ω_1, integrate over Ω, and exploit boundary conditions. Then, we have

$$\frac{d}{dt}|\omega_1|^2_{2,\Omega} + \nu|\omega_1|^2_{2,\Omega} \le c|u_1|^4_{4,\Omega} + c|F_1|^2_{2,\Omega}. \tag{A4}$$

We introduce the quantity

$$X(t) = |u_1(t)|^4_{4,\Omega} + |\omega_1(t)|^2_{2,\Omega}. \tag{A5}$$

Then, (A3) and (A4) imply

$$\frac{d}{dt}X + \nu X \le c_0X^2 + G(t), \tag{A6}$$

where

$$G(t) = c(|f_1(t)|^4_{4,\Omega} + |F_1(t)|^2_{2,\Omega}). \tag{A7}$$

We consider (A6) on the time interval $(0,T)$. We assume that for $t\in(0,T)$, the following inequality holds

$$G(t) \le c_0k_0. \tag{A8}$$

Then, (A6) takes the form

$$\frac{d}{dt}X \le c_0k_0\left(\frac{1}{k_0}X^2 + 1\right). \tag{A9}$$

Let $X = \alpha X'$. Then,

$$\frac{d}{dt}X' \le \frac{c_0 k_0}{\alpha}\left(\frac{\alpha^2}{k_0}X'^2 + 1\right).$$

Setting $\alpha^2 = k_0$ yields

$$\frac{d}{dt}X' \le c_0\sqrt{k_0}(X'^2 + 1). \tag{A10}$$

Integrating (A10) with respect to time implies

$$\text{arctg}\, X'(t) - \text{arctg}\, X'(0) \le c_0\sqrt{k_0}t.$$

Hence,

$$X'(t) \le \frac{\text{tg}\,(c_0\sqrt{k_0}t) + X'(0)}{1 - X'(0)\text{tg}\,(c_0\sqrt{k_0}t)}.$$

Recalling that $X'(0) = \frac{X(0)}{\sqrt{k_0}}$ and setting $y = c_0\sqrt{k_0}t$, we obtain

$$X(t) \le \frac{(\text{tg}\, y + \frac{X(0)}{\sqrt{k_0}})\sqrt{k_0}}{1 - X(0)c_0 t\frac{\text{tg}\, y}{y}} \le \beta(T), \tag{A11}$$

where $t \le T$. Hence, for large T, (A11) holds for sufficiently small $X(0)$ and k_0.

Consider (A6) in the interval $(0, T)$. Using (A11), we can write (A6) in the form

$$\frac{d}{dt}X + \nu_* X \le G(t), \tag{A12}$$

where $\nu_* = \nu - c_0\beta$.

Integrating (A12) with respect to time yields

$$X(t) \le e^{-\nu_* t}\int_0^t G(t')e^{\nu_* t'}dt' + e^{-\nu_8 t}X(0). \tag{A13}$$

Setting $t = T$ implies

$$X(T) \le \int_0^T G(t)dt + e^{-\nu_* T}X(0). \tag{A14}$$

For sufficiently small $X(0)$ and k_0, the time interval $(0, T)$ can be chosen to be large. Then, (A14) can imply that

$$X(T) \le X(0). \tag{A15}$$

Therefore, the previous considerations can be performed for any time interval $(kT, (k+1)T)$, $k \in \mathbb{N}$.

References

1. Ladyzhenskaya, O.A. Uniquae global solvability of the three-dimensional Cauchy problem for the Navier–Stokes equations in the presence of axial symmetry. *Zap. Nauchn. Sem. LOMI* **1968**, *7*, 155–177. (In Russian)
2. Ukhovskii, M.R.; Yudovich, V.I. Axially symmetric flows of ideal and viscous fluids filling the whole space. *J. Appl. Math. Mech.* **1968**, *32*, 52–61.
3. Nowakowski, B.; Zajączkowski, W.M. Global regular axially-symmetric solutions to the Navier-Stokes equations with small swirl. *J. Math. Fluid Mech.* **2023**, *25*, 73. [CrossRef]
4. Liu, Y. Solving the axisymmetric Navier-Stokes equations in critical spaces (I): The case with small swirl component. *J. Diff. Equ.* **2022**, *314*, 287–315. [CrossRef]
5. Lei, Z.; Zhang, Q.S. Criticality of the axially symmetric Navier-Stokes equations. *Pacific J. Math.* **2017**, *289*, 169–187. [CrossRef]
6. Zhang, P.; Zhang, T. Global axisymmetric solutions to three-dimensional Navier-Stokes system. *Int. Math. Res. Not.* **2014**, *2014*, 610–642. [CrossRef]
7. Wei, D. Regularity criterion to the axially symmetric Navier-Stokes equations. *J. Math. Anal. Appl.* **2016**, *435*, 402–413. [CrossRef]

8. Chen, H.; Fang, D.; Zhang, T. Regularity of 3d axisymmetric Navier-Stokes equations. *Disc. Cont. Dyn. Syst.* **2017**, *37*, 1923–1939. [CrossRef]
9. Liu, J.G.; Wang, W.C. Characterization and regularity for axisymmetric solenoidal vector fields with application to Navier-Stokes equations. *SIAM J. Math. Anal.* **2009**, *41*, 1825–1850. [CrossRef]
10. Kondrat'ev, V. Boundary value problems for elliptic equations in domains with conical or angular points. *Trudy Moskov. Mat. Obshch.* **1967**, *16*, 209–292. (In Russian); English Translation *Trans. Mosc. Math. Soc.* **1967**, *16*, 227–313.
11. Nowakowski, B.; Zajączkowski, W.M. On weighted estimates for the stream function of axially-symmetric solutions to the Navier-Stokes equations in a bounded cylinder. *arXiv* **2022**, arXiv:2210.15729.
12. Hou, T.Y.; Li, C. Dynamic stability of the three-dimensional axisymmetric Navier-Stokes equations with swirl. *Comm. Pure Appl. Math.* **2008**, *61*, 661–697. [CrossRef]
13. Ladyzhenskaya, O.A. *Mathematical Theory of Viscous Incompressible Flow, Nauka, Moscow 1970*, 2nd English ed.; Silverman, R.A., Chu, J., Eds.; Mathematics and Its Applications; Gordon and Breach, Science Publishers: New York, NY, USA, 1970; Volume 2, p. xviii+224. (In Russian)
14. Besov, O.V.; Il'in, V.P.; Nikolskii, S.M. *Integral Representations of Functions and Imbedding Theorems, Nauka, Moscow (1975)*; English Translation; Scripta Series in Mathematics; V. H. Winston: New York, NY, USA, 1978; Volume I. (In Russian)
15. Maremonti, P.; Solonnikov, V.A. On the estimates of evolution Stokes problem in anisotropic Sobolev spaces with mixed norm. *Zap. Nauchn. Sem. LOMI* **1994**, *223*, 124–150.
16. Ladyzhenskaya, O.A.; Solonnikov, V.A.; Uraltseva, N.N. *Linear and Quasilinear Equations of Parabolic Type, Nauka, Moscow (1967)*; American Mathematical Society: Providence, RI, USA, 1968. (In Russian)