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Article

An Application for Bi-Concave Functions Associated with q-Convolution

Sheza M. El-Deeb [1,2] and Adriana Catas [3,*]

[1] Department of Mathematics, College of Science and Arts, Al-Badaya, Qassim University, Buraidah 52571, Saudi Arabia; s.eldeeb@qu.edu.sa
[2] Department of Mathematics, Faculty of Science, Damietta University, New Damietta 34517, Egypt
[3] Department of Mathematics and Computer Science, University of Oradea, 1 University Street, 410087 Oradea, Romania
* Correspondence: acatas@uoradea.ro

Abstract: The aim of this paper is to introduce and investigate some new subclasses of bi-concave functions using q-convolution and some applications. These special cases are obtaining by making use of a q- derivative linear operator. For the new introduced subclasses, the authors obtain the first two initial Taylor–Maclaurin coefficients $|c_2|$ and $|c_3|$ of bi-concave functions. For certain values of the parameters, the authors deduce interesting corollaries for coefficient bounds which imply special cases of the new introduced operator. Also, we develop two examples for coefficients $|c_2|$ and $|c_3|$ for certain functions.

Keywords: bi-concave; convolution; fractional derivative; q-derivative; q-analogue of poisson operator

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1. Introduction and Preliminaries

Assuming that $\mathcal{A}$ denote the class of functions with the given form:

$$\mathcal{F}(\varsigma) = \varsigma + \sum_{n=2}^{+\infty} c_n \varsigma^n, \quad (\varsigma \in \mathcal{U}), \tag{1}$$

which in the unit disc $\mathcal{U} = \{\varsigma : |\varsigma| < 1\}$ are analytic and univalent, and let the function $\mathcal{H} \in \mathcal{A}$ be given by

$$\mathcal{H}(\varsigma) := \varsigma + \sum_{n=2}^{+\infty} d_n \varsigma^n, \ \ \varsigma \in \mathcal{U}. \tag{2}$$

Then, $\mathcal{F}$ and $\mathcal{H}$ Hadamard (or convolution) product is given by

$$(\mathcal{F} * \mathcal{H})(\varsigma) := \varsigma + \sum_{n=2}^{+\infty} c_n d_n \varsigma^n, \ \ \varsigma \in \mathcal{U}.$$

For each univalent function $\mathcal{F} \in \mathcal{A}$, the Koebe one-quarter Theorem ([1]) establishes that the disk with radius $(1/4)$ belongs to the image of $\mathcal{U}$. An inverse $\mathcal{F}^{-1}$ of a function $\mathcal{F} \in \mathcal{A}$ exists and is satisfied.

$$\mathcal{F}(\mathcal{F}^{-1}(\varpi)) = \varpi, \quad \left(|\varpi| < \mathcal{R}_0(\mathcal{F}), \ \mathcal{R}_0(\mathcal{F}) \geq \frac{1}{4}\right),$$

where

$$\mathcal{G}(\varpi) = \mathcal{F}^{-1}(\varpi) = \varpi - c_2\varpi^2 + \left(2c_2^2 - c_3\right)\varpi^3 -$$

$$\left(5c_2^3 - 5c_2c_3 + c_4\right)\varpi^4 + \ldots,\ \varpi \in \mathcal{U}. \tag{3}$$

The study of analytic and bi-univalent functions is revitalized in 2010 by Srivastava et al., and the literature has since been supplemented with many sequels to their paper (see [2]). A function $\mathcal{F} \in \mathcal{A}$ is called to as bi-univalent in $\mathcal{U}$ if $\mathcal{F}$ and $\mathcal{F}^{-1}$ are both univalent in $\mathcal{U}$.

Let Σ denote the class of bi-univalent functions in $\mathcal{U}$ given by (1). Note that the functions $\mathcal{F}_1(\varsigma) = \frac{\varsigma}{1-\varsigma}$, $\mathcal{F}_2(\varsigma) = \frac{1}{2}\log\frac{1+\varsigma}{1-\varsigma}$, $\mathcal{F}_3(\varsigma) = -\log(1-\varsigma)$, with their corresponding inverses $\mathcal{F}_1^{-1}(\varpi) = \frac{\varpi}{1+\varpi}$, $\mathcal{F}_2^{-1}(\varpi) = \frac{e^{2\varpi}-1}{e^{2\varpi}+1}$, $\mathcal{F}_3^{-1}(\varpi) = \frac{e^{\varpi}-1}{e^{\varpi}}$, are elements of Σ (see [2]). For a brief history and interesting examples in the class Σ, see [3]. Brannan and Taha [4] (see also [2]) introduced certain subclasses of the bi-univalent functions class Σ similar to the familiar subclasses $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ of starlike and convex functions of order α $(0 \le \alpha < 1)$, respectively (see [3]), where the function $\mathcal{G}$ is the analytic extension of $\mathcal{F}^{-1}$ to $\mathcal{U}$ given by (3).

The function $\mathcal{F} : \mathcal{U} \to \mathbb{C}$ is said to belong to the family $\mathcal{C}_0(\alpha)$ if $\mathcal{F}$ satisfies the following conditions:

(i) $\mathcal{F}$ is analytic in $\mathcal{U}$ with the standard normalization $\mathcal{F}(0) = \mathcal{F}'(0) - 1 = 0$;
(ii) $\mathcal{F}$ maps $\mathcal{U}$ conformally onto a set whose complement with respect to $\mathbb{C}$ is convex;
(iii) The opening angle of $\mathcal{F}(\mathcal{U})$ at ∞ is less than or equal to $\pi\alpha$, $\alpha \in (1,2]$.

Concave univalent functions are referred to as the class $\mathcal{C}_0(\alpha)$, and for a thorough study of concave functions (see [5,6]). In particular, the inequality

$$\Re\left(1 + \frac{\varsigma\,\mathcal{F}''(\varsigma)}{\mathcal{F}'(\varsigma)}\right) < 0, \quad (\varsigma \in \mathcal{U}),$$

is used. Bhowmik et al. [7] showed that an analytic function $\mathcal{F}$ maps $\mathcal{U}$ onto a concave domain of angle $\pi\alpha$, if and only if $\Re(P_{\mathcal{F}}(\varsigma)) > 0$, where

$$P_{\mathcal{F}}(\varsigma) = \frac{2}{\alpha-1}\left[\frac{(\alpha+1)(1+\varsigma)}{2(1-\varsigma)} - 1 - \frac{\varsigma\,\mathcal{F}''(\varsigma)}{\mathcal{F}'(\varsigma)}\right].$$

There have been a number of investigations on basic subclasses of concave univalent functions (see [8,9]).

For $0 < q < 1$, the q-derivative operator [10,11] (see also [12]) for $\mathcal{F} * \mathcal{H}$ is defined by

$$\begin{aligned}\mathcal{D}_q(\mathcal{F} * \mathcal{H})(\varsigma) &:= \mathcal{D}_q\left(\varsigma + \sum_{n=2}^{+\infty} c_n d_n \varsigma^n\right)\\ &= \frac{(\mathcal{F} * \mathcal{H})(\varsigma) - (\mathcal{F} * \mathcal{H})(q\varsigma)}{\varsigma(1-q)} = 1 + \sum_{n=2}^{+\infty}[n]_q c_n d_n \varsigma^{n-1},\ \varsigma \in \mathcal{U},\end{aligned}$$

where

$$[n]_q := \frac{1-q^n}{1-q} = 1 + \sum_{j=1}^{n-1} q^j, \qquad [0]_q := 0. \tag{4}$$

For $\alpha > -1$ and $0 < q < 1$, El-Deeb et al. [12] defined the linear operator $\mathcal{R}_{\mathcal{H}}^{\alpha,q} : \mathcal{A} \to \mathcal{A}$ as follows

$$\mathcal{R}_{\mathcal{H}}^{\alpha,q}\mathcal{F}(\varsigma) * \mathcal{I}_{q,\alpha+1}(\varsigma) = \varsigma\,\mathcal{D}_q(\mathcal{F} * \mathcal{H})(\varsigma),\ \varsigma \in \mathcal{U},$$

where the function $\mathcal{I}_{q,\alpha+1}$ is given by

$$\mathcal{I}_{q,\alpha+1}(\varsigma) := \varsigma + \sum_{n=2}^{+\infty} \frac{[\alpha+1]_{q,n-1}}{[n-1]_q!}\varsigma^n,\ \varsigma \in \mathcal{U}.$$

A simple computation shows that

$$\mathcal{R}_{\mathcal{H}}^{\alpha,q}\mathcal{F}(\varsigma) := \varsigma + \sum_{n=2}^{+\infty} \frac{[n]_q!}{[\alpha+1]_{q,n-1}} c_n d_n \varsigma^n \quad (\alpha > -1,\ 0 < q < 1,\ \varsigma \in \mathcal{U}). \tag{5}$$

By using the operator $\mathcal{R}_{\mathcal{H}}^{\alpha,q}$, we define a new operator as follows:

$$\begin{aligned} D_{\mathcal{H},\delta}^{\alpha,q,0}\mathcal{F}(\varsigma) &= \mathcal{R}_{\mathcal{H}}^{\alpha,q}\mathcal{F}(\varsigma) \\ D_{\mathcal{H},\delta}^{\alpha,q,1}\mathcal{F}(\varsigma) &= \delta\varsigma^3\Big(\mathcal{R}_{\mathcal{H}}^{\alpha,q}\mathcal{F}(\varsigma)\Big)''' + (1+2\delta)\varsigma^2\Big(\mathcal{R}_{\mathcal{H}}^{\alpha,q}\mathcal{F}(\varsigma)\Big)'' + \varsigma\Big(\mathcal{R}_{\mathcal{H}}^{\alpha,q}\mathcal{F}(\varsigma)\Big)' \\ &\ . \\ &\ . \\ D_{\mathcal{H},\delta}^{\alpha,q,m}\mathcal{F}(\varsigma) &= \delta\varsigma^3\Big(D_{\mathcal{H},\delta}^{\alpha,q,m-1}\mathcal{F}(\varsigma)\Big)''' + (1+2\delta)\varsigma^2\Big(D_{\mathcal{H},\delta}^{\alpha,q,m-1}\mathcal{F}(\varsigma)\Big)'' + \varsigma\Big(D_{\mathcal{H},\delta}^{\alpha,q,m-1}\mathcal{F}(\varsigma)\Big)' \\ &= \varsigma + \sum_{n=2}^{+\infty} n^{2m}(\delta(n-1)+1)^m \frac{[n]_q!}{[\alpha+1]_{q,n-1}} c_n d_n \varsigma^n \\ &= \varsigma + \sum_{n=2}^{+\infty} \rho_n c_n \varsigma^n \ \ (\alpha > -1,\ 0 < q < 1,\ m,\delta \in \mathbb{N}_0 = \mathbb{N} \cup \{0\},\ \varsigma \in \mathcal{U}), \end{aligned}$$

where

$$\rho_n = n^{2m}(\delta(n-1)+1)^m \frac{[n]_q!}{[\alpha+1]_{q,n-1}} d_n. \tag{6}$$

From the Definition relation (5), we can easily verify that the next relations hold for all $\mathcal{F} \in \mathcal{A}$:

(i) $[\alpha+1]_q D_{\mathcal{H},\delta}^{\alpha,q,m}\mathcal{F}(\varsigma) = [\alpha]_q D_{\mathcal{H},\delta}^{\alpha+1,q,m}\mathcal{F}(\varsigma) + q^{\mu}\varsigma\,\mathcal{D}_q\Big(D_{\mathcal{H},\delta}^{\alpha+1,q,m}\mathcal{F}(\varsigma)\Big)$, $\varsigma \in \mathcal{U}$;

(ii) $\mathcal{I}_{\mathcal{H},\delta}^{\alpha,m}\mathcal{F}(\varsigma) := \lim_{q\to 1^-} D_{\mathcal{H},\delta}^{\alpha,q,m}\mathcal{F}(\varsigma) = \varsigma + \sum_{n=2}^{+\infty} n^{2m}(\delta(n-1)+1)^m \frac{n!}{(\alpha+1)_{n-1}} c_n d_n \varsigma^n$, $\varsigma \in \mathcal{U}$.

Remark 1. *Taking different particular cases for the coefficients d_n, we obtain the next special cases for the operator $\mathcal{D}_h^{\mu,q}$:*

(i) For $d_n = 1$ and $m = 0$, we obtain the operator $\mathcal{B}_q^{\alpha}$ defined by Srivastava et al. [13] as follows

$$\mathcal{B}_q^{\alpha}\mathcal{F}(\varsigma) := \varsigma + \sum_{n=2}^{+\infty} \frac{[n]_q!}{[\alpha+1]_{q,n-1}} c_n \varsigma^n,\ (\alpha > -1,\ 0 < q < 1,\ \varsigma \in \mathcal{U}); \tag{7}$$

(ii) For $d_n = \dfrac{(-1)^{n-1}\Gamma(\rho+1)}{4^{n-1}(n-1)!\Gamma(n+\rho)}$, $\rho > 0$ and $m = 0$, we obtain the operator $\mathcal{N}_{\rho,q}^{\alpha}$ defined by El-Deeb [14] as follows

$$\begin{aligned} \mathcal{N}_{\rho,q}^{\alpha}\mathcal{F}(\varsigma) &:= \varsigma + \sum_{n=2}^{+\infty} \frac{(-1)^{n-1}\Gamma(\rho+1)}{4^{n-1}(n-1)!\Gamma(n+\rho)} \cdot \frac{[n]_q!}{[\alpha+1]_{q,n-1}} c_n \varsigma^n \\ &= \varsigma + \sum_{n=2}^{+\infty} \frac{[n]_q!}{[\alpha+1]_{q,n-1}} \phi_n c_n \varsigma^n,\ (\rho > 0,\ \alpha > -1,\ 0 < q < 1,\ \varsigma \in \mathcal{U}), \end{aligned} \tag{8}$$

where

$$\phi_n := \frac{(-1)^{n-1}\Gamma(\rho+1)}{4^{n-1}(n-1)!\Gamma(n+\rho)}; \tag{9}$$

(iii) For $d_n = \left(\dfrac{t+1}{t+n}\right)^{\nu}$, $\nu > 0$, $t \geq 0$ and $m = 0$, we obtain the operator $\mathcal{M}_{t,q}^{\alpha,\nu}$ as follows

$$\mathcal{M}_{t,q}^{\alpha,\nu}\mathcal{F}(\varsigma) := \varsigma + \sum_{n=2}^{+\infty} \left(\frac{t+1}{t+n}\right)^{\nu} \cdot \frac{[n]_q!}{[\alpha+1]_{q,n-1}} c_n \varsigma^n,\ \varsigma \in \mathcal{U}; \tag{10}$$

(iv) *For* $d_n = \dfrac{\sigma^{n-1}}{(n-1)!}e^{-\sigma}$, $\sigma > 0$ *and* $m = 0$, *we obtain the q-analogue of Poisson operator* $\mathcal{I}_q^{\alpha,\sigma}$ *defined by El-Deeb et al. [12] as follows*

$$\mathcal{I}_q^{\alpha,\sigma}\mathcal{F}(\varsigma) := \varsigma + \sum_{m=2}^{+\infty} \frac{\sigma^{n-1}}{(n-1)!}e^{-\sigma} \cdot \frac{[n]_q!}{[\alpha+1]_{q,n-1}} c_n \varsigma^n, \ \varsigma \in \mathcal{U}. \tag{11}$$

Definition 1. *Let the functions* $R,\ S : \mathcal{U} \to \mathbb{C}$ *be so constrained that*

$$\min\{\Re(R(\varsigma),\ \Re(S(\varsigma)\} > 0,$$

and

$$R(0) = S(0) = 1.$$

In recent years, using the idea of analytic and bi-univalent functions, many ideas have been developed by different well-known authors. Also, the fractional q-calculus was applied in the geometric function theory, which has a new generalization of the classical operators. The concept of q-calculus operator has been broadly been applied in various fields, including optimal control, quantum physics, q-difference, fractional sub-diffusion equations, hypergeometric series and q-integral equations.

Now, we define the following subclass of bi-concave functions $\mathcal{B}_{\mathcal{H},\delta}^{\alpha,q,m}\mathcal{C}_0(\lambda)$:

Definition 2. *Let the function* $\mathcal{F}$ *have the form (1), which is said to be in the class* $\mathcal{B}_{\mathcal{H},\delta}^{\alpha,q,m}\mathcal{C}_0(\lambda)$ *if the following conditions are satisfied:*

$$\mathcal{F} \in \Sigma, \quad \textit{with} \quad \frac{2}{\lambda-1}\left[\frac{(1+\lambda)(1+\varsigma)}{2(1-\varsigma)} - 1 - \frac{\varsigma\left(D_{\mathcal{H},\delta}^{\alpha,q,m}\mathcal{F}(\varsigma)\right)''}{\left(D_{\mathcal{H},\delta}^{\alpha,q,m}\mathcal{F}(\varsigma)\right)'}\right] \in R(\mathcal{U}), \tag{12}$$

and

$$\frac{2}{\lambda-1}\left[\frac{(1+\lambda)(1+\varpi)}{2(1-\varpi)} - 1 - \frac{\varpi\left(D_{\mathcal{H},\delta}^{\alpha,q,m}\mathcal{G}(\varpi)\right)''}{\left(D_{\mathcal{H},\delta}^{\alpha,q,m}\mathcal{G}(\varpi)\right)'}\right] \in S(\mathcal{U}), \tag{13}$$

with $\alpha > -1$, $0 < q < 1$, $m, \delta \in \mathbb{N}_0$ *and* $\lambda \in (1, 2]$, *where (3) denotes the function* $\mathcal{G}$, *which is the analytic extension of* $\mathcal{F}^{-1}$ *to* $\mathcal{U}$.

Remark 2.

(i) *Putting* $d_n = \dfrac{(-1)^{n-1}\Gamma(\rho+1)}{4^{n-1}(n-1)!\Gamma(n+\rho)}$, $\rho > 0$ *and* $m = 0$, *we find that* $\mathcal{N}_\rho^{\alpha,q}\mathcal{C}_0(\lambda)$ *indicates the functions* $\mathcal{F} \in \Sigma$ *that satisfy (12) and (13) for* $D_{\mathcal{H},\delta}^{\alpha,q,m}$ *substituted with* $\mathcal{N}_{\rho,q}^{\alpha}$ *(8);*

(ii) *Putting* $\left(\dfrac{t+1}{t+n}\right)^{\nu}$, $\nu > 0$, $t \geq 0$ *and* $m = 0$, *we obtain that* $\mathcal{M}_{t,\nu}^{\alpha,q}\mathcal{C}_0(\lambda)$ *indicates the functions* $\mathcal{F} \in \Sigma$ *that satisfy (12) and (13) for* $D_{\mathcal{H},\delta}^{\alpha,q,m}$ *substituted with* $\mathcal{M}_{t,q}^{\alpha,\nu}$ *(10);*

(iii) *Putting* $\dfrac{\sigma^{n-1}}{(n-1)!}e^{-\sigma}$, $\sigma > 0$ *and* $m = 0$, *we obtain that* $\mathcal{I}_\sigma^{\alpha,q}\mathcal{C}_0(\lambda)$ *indicates the functions* $\mathcal{F} \in \Sigma$ *that satisfy (12) and (13) for* $D_{\mathcal{H},\delta}^{\alpha,q,m}$ *substituted with* $\mathcal{I}_q^{\alpha,\sigma}$ *(11).*

We established some results for coefficients bounds for bi-concave functions belonging to the class $\mathcal{B}_{\mathcal{H},\delta}^{\alpha,q,m}\mathcal{C}_0(\lambda)$.

2. Coefficient Bounds for the Function Class $\mathcal{B}_{\mathcal{H},\delta}^{\alpha,q,m}\mathcal{C}_0(\lambda)$

In this section, we discuss a class of bi-univalent analytic functions by applying a principle of convolution. In this sense, we establish in advance a new q-linear differential operator. Further we provide an estimate for the function coefficients $|c_2|$ and $|c_3|$ of the new classes.

Theorem 1. *If the function $\mathcal{F}$ given by (1) belongs to the class $\mathcal{B}^{\alpha,q,m}_{\mathcal{H},\delta}\mathcal{C}_0(\lambda)$, and $\lambda \in (1,2]$, then*

$$|c_2| \leq \min\left\{\sqrt{\frac{(\lambda+1)^2}{4\rho_2^2} + \frac{(\lambda-1)^2\left(\left|R'(0)\right|^2+\left|S'(0)\right|^2\right)}{32\rho_2^2} + \frac{(\lambda^2-1)\left(\left|R'(0)\right|+\left|S'(0)\right|\right)}{8\rho_2^2}},\right.$$
$$\left.\sqrt{\frac{(\lambda-1)\left(\left|R''(0)\right|+\left|S''(0)\right|\right)}{16\left(2\rho_2^2-3\rho_3\right)} + \frac{(\lambda+1)}{2\left(2\rho_2^2-3\rho_3\right)}}\right\}, \tag{14}$$

and

$$|c_3| \leq \min\left\{\frac{8(\lambda+1)^2+(\lambda-1)^2\left(\left|R'(0)\right|^2+\left|S'(0)\right|^2\right)}{32\rho_2^2} + \frac{(\lambda^2-1)\left(\left|R'(0)\right|+\left|S'(0)\right|\right)}{8\rho_2^2} + \frac{(\lambda-1)\left(\left|R''(0)\right|+\left|S''(0)\right|\right)}{48\rho_3},\right.$$
$$\left.\frac{(\lambda-1)\left(3\rho_3-\rho_2^2\right)\left|R''(0)\right|+(\lambda-1)\rho_2^2\left|S''(0)\right|}{24\rho_3\left(2\rho_2^2-3\rho_3\right)} + \frac{(\lambda+1)}{2\left(2\rho_2^2-\rho_3\right)}\right\}, \tag{15}$$

where ρ_n $(n \in \{2,3\})$ are defined by (6).

Proof. If $\mathcal{F} \in \mathcal{B}^{\alpha,q,m}_{\mathcal{H},\delta}\mathcal{C}_0(\lambda)$, from (12) and (13), respectively. Hence, it follows that

$$\frac{2}{\lambda-1}\left[\frac{(1+\lambda)(1+\varsigma)}{2(1-\varsigma)} - 1 - \frac{\varsigma\left(D^{\alpha,q,m}_{\mathcal{H},\delta}\mathcal{F}(\varsigma)\right)''}{\left(D^{\alpha,q,m}_{\mathcal{H},\delta}\mathcal{F}(\varsigma)\right)'}\right] = R(\varsigma), \tag{16}$$

and

$$\frac{2}{\lambda-1}\left[\frac{(1+\lambda)(1+\varpi)}{2(1-\varpi)} - 1 - \frac{\varpi\left(D^{\alpha,q,m}_{\mathcal{H},\delta}\mathcal{G}(\varpi)\right)''}{\left(D^{\alpha,q,m}_{\mathcal{H},\delta}\mathcal{G}(\varpi)\right)'}\right] = S(\varpi), \tag{17}$$

where R and S satisfy the conditions of Definition 1. Then, the functions $R(\varsigma)$ and $S(\varpi)$ have the following Taylor–Maclaurin series expansions:

$$R(\varsigma) = 1 + r_1\varsigma + r_2\varsigma^2 + \ldots,$$

and

$$S(\varpi) = 1 + s_1\varpi + s_2\varpi^2 + \ldots,$$

respectively. By equalizing according to the coefficients of ς and ϖ in (14) and (15), it is obvious that

$$\frac{2[(1+\lambda) - 2\rho_2 c_2]}{\lambda-1} = r_1, \tag{18}$$

$$\frac{2\left[(1+\lambda) + 4\rho_2^2 c_2^2 - 6\rho_3 c_3\right]}{\lambda-1} = r_2, \tag{19}$$

$$-\frac{2[(1+\lambda) - 2\rho_2 c_2]}{\lambda-1} = s_1, \tag{20}$$

and

$$\frac{2\left[(\lambda+1)+4\rho_2^2c_2^2-6\rho_3\left(2c_2^2-c_3\right)\right]}{\lambda-1}=s_2. \tag{21}$$

Using (18) and (20), we obtain

$$r_1=-s_1, \tag{22}$$

and from (18), we can write

$$c_2=\frac{(\lambda+1)}{2\rho_2}-\frac{(\lambda-1)}{4\rho_2}r_1. \tag{23}$$

Squaring (18) and (20), after adding relations, we obtain

$$c_2^2=\frac{(\lambda+1)^2}{4\rho_2^2}+\frac{(\lambda-1)^2\left(r_1^2+s_1^2\right)}{32\rho_2^2}-\frac{\left(\lambda^2-1\right)}{8\rho_2^2}(r_1-s_1). \tag{24}$$

Adding (19) and (21), we have

$$c_2^2=\frac{(\lambda-1)(r_2+s_2)}{8\left(2\rho_2^2-3\rho_3\right)}-\frac{(\lambda+1)}{2\left(2\rho_2^2-3\rho_3\right)}. \tag{25}$$

Taking the absolute value of (24) and (25), we conclude that

$$|c_2|\leq\sqrt{\frac{(\lambda+1)^2}{4\rho_2^2}+\frac{(\lambda-1)^2\left(\left|R'(0)\right|^2+\left|S'(0)\right|^2\right)}{32\rho_2^2}+\frac{\left(\lambda^2-1\right)\left(\left|R'(0)\right|+\left|S'(0)\right|\right)}{8\rho_2^2}},$$

and

$$|c_2|\leq\sqrt{\frac{(\lambda-1)\left(\left|R''(0)\right|+\left|S''(0)\right|\right)}{16\left(2\rho_2^2-3\rho_3\right)}+\frac{(\lambda+1)}{2\left(2\rho_2^2-3\rho_3\right)}},$$

which gives the bound for $|c_2|$ as we asserted in our Theorem.

To find the bound for $|c_3|$, by subtracting (21) from (19), we obtain

$$c_3=c_2^2-\frac{(\lambda-1)(r_2-s_2)}{24\rho_2}. \tag{26}$$

Also, upon substituting the value of in view of c_2^2 from (24) and (25) into (26), we obtain

$$c_3=\frac{(1+\lambda)^2}{4\rho_2^2}+\frac{(\lambda-1)^2\left(r_1^2+s_1^2\right)}{32\rho_2^2}-\frac{\left(\lambda^2-1\right)}{8\rho_2^2}(r_1-s_1)-\frac{(\lambda-1)(r_2-s_2)}{24\rho_3}, \tag{27}$$

and

$$c_3=\frac{(\lambda-1)(r_2+s_2)}{8\left(2\rho_2^2-3\rho_3\right)}-\frac{(1+\lambda)}{2\left(2\rho_2^2-3\rho_3\right)}-\frac{(\lambda-1)(r_2-s_2)}{24\rho_3}. \tag{28}$$

Taking the absolute value of (27) and (28), we obtain

$$|c_3|\leq\frac{8(\lambda+1)^2+(\lambda-1)^2\left(\left|R'(0)\right|^2+\left|S'(0)\right|^2\right)}{32\rho_2^2}+\frac{\left(\lambda^2-1\right)\left(\left|R'(0)\right|+\left|S'(0)\right|\right)}{8\rho_2^2}+\frac{(\lambda-1)\left(\left|R''(0)\right|+\left|S''(0)\right|\right)}{48\rho_3},$$

and

$$|c_3|\leq\frac{(\lambda-1)\left(3\rho_3-\rho_2^2\right)\left|R''(0)\right|+(\lambda-1)\rho_2^2\left|S''(0)\right|}{24\rho_3\left(2\rho_2^2-3\rho_3\right)}+\frac{(\lambda+1)}{2\left(2\rho_2^2-\rho_3\right)}.$$

This completes the proof of the Theorem. □

Putting $m=d_n=1$, we determine that $\mathcal{S}^{\alpha,q}\mathcal{C}_0(\lambda)$ in Theorem 1, we obtain the following Corollary:

Corollary 1. *For the function $\mathcal{F}$ given by* (1) *belonging to the class $\mathcal{S}^{\alpha,q}\mathcal{C}_0(\lambda)$, and $\lambda \in (1,2]$, then*

$$|c_2| \le \min\left\{ \sqrt{\frac{(\lambda+1)^2\left([\alpha+1]_q\right)^2}{4\left([2]_q!\right)^2} + \frac{(\lambda-1)^2\left([\alpha+1]_q\right)^2\left(\left|R'(0)\right|^2+\left|S'(0)\right|^2\right)}{32\left([2]_q!\right)^2} + \frac{(\lambda^2-1)\left([\alpha+1]_q\right)^2\left(\left|R'(0)\right|+\left|S'(0)\right|\right)}{8\left([2]_q!\right)^2}}, \right.$$

$$\left. \sqrt{\frac{(\lambda-1)\left(\left|R''(0)\right|+\left|S''(0)\right|\right)}{16\left(2\left(\frac{[2]_q!}{[\alpha+1]_q}\right)^2-\frac{3[3]_q!}{[\alpha+1]_{q,2}}\right)} + \frac{(\lambda+1)}{2\left(2\left(\frac{[2]_q!}{[\alpha+1]_q}\right)^2-\frac{3[3]_q!}{[\alpha+1]_{q,2}}\right)}} \right\},$$

and

$$|c_3| \le \min\left\{ \frac{\left([\alpha+1]_q\right)^2\left[8(\lambda+1)^2+(\lambda-1)^2\left(\left|R'(0)\right|^2+\left|S'(0)\right|^2\right)\right]}{32\left([2]_q!\right)^2} + \right.$$

$$\frac{(\lambda^2-1)\left([\alpha+1]_q\right)^2\left(\left|R'(0)\right|+\left|S'(0)\right|\right)}{8\left([2]_q!\right)^2} + \frac{(\lambda-1)[\alpha+1]_{q,2}\left(\left|R''(0)\right|+\left|S''(0)\right|\right)}{48[3]_q!},$$

$$\left. \frac{(\lambda-1)\left(3\frac{[3]_q!}{[\alpha+1]_{q,2}}-\left(\frac{[2]_q!}{[\alpha+1]_q}\right)^2\right)\left|R''(0)\right|+(\lambda-1)\left(\frac{[2]_q!}{[\alpha+1]_q}\right)^2\left|S''(0)\right|}{\frac{24[3]_q!}{[\alpha+1]_{q,2}}\left(2\left(\frac{[2]_q!}{[\alpha+1]_q}\right)^2-\frac{3[3]_q!}{[\alpha+1]_{q,2}}\right)} + \frac{(\lambda+1)}{2\left(2\left(\frac{[2]_q!}{[\alpha+1]_q}\right)^2-\frac{3[3]_q!}{[\alpha+1]_{q,2}}\right)} \right\}.$$

Putting $d_n = \dfrac{(-1)^{n-1}\Gamma(\rho+1)}{4^{n-1}(n-1)!\Gamma(n+\rho)}$, $\rho > 0$ and $m = 0$ in Theorem 1, we obtain the following Corollary:

Corollary 2. *If the function $\mathcal{F}$ given by* (1) *belongs to the class $\mathcal{N}_{\rho}^{\alpha,q}\mathcal{C}_0(\lambda)$, and $\lambda \in (1,2]$, then*

$$|c_2| \le \min\left\{ \sqrt{\frac{(\lambda+1)^2\left([\alpha+1]_q\right)^2}{4\left([2]_q!\right)^2\phi_2^2} + \frac{(\lambda-1)^2\left([\alpha+1]_q\right)^2\left(\left|R'(0)\right|^2+\left|S'(0)\right|^2\right)}{32\left([2]_q!\right)^2\phi_2^2} + \frac{(\lambda^2-1)\left([\alpha+1]_q\right)^2\left(\left|R'(0)\right|+\left|S'(0)\right|\right)}{8\left([2]_q!\right)^2\phi_2^2}}, \right.$$

$$\left. \sqrt{\frac{(\lambda-1)\left(\left|R''(0)\right|+\left|S''(0)\right|\right)}{16\left(2\left(\frac{[2]_q!}{[\alpha+1]_q}\right)^2\phi_2^2-\frac{3[3]_q!}{[\alpha+1]_{q,2}}\phi_3\right)} + \frac{(\lambda+1)}{2\left(2\left(\frac{[2]_q!}{[\alpha+1]_q}\right)^2\phi_2^2-\frac{3[3]_q!}{[\alpha+1]_{q,2}}\phi_3\right)}} \right\},$$

and

$$|c_3| \le \min\left\{ \frac{\left([\alpha+1]_q\right)^2\left[8(\lambda+1)^2+(\lambda-1)^2\left(\left|R'(0)\right|^2+\left|S'(0)\right|^2\right)\right]}{32\left([2]_q!\right)^2\phi_2^2} + \right.$$

$$\frac{(\lambda^2-1)\left([\alpha+1]_q\right)^2\left(\left|R'(0)\right|+\left|S'(0)\right|\right)}{8\left([2]_q!\right)^2\phi_2^2} + \frac{(\lambda-1)[\alpha+1]_{q,2}\left(\left|R''(0)\right|+\left|S''(0)\right|\right)}{48[3]_q!\phi_3},$$

$$\left. \frac{(\lambda-1)\left(3\frac{[3]_q!}{[\alpha+1]_{q,2}}d_3-\left(\frac{[2]_q!}{[\alpha+1]_q}\right)^2 d_2^2\right)\left|R''(0)\right|+(\lambda-1)\left(\frac{[2]_q!}{[\alpha+1]_q}\right)^2 d_2^2\left|S''(0)\right|}{\frac{24[3]_q!}{[\alpha+1]_{q,2}}\phi_3\left(2\left(\frac{[2]_q!}{[\alpha+1]_q}\right)^2\phi_2^2-\frac{3[3]_q!}{[\alpha+1]_{q,2}}\phi_3\right)} + \frac{(\lambda+1)}{2\left(2\left(\frac{[2]_q!}{[\alpha+1]_q}\right)^2\phi_2^2-\frac{3[3]_q!}{[\alpha+1]_{q,2}}\phi_3\right)} \right\},$$

where ϕ_n $(n \in \{2,3\})$ are defined by (9).

Putting $d_n = \left(\dfrac{t+1}{t+n}\right)^{\nu}$, $\nu > 0$, $t \ge 0$ and $m = 0$ in Theorem 1, we obtain the following Corollary:

Corollary 3. *Let the function $\mathcal{F}$ given by* (1) *belongs to the class $\mathcal{M}_{t,\nu}^{\alpha,q}\mathcal{C}_0(\lambda)$, and $\lambda \in (1,2]$, then*

$$|c_2| \leq \min\left\{ \sqrt{\frac{(\lambda+1)^2(t+2)^{2\alpha}\left([\alpha+1]_q\right)^2}{4\left([2]_q!\right)^2(t+1)^{2\nu}} + \frac{(\lambda-1)^2(t+2)^{2\alpha}\left([\alpha+1]_q\right)^2\left(\left|R'(0)\right|^2+\left|S'(0)\right|^2\right)}{32\left([2]_q!\right)^2(t+1)^{2\nu}} + \frac{\left(\lambda^2-1\right)(t+2)^{2\nu}\left([\alpha+1]_q\right)^2\left(\left|R'(0)\right|+\left|S'(0)\right|\right)}{8\left([2]_q!\right)^2(t+1)^{2\nu}}},\right.$$

$$\left.\sqrt{\frac{(\lambda-1)\left(\left|R''(0)\right|+\left|S''(0)\right|\right)}{16\left(2\left(\frac{[2]_q!}{[\alpha+1]_q}\right)^2\left(\frac{t+1}{t+2}\right)^{2\nu}-\frac{3[3]_q!}{[\alpha+1]_{q,2}}\left(\frac{t+1}{t+3}\right)^{\nu}\right)} + \frac{(\lambda+1)}{2\left(2\left(\frac{[2]_q!}{[\alpha+1]_q}\right)^2\left(\frac{t+1}{t+2}\right)^{2\nu}-\frac{3[3]_q!}{[\alpha+1]_{q,2}}\left(\frac{t+1}{t+3}\right)^{\nu}\right)}}\right\},$$

and

$$|c_3| \leq \min\left\{ \frac{\left([\alpha+1]_q\right)^2(t+2)^{2\nu}\left[8(\lambda+1)^2+(\lambda-1)^2\left(\left|R'(0)\right|^2+\left|S'(0)\right|^2\right)\right]}{32\left([2]_q!\right)^2(t+1)^{2\nu}} + \right.$$

$$\frac{\left(\lambda^2-1\right)(t+2)^{2\nu}\left([\alpha+1]_q\right)^2\left(\left|R'(0)\right|+\left|S'(0)\right|\right)}{8\left([2]_q!\right)^2(t+1)^{2\nu}} + \frac{(\lambda-1)(t+3)^{\nu}[\alpha+1]_{q,2}\left(\left|R''(0)\right|+\left|S''(0)\right|\right)}{48[3]_q!(t+1)^{\nu}},$$

$$\frac{(\lambda-1)\left(3\frac{[3]_q!}{[\alpha+1]_{q,2}}\left(\frac{t+1}{t+3}\right)^{\nu}-\left(\frac{[2]_q!}{[\alpha+1]_q}\right)^2\left(\frac{t+1}{t+2}\right)^{2\nu}\right)\left|R''(0)\right|+(\lambda-1)\left(\frac{[2]_q!}{[\alpha+1]_q}\right)^2\left(\frac{t+1}{t+2}\right)^{2\nu}\left|S''(0)\right|}{\frac{24[3]_q!}{[\alpha+1]_{q,2}}\left(\frac{t+1}{t+3}\right)^{\nu}\left(2\left(\frac{[2]_q!}{[\alpha+1]_q}\right)^2\left(\frac{t+1}{t+2}\right)^{2\nu}-\frac{3[3]_q!}{[\alpha+1]_{q,2}}\left(\frac{t+1}{t+3}\right)^{\nu}\right)}$$

$$\left. + \frac{(\lambda+1)}{2\left(2\left(\frac{[2]_q!}{[\alpha+1]_q}\right)^2\left(\frac{t+1}{t+2}\right)^{2\nu}-\frac{3[3]_q!}{[\alpha+1]_{q,2}}\left(\frac{t+1}{t+3}\right)^{\nu}\right)}\right\}.$$

Putting $d_n = \dfrac{\sigma^{n-1}}{(n-1)!}e^{-\sigma}$, $\sigma > 0$ and $m = 0$ in Theorem 1, we obtain the following Corollary:

Corollary 4. *Let the function $\mathcal{F}$ given by* (1) *belong to the class $\mathcal{I}_{\sigma}^{\alpha,q}\mathcal{C}_0(\lambda)$, and $\lambda \in (1,2]$; then,*

$$|c_2| \leq \min\left\{ \sqrt{\frac{(\lambda+1)^2\left([\alpha+1]_q\right)^2}{4\sigma^2\left([2]_q!\right)^2e^{-2\sigma}} + \frac{(\lambda-1)^2\left([\alpha+1]_q\right)^2\left(\left|R'(0)\right|^2+\left|S'(0)\right|^2\right)}{32\sigma^2\left([2]_q!\right)^2e^{-2\sigma}} + \frac{\left(\lambda^2-1\right)\left([\alpha+1]_q\right)^2\left(\left|R'(0)\right|+\left|S'(0)\right|\right)}{8\sigma^2\left([2]_q!\right)^2e^{-2\sigma}}},\right.$$

$$\left.\sqrt{\frac{(\lambda-1)\left(\left|R''(0)\right|+\left|S''(0)\right|\right)}{16\left(2\left(\frac{[2]_q!}{[\alpha+1]_q}\right)^2\sigma^2e^{-2\sigma}-\left(\frac{3[3]_q!}{2[\alpha+1]_{q,2}}\right)\sigma^2e^{-\sigma}\right)} + \frac{(\lambda+1)}{2\left(2\left(\frac{[2]_q!}{[\alpha+1]_q}\right)^2\sigma^2e^{-2\sigma}-\left(\frac{3[3]_q!}{2[\alpha+1]_{q,2}}\right)\sigma^2e^{-\sigma}\right)}}\right\},$$

and

$$|c_3| \leq \min\left\{ \frac{\left([\alpha+1]_q\right)^2\left[8(\lambda+1)^2+(\lambda-1)^2\left(\left|R'(0)\right|^2+\left|S'(0)\right|^2\right)\right]}{32\left([2]_q!\right)^2\sigma^2e^{-2\sigma}} + \right.$$

$$\frac{\left(\lambda^2-1\right)\left([\alpha+1]_q\right)^2\left(\left|R'(0)\right|+\left|S'(0)\right|\right)}{8\left([2]_q!\right)^2\sigma^2e^{-2\sigma}} + \frac{(\lambda-1)[\alpha+1]_{q,2}\left(\left|R''(0)\right|+\left|S''(0)\right|\right)}{24[3]_q!\sigma^2e^{-\sigma}},$$

$$\frac{(\lambda-1)\left(\frac{3[3]_q!}{2[\alpha+1]_{q,2}}\sigma^2 e^{-\sigma}-\left(\frac{[2]_q!}{[\alpha+1]_q}\right)^2\sigma^2 e^{-2\sigma}\right)\left|R''(0)\right|+(\lambda-1)\left(\frac{[2]_q!}{[\alpha+1]_q}\right)^2\sigma^2 e^{-2\sigma}\left|S''(0)\right|}{\frac{12[3]_q!\sigma^2 e^{-\sigma}}{[\alpha+1]_{q,2}}\left(2\left(\frac{[2]_q!}{[\alpha+1]_q}\right)^2\sigma^2 e^{-2\sigma}-\left(\frac{3[3]_q!}{2[\alpha+1]_{q,2}}\right)\sigma^2 e^{-\sigma}\right)}$$

$$+\frac{(\lambda+1)}{2\left(2\left(\frac{[2]_q!}{[\alpha+1]_q}\right)^2\sigma^2 e^{-2\sigma}-\left(\frac{3[3]_q!}{2[\alpha+1]_{q,2}}\right)\sigma^2 e^{-\sigma}\right)}\Bigg\}.$$

By specializing the functions R and S in Theorem 1, we obtain the following examples:

Example 1. *Having the functions*

$$R(\varsigma)=\left(\frac{1+\varsigma}{1-\varsigma}\right)^\beta=1+2\beta\varsigma+2\beta^2\varsigma^2+\ldots,\quad (0<\beta\le 1),$$

$$S(\varpi)=\left(\frac{1-\varpi}{1+\varpi}\right)^\beta=1-2\beta\varpi+2\beta^2\varpi^2-\ldots,\quad (0<\beta\le 1),$$

and $\mathcal{F}$ given by (1), which belongs to the class $\mathcal{B}^{\alpha,q,0}_{\mathcal{H},\delta}\mathcal{C}_0(\lambda)$, and $\lambda\in(1,2]$, then

$$|c_2|\le\min\left\{\sqrt{\frac{(\lambda+1)^2\left([\alpha+1]_q\right)^2}{4\left([2]_q!\right)^2 d_2^2}+\frac{\beta^2(\lambda-1)^2\left([\alpha+1]_q\right)^2}{4\left([2]_q!\right)^2 d_2^2}+\frac{\beta\left(\lambda^2-1\right)\left([\alpha+1]_q\right)^2}{2\left([2]_q!\right)^2 d_2^2}},\right.$$

$$\left.\sqrt{\frac{\beta^2(\lambda-1)}{2\left(2\left(\frac{[2]_q!}{[\alpha+1]_q}\right)^2 d_2^2-\frac{3[3]_q!}{[\alpha+1]_{q,2}}d_3\right)}+\frac{(\lambda+1)}{2\left(2\left(\frac{[2]_q!}{[\alpha+1]_q}\right)^2 d_2^2-\frac{3[3]_q!}{[\alpha+1]_{q,2}}d_3\right)}}\right\},$$

and

$$|c_3|\le\min\left\{\frac{\beta^2\left([\alpha+1]_q\right)^2\left[8(\lambda+1)^2+(\lambda-1)^2\right]}{4\left([2]_q!\right)^2 d_2^2}+\frac{\beta\left(\lambda^2-1\right)\left([\alpha+1]_q\right)^2}{2\left([2]_q!\right)^2 d_2^2}+\frac{\beta^2(\lambda-1)[\alpha+1]_{q,2}}{6[3]_q! d_3},\right.$$

$$\left.\frac{(\lambda-1)\beta^2\left(3\frac{[3]_q!}{[\alpha+1]_{q,2}}d_3-\left(\frac{[2]_q!}{[\alpha+1]_q}\right)^2 d_2^2\right)+(\lambda-1)\beta^2\left(\frac{[2]_q!}{[\alpha+1]_q}\right)^2 d_2^2}{\frac{6[3]_q!}{[\alpha+1]_{q,2}}d_3\left(2\left(\frac{[2]_q!}{[\alpha+1]_q}\right)^2 d_2^2-\frac{3[3]_q!}{[\alpha+1]_{q,2}}d_3\right)}+\frac{(\lambda+1)}{2\left(2\left(\frac{[2]_q!}{[\alpha+1]_q}\right)^2 d_2^2-\frac{3[3]_q!}{[\alpha+1]_{q,2}}d_3\right)}\right\}.$$

Example 2. *Considering the functions*

$$R(\varsigma)=\frac{1+(1-2\sigma)\varsigma}{1-\varsigma}=1+2(1-\sigma)\varsigma+2(1-\sigma)\varsigma^2+\ldots,\quad (0\le\sigma<1),$$

$$S(\varpi)=\frac{1-(1-2\sigma)\varpi}{1+\varpi}=1-2(1-\sigma)\varpi+2(1-\sigma)\varpi^2+\ldots,\quad (0\le\sigma<1),$$

and $\mathcal{F}$ given by (1), which belongs to the class $\mathcal{B}^{\alpha,q,0}_{\mathcal{H},\delta}\mathcal{C}_0(\lambda)$, and $\lambda\in(1,2]$, then

$$|c_2| \leq \min\left\{ \sqrt{\frac{(\lambda+1)^2\left([\alpha+1]_q\right)^2}{4\left([2]_q!\right)^2 d_2^2} + \frac{(\lambda-1)^2(1-\sigma)^2\left([\alpha+1]_q\right)^2}{3\left([2]_q!\right)^2 d_2^2} + \frac{\left(\lambda^2-1\right)(1-\sigma)\left([\alpha+1]_q\right)^2}{2\left([2]_q!\right)^2 d_2^2}}, \right.$$

$$\left. \sqrt{\frac{(\lambda-1)(1-\sigma)^2}{2\left(2\left(\frac{[2]_q!}{[\alpha+1]_q}\right)^2 d_2^2 - \frac{3[3]_q!}{[\alpha+1]_{q,2}} d_3\right)} + \frac{(\lambda+1)}{2\left(2\left(\frac{[2]_q!}{[\alpha+1]_q}\right)^2 d_2^2 - \frac{3[3]_q!}{[\alpha+1]_{q,2}} d_3\right)}} \right\},$$

and

$$|c_3| \leq \min\left\{ \frac{\left([\alpha+1]_q\right)^2\left[8(\lambda+1)^2+(\lambda-1)^2(1-\sigma)^2\right]}{4\left([2]_q!\right)^2 d_2^2} + \frac{\left(\lambda^2-1\right)(1-\sigma)\left([\alpha+1]_q\right)^2}{2\left([2]_q!\right)^2 d_2^2} + \frac{(\lambda-1)(1-\sigma)^2[\alpha+1]_{q,2}}{6[3]_q! d_3}, \right.$$

$$\left. \frac{(\lambda-1)(1-\sigma)^2\left(3\frac{[3]_q!}{[\alpha+1]_{q,2}} d_3 - \left(\frac{[2]_q!}{[\alpha+1]_q}\right)^2 d_2^2\right) + (\lambda-1)(1-\sigma)^2\left(\frac{[2]_q!}{[\alpha+1]_q}\right)^2 d_2^2}{\frac{6[3]_q!}{[\alpha+1]_{q,2}} d_3\left(2\left(\frac{[2]_q!}{[\alpha+1]_q}\right)^2 d_2^2 - \frac{3[3]_q!}{[\alpha+1]_{q,2}} d_3\right)} + \frac{(\lambda+1)}{2\left(2\left(\frac{[2]_q!}{[\alpha+1]_q}\right)^2 d_2^2 - \frac{3[3]_q!}{[\alpha+1]_{q,2}} d_3\right)} \right\}.$$

3. Concluding Remarks and Observations

In this study, we used the q-derivative operator $D_{\mathcal{H},\delta}^{\alpha,q,m}$ to introduce and examine the properties of a few new subclasses of the class of analytic and bi-concave functions in the open unit disk $\mathcal{U}$. We derived estimates for the initial Taylor–Maclaurin coefficients $|c_2|$ and $|c_3|$ for functions belonging to the bi-concave function classes that are presented in this study, among other features and results. In addition, we chose to deduce a few corollaries and implications of our main points (see Theorem 1). Future studies may uncover special features of the defined subclasses of analytic and bi-concave functions.

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