

Article

# Concerning Fuzzy $b$ -Metric Spaces <sup>†</sup>

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<sup>†</sup> Dedicated to Professor Valentín Gregori on the occasion of their 70th birthday.

**Abstract:** In an article published in 2015, Hussain et al. introduced a notion of a fuzzy  $b$ -metric space and obtained some fixed point theorems for this kind of space. Shortly thereafter, Nădăban presented a notion of a fuzzy  $b$ -metric space that is slightly different from the one given by Hussain et al., and explored some of its topological properties. Related to Nădăban's study, Sedghi and Shobe, Saadati, and Šostak independently conducted investigations in articles published in 2012, 2015, and 2018, respectively, about another class of spaces that Sedgi and Shobe called  $b$ -fuzzy metric spaces, Saadati, fuzzy metric type spaces, and Šostak, fuzzy  $k$ -metric spaces. The main contributions of our paper are the following: First, we propose a notion of fuzzy  $b$ -metric space that encompasses and unifies the aforementioned types of spaces. Our approach, which is based on Gabriel's notion of a fuzzy metric space, allows us to simultaneously cover two interesting classes of spaces, namely, the 01-fuzzy  $b$ -metric spaces and the  $K$ -stationary fuzzy  $b$ -metric spaces. Second, we show that each fuzzy  $b$ -metric space, in our sense, admits uniformity with a countable base. From this fact, we derive, among other consequences, that the topology induced by means of its "open" balls is metrizable. Finally, we obtain a characterization of complete fuzzy  $b$ -metric spaces with the help of a fixed point result which is also proved here. In support of our approach, several examples, including an application to a type of difference equations, are discussed.

**Keywords:** fuzzy  $b$ -metric space; uniformity; metrizable; complete; fixed point; difference equation

**MSC:** 54A40; 54E15; 54E50; 54H25; 47H10



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## 1. Introduction

We will assume that readers are familiar with the concept of a uniform space, as well as its basic properties. The classical books [1] and [2] are instances of excellent references for general topology and, in particular, for the study of uniformities.

In this paper, we will denote by  $\mathbb{N}$  and  $\mathbb{R}^+$  the set of natural numbers and the set of non-negative real numbers, respectively.

Given a metric  $d$  on a set  $\mathfrak{X}$ , we denote by  $\tau_d$  and  $\mathfrak{U}_d$ , respectively, the topology and the uniformity induced by  $d$ . We remind the reader that  $\mathfrak{U}_d$  has as a base the countable family

$$\{ \{ (x, y) \in \mathfrak{X} \times \mathfrak{X} : d(x, y) < 2^{-n} \} : n \in \mathbb{N} \} .$$

It is known that the so-called  $b$ -metric spaces provide a simultaneous and appealing generalization of metric spaces and quasi-normed spaces (a detailed and careful exposition in this regard can be found in [3] (Section 2)). The name  $b$ -metric space is due to Czerwik [4,5], who explored some topological properties of these spaces and obtained some fixed point theorems. However,  $b$ -metric spaces were also considered by various authors under other denominations and from other points of view. In fact, there are numerous publications on both topological properties and fixed point theory in such spaces, so, both for our goals here and for the pertinent updates it will suffice to refer to the recent articles [3,6,7], and the references therein.



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Let us recall that a  $b$ -metric on a set  $\mathfrak{X}$  is a pair  $(d, K)$ , where  $d$  is a function from  $\mathfrak{X} \times \mathfrak{X}$  to  $\mathbb{R}^+$  and  $K$  is a real constant with  $K \geq 1$ , fulfilling the next conditions for every  $x, y, z \in \mathfrak{X}$ :

$$(bM1) \quad d(x, y) = 0 \text{ if and only if } x = y;$$

$$(bM2) \quad d(x, y) = d(y, x);$$

$$(bM3) \quad d(x, y) \leq K(d(x, z) + d(z, y)).$$

Thus, a  $b$ -metric space is a 3-tuple  $(\mathfrak{X}, d, K)$  such that  $\mathfrak{X}$  is a set and  $(d, K)$  is a  $b$ -metric on  $\mathfrak{X}$ .

Note that every metric space can be considered a  $b$ -metric space for any  $K \geq 1$ .

**Remark 1.** See, e.g., [3,8–10] to find noteworthy examples of  $b$ -metric spaces. In particular, it is well known that if  $(\mathfrak{X}, e)$  is a metric space, the pair  $(d, K)$  is a  $b$ -metric on  $\mathfrak{X}$ , where  $d(x, y) = (e(x, y))^r$  for all  $x, y \in \mathfrak{X}$ , with  $r > 1$  and  $K = 2^{r-1}$ . Moreover,  $d$  is not a metric on  $\mathfrak{X}$ , in general.

Just as in the metric case, each  $b$ -metric  $d$  on a set  $\mathfrak{X}$  induces naturally a topology  $\tau_d$  on  $\mathfrak{X}$ : A subset  $\mathfrak{Y}$  of  $\mathfrak{X}$  is declared  $\tau_d$ -open if for each  $x \in \mathfrak{Y}$  there is an  $\varepsilon_x > 0$  such that the “open” ball  $B_d(x, \varepsilon_x) := \{y \in \mathfrak{X} : d(x, y) < \varepsilon_x\}$  is a subset of  $\mathfrak{Y}$ . Then, it is easily checked that  $\tau_d$  fulfills the axioms of a topology.

It is interesting to point out that, in contrast to the classical metric case, not every “open” ball is a  $\tau_d$ -open set (see, e.g., [10] (Example 3.9)). Nevertheless, the topological space  $(\mathfrak{X}, \tau_d)$  is metrizable (see, e.g., [6,7]).

We also have (see, e.g., [6,7]) that a sequence  $(x_n)_{n \in \mathbb{N}}$  in a  $b$ -metric space  $(\mathfrak{X}, d, K)$  is  $\tau_d$ -convergent to an  $x \in \mathfrak{X}$  if and only if  $d(x, x_n) \rightarrow 0$  as  $n \rightarrow +\infty$ , and, by definition, the  $b$ -metric space  $(\mathfrak{X}, d, K)$  is complete, provided that every Cauchy sequence in  $(\mathfrak{X}, d, K)$  is  $\tau_d$ -convergent, where Cauchy sequences are defined exactly as for metric spaces.

Generalizing naturally the concept of a fuzzy metric space (in the sense of Kramosil and Michálek [11]) to a  $b$ -metric context, Nădăban introduced in [12] the notion of a fuzzy  $b$ -metric space and analyzed some of its basic topological properties. A year earlier, Hussain et al. [13] described a somewhat different notion to the one given by Nădăban, and obtained some fixed point theorems for those spaces. Under the names of a  $b$ -fuzzy metric space, a fuzzy metric type space, and a fuzzy  $k$ -metric space, respectively, Sedghi and Shobe [14], Saadati [15] and Šostak [16] discussed in detail many topological properties of a close structure, based on the notion of a fuzzy metric space due to George and Veeramani [17,18]. In particular, several relevant examples were furnished in [15,16]. Later on, and from the perspective of Kramosil and Michálek, Zhong and Šostak [19] generalized the notion of a fuzzy  $k$ -metric space, giving motivating examples and deeply discussing the properties of this new structure. In parallel, various authors have contributed to the development of a theory of a fixed point for these structures (see, e.g., [13,14,20–24]), while generalizations of the notion of a fuzzy  $b$ -metric space to the metric-like and the quasi-metric setting have been initiated and studied in [25,26], respectively.

Based on the notion of fuzzy metric space introduced by Gabriec in [27], we here propose a definition of fuzzy  $b$ -metric space that generalizes and unifies the types of spaces mentioned in the preceding paragraph. Our approach enables us to simultaneously cover two important classes of spaces, namely, the 01-fuzzy  $b$ -metric spaces and the stationary fuzzy ( $b$ )-metric spaces. We show that each fuzzy  $b$ -metric space, in our sense, admits a uniformity with a countable base, and deduce from this fact that the topology induced by means of its “open” balls is metrizable as well as a fuzzy  $b$ -metric counterpart of a renowned and classical characterization of compact metric spaces due to Niemytzki and Tychonoff [28]. We also characterize complete fuzzy  $b$ -metric spaces by means of a fixed point theorem that is proved here. Various examples illustrating our point of view are included.

## 2. On the Notion of Fuzzy $b$ -Metric Space: Remarks and Examples

We begin this section by recalling the following important concepts.

A binary operation  $*$  :  $[0, 1] \times [0, 1] \rightarrow [0, 1]$  is a continuous triangular norm provided that it verifies the next conditions: (i)  $*$  is associative and commutative; (ii)  $*$  is continuous; (iii)  $u * 1 = u$  for all  $u \in [0, 1]$ ; (iv)  $u * w \leq v * w$  if  $u \leq v$ , where  $u, v, w \in [0, 1]$ .

The book [29] constitutes an excellent source for the study of continuous triangular norms.

We remind that  $* \leq \wedge$  for any continuous triangular norm  $*$ , where  $\wedge$  designates the continuous triangular norm defined as  $u \wedge v = \min\{u, v\}$ .

To help the reader understand the different notions of fuzzy  $b$ -metric space and other related ones that are of interest in our context, we consider the next conditions for a set  $\mathfrak{X}$ , a fuzzy set  $\mathfrak{F}$  in  $\mathfrak{X} \times \mathfrak{X} \times \mathbb{R}^+$ , a continuous triangular norm  $*$ , a real constant with  $K \geq 1$ , and  $x, y, z \in \mathfrak{X}$ :

- (FbM1)  $\mathfrak{F}(x, y, 0) = 0$ ;
- (FbM1')  $\mathfrak{F}(x, y, t) > 0$  for all  $t > 0$ ;
- (FbM2)  $x = y$  if and only if  $\mathfrak{F}(x, y, t) = 1$  for all  $t > 0$ ;
- (FbM2')  $\mathfrak{F}(x, x, t) = 1$  for all  $t > 0$ , and  $\mathfrak{F}(x, y, t) < 1$  whenever  $y \neq x$ ;
- (FbM3)  $\mathfrak{F}(x, y, t) = \mathfrak{F}(y, x, t)$  for all  $t > 0$ ;
- (FbM4)  $\mathfrak{F}(x, z, K(t + s)) \geq \mathfrak{F}(x, y, t) * \mathfrak{F}(y, z, s)$  for all  $t, s \geq 0$ ;
- (FbM5) the function  $\mathfrak{F}(x, y, \cdot) : \mathbb{R}^+ \rightarrow [0, 1]$  is left continuous.
- (FbM5') the function  $\mathfrak{F}(x, y, \cdot) : \mathbb{R}^+ \rightarrow [0, 1]$  is continuous.
- (FbM6)  $\lim_{t \rightarrow +\infty} \mathfrak{F}(x, y, t) = 1$ .

If conditions (FbM1'), (FbM2'), (FbM3), (FbM4), and (FbM5') are satisfied, the 4-tuple  $(\mathfrak{X}, \mathfrak{F}, *, K)$  was called, by Sedghi and Shobe a  $b$ -fuzzy metric space, by Saadati [15], a fuzzy metric type space and, by Šostak [16], a fuzzy  $k$ -metric space. Observe that when  $K = 1$ , one has the notion of a fuzzy metric space in the sense of [17,18].

If conditions (FbMb1'), (FbM2'), (FbM3), (FbM4), and (FbM5) are satisfied, the 4-tuple  $(\mathfrak{X}, \mathfrak{F}, *, K)$  was called, by Hussain et al. [13], a fuzzy  $b$ -metric space.

Note that in the preceding notions, the domain of  $\mathfrak{F}$  is the set  $\mathfrak{X} \times \mathfrak{X} \times (\mathbb{R}^+ \setminus \{0\})$ . Nevertheless, and without loss of generality, we can extend its domain to  $\mathfrak{X} \times \mathfrak{X} \times \mathbb{R}^+$  and define  $\mathfrak{F}(x, y, 0) = 0$  for all  $x, y \in \mathfrak{X}$ .

In [12], Nădăban defined a fuzzy  $b$ -metric space as 4-tuple  $(\mathfrak{X}, \mathfrak{F}, *, K)$  for which conditions (FbMb1), (FbM2), (FbM3), (FbM4), (FbM5) and (FbM6) are satisfied (Zhong and Šostak [19] also called fuzzy  $k$ -metric spaces to these spaces). Observe that when  $K = 1$  one has the notion of a fuzzy metric space in the sense of [11].

Now, we give the following.

**Definition 1.** A fuzzy  $b$ -metric on a set  $\mathfrak{X}$  is a 3-tuple  $(\mathfrak{F}, *, K)$ , where  $\mathfrak{F}$  is a fuzzy set in  $\mathfrak{X} \times \mathfrak{X} \times \mathbb{R}^+$ ,  $*$  is a continuous triangular norm and  $K$  is a real constant with  $K \geq 1$ , fulfilling conditions (FbM1), (FbM2), (FbM3), (FbM4) and (FbM5) above.

A fuzzy  $b$ -metric space is a 4-tuple  $(\mathfrak{X}, \mathfrak{F}, *, K)$  such that  $\mathfrak{X}$  is a set and  $(\mathfrak{F}, *, K)$  is a fuzzy  $b$ -metric on  $\mathfrak{X}$ .

Note that a fuzzy  $b$ -metric space is a fuzzy metric space, in the sense of Gabriel [27], when  $k = 1$ . Moreover, every fuzzy metric space  $(\mathfrak{X}, \mathfrak{F}, *)$ , in the sense of [27], can be considered to be a fuzzy  $b$ -metric space, in the sense of Definition 1, for any  $K \geq 1$ , because, by [27] (Lemma 4),  $\mathfrak{F}(x, y, K(t + s)) \geq \mathfrak{F}(x, y, t + s)$  for all  $x, y \in \mathfrak{X}$  and all  $t, s > 0$ .

It is clear that the notion proposed in Definition 1 generalizes the ones given in [12–16,19]. Examples 1 and 3 below show that, in fact, it provides a real generalization of such notions.

In the rest of this paper, the terms fuzzy  $b$ -metric and fuzzy  $b$ -metric space will be used in the sense of Definition 1, while the terms fuzzy metric and fuzzy metric space will be used in the sense of [27].

The next observations and properties, due to Nădăban [12] (see also [15,16]), remain valid for fuzzy  $b$ -metric spaces (indeed, condition (FbM6) is not necessary for showing that facts).

**Remark 2.** Let  $(\mathfrak{X}, \mathfrak{F}, *, K)$  be a fuzzy  $b$ -metric space and let  $t > s \geq 0$ . Then,  $\mathfrak{F}(x, y, Kt) \geq \mathfrak{F}(x, y, s)$  for all  $x, y \in \mathfrak{X}$ .

Let  $(\mathfrak{X}, \mathfrak{F}, *, K)$  be a fuzzy  $b$ -metric space. For each  $x \in \mathfrak{X}$ ,  $\varepsilon \in (0, 1)$  and  $t > 0$ , define

$$B_{\mathfrak{F}}(x, \varepsilon, t) := \{y \in X : \mathfrak{F}(x, y, t) > 1 - \varepsilon\}.$$

Now, set

$$\tau_{\mathfrak{F}} = \{\mathfrak{O} \subseteq \mathfrak{X} : \text{for each } x \in \mathfrak{O} \text{ there are } \varepsilon_x \in (0, 1), t > 0, \text{ such that } B_{\mathfrak{F}}(x, \varepsilon_x, t) \subseteq \mathfrak{O}\}.$$

Then,  $\tau_{\mathfrak{F}}$  is a topology on  $\mathfrak{X}$ , and, analogously to  $b$ -metric spaces, the “open” balls  $B_{\mathfrak{F}}(x, \varepsilon, t)$  are not necessarily  $\tau_{\mathfrak{F}}$ -open sets. In this direction, Example 5 below provides a method to easily construct fuzzy  $b$ -metric spaces for which there exist “open” balls that are not open sets.

From the construction of  $\tau_{\mathfrak{F}}$  it follows that a sequence  $(x_n)_{n \in \mathbb{N}}$  in  $\mathfrak{X}$  is  $\tau_{\mathfrak{F}}$ -convergent to  $x \in \mathfrak{X}$  if and only if  $\lim_{n \rightarrow +\infty} \mathfrak{F}(x, x_n, t) = 1$  for all  $t > 0$ .

A sequence  $(x_n)_{n \in \mathbb{N}}$  in  $\mathfrak{X}$  is called a Cauchy sequence if for each  $\varepsilon \in (0, 1)$  and  $t > 0$  there exists an  $n_{\varepsilon, t} \in \mathbb{N}$  such that  $\mathfrak{F}(x_n, x_m, t) > 1 - \varepsilon$  for all  $n, m \geq n_{\varepsilon, t}$ , and  $(\mathfrak{X}, \mathfrak{F}, *, K)$  is called complete if every Cauchy sequence is  $\tau_{\mathfrak{F}}$ -convergent.

Now, we shall give several examples and observations.

**Example 1** (compare [19]). Given a  $b$ -metric space  $(\mathfrak{X}, d, K)$  denoted by  $\mathfrak{F}_{d,01}$  the fuzzy set in  $\mathfrak{X} \times \mathfrak{X} \times \mathbb{R}^+$  defined as  $\mathfrak{F}_{d,01}(x, y, t) = 1$  if  $d(x, y) < t$ , and  $\mathfrak{F}_{d,01}(x, y, t) = 0$  if  $d(x, y) \geq t$ . Then,  $(\mathfrak{F}_{d,01}, *, K)$  is a fuzzy  $b$ -metric on  $\mathfrak{X}$  for any continuous triangular norm  $*$ , which we call a 01-fuzzy  $b$ -metric, and the 4-tuple  $(\mathfrak{X}, \mathfrak{F}, *, K)$ , a 01-fuzzy  $b$ -metric space. Furthermore, the topology  $\tau_{\mathfrak{F}_{d,01}}$  agrees with the topology  $\tau_d$  induced by  $d$ , i.e.,  $\tau_{\mathfrak{F}_{d,01}} = \tau_d$ . We also have that  $(\mathfrak{X}, \mathfrak{F}_{d,01}, *, K)$  is complete if and only if  $(\mathfrak{X}, d, K)$  is complete.

Note that this type of space is not covered by the notions established in [13–16] because such spaces do not verify condition (FbM1') whenever  $|\mathfrak{X}| > 1$ .

**Example 2** (compare [12,15,16]). Let  $(d, K)$  be a  $b$ -metric on a set  $\mathfrak{X}$ . Define a fuzzy set  $\mathfrak{F}_d$  in  $\mathfrak{X} \times \mathfrak{X} \times \mathbb{R}^+$  as  $\mathfrak{F}_d(x, y, 0) = 0$  for all  $x, y \in \mathfrak{X}$ ,  $\mathfrak{F}_d(x, x, t) = 1$  for all  $x \in \mathfrak{X}$  and all  $t > 0$ , and  $\mathfrak{F}_d(x, y, t) = t / (t + d(x, y))$  otherwise. Then,  $(\mathfrak{F}_d, *, K)$  is a fuzzy  $b$ -metric for any continuous triangular norm  $*$ , named the standard fuzzy  $b$ -metric. In addition,  $\tau_{\mathfrak{F}_d} = \tau_d$ , and  $(\mathfrak{X}, \mathfrak{F}_d, *, K)$  is complete if and only if  $(\mathfrak{X}, d, K)$  is complete.

It is well known [15,16] that, by deleting condition (FbM1), then,  $(\mathfrak{X}, \mathfrak{F}_d, *, K)$  is a  $b$ -fuzzy metric space, i.e., a fuzzy metric type space, i.e., a fuzzy  $k$ -metric space.

In one paper [30], the notion of a stationary fuzzy metric was introduced: A fuzzy metric  $(\mathfrak{F}, *)$  on a set  $\mathfrak{X}$  is stationary provided that there is a function  $\varphi : \mathfrak{X} \times \mathfrak{X} \rightarrow [0, 1]$  such that  $\mathfrak{F}(x, y, t) = \varphi(x, y)$  for all  $x, y \in \mathfrak{X}$  and all  $t > 0$ .

**Example 3** (compare [31]). Let  $\alpha \in \mathbb{R}^+$  be a constant and let  $\mathfrak{X} = \mathbb{R}^+$ . Denote by  $*_P$  the product continuous triangular norm; thus,  $u *_P v = uv$  for all  $u, v \in [0, 1]$ . Then, the pair  $(\mathfrak{F}, *_P)$  is a stationary fuzzy metric on  $\mathfrak{X}$ , where the fuzzy set  $\mathfrak{F}$  is defined as  $\mathfrak{F}(x, y, 0) = 0$  for all  $x, y \in \mathfrak{X}$ ,  $\mathfrak{F}(x, x, t) = 1$  for all  $x \in \mathfrak{X}$  and all  $t > 0$ , and

$$\mathfrak{F}(x, y, t) = \frac{\min\{x, y\} + \alpha}{\max\{x, y\} + \alpha},$$

otherwise (see [31] (Example 3 (C))).

Furthermore,  $(\mathfrak{X}, \mathfrak{F}, *_P)$  is complete when  $\alpha > 0$ , and it is not complete for  $\alpha = 0$  (indeed, notice that  $(1/n)_{n \in \mathbb{N}}$  is a non- $\tau_{\mathfrak{F}}$ -convergent Cauchy sequence).

Note that this type of space is not covered by the notions established in [12,19] because such spaces do not verify condition (FbM6) whenever  $|\mathfrak{X}| > 1$ .

It is interesting to emphasize that the kind of stationary fuzzy (*b*-)metrics given in Example 3 constitutes the substrate to constructing suitable fuzzy metrics that are successfully applied to questions concerning color image processing (see, e.g., [31] (Section 4).

Observe that if we define the concept of a stationary fuzzy *b*-metric in a natural way, we obviously have that every stationary fuzzy *b*-metric is a stationary fuzzy metric. For this reason, we are going to construct a type of fuzzy *b*-metrics that is stationary only when one has a fuzzy metric.

We say that a fuzzy *b*-metric  $(\mathfrak{F}, *)$  on a set  $\mathfrak{X}$  is *K*-stationary if there is a *b*-metric  $(d, K)$  on  $\mathfrak{X}$  and a function  $\varphi : \mathfrak{X} \times \mathfrak{X} \rightarrow [0, 1]$  such that

$$\mathfrak{F}(x, y, t) = \exp\left(-\frac{(K-1)d(x, y)}{t}\right) \varphi(x, y),$$

for all  $x, y \in \mathfrak{X}$  and all  $t > 0$ .

Notice that if  $K = 1$ ,  $(\mathfrak{F}, *)$  is a stationary fuzzy metric on  $\mathfrak{X}$ .

**Example 4.** Let  $\mathfrak{X} = \mathbb{R}^+$  and let  $(d, K)$  be any *b*-metric on  $\mathfrak{X}$  (for instance, and according to Remark 1, define  $d(x, y) = |x - y|^r$  for all  $x, y \in \mathfrak{X}$ , with  $r > 1$  and  $K = 2^{r-1}$ ). Then, the pair  $(d, K)$  is a *b*-metric on  $\mathfrak{X}$  such that  $d$  is not a metric).

Now, define a fuzzy set  $\mathfrak{F}$  in  $\mathfrak{X} \times \mathfrak{X} \times \mathbb{R}^+$  as  $\mathfrak{F}(x, y, 0) = 0$  for all  $x, y \in \mathfrak{X}$ ,  $\mathfrak{F}(x, x, t) = 1$  for all  $x \in \mathfrak{X}$  and all  $t > 0$ , and

$$\mathfrak{F}(x, y, t) = \exp\left(-\frac{(K-1)d(x, y)}{t}\right) \frac{\min\{x, y\} + \alpha}{\max\{x, y\} + \alpha},$$

otherwise ( $\alpha \in \mathbb{R}^+$  is a constant).

We show that  $(\mathfrak{F}, *_p, K)$  is a *K*-stationary fuzzy *b*-metric on  $\mathfrak{X}$ . To this end, it suffices to check condition (FbM4) because the rest of the conditions of Definition 1 are obvious.

Indeed, Saadati proved in [15] (Example 1.5) that the 3-tuple  $(\mathfrak{G}, *_p, K)$  is a fuzzy *b*-metric on  $\mathfrak{X}$  where  $\mathfrak{G}(x, y, 0) = 0$  for all  $x, y \in \mathfrak{X}$ ,  $\mathfrak{G}(x, x, t) = 1$  for all  $x \in \mathfrak{X}$  and all  $t > 0$ , and

$$\mathfrak{G}(x, y, t) = \exp\left(-\frac{(K-1)d(x, y)}{t}\right),$$

otherwise. Taking also into account Example 3, we obtain

$$\begin{aligned} \mathfrak{F}(x, y, K(t+s)) &= \mathfrak{G}(x, y, K(t+s)) \frac{\min\{x, y\} + \alpha}{\max\{x, y\} + \alpha} \\ &\geq \mathfrak{G}(x, z, t) \mathfrak{G}(z, y, s) \frac{\min\{x, z\} + \alpha}{\max\{x, z\} + \alpha} \frac{\min\{z, y\} + \alpha}{\max\{z, y\} + \alpha} \\ &= \mathfrak{F}(x, z, t) \mathfrak{F}(z, y, s). \end{aligned}$$

We point out that for any  $x, y \in \mathfrak{X}$ , with  $x \neq y$ , one has  $\lim_{t \rightarrow +\infty} \mathfrak{F}(x, y, t) < 1$ , so, condition (FbM6) is not satisfied, in general.

Zhong and Šostak gave in [19] an example of a fuzzy *b*-metric space that is not a fuzzy metric space. Next, we offer a method to construct fuzzy *b*-metric spaces that are not fuzzy metric spaces.

**Proposition 1.** Let  $(\mathfrak{X}, d, K)$  be a *b*-metric space such that  $(\mathfrak{X}, d)$  is not a metric space. Then, the fuzzy *b*-metric space  $(\mathfrak{X}, \mathfrak{F}_{d,01}, *, K)$  of Example 1 is not a fuzzy metric space.

**Proof.** Since  $(\mathfrak{X}, d)$  is not a metric space, there exists  $x, y, z \in \mathfrak{X}$  and  $\varepsilon > 0$  such that

$$d(x, y) > 2\varepsilon + d(x, z) + d(z, y).$$

Put  $t = d(x, z) + \varepsilon$  and  $s = d(z, y) + \varepsilon$ . Then,  $\mathfrak{F}_{d,01}(x, z, t) = 1$  and  $\mathfrak{F}_{d,01}(z, y, s) = 1$ . So,

$$\mathfrak{F}_{d,01}(x, z, t) * \mathfrak{F}_{d,01}(z, y, s) = 1.$$

However,  $\mathfrak{F}_{d,01}(x, y, t + s) = 0$  because  $d(x, y) > t + s$ .  $\square$

**Example 5.** Let  $(\mathfrak{X}, d, K)$  be a  $b$ -metric space for which there is an “open” ball  $B_d(x, r)$  which is not a  $\tau_d$ -open set. Then, there exists a sequence  $(x_n)_{n \in \mathbb{N}}$  whose elements are in  $\mathfrak{X} \setminus B_d(x, r)$  and a  $y \in B_d(x, r)$  such that  $d(y, x_n) \rightarrow 0$  as  $n \rightarrow +\infty$ .

(A) Let  $(\mathfrak{X}, \mathfrak{F}_{d,01}, *, K)$  be the fuzzy metric space of Example 1. Fix an  $\varepsilon \in (0, 1)$ . We show that  $B_{\mathfrak{F}_{d,01}}(x, \varepsilon, r)$  is not a  $\tau_{\mathfrak{F}_{d,01}}$ -open set.

Indeed, since  $d(x, y) < r$  we obtain  $\mathfrak{F}_{d,01}(x, y, r) = 1$ , so  $y \in B_{\mathfrak{F}_{d,01}}(x, \varepsilon, r)$ .

On the other hand, the elements of the sequence  $(x_n)_{n \in \mathbb{N}}$  are in  $\mathfrak{X} \setminus B_{\mathfrak{F}_{d,01}}(x, \varepsilon, r)$  because  $\mathfrak{F}_{d,01}(x, x_n, r) = 0$  for all  $n \in \mathbb{N}$ .

Finally, since  $\tau_d = \tau_{\mathfrak{F}_{d,01}}$  (see Example 1), we obtain that the sequence  $(x_n)_{n \in \mathbb{N}}$  is  $\tau_{\mathfrak{F}_{d,01}}$ -convergent to  $y$ . Consequently,  $B_{\mathfrak{F}_{d,01}}(x, \varepsilon, r)$  is not a  $\tau_{\mathfrak{F}_{d,01}}$ -open set.

(B) Let  $(\mathfrak{X}, \mathfrak{F}_d, *, K)$  be the  $b$ -fuzzy metric space of Example 2. Then, we obtain

$$\mathfrak{F}_d(x, y, r) = \frac{r}{r + d(x, y)} > \frac{r}{2r} = \frac{1}{2},$$

and

$$\mathfrak{F}_d(x, x_n, r) = \frac{r}{r + d(x, x_n)} \leq \frac{r}{2r} = \frac{1}{2},$$

for all  $n \in \mathbb{N}$ , which implies that  $y \in B_{\mathfrak{F}_d}(x, 1/2, r)$  but  $x_n \in \mathfrak{X} \setminus B_{\mathfrak{F}_d}(x, 1/2, r)$ . Since  $\tau_d = \tau_{\mathfrak{F}_d}$  we obtain that the sequence  $(x_n)_{n \in \mathbb{N}}$  is  $\tau_{\mathfrak{F}_d}$ -convergent to  $y$ . So,  $\mathfrak{X} \setminus B_{\mathfrak{F}_d}(x, 1/2, r)$  is not  $\tau_{\mathfrak{F}_d}$ -closed and, consequently, the “open” ball  $B_{\mathfrak{F}_d}(x, 1/2, r)$  is not  $\tau_{\mathfrak{F}_d}$ -open.

In connection with the preceding example, we remind the reader that Kirk and Shahzad [9] introduced the notion of a strong  $b$ -metric space to obtain a class of  $b$ -metric spaces for which every “open” ball is an open set. Let us recall that a  $b$ -metric space  $(\mathfrak{X}, d, K)$  is a strong  $b$ -metric space if it fulfills conditions (bM1) and (bM2) above, and condition (bM3) is replaced with the following one:

$$d(x, y) \leq d(x, z) + Kd(z, y), \text{ for all } x, y, z \in \mathfrak{X}.$$

Extending the idea of Kirk and Shahzad to the fuzzy framework, Öner introduced and examined in [32] the concept of a fuzzy strong  $b$ -metric space. A  $b$ -fuzzy metric space (equivalently, a fuzzy metric type space, and a fuzzy  $k$ -metric space)  $(\mathfrak{X}, \mathfrak{F}, *, K)$  is said to be a fuzzy strong  $b$ -metric space if it satisfies conditions (FbM1’), (FbM2’), (FbM3), (FbM5’), and condition (FbM4) is replaced with the following stronger one:

$$\mathfrak{F}(x, z, t + Ks) \geq \mathfrak{F}(x, y, t) * \mathfrak{F}(y, z, s) \text{ for all } x, y, z \in \mathfrak{X} \text{ and all } t, s \geq 0.$$

Then, Öner showed that every “open” ball in a fuzzy strong  $b$ -metric space is an open set. Since then, fuzzy strong  $b$ -metric spaces have received the attention of some researchers (see, e.g., [33–35]).

**Remark 3.** In a recent result [24] (Theorem 2.1), the authors claimed that every “open” ball in a  $b$ -fuzzy metric space  $(\mathfrak{X}, \mathfrak{F}, *, K)$  is a  $\tau_{\mathfrak{F}}$ -open set. Example 5 (B) above shows that, unfortunately, this result is not correct (the error occurs because, in the proof of that theorem, the inequality  $t_0/b < t$  does not imply, in general, that  $t > t_0$ , so  $B(y, 1 - r_1, (t - t_0)/r)$  may be undefined).

### 3. Uniform Properties and Metrizability of Fuzzy $b$ -Metric Spaces

We begin by establishing the following version of the famous Kelley’s metrization Lemma [1] (p. 185), which will be a key to proving the main result of this section.

**Lemma 1.** Let  $\mathfrak{U}$  be a uniformity on a set  $\mathfrak{X}$  and let  $\mathcal{V} = \{V_n : n \in \mathbb{N}\}$  be a countable family of symmetric members of  $\mathfrak{U}$  such that  $\Delta = \bigcap_{n \in \mathbb{N}} V_n$  and  $V_{n+1}^2 \subseteq V_n$  for all  $n \in \mathbb{N}$ . Then, there exists a metric  $d$  on  $X$  such that

$$V_{n+1} \subseteq \{(x, y) \in \mathfrak{X} \times \mathfrak{X} : d(x, y) < 2^{-n}\} \subseteq V_n,$$

for all  $n \in \mathbb{N}$ .

(As usual, by  $\Delta$  we have indicated the diagonal of  $X \times X$ ).

We also need the next auxiliary result.

**Proposition 2.** Let  $(\mathfrak{X}, \mathfrak{F}, *, K)$  be a fuzzy  $b$ -metric space. For any  $x, y \in \mathfrak{X}$  the following are equivalent:

- (i)  $\mathfrak{F}(x, y, t) > 1 - t$  for all  $t > 0$ .
- (ii)  $\mathfrak{F}(x, y, 1/n) > 1 - 1/n$  for all  $n \in \mathbb{N}$ .
- (iii)  $x = y$ .

**Proof.** (i) $\Rightarrow$ (ii) is obvious and (iii) $\Rightarrow$ (i) follows from Definition 1, (FbM2).

(ii) $\Rightarrow$ (iii) Let  $x, y \in \mathfrak{X}$  be such that  $\mathfrak{F}(x, y, 1/n) > 1 - 1/n$  for all  $n \in \mathbb{N}$ . Fix  $t > 0$ . Choose an  $n_0 \in \mathbb{N}$  verifying  $n_0 t > K$ . Then, for every  $n \geq n_0$  we obtain  $t/K > 1/n$ , so, by Remark 2,

$$\mathfrak{F}(x, y, t) = \mathfrak{F}(x, y, K \frac{t}{K}) \geq \mathfrak{F}(x, y, \frac{1}{n}) > 1 - \frac{1}{n},$$

for all  $n \geq n_0$ , which implies that  $\mathfrak{F}(x, y, t) = 1$ . Since  $t$  is arbitrary we deduce that  $x = y$ , by Definition 1, (FbM2).  $\square$

**Theorem 1.** Let  $(\mathfrak{X}, \mathfrak{F}, *, b)$  be a fuzzy  $b$ -metric space. For each  $n \in \mathbb{N}$  put

$$U_n = \{(x, y) \in \mathfrak{X} \times \mathfrak{X} : \mathfrak{F}(x, y, 1/n) > 1 - 1/n\}.$$

Then, the following statements hold:

(st1) The family

$$\mathfrak{U}_{\mathfrak{F}} = \{U \subseteq \mathfrak{X} \times \mathfrak{X} : \text{there is } n \in \mathbb{N} \text{ with } U_n \subseteq U\},$$

is a uniformity on  $\mathfrak{X}$  for which the countable family  $\mathfrak{B}_{\mathfrak{F}} = \{U_n : n \in \mathbb{N}\}$  is a base.

(st2) There is a metric  $d_{\mathfrak{F}}$  on  $\mathfrak{X}$  whose induced uniformity coincides with  $\mathfrak{U}_{\mathfrak{F}}$ .

(st3)  $\tau_{d_{\mathfrak{F}}} = \tau_{\mathfrak{F}}$  on  $\mathfrak{X}$ .

(st4)  $(\mathfrak{X}, \tau_{\mathfrak{F}})$  is a metrizable topological space.

(st5)  $(\mathfrak{X}, \mathfrak{F}, *, K)$  is complete if and only if  $(\mathfrak{X}, d_{\mathfrak{F}})$  is so.

**Proof.** (st1): We first show that  $\mathfrak{B}_{\mathfrak{F}}$  is a (countable) base for uniformity on  $\mathfrak{X}$ .

We have  $\bigcap_{n \in \mathbb{N}} U_n = \Delta$  because if  $(x, y) \in \bigcap_{n \in \mathbb{N}} U_n$ , we obtain  $\mathfrak{F}(x, y, 1/n) > 1 - 1/n$  for all  $n \in \mathbb{N}$ , so  $x = y$  by Proposition 2. Furthermore, it is obvious that  $U_n^{-1} = U_n$  for all  $n \in \mathbb{N}$ , so each  $U_n$  is symmetric.

Next, we show that for each  $n, m \in \mathbb{N}$  there is  $p \in \mathbb{N}$  such that  $U_p^2 \subseteq U_n \cap U_m$ .

Indeed, suppose without loss of generality that  $m \geq n$ . By the continuity of  $*$ , there is  $p \in \mathbb{N}$  such that

$$p > 2K^2m \quad \text{and} \quad (1 - 1/p) * (1 - 1/p) > 1 - 1/m.$$

Let  $(x, y) \in U_p^2$ . Then, there is  $z \in X$  such that  $(x, z) \in U_p$  and  $(z, y) \in U_p$ .

So, from Remark 2 (note that  $1/Kn \geq 1/Km > 2K/p$ ) and Definition 1, (FbM4), we deduce that

$$\begin{aligned} \mathfrak{F}(x, y, \frac{1}{n}) &= \mathfrak{F}(x, y, K\frac{1}{Kn}) \geq \mathfrak{F}(x, y, \frac{2K}{p}) \\ &\geq \mathfrak{F}(x, z, \frac{1}{p}) * \mathfrak{F}(z, y, \frac{1}{p}) \geq (1 - \frac{1}{p}) * (1 - \frac{1}{p}) \\ &> 1 - \frac{1}{m} \geq 1 - \frac{1}{n}, \end{aligned}$$

and, similarly,

$$\mathfrak{F}(x, y, \frac{1}{m}) > 1 - \frac{1}{m},$$

which implies that  $(x, y) \in U_n \cap U_m$ .

We conclude that  $\mathfrak{U}_{\mathfrak{F}}$  is a uniformity on  $\mathfrak{X}$  for which the (countable) family  $\mathfrak{B}_{\mathfrak{F}}$  is a base.

(st2): In the statement (st1) we have shown that each member  $U_n$  of the (countable) base  $\mathfrak{B}_{\mathfrak{F}}$  is symmetric and verifies that  $U_{n+1}^2 \subseteq U_n$ . Furthermore,  $\Delta = \bigcap_{n \in \mathbb{N}} U_n$ . Hence, all conditions of Lemma 1 are satisfied, so there exists a metric  $d_{\mathfrak{F}}$  on  $\mathfrak{X}$  such that

$$U_{n+1} \subseteq \{(x, y) \in \mathfrak{X} \times \mathfrak{X} : d_{\mathfrak{F}}(x, y) < 2^{-n}\} \subseteq U_n, \tag{1}$$

for all  $n \in \mathbb{N}$ , which implies that the uniformities  $\mathfrak{U}_{d_{\mathfrak{F}}}$  and  $\mathfrak{U}_{\mathfrak{F}}$  coincide.

(st3) It is a trivial consequence of (st2), and (st4) is a trivial consequence of (st3).

(st5): From the inclusion relations (1) obtained in statement (st2), it follows that a sequence in  $\mathfrak{X}$  is a Cauchy sequence in  $(\mathfrak{X}, \mathfrak{F}, *, K)$  if and only if it is a Cauchy sequence in  $(X, d_{\mathfrak{F}})$ , so, by the statement (st3), we deduce that  $(\mathfrak{X}, \mathfrak{F}, *, K)$  is complete if and only if  $(X, d_{\mathfrak{F}})$  is so.  $\square$

**Remark 4.** Using a technique similar to the employed one in the proof of (st1) above, Gregori and Romaguera showed in [36] that every fuzzy metric space (in the sense of George and Veeramani) is metrizable, and Öner and Sostak showed in [34] that every fuzzy strong  $b$ -metric space is metrizable.

It follows from Theorem 1 that several classical theorems of the theory of metric spaces can be easily translated to the fuzzy  $b$ -metric framework.

For instance, Niemytzki and Tychonoff proved, in [28], their famous theorem that a metrizable topological space  $(\mathfrak{X}, \tau)$  is compact if and only if every metric  $d$  on  $\mathfrak{X}$  such that  $\tau_d = \tau$  is complete.

We conclude this section by obtaining the following fuzzy  $b$ -metric version of Niemytzki-Tychonoff’s theorem.

**Theorem 2.** Let  $(\mathfrak{X}, \mathfrak{F}, *, K)$  be a fuzzy  $b$ -metric space. Then, the following statements are equivalent:

- (1) The topological space  $(\mathfrak{X}, \tau_{\mathfrak{F}})$  is compact.
- (2) Every fuzzy  $b$ -metric  $(\mathfrak{G}, *,', K')$  on  $\mathfrak{X}$  such that  $\tau_{\mathfrak{G}} = \tau_{\mathfrak{F}}$  is complete.
- (3) Every fuzzy  $b$ -metric  $(\mathfrak{G}, *, K)$  on  $\mathfrak{X}$  such that  $\tau_{\mathfrak{G}} = \tau_{\mathfrak{F}}$  is complete.

**Proof.** (1)  $\Rightarrow$  (2) From our assumption and Theorem 1, (st3), it follows that  $(\mathfrak{X}, \tau_{\mathfrak{F}})$  is a compact metrizable space. So, by the Niemytzki-Alexandorff theorem, every metric  $d$  on  $\mathfrak{X}$  such that  $\tau_d = \tau_{\mathfrak{F}}$  is complete. Let  $(\mathfrak{G}, *,', K')$  be a fuzzy  $b$ -metric on  $\mathfrak{X}$  such that  $\tau_{\mathfrak{G}} = \tau_{\mathfrak{F}}$ . By Theorem 1, there is a metric  $e$  on  $\mathfrak{X}$  fulfilling, in particular, (st2) and (st5) with respect to  $(\mathfrak{X}, \mathfrak{G}, *,', K')$ . Hence, by (st2),  $\tau_e = \tau_{\mathfrak{G}}$ , and, thus,  $\tau_e = \tau_{\mathfrak{F}}$ . Moreover, by (st5),  $(\mathfrak{G}, *,', K')$  is complete because  $e$  is so.

(2)  $\Rightarrow$  (3) Obvious.

(3)  $\Rightarrow$  (1) Let  $d$  be a metric on  $\mathfrak{X}$  such that  $\tau_d = \tau_{\mathfrak{F}}$ . By Example 1, the 3-tuple  $(\mathfrak{F}_{d,01}, *, K)$  is a fuzzy  $b$ -metric on  $\mathfrak{X}$  such that  $\tau_{\mathfrak{F}_{d,01}} = \tau_d$ . Hence,  $\tau_{\mathfrak{F}} = \tau_{\mathfrak{F}_{d,01}}$ . Since,

by assumption,  $(\mathfrak{F}_{d,01}, *, K)$  is complete, we deduce that  $d$  is complete. By applying the Niemytzki-Tychonoff theorem we conclude that  $(\mathfrak{X}, \tau_{\mathfrak{F}})$  is compact.  $\square$

#### 4. A Characterization of Complete Fuzzy $b$ -Metric Spaces and Further Examples

Hicks [37] introduced and discussed a relevant type of contractive mappings under the name of  $C$ -contractions (see also [38]). In one paper [39], the usefulness of contractions of the Hicks' type to characterize the completeness of fuzzy metric spaces was shown. We extend and adapt Hicks' notion to the fuzzy  $b$ -metric framework as follows.

**Definition 2.** Let  $(\mathfrak{X}, \mathfrak{F}, *, K)$  be a fuzzy  $b$ -metric space. A self-map  $\mathcal{T}$  of  $\mathfrak{X}$  is said to be a Hicks contraction (on  $\mathfrak{X}$ ) if there is a constant  $c \in (0, 1)$  for which the following contraction condition holds, for any  $x, y \in \mathfrak{X}$ , and  $t > 0$ ,

$$\mathfrak{F}(x, y, t) > 1 - t \implies \mathfrak{F}(\mathcal{T}x, \mathcal{T}y, ct) > 1 - ct. \tag{2}$$

If  $\mathcal{C}$  is a subset of  $\mathfrak{X}$ , the notion of Hicks contraction on  $\mathcal{C}$  is defined in the obvious fashion, by considering the restriction of  $\mathfrak{F}$  to  $\mathcal{C} \times \mathcal{C} \times \mathbb{R}^+$ .

In Theorem 3 below, we obtain a characterization of complete fuzzy  $b$ -metric spaces by means of Hicks contractions having a fixed point. This result, which generalizes [39] (Theorem 4), may be considered a fuzzy alternative for the characterization of complete  $b$ -metric spaces obtained in [7] (Theorem 2).

**Theorem 3.** Let  $(\mathfrak{X}, \mathfrak{F}, *, K)$  be a fuzzy  $b$ -metric space. Then, the following statements are equivalent:

- (1)  $(\mathfrak{X}, \mathfrak{F}, *, K)$  is complete.
- (2) Every Hicks contraction on any of the closed subsets of  $(\mathfrak{X}, \mathfrak{F}, *, K)$  has a (unique) fixed point.

**Proof.** (1)  $\implies$  (2) The first part of the proof of this implication uses *mutatis mutandis* a bright idea from Radu [40] (see also the proof of [7] (Theorem 2)), so some details are omitted.

Let  $\mathcal{T}$  be a Hicks contraction on a  $\tau_{\mathfrak{F}}$ -closed subset  $\mathcal{C}$  of  $\mathfrak{X}$ , with constant  $c \in (0, 1)$ .

Fix  $t_0 > 1$ . Then, for any  $x, y \in \mathcal{C}$ ,  $\mathfrak{F}(x, y, t_0) > 1 - t_0$ , so  $\mathfrak{F}(\mathcal{T}x, \mathcal{T}y, ct_0) > 1 - ct_0$ . Repeating this process we obtain

$$\mathfrak{F}(\mathcal{T}^n x, \mathcal{T}^n y, c^n t_0) > 1 - c^n t_0, \tag{3}$$

for all  $x, y \in \mathcal{C}$  and all  $n \in \mathbb{N} \cup \{0\}$ .

Now, choose any  $x \in \mathcal{C}$ . We show that  $(\mathcal{T}^n x)_{n \in \mathbb{N}}$  is a Cauchy sequence in  $(\mathfrak{X}, \mathfrak{F}, *, K)$ . Indeed, given  $\varepsilon \in (0, 1)$ , take  $n(\varepsilon) \in \mathbb{N}$  such that  $c^{n(\varepsilon)} t_0 < \varepsilon$ . Let  $m > n > n(\varepsilon)$ . Thus,  $m = n + j$  for some  $j \in \mathbb{N}$ , and putting  $y = \mathcal{T}^j x$ , the inequality (3) provides

$$\mathfrak{F}(\mathcal{T}^n x, \mathcal{T}^m x, c^n t_0) > 1 - c^n t_0 > 1 - \varepsilon.$$

Therefore, there exists  $z \in \mathfrak{X}$  such that  $(x_n)_{n \in \mathbb{N}}$   $\tau_{\mathfrak{F}}$ -converges to  $z$ . Since  $\mathcal{C}$  is  $\tau_{\mathfrak{F}}$ -closed,  $z \in \mathcal{C}$ . Furthermore, contraction condition (2) immediately implies that  $z = \mathcal{T}z$ .

Finally, let  $u \in \mathcal{C}$  be such that  $u = \mathcal{T}u$ . Then, inequality (3) implies that  $\mathfrak{F}(u, z, t) > 1 - t$  for all  $t > 0$ . So  $z$  is the unique fixed point of  $\mathcal{T}$  in  $\mathcal{C}$ .

(2)  $\implies$  (1) Suppose that  $(\mathfrak{X}, \mathfrak{F}, *, K)$  is not complete. Then, there is a Cauchy sequence  $(x_n)_{n \in \mathbb{N}}$  of distinct points, in  $\mathfrak{X}$ , which is not  $\tau_{\mathfrak{F}}$ -convergent.

By Theorem 1 ((st3) and the proof of (st5)),  $(x_n)_{n \in \mathbb{N}}$  is a non- $\tau_{d_{\mathfrak{F}}}$ -convergent Cauchy sequence in the metric space  $(\mathfrak{X}, d_{\mathfrak{F}})$ . Hence, there is a sequence  $(\eta_n)_{n \in \mathbb{N}}$  of real numbers, such that  $\eta_n \in (0, 1)$  for all  $n \in \mathbb{N}$ , and  $d_{\mathfrak{F}}(x_n, x_m) \geq \eta_n$  whenever  $m \neq n$ . This, in turn,

and by virtue of the inclusion relations (1), implies the existence of a sequence of real numbers  $(\mu_n)_{n \in \mathbb{N}}$  such that  $\mu_n \in (0, 1)$  for all  $n \in \mathbb{N}$ , satisfying

$$\mathfrak{F}(x_n, x_m, \mu_n) \leq 1 - \mu_n, \tag{4}$$

whenever  $m \neq n$ .

Fix  $c \in (0, 1)$ . As  $(x_n)_{n \in \mathbb{N}}$  is a Cauchy sequence, for each  $n \in \mathbb{N}$  we find a  $\lambda(n) \in \mathbb{N}$  such that  $\lambda(1) > 1$  and  $\mathfrak{F}(x_i, x_j, c\mu_1/2K^2) > 1 - (c\mu_1/2K^2)$  for all  $i, j \geq \lambda(1)$ , and for  $n \geq 2$ ,  $\lambda(n) > \max\{n, \lambda(n - 1)\}$  and

$$\mathfrak{F}(x_i, x_j, \frac{c\mu_n}{2K^2}) > 1 - \frac{c\mu_n}{2K^2}, \tag{5}$$

for all  $i, j \geq \lambda(n)$ .

Now we define a self-map  $\mathcal{T}$  from the  $\tau_{\mathfrak{F}}$ -closed subset  $\mathcal{C} := \{x_n : n \in \mathbb{N}\}$  into itself as follows:  $\mathcal{T}x_n = x_{\lambda(n)}$  for all  $n \in \mathbb{N}$ .

Obviously,  $\mathcal{T}$  has no fixed points because  $\lambda(n) > n$  for all  $n \in \mathbb{N}$ . We are going to prove that, nevertheless,  $\mathcal{T}$  is a Hicks contraction on  $\mathcal{C}$  (with constant  $c$ ).

Indeed, suppose that  $\mathfrak{F}(x_n, x_m, t) > 1 - t$ , with  $t > 0$ , and assume, without loss of generality, that  $n > m$ . Then, it follows from inequality (4) that  $\mu_n \leq Kt$ , so  $\mu < 2Kt$ .

Hence, taking into account inequality (5), we obtain

$$\begin{aligned} \mathfrak{F}(\mathcal{T}x_n, \mathcal{T}x_m, ct) &= \mathfrak{F}(x_{\lambda(n)}, x_{\lambda(m)}, ct) \geq \mathfrak{F}(x_{\lambda(n)}, x_{\lambda(m)}, \frac{c\mu_n}{2K^2}) \\ &> 1 - \frac{c\mu_n}{2K^2} > 1 - ct. \end{aligned}$$

We have reached a contradiction that finishes the proof.  $\square$

To finish we give two examples, including an application to difference equations, which complement and illustrate Theorem 3.

**Example 6.** Consider the following difference equation, which can be seen as a simultaneous variant and extension of a classical one posed, e.g., in [41] (p. 515), [42] (Equation (5)), [43] (Equation #23):  $x_1 = 1$  and

$$x_{n+1} = \left( \frac{qx_n}{s + rx_n} \right)^4,$$

for all  $n \in \mathbb{N}$ , where  $0 < q < 2^{5/2}s$  and  $r > 0$ .

By applying Theorem 3 to a suitable and appealing complete fuzzy b-metric space which is not a fuzzy metric, we will show the existence and uniqueness of the solution for the above equation.

To this end, we first remind some well-known concepts and properties.

A sequence  $(x_n)_{n \in \mathbb{N}}$  of real numbers, will be denote by  $\mathbf{x} := (x_n)_{n \in \mathbb{N}}$ , or simply by  $\mathbf{x}$  if there is no confusion. As usual, let

$$l_p := \{ \mathbf{x} : \sum_{n=1}^{+\infty} |x_n|^p < +\infty \}.$$

where  $p$  is a constant with  $p \in (0, 1)$ .

Then, the pair  $(d_p, 2^{1/p})$  is a complete b-metric on  $l_p$  where  $d_p : l_p \times l_p \rightarrow \mathbb{R}^+$  is given by

$$d_p(\mathbf{x}, \mathbf{y}) = \left( \sum_{n=1}^{+\infty} |x_n - y_n|^p \right)^{1/p},$$

for all  $\mathbf{x}, \mathbf{y} \in l_p$  (see [8] (Example 1.3), [9] (Example 12.1)). Furthermore,  $d_p$  is not a metric on  $l_p$ .

According to Pietsch [44] (p. 67), the first example of a space of this kind was addressed by Tychonoff [45] (p. 768). From Tychonoff's result, it follows that the 3-tuple  $(l_{1/2}, d_{1/2}, 2)$  is a complete b-metric space, but not a metric space (see also [3] (p. 526)).

Observe that if we extend in a natural way the preceding definitions of  $l_p$  and  $d_p$  to  $p = 1$ , we yield the classical complete metric space  $(l_1, d_1)$ .

Due to the shape of the elements of our difference equation, we select, because of its apparent simplicity, the complete 01-fuzzy b-metric space  $(l_{1/2}, \mathfrak{F}_{d_{1/2,01}}, *, 2)$  (see Example 1) as a suitable mathematical setting for our purpose (we outline that the complete fuzzy metric space  $(l_1, \mathfrak{F}_{d_{1,01}}, *)$  also could be a good candidate to our study. However, we will use the above-chosen space to emphasize the usefulness of Theorem 3 and its applicability when fuzzy b-metric spaces that are not fuzzy metrics are employed).

Let  $C$  be the subset of  $l_{1/2}$  defined as

$$C := \{x \in l_{1/2} : x_1 = 1 \text{ and } 0 \leq x_n \leq 1/(n + 1)^4 \text{ for all } n \geq 2\}.$$

It is straightforward to verify that  $C$  is a closed subset of  $(l_{1/2}, d_{1/2}, 2)$ . Therefore, it is also a closed subset of  $(l_{1/2}, \mathfrak{F}_{d_{1/2,01}}, *, 2)$ .

Now we prove that  $d_2(x, y) < 1$  for all  $x, y \in C$  (this fact will be worthy later on).

Indeed, for any pair  $x, y \in C$  we have:

$$d_{1/2}(x, y) = \left(\sum_{n=2}^{+\infty} |x_n - y_n|^{1/2}\right)^2 \leq \left(\sum_{n=2}^{+\infty} \frac{1}{(n + 1)^2}\right)^2 < \left(\sum_{n=2}^{+\infty} \frac{1}{n^2}\right)^2 = \left(\frac{\pi^2}{6} - 1\right)^2 < 1.$$

Hence,  $d_{1/2}(x, y) < 1$ .

Next, we define a map  $\mathcal{T}$  on  $C$  as follows: for each  $x := (x_n)_{n \in \mathbb{N}} \in C$ ,  $(\mathcal{T}x)_1 = 1$ , and

$$(\mathcal{T}x)_{n+1} = \left(\frac{qx_n}{s + rx_n}\right)^4,$$

for all  $n \in \mathbb{N}$ .

Actually,  $\mathcal{T}$  is a self-map of  $C$ : Indeed, for each  $x := (x_n)_{n \in \mathbb{N}} \in C$ ,  $(\mathcal{T}x)_1 = 1$ , and

$$(\mathcal{T}x)_{n+1} = \left(\frac{qx_n}{s + rx_n}\right)^4 \leq \left(\frac{q}{s(n + 1)^4}\right)^4 < \frac{2^{10}}{(n + 1)^{16}} < \frac{1}{(n + 1)^4},$$

for all  $n \in \mathbb{N}$ .

It remains to prove that  $\mathcal{T}$  has a unique fixed point in  $C$ , which, evidently, will be the solution of the given difference equation.

We shall show that  $\mathcal{F}$  is a Hicks contraction on  $C$ , with constant  $c = q^4/2^{10}s^4 < 1$ .

Let  $x, y \in C$  and  $t > 0$  such that  $\mathfrak{F}_{d_{1/2,01}}(x, y, t) > 1 - t$ .

If  $\mathfrak{F}_{d_{1/2,01}}(x, y, t) = 0$ , we obtain  $d_{1/2}(x, y) \geq t > 1$ , which contradicts the statement showed above that  $d_{1/2}(x, y) < 1$ .

Consequently, we only need to examine the case that  $\mathfrak{F}_{d_{1/2,01}}(x, y, t) = 1$ .

In such a case, we obtain  $d_{1/2}(x, y) < t$ , and we shall show that  $\mathfrak{F}_{d_{1/2,01}}(\mathcal{T}x, \mathcal{T}y, ct) = 1$ .

To achieve this, the next easy relation will be useful:

$$|A^4 - B^4| = |A^2 + B^2||A + B||A - B|,$$

for all pairs  $A, B$ , of real numbers.

Combining the preceding relation with the following ones:

$$\left|\left(\frac{qx_n}{s + rx_n}\right)^2 + \left(\frac{qy_n}{s + ry_n}\right)^2\right| \leq \frac{2q^2}{s^2(n + 1)^8},$$

and

$$\left|\frac{qx_n}{s + rx_n} + \frac{qy_n}{s + ry_n}\right| \leq \frac{2q}{s(n + 1)^4},$$

we obtain

$$\begin{aligned}
 |(\mathcal{T}x)_{n+1} - (\mathcal{T}y)_{n+1}| &= \left| \left( \frac{qx_n}{s + rx_n} \right)^4 - \left( \frac{qy_n}{s + ry_n} \right)^4 \right| \\
 &\leq \frac{4q^3}{s^3(n+1)^{12}} \left| \frac{qx_n}{s + rx_n} - \frac{qy_n}{s + ry_n} \right| \\
 &\leq \frac{4q^3}{s^3(n+1)^{12}} \cdot \frac{qs|x_n - y_n|}{s^2} \leq \frac{q^4}{2^{10}s^4} |x_n - y_n| \\
 &= c|x_n - y_n|,
 \end{aligned}$$

for all  $n \in \mathbb{N}$ . Therefore,

$$\begin{aligned}
 d_{1/2}(\mathcal{T}x, \mathcal{T}y) &= \left( \sum_{n=2}^{+\infty} |(\mathcal{T}x)_n - (\mathcal{T}y)_n|^{1/2} \right)^2 \leq c \left( \sum_{n=1}^{+\infty} |x_n - y_n|^{1/2} \right)^2 \\
 &= cd_{1/2}(x, y) < ct,
 \end{aligned}$$

which implies that  $\mathfrak{F}_{d_{1/2}, 01}(\mathcal{T}x, \mathcal{T}y, ct) = 1$ .

We conclude that  $\mathcal{T}$  is a Hicks contraction on the closed subset  $\mathcal{C}$  of  $l_{1/2}$ . So, by Theorem 3,  $\mathcal{T}$  has a unique fixed point  $z \in \mathcal{C}$ , which is a solution for our difference equation.

**Remark 5.** Although the index  $K = 2^{1/p}$ ,  $0 < p < 1$ , is valid in Example 6, Reviewer 4 has shown in their report that a better index is the one given by  $K = 2^{(1/p)-1}$ .

**Example 7.** Let  $\mathfrak{X} = (a, 1]$ , with  $a \in (0, 1)$  constant, let  $*$  be a continuous triangular norm and let  $\mathfrak{F}$  be the fuzzy set in  $\mathfrak{X} \times \mathfrak{X} \times \mathbb{R}^+$  defined as  $\mathfrak{F}(x, y, 0) = 0$  for all  $x, y \in \mathfrak{X}$ ,  $\mathfrak{F}(x, x, t) = 1$  for all  $x \in \mathfrak{X}$  and all  $t > 0$ , and  $\mathfrak{F}(x, y, t) = \min\{x * y, t\}$ , otherwise.

We first show that  $(\mathfrak{F}, *)$  is a fuzzy metric (hence, a fuzzy b-metric for all  $K \geq 1$ ) on  $\mathfrak{X}$ .

Since conditions (FbM1), (FbM2), (FbM3) and (FbM5), in Definition 1, are almost trivially fulfilled, we only show that  $\mathfrak{F}(x, z, t + s) \geq \mathfrak{F}(x, y, t) * \mathfrak{F}(y, z, s)$ , for all  $x, y, z \in \mathfrak{X}$  and all  $t, s > 0$ .

Indeed, we have

$$x * z \geq (x * y) * (y * z) \geq \min\{x * y, t\} * \min\{y * z, s\},$$

and, if  $t + s \leq 1$ ,

$$t + s \geq t * s \geq \min\{x * y, t\} * \min\{y * z, s\}.$$

Consequently,

$$\begin{aligned}
 \mathfrak{F}(x, z, t + s) &= \min\{x * z, t + s\} \geq \min\{x * y, t\} * \min\{y * z, s\} \\
 &= \mathfrak{F}(x, y, t) * \mathfrak{F}(y, z, s).
 \end{aligned}$$

Now, we shall deduce that  $(\mathfrak{X}, \mathfrak{F}, *)$  is complete, as a consequence of Theorem 3.

To achieve it, let  $\mathcal{C}$  be a  $\tau_{\mathfrak{F}}$ -closed subset of  $\mathfrak{X}$  and let  $\mathcal{T}$  be a Hicks contraction on  $\mathcal{C}$  (with contraction constant  $c$ ). Fix  $x \in \mathfrak{X}$ ,  $t_0 > 1$ , and suppose that  $\mathcal{T}^n x \neq \mathcal{T}^{n+1} x$  for all  $n \in \mathbb{N} \cup \{0\}$ . Then (see the proof of Theorem 3, (1)  $\Rightarrow$  (2)),

$$\mathfrak{F}(\mathcal{T}^n x, \mathcal{T}^{n+1} x, c^n t_0) > 1 - c^n t_0,$$

for all  $n \in \mathbb{N} \cup \{0\}$ .

By the definition of  $\mathfrak{F}$  we obtain  $c^n t_0 \geq \mathfrak{F}(\mathcal{T}^n x, \mathcal{T}^{n+1} x, c^n t_0)$ . Hence,  $c^n t_0 > 1/2$  for all  $n \in \mathbb{N} \cup \{0\}$ . Thus, we have reached a contradiction. Therefore, there is  $k \in \mathbb{N}$  such that  $\mathcal{T}^k x = \mathcal{T}^{k+1} x$ , which implies that  $\mathcal{T}^k x$  is a fixed point of  $\mathcal{T}$ . From Theorem 3, (2)  $\Rightarrow$  (1), we deduce that  $(\mathfrak{X}, \mathfrak{F}, *)$  is complete, i.e.,  $(\mathfrak{X}, \mathfrak{F}, *, K)$  is complete for any  $K \geq 1$ .

## 5. Conclusions

Based on the notions of Kramosil and Michálek, and Grabiec of a fuzzy metric space, we have proposed a notion of fuzzy  $b$ -metric space that generalizes and unifies previous notions given and, independently, analyzed by Sedghi and Shobe, Hussain et al., Saadati, Nădăban, and Šostak. Our approach simultaneously covers two interesting classes of spaces, namely, the 01-fuzzy  $b$ -metric spaces and the  $K$ -stationary fuzzy  $b$ -metric spaces. We have proved that each fuzzy  $b$ -metric space, in our sense, admits a uniformity with a countable base. From this fact, we derive, among other consequences, that the topology induced by means of its “open” balls is metrizable. A characterization of complete fuzzy  $b$ -metric spaces with the help of a fixed point result is also obtained. Several examples, including an application to a type of difference equations, were also discussed.

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