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Article

Blow-Up of Solution of Lamé Wave Equation with Fractional Damping and Logarithmic Nonlinearity Source Terms

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Abstract: In this work, by the use of a semigroup theory approach, we provide a global solution for an initial boundary value problem of the wave equation with logarithmic nonlinear source terms and fractional boundary dissipation. In addition to this, we establish a blow-up result for the solution under the condition of non-positive initial energy.

Keywords: blow-up; fractional boundary dissipation; logarithmic Lamé system; partial differential equations; mathematical operators

MSC: 35L70; 35L35; 35B40

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1. Introduction

Fractional calculus has found applications in various fields because of its ability to describe phenomena involving non-integer order derivatives and integrals. Fractional calculus is a powerful and versatile tool that extends the capabilities of traditional calculus, enabling a more accurate representation of complex systems across a wide range of scientific and engineering domains. Its importance lies in its ability to bridge the gap between theory and real-world observations, providing a more effective representation of natural problems. Recent times have witnessed the emergence of novel definitions for fractional derivatives and integrals, extending the classical formulations in various ways. Moreover, a dynamic realm of research in mathematical analysis has been dedicated to the meticulous examination of the functional properties inherent in these new definitions. Extensive exploration of systems involving partial differential equations with fractional-order operators has been conducted from both analytical and numerical perspectives. These systems find widespread applications in science and technology, seamlessly modeling phenomena in diverse fields, such as biology, ecology, and chemistry, among others. The physical meaning of using a fractional derivative in the boundary condition is a complex topic that may be considered in our future work. However, fractional derivatives are non-local, which makes fractional calculus more attractive than the classical derivatives for real-world problems.

In [1], the authors delved into the realm of ultra-parabolic equations. Specifically, they explored equations with a singular lower-order term, showing the applicability of Harnack inequalities in this context. The paper contributed to the understanding of hypoelliptic ultra-parabolic equations, providing insights that enhance our comprehension of their behavior and laying the groundwork for further developments in the field of mathematical analysis.

In the light of the work of [2], we propose the following problem:

$$\begin{cases} w_{tt} - \mu\Delta w - (\lambda+\mu)\nabla(divw) & \\ +w_t = w|w|^{p-2}ln|w|^k & y\in\Omega,\, t>0, \\ \frac{\partial w}{\partial v} = -b\partial_t^{\alpha,\eta} w, & y\in\Gamma_0, t>0, \\ w(y,t)=0 & y\in\Gamma_1,\, t>0, \\ w(y,0)=w_0(y), \quad w_t(y,0)=w_1(y) & y\in\Omega, \end{cases} \tag{1}$$

in which Ω denotes a bounded domain in $\mathbb{R}^n$, where n is a positive integer. The domain possesses a smooth boundary, denoted as $\partial\Omega$, characterized by a C^2 class. Additionally, ν represents the unit outward normal vector to $\partial\Omega$, which can be expressed as the union of closed subsets Γ_0 and Γ_1. Both Γ_0 and Γ_1 are subsets of $\partial\Omega$ and satisfy the condition that their union, $\Gamma_0\cup\Gamma_1$, is an empty set. Here, nonlinearities occur, which are needed to obtain "blow-up" solutions. Nonlinear equations are usually difficult to analyze, and local existence can be established by standard arguments for most reasonable PDEs. However, global existence is not guaranteed, and blow-up can occur because of the presence of nonlinearities.

Let μ and λ denote the Lamé constants, satisfying $\mu>0$ and $\lambda+\mu\geq 0$. The parameter p is greater than two, and the constant k is a small non-negative real number. The space $L^2(\mathcal{D})$ comprises square integrable functions on $\mathcal{D}$ with the inner product $\langle\cdot,\cdot\rangle$ and its associated norm $|\cdot|_2$. Here, b is a nonnegative real number, and $\partial_t^{\alpha,\eta}$ represents Caputo's generalized fractional derivative with $0<\alpha<1$. This derivative is defined in [2,3] as given below:

$$\partial_t^{\alpha,\eta} w(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-r)^{-\alpha}e^{-\eta(t-r)}w_r(r)dr,\ \ \eta\geq 0,$$

in which the Γ is the Euler gamma function. Further, we have the following:

$$\partial_t^{\alpha,\eta} w(t) = I^{1-\alpha,\eta}w_t(t), \tag{2}$$

in which $I^{\alpha,\eta}$ is the exponential integro-differential operator of fractional derivative, given by

$$\partial_t^{\alpha,\eta} w(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-r)^{\alpha-1}e^{-\eta(t-r)}w(r)dr,\ \ \eta\geq 0.$$

In the literature, several researchers explored problems of this nature from different aspects [4,5]. Utilizing the Lyapunov functional, they conducted a comprehensive study on the global existence of solutions and the overall decay within a bounded domain for a nonlinear wave equation featuring fractional derivative boundary conditions. Additionally, these studies delved into the examination of solutions, investigating both non-positive and positive initial energy and exploring the potential occurrence of blow-up phenomena. The blow-up issue in extraordinary problems has caused a lot of ink to flow. For instance, one can mention the paper by Liquing Lu and Shengjia Li [6]. Other authors, like [7], were interested in the case of such a problem in the frame of a Lamé system since it is found in quite a number of applications. In [8], the author established the global nonexistence of solutions for logarithmic wave equations with nonlinear damping and distributed delay terms. The findings contributed valuable insights into the limitations and constraints of such equations, enriching our understanding of their dynamic behavior. In this scenario, Yüksekkaya et al. [9], employing semigroup theory, addressed and established the well-posedness of an initial-boundary value problem for a logarithmic Lamé system with a time delay within a bounded domain. They further demonstrated the system's possession of global solutions using the well-depth method, subject to suitable assumptions on the weights of both the time delay and frictional damping. Additionally, they provided an exponential stability decay result. This work is organized as follows: Section 1 provides a preliminary discussion of the requisite definitions and statements. Our focus in Section 2 is to illustrate the global existence and uniqueness of solutions in (1), and Section 3 is dedicated to presenting blow-up results.

2. Preliminaries

In this section, some basic results and concepts are introduced that are used in the results of our work. Assume

$$H^1_{\Gamma_1}(\Omega) = \{w \in H^1(\Omega),\, w = 0 \text{ on } \Gamma_1 = 0\}.$$

Lemma 1 ([10], Sobolev–Poincaré' s inequality)**.** *Assume a number m in a way that*

$$1 \le m \le +\infty (n = 1,2) \text{ or } 1 \le m \le (n+2)/(n-2),\ (n \ge 3).$$

Then, one can find a constant $C_s > 0$ in a manner that

$$\|w\|_{m+1} \le C_s \|\nabla w\|_2 \text{ for } w \in H^1_0(\Omega).$$

Lemma 2 (Trace-Sobolev embedding. See [11])**.** *The following holds true:*

$$H^1_{\Gamma_1}(\Omega) \hookrightarrow L^{q+1}(\Omega),$$

if

$$1 \le q \le \infty, (n = 1,2) \text{ or } 1 \le q \le \frac{n+2}{n-2},\ (n \ge 3), \tag{3}$$

i.e.,

$$\|u\|_{q+1} \le B_{q,\Omega} \|\nabla u\|_2,\ \forall u \in H^1_{\Gamma_1}(\Omega),$$

where $B_{q,\Omega}$ is the best constant fulfilling the trace-Sobolev embedding.

Definition 1. *We define w as a blow-up solution of (1) at a finite time T^* if*

$$\lim_{t \to T^{*-}} \|\nabla w\|_2 = +\infty.$$

Lemma 3 ([12], Lemma 4.2)**.** *Let $t_0 \ge 0$ and $F(t)$ be a function of non-increasing nature on $[t_0, \infty)$ satisfying*

$$\left(F'(t)\right)^2 \ge m(F(t))^{2+1/\gamma} + \sigma, \quad t_0 \le t,\ 0 < \gamma, \tag{4}$$

in which $m < 0$ and $\sigma > 0$. Then, there is a finite time T^* in a way that

$$\lim_{t \to T^{*-}} F(t) = 0, \tag{5}$$

$F(t_0) < min\left\{1, \sqrt{-\frac{\sigma}{m}}\right\}$,

$$T^* \le t_0 + \frac{1}{\sqrt{-m}} ln \frac{\sqrt{-\sigma/m}}{\sqrt{-\sigma/m} - F(t_0)}. \tag{6}$$

Theorem 1 ([2], Theorem 2.6)**.** *Assume a function ψ as follows:*

$$\psi(\xi) = |\xi|^{\frac{(2\alpha-1)}{2}},\ 1 > \alpha > 0,\ \xi \in \mathbb{R}.$$

$$\psi(\xi) = |\xi|^{\frac{(2\alpha-1)}{2}},\ 1 > \alpha > 0,\ \xi \in \mathbb{R}. \tag{7}$$

Then, we have

$$I^{1-\alpha,\eta} U = O, \tag{8}$$

which shows a relation between the input U and the output O of the below system:

$$\partial_t \Phi(\xi, t) + (\xi^2 + \eta)\Phi(\xi, t) - U(L, t)\psi(\xi) = 0,\ 0 < t,\ \eta \ge 0,\ \xi \in \mathbb{R}, \tag{9}$$

$$\Phi(\xi,0)=0, \tag{10}$$

$$O(t)=\frac{sin(\alpha\pi)}{\pi}\int_{-\infty}^{+\infty}\Phi(\xi,t)\psi(\xi)d\xi,\ \xi\in\mathbb{R},\ t>0. \tag{11}$$

As a consequence of (2) and Theorem 1, we can obtain the augmented system for system (1) as follows:

$$\begin{cases} w_{tt}-\mu\Delta w-(\lambda+\mu)\nabla(divw) & \\ +w_t=w|w|^{p-2}ln|w|^k & y\in\Omega,\ 0<t,\\ \partial_t\Phi(\xi,t)+(\xi^2+\eta)\Phi(\xi,t)-w_t(y,t)\psi(\xi)=0 & y\in\Gamma_0,\ \xi\in\mathbb{R},\ 0<t,\\ \frac{\partial w}{\partial v}=-b_1\int_{-\infty}^{+\infty}\Phi(\xi,t)\psi(\xi)d\xi, & y\in\Gamma_0, 0<t,\\ w(y,t)=0 & y\in\Gamma_1,\ 0<t,\\ w(y,0)=w_0(y),\quad w_t(y,0)=w_1(y) & y\in\Omega,\\ \Phi(\xi,0)=0 & \xi\in\mathbb{R}, \end{cases} \tag{12}$$

in which $b_1=b\left(\frac{sin(\alpha\pi)}{\pi}\right)$.

Lemma 4 ([2]). *Let $\beta\in D_\eta=\{\beta\in\mathbb{C}:\mathcal{I}m\beta\neq0\}\cup\{\beta\in\mathbb{C}:\mathcal{R}e\beta+\eta>0\}$. Then, we have*

$$A_\beta=\int_{-\infty}^{+\infty}\frac{\psi^2(\xi)}{\eta+\beta+\xi^2}d\xi=\frac{\pi}{sin(\alpha\pi)}(\eta+\beta)^{\alpha-1}.$$

Let $E(t)$ be the energy functional related to (12) given by

$$\begin{aligned} E(t)=&\frac{1}{2}\|w_t\|_2^2+\frac{\mu}{2}\|\nabla w\|_2^2+\frac{\lambda+\mu}{2}\|divw\|_2^2+\frac{k}{p^2}\|w\|_p^p\\ &-\frac{1}{p}\int_\Omega|w|^pln|w|^kdy+\frac{b_1}{2}\int_{\Gamma_0}\int_{-\infty}^{+\infty}|\Phi(\xi,t)|^2d\xi d\rho. \end{aligned} \tag{13}$$

Lemma 5. *Assume $(u,\,\Phi)$ is a regular solution to the problem (12). Then, $E(t)$ given in (13) is a non-increasing function, and*

$$\frac{d}{dt}E(t)=-\|w_t\|_2^2-b_1\int_{\Gamma_0}\int_{-\infty}^{+\infty}(\xi^2+\eta)|\Phi(\xi,t)|^2d\xi d\rho. \tag{14}$$

Proof. Multiply u_t with the first equation in (12), and integrating by parts over Ω, we have

$$\int_\Omega w_{tt}w_tdy-\mu\int_\Omega\Delta ww_tdy-(\lambda+\mu)\int_\Omega\nabla(divw)w_tdy+\|w_t\|_2^2=\int_\Omega w|w|^{p-2}ln|w|^kw_tdy.$$

We obtain

$$\begin{aligned} &\frac{d}{dt}\left(\frac{1}{2}\|\nabla w_t\|_2^2+\frac{1}{2}\|w_t\|_2^2+\frac{1}{2}\|divw\|_2^2-\frac{1}{p}\int_\Omega|w|^pln|w|^k+\frac{k}{p^2}\|w\|_p^p\right)\\ &=-b_1\int_{\Gamma_0}w_t(y,t)\int_{-\infty}^{+\infty}(\xi^2+\eta)|\Phi(\xi,t)|^2d\xi d\rho-\|w_t\|_2^2. \end{aligned} \tag{15}$$

Multiply $b_1\Phi$ with the second equation in (12), and integrating over $\Gamma_0\times(-\infty,+\infty)$, we have

$$\begin{aligned} b_1\int_{\Gamma_0}w_t(y,t)\int_{-\infty}^{+\infty}\Psi(\xi)\Phi(\xi,t)d\xi d\rho=&\frac{b_1}{2}\frac{d}{dt}\int_{\Gamma_0}\int_{-\infty}^{+\infty}|\Phi(\xi,t)|^2d\xi d\rho\\ &+b_1\int_{\Gamma_0}\int_{-\infty}^{+\infty}(\xi^2+\eta)|\Phi(\xi,t)|^2d\xi d\rho. \end{aligned} \tag{16}$$

Using (13), (15), and (16) leads to

$$\frac{d}{dt}E(t) = -\left(b_1 \int_{\Gamma_0} \int_{-\infty}^{+\infty} (\xi^2 + \eta)|\Phi(\xi,t)|^2 d\xi d\rho + \|w_t\|_2^2\right).$$

Consequently, the energy functional given in (13) is a non-increasing function. □

Lemma 6 ([2])**.** *Assume $(w,\, \Phi)$ is a regular solution of (12). Then, we have the following:*

$$\int_{\Gamma_0} \int_{-\infty}^{+\infty} (\xi^2 + \eta)\Phi(\xi,t) \int_0^t \Phi(\xi,s) ds d\xi d\rho = \int_{\Gamma_0} w(y,t) \int_{-\infty}^{+\infty} \Phi(\xi,t)\Psi(\xi) d\xi d\rho - \int_{\Gamma_0} \int_{-\infty}^{+\infty} |\Phi(\xi,t)|^2 d\xi d\rho.$$

Now, we rewrite the system (12) in the following related system:

$$\begin{cases} \mathcal{Z}_t(t) = \mathcal{A}\mathcal{Z}(t) + \mathcal{G}(\mathcal{Z}), \\ \mathcal{Z}(0) = \mathcal{Z}_0,\; 0 < t, \end{cases} \tag{17}$$

where $\mathcal{Z} = (w,\, v,\, \Phi)^T$, $\mathcal{Z}_0 = (w_0,\, v_0,\, \Phi_0)^T$. Furthermore, the operator $\mathcal{A}$ given by

$$\mathcal{A} = \begin{pmatrix} 0 & 1 & 0 \\ \mu\Delta & -1 & 0 \\ 0 & \Psi(\xi) & -(\xi^2+\eta) \end{pmatrix},$$

$$\mathcal{G}(\mathcal{Z}) = \begin{pmatrix} 0 \\ (\lambda+\mu)\nabla(divw) + w|w|^{p-2} ln|w|^k \\ 0 \end{pmatrix},$$

The domain $D(\mathcal{A})$ of $\mathcal{A}$ is given by

$$D(\mathcal{A}) = \left\{ \begin{array}{l} \mathcal{Z} = (w,\, v,\, \Phi)^T \in H : w \in (H^2(\Omega) \cap H^1_{\Gamma_1}(\Omega)),\; v \in H^1(\Omega),\; \frac{\partial \Phi}{\partial t} + (\xi^2+\eta)\Phi - v\psi(\xi) = 0\Big|_{\Gamma_0} \\ \frac{\partial w}{\partial \nu} + b_1 \int_{-\infty}^{+\infty} \Phi(\xi,t)\psi(\xi) d\xi = 0\Big|_{\Gamma_0},\quad |\xi|\Phi \in L^2(\Omega \times (-\infty, +\infty)) \end{array} \right\}.$$

Demonstrating characteristics of a sectorial operator, it can be established that $-\mathcal{A}$ holds such properties. Additionally, $\mathcal{A}$ gives rise to an analytic contraction semigroup, denoted as $T(t) : t \geq 0$. Furthermore, $\mathcal{A}$ exhibits a compact resolvent. The nonlinear mapping $\mathcal{G}$ from the space E to itself is locally Lipschitz-continuous and possesses the property of mapping bounded sets to bounded sets.

Lemma 7 ([13], Lemma 1)**.** *Let $\mathcal{Z}_0 \in D(\mathcal{A})$ and $t_0 = t_0(\mathcal{Z}_0) > 0$ in a manner that the mild solution $\mathcal{Z}(t)$ of (17) with $\mathcal{Z}(0) = \mathcal{Z}_0$ uniquely exists for $t \in [0,\, t_0]$ and*

$$\mathcal{Z} \in C((0,t_0); D(\mathcal{A})) \cap C^1((0,t_0); D(\mathcal{A})).$$

For $t \in [0, t_0]$, this mild solution is a classical solution of (17), if $\mathcal{Z}_0 \in D(\mathcal{A})$.

3. Global Existence of Solutions

Theorem 2. *For every $\mathcal{Z}_0 \in D(\mathcal{A})$ and $\xi^2 < \eta$, one can find an exclusive global mild solution $\mathcal{Z}(t)$ of (17) for $t \geq 0$. This solution exhibits the regularity properties specified in Lemma 7. The related solution semigroup $S(t)$, with $t \geq 0$, demonstrates dissipative characteristics, as evidenced by the existence of an absorbing set in $D(\mathcal{A})$.*

Proof. First, notice that if we take the inner product of (1) by $2v$ in H, we obtain

$$\frac{d}{dt}\left[\|v\|^2 + \mu\|\nabla w\|^2 + (\lambda+\mu)\nabla\|divw\|^2 - \frac{2}{p}\int_\Omega |w|^p ln|w|^k dy + \frac{2k}{p^2}\int_\Omega |w|^p dy\right] = -2\|v\|^2. \tag{18}$$

Thin, integration (18) over $[0, t]$ for any t positive entails that

$$\|v\|^2 + \mu\|\nabla w\|^2 + (\lambda+\mu)\nabla\|divw\|^2 - \frac{2}{p}\int_\Omega |w|^p ln|w|^k + \frac{2k}{p^2}\|w\|^p - 2\int_0^t \|v\|^2 ds = \|w_1\|^2$$
$$+ \mu\|\nabla w_0\|^2 + (\lambda+\mu)\nabla\|divw_0\|^2 - \frac{2}{p}\int_\Omega |w_0|^p ln|w_0|^k dy + \frac{2k}{p^2}\|w_0\|^p. \tag{19}$$

Next, multiplying the second equation of (12) by $\frac{b_1\Phi}{\xi^2+\eta}$ and integrating over $\Gamma_0 \times (-\infty, +\infty)$, we have

$$\frac{b_1}{2}\frac{d}{dt}\int_{\Gamma_0}\int_{-\infty}^{+\infty}\frac{|\Phi(\xi,t)|^2}{\xi^2+\eta}d\xi d\rho + b_1\|\Phi\|^2 - b_1\int_{\Gamma_0}\int_{-\infty}^{+\infty}\frac{v(y,t)}{(\xi^2+\eta)}\psi(\xi)\Phi(\xi,t)d\xi d\rho = 0. \tag{20}$$

Now, taking a non-negative number ε and multiplying εu with the first equation of (12), we obtain

$$\varepsilon w_{tt}w - \varepsilon\mu\Delta w\, w - \varepsilon(\lambda+\mu)\nabla(divw)\, w + \varepsilon v w - \varepsilon|w|^p ln|w|^k = 0. \tag{21}$$

By integrating over Ω, we obtain

$$\frac{d}{dt}\left[\varepsilon\int_\Omega vudx + \frac{\varepsilon}{2}\|u\|^2\right] - \varepsilon\|v\|^2 + \varepsilon\mu\|\nabla u\|^2 + \varepsilon(\lambda+\mu)\|(div\, u)\|^2 - \varepsilon\int_\Omega |u|^p ln|u|^k dx = 0. \tag{22}$$

From (18), (20), and (22), we have the following:

$$\frac{d}{dt}\mathcal{F}_1(t) + \mathcal{F}_2(t) = 0, \tag{23}$$

where

$$\mathcal{F}_1(t) = \|v\|^2 + \mu\|\nabla w\|^2 + (\lambda+\mu)\|divw\|^2 - \frac{2}{p}\int_\Omega |w|^p ln|w|^k dy$$
$$+ \frac{2k}{p^2}\int_\Omega |y|^p dx + \frac{b_1}{2}\int_{\Gamma_0}\int_{-\infty}^{+\infty}\frac{|\Phi(\xi,t)|^2}{\xi^2+\eta}d\xi d\rho + \varepsilon\int_\Omega vwdy + \frac{\varepsilon}{2}\|w\|^2 \tag{24}$$

and

$$\mathcal{F}_2(t) = 2\|v\|^2 + b_1\|\Phi\|^2 - b_1\int_{\Gamma_0}\int_{-\infty}^{+\infty}\frac{v(y,t)}{(\xi^2+\eta)}\psi(\xi)\Phi(\xi,t)d\xi d\rho - \varepsilon\|v\|^2 + \varepsilon\mu\|\nabla w\|^2$$
$$+ \varepsilon(\lambda+\mu)\|(div\, w)\|^2 - \varepsilon\int_\Omega |w|^p ln|w|^k dy.$$

Nevertheless, we can estimate $\frac{2}{\varepsilon}\mathcal{F}_2(t) - \mathcal{F}_1(t)$ by

$$\frac{2}{\varepsilon}(2-\varepsilon)\|v\|^2 + \frac{2}{\varepsilon}b_1\|\Phi\|^2 - \frac{b_1}{2}\int_{\Gamma_0}\int_{-\infty}^{+\infty}\frac{|\Phi(\xi,t)|^2}{\xi^2+\eta}d\xi d\rho + (\lambda+\mu)\|(div\, w)\|^2$$
$$+ \mu\|\nabla w\|^2 - \frac{\varepsilon}{2}\|w\|^2 - \frac{2}{\varepsilon}b_1\int_{\Gamma_0}\int_{-\infty}^{+\infty}\frac{v(y,t)}{(\xi^2+\eta)}\psi(\xi)\Phi(\xi,t)d\xi d\rho$$
$$- \varepsilon\int_\Omega vwdy - \frac{2k}{p^2}\|w\|^p - \frac{p-1}{p}\int_\Omega |w|^p ln|w|^k dy.$$

Then, through Lemma 2 and Young's inequality, we have, for all non-negative constant δ,

$$\int_\Omega vwdy \le \frac{1}{4\delta}\|v\|_2^2 + B_{1,\Omega}^2\delta\|\nabla w\|_2^2. \tag{25}$$

According to the Lemma 2, we obtain

$$\|w\|_2^2 \le B_{1,\Omega}^2\|\nabla w\|_2^2. \tag{26}$$

$$\|w\|_p^p \leq B_{p-1,\Omega}^p \left(\frac{2pE(0)}{p-2} \right)^{\frac{p-2}{2}} \|\nabla w\|_2^2. \tag{27}$$

Again, through Lemma 4 and Young's inequality, we obtain

$$\begin{aligned}
\int_{\Gamma_0} \int_{-\infty}^{+\infty} \frac{v(y,t)}{(\xi^2+\eta)} \psi(\xi) \Phi(\xi,t) d\xi d\rho \leq{}& \frac{1}{4\delta} \int_{\Gamma_0} \int_{-\infty}^{+\infty} \frac{|\Phi(\xi,t)|^2}{(\xi^2+\eta)} d\xi d\rho \\
&+ \delta \int_{\Gamma_0} |v(y,t)|^2 \int_{-\infty}^{+\infty} \frac{|\Psi^2(\xi)|^2}{(\xi^2+\eta)} d\xi d\rho \\
\leq{}& \frac{1}{4\delta} \int_{\Gamma_0} \int_{-\infty}^{+\infty} \frac{|\Phi(\xi,t)|^2}{(\xi^2+\eta)} d\xi d\rho + \delta \int_{\Gamma_0} |v(y,t)|^2 A_0 d\rho \\
\leq{}& \frac{1}{4\delta} \int_{\Gamma_0} \int_{-\infty}^{+\infty} \frac{|\Phi(\xi,t)|^2}{(\xi^2+\eta)} d\xi d\rho + \delta \|v(y,t)\|^2 A_0.
\end{aligned} \tag{28}$$

By using inequality of [8] and Lemma 2, we obtain

$$\int_\Omega |w|^p ln|w|^k dy \leq k\|\nabla w\|_2^2. \tag{29}$$

From (25)–(29), we obtain

$$\begin{aligned}
\frac{2}{\varepsilon}\mathcal{F}_2(t) - \mathcal{F}_1(t) \geq{}& \left(\mu - \frac{\varepsilon}{2} B_{1,\Omega}^2 - \varepsilon\delta B_{1,\Omega}^2 - \frac{2k}{p^2} B_{p-1,\Omega}^p \left(\frac{2pE(0)}{p-2} \right)^{\frac{p-2}{2}} - \frac{p-1}{p} k \right) \|\nabla w\|_2^2 \\
&+ \left(-2 + \frac{4}{\varepsilon} - 2\frac{b_1}{\varepsilon}\delta A_0 - \frac{\varepsilon}{4\delta} \right) \|v\|_2^2 \\
&+ \frac{2b_1}{\varepsilon} \|\Phi\|_2^2 - \left(\frac{b_1}{2\varepsilon\delta} + \frac{b_1}{2} \right) \int_{\Gamma_0} \int_{-\infty}^{+\infty} \frac{|\Phi(\xi,t)|^2}{(\xi^2+\eta)} d\xi d\rho.
\end{aligned} \tag{30}$$

We choose $\varepsilon > 0$ such that

$$\begin{cases}
\mu - \frac{\varepsilon}{2} B_{1,\Omega}^2 - \varepsilon\delta B_{1,\Omega}^2 - \frac{2k}{p^2} B_{p-1,\Omega}^p \left(\frac{2pE(0)}{p-2} \right)^{\frac{p-2}{2}} - \frac{p-1}{p} k \geq 0, \\
-2 + \frac{4}{\varepsilon} - \frac{2b_1}{\varepsilon}\delta A_0 - \frac{\varepsilon}{4\delta} \geq 0, \\
\varepsilon - 1 < 0, \\
\frac{2}{p}k - \mu + \varepsilon C_s^2 \leq 0.
\end{cases}$$

Consequently,

$$\frac{2}{\varepsilon}\mathcal{F}_2(t) - \mathcal{F}_1(t) \geq -\left(\frac{b_1}{2\varepsilon\delta} + \frac{b_1}{2} \right) \int_{\Gamma_0} \int_{-\infty}^{+\infty} \frac{|\Phi(\xi,t)|^2}{(\xi^2+\eta)} d\xi d\rho; \tag{31}$$

then,

$$\mathcal{F}_2(t) \geq \frac{\varepsilon}{2}\mathcal{F}_1(t) - \frac{\varepsilon b_1}{4} \left(\frac{1}{\varepsilon\delta} + 1 \right) \int_{\Gamma_0} \int_{-\infty}^{+\infty} \frac{|\Phi(\xi,t)|^2}{(\xi^2+\eta)} d\xi d\rho. \tag{32}$$

By substituting (31) into (23), we obtain a differential inequality:

$$
\begin{aligned}
\frac{d}{dt}\mathcal{F}_1(t)+\frac{\varepsilon}{2}\mathcal{F}_1(t) \le{}& \frac{\varepsilon b_1}{4}\left(\frac{1}{\varepsilon\delta}+1\right)\int_{\Gamma_0}\int_{-\infty}^{+\infty}\frac{|\Phi(\xi,t)|^2}{(\xi^2+\eta)}d\xi d\rho \\
&\le \gamma\mathcal{F}_1(t)-\gamma\Big(\|v\|^2+\mu\|\nabla w\|^2+(\lambda+\mu)\|divw\|^2-\frac{2}{p}\int_\Omega |w|^p ln|w|^k dy \\
&+\frac{2k}{p^2}\int_\Omega |w|^p dy+\frac{b_1}{2}\int_{\Gamma_0}\int_{-\infty}^{+\infty}\frac{|\Phi(\xi,t)|^2}{\xi^2+\eta}d\xi d\rho+\varepsilon\int_\Omega vwdy+\frac{\varepsilon}{2}\|w\|^2\Big) \\
&+\frac{\varepsilon b_1}{4}\left(\frac{1}{\varepsilon\delta}+1\right)\int_{\Gamma_0}\int_{-\infty}^{+\infty}\frac{|\Phi(\xi,t)|^2}{(\xi^2+\eta)}d\xi d\rho \\
&\le \gamma\mathcal{F}_1(t)-\gamma\|v\|^2-\gamma\mu\|\nabla w\|^2+\frac{2\gamma}{p}\int_\Omega |w|^p ln|w|^k dy-\gamma\varepsilon\int_\Omega vwdy \\
&+\left[\frac{\varepsilon b_1}{4}\left(\frac{1}{\varepsilon\delta}+1\right)-\frac{\gamma b_1}{2}\right]\int_{\Gamma_0}\int_{-\infty}^{+\infty}\frac{|\Phi(\xi,t)|^2}{(\xi^2+\eta)}d\xi d\rho.
\end{aligned} \tag{33}
$$

On the other hand,

$$\int_\Omega vwdy \ge -\|w\|^2-\|v\|^2. \tag{34}$$

Combining with (29), (34), and (33), we have

$$
\begin{aligned}
\frac{d}{dt}\mathcal{F}_1(t)+\left(\frac{\varepsilon}{2}-\gamma\right)\mathcal{F}_1(t) \le{}& \gamma\left(\frac{2k}{p}-\mu+\varepsilon C_s\right)\|\nabla w\|^2-\gamma(\varepsilon-1)\|v\|^2 \\
&+\left[\frac{\varepsilon b_1}{4}\left(\frac{1}{\varepsilon\delta}+1\right)-\frac{\gamma b_1}{2}\right]\int_{\Gamma_0}\int_{-\infty}^{+\infty}\frac{|\Phi(\xi,t)|^2}{(\xi^2+\eta)}d\xi d\rho.
\end{aligned} \tag{35}
$$

We choose γ such that

$$
\begin{cases}
\frac{\varepsilon}{2}-\gamma>0, \\
\frac{\varepsilon b_1}{4}\left(\frac{1}{\varepsilon\delta}+1\right)-\frac{\gamma b_1}{2}<0.
\end{cases}
$$

Hence,

$$\frac{d}{dt}\mathcal{F}_1(t)+\left(\frac{\varepsilon}{2}-\gamma\right)\mathcal{F}_1(t)\le 0\le\frac{\varepsilon}{2}-\gamma, \tag{36}$$

Next, a simple integration of (36) yields

$$\mathcal{F}_1(t)\le(\mathcal{F}_1(0)-1)\exp^{-(\frac{\varepsilon}{2}-\gamma)t}+1. \tag{37}$$

Since $\xi^2<\eta$, we obtain $\frac{1}{\xi^2+\eta}>\frac{1}{2\eta}$ and deduce that

$$\int_{\Gamma_0}\int_{-\infty}^{+\infty}\frac{|\Phi(\xi,t)|^2}{\xi^2+\eta}d\xi d\rho\ge\frac{1}{2\eta}\|\Phi\|_2^2. \tag{38}$$

From (29), (34), (38), and (24),

$$
\begin{aligned}
\mathcal{F}_1(t) &\ge \|v\|_2^2+\frac{b_1}{2\eta}\|\Phi\|_2^2+(\mu-\frac{2k}{p})\|\nabla w\|_2^2-\varepsilon\|w(t)\|_2^2-\varepsilon\|v\|_2^2+\frac{\varepsilon}{2}\|w(t)\|_2^2 \\
&\ge \tilde{c}\|w(t),v(t),\Phi(t)\|^2,
\end{aligned} \tag{39}
$$

in which $\tilde{c}=\min\left\{\frac{1}{2},\frac{b_1}{2\eta},(\mu-\frac{2k}{p})C_s-\frac{\varepsilon}{2}\right\}$.

So, by combining (39) with (12), we obtain

$$\tilde{c}\|w(t),v(t),\Phi(t)\|^2\le\mathcal{F}_1(t)\le(\mathcal{F}_1(0)-1)\exp^{-(\frac{\varepsilon}{2}-\gamma)t}+1. \tag{40}$$

From the aforementioned inequality, it is deduced that no mild solution $\mathcal{Z}(t) = (u(t), v(t), \Phi(t))$ can experience blow-up. Hence, for $t \geq 0$, all the solutions exist globally. In addition to this, the following is obtained:

$$_{t\to\infty}\|(w(t),\, v(t),\, \Phi(t))\|_E^2 \leq \frac{1}{\tilde{c}}. \tag{41}$$

Then, we have

$$B_r = \{z \in E : \|z\|_E \leq r\},$$

which is an absorbing set with constant $r > \sqrt{\frac{1}{\tilde{c}}}$. It is important to observe that the solution semigroup is characterized by the definition $S(t)\mathcal{Z}_0 = \mathcal{Z}(t;\mathcal{Z}_0)$, where t is a positive constant. □

4. Blow-Up

Here, our focus is on the blow-up in the case of negative energies. We suppose that

$$J(t) = \int_0^t \|w\|_2^2 ds + \|w\|_2^2 + b_1 L(t), \tag{42}$$

where

$$L(t) = \int_0^t \int_{\Gamma_0} \int_{-\infty}^{+\infty} (\xi^2+\eta)\left(\int_0^s \Phi(\xi,z)\right)^2 d\xi d\rho ds. \tag{43}$$

Lemma 8 ([2], Lemma 5.1)**.** *Let us assume that* $\|\nabla w\|_2^2$ *is bounded on* $[0,\, T)$*,; then,*

$$L(t) \leq C < +\infty. \tag{44}$$

More accurately,

$$L(t) \leq \frac{1}{2} C_1 B_{1,\Gamma_0}^2 e^{-\eta C_2}\left[C_2^{-\alpha-1}\alpha + C_2^{-\alpha}\eta\right]\Gamma(\alpha)T^4, \tag{45}$$

with

$$C_1 =_{t\in[0,\,T)} \left\{1,\, \|\nabla w\|_2^2\right\},$$

where C_2 *is a positive constant.*

Lemma 9. *Assume that* $2 < p$*, then,*

$$J''(t) \geq p\left(\|v\|_2^2 - 2E(0) + \int_0^t \|w_s\|_2^2 ds + b_1 \int_0^t \int_{\Gamma_0} \int_{-\infty}^{+\infty} (\xi^2+\eta)|\Phi(\xi,s)|^2 d\xi d\rho ds\right). \tag{46}$$

Proof. The derivative of $J(t)$ is given by

$$J'(t) = 2\int_\Omega w\, v\, dy + \|w\|_2^2 + b_1 L'(t), \tag{47}$$

where

$$L'(t) = 2\int_0^t \int_{\Gamma_0} \int_{-\infty}^{+\infty} (\xi^2+\eta)\Phi(\xi,s) \int_0^s \Phi(\xi,z) dz d\xi d\rho ds.$$

Then, its second derivative is given by

$$J''(t) = 2\|v\|_2^2 + 2\int_\Omega (w_{tt}+v)w\, dy + b_1 L''(t), \tag{48}$$

where

$$L''(t) = 2\int_{\Gamma_0} \int_{-\infty}^{+\infty} (\xi^2+\eta)\Phi(\xi,t) \int_0^s \Phi(\xi,s) ds d\xi d\rho.$$

From (12), we obtain

$$
\begin{aligned}
J''(t) = 2\Big(&\|v\|_2^2 - \mu\|\nabla w\|_2^2 - (\lambda+\mu)\|divw\|_2^2 + \int_\Omega |w|^p ln|w|^k dy\Big) \\
&- 2b_1 \int_{\Gamma_0} w(y,t) \int_{-\infty}^{+\infty} \Psi(\xi)\Phi(\xi,t)d\xi d\rho + b_1 L''(t).
\end{aligned} \tag{49}
$$

So, if we integrate (14) over $(0,t)$, then

$$
E(t) = E(0) - \int_0^t \|w_s\|_2^2 ds - b_1 \int_0^t \int_{\Gamma_0} \int_{-\infty}^{+\infty} (\xi^2+\eta)|\Phi(\xi,s)|^2 d\xi d\rho ds. \tag{50}
$$

Then, from the definition of $E(t)$, the following is obtained:

$$
\begin{aligned}
2\int_\Omega |w|^p ln|w|^k dy = 2p\Big(&-E(0) + \int_0^t \|w_s\|_2^2 ds + b_1 \int_0^t \int_{\Gamma_0} \int_{-\infty}^{+\infty} (\xi^2+\eta)|\Phi(\xi,s)|^2 d\xi d\rho ds\Big) \\
&+ p\|v\|_2^2 + \mu p\|\nabla w\|_2^2 + p(\lambda+\mu)\|divw\|_2^2 + 2\frac{k}{p}\|w\|_p^p \\
&+ b_1 p \int_{\Gamma_0} \int_{-\infty}^{+\infty} |\Phi(\xi,t)|^2 d\xi d\rho.
\end{aligned} \tag{51}
$$

By substituting (51) into (49), we obtain

$$
\begin{aligned}
J''(t) = (p+2)\|v\|_2^2 &+ \mu(p-2)\|\nabla w\|_2^2 + (p-2)(\lambda+\mu)\|divw\|_2^2 + -2pE(0) \\
&+ 2p\int_0^t \|w_s\|_2^2 ds + 2\frac{k}{p}\|w\|_p^p - 2b_1 \int_{\Gamma_0} w(y,t) \int_{-\infty}^{+\infty} \Psi(\xi)\Phi(\xi,t)d\xi d\rho + b_1 L''(t) \\
&+ 2pb_1 \int_0^t \int_{\Gamma_0} \int_{-\infty}^{+\infty} (\xi^2+\eta)|\Phi(\xi,s)|^2 d\xi d\rho ds + b_1 p \int_{\Gamma_0} \int_{-\infty}^{+\infty} |\Phi(\xi,t)|^2 d\xi d\rho.
\end{aligned} \tag{52}
$$

Using Lemma 6, one can prove that

$$
\begin{aligned}
J''(t) =&(p+2)\|v\|_2^2 + \mu(p-2)\|\nabla w\|_2^2 + (p-2)(\lambda+\mu)\|divw\|_2^2 + -2pE(0) \\
&+ 2p\int_0^t \|w_s\|_2^2 ds + 2\frac{k}{p}\|w\|_p^p + b_1(p-2)\int_{\Gamma_0} \int_{-\infty}^{+\infty} |\Phi(\xi,t)|^2 d\xi d\rho \\
&+ 2pb_1 \int_0^t \int_{\Gamma_0} \int_{-\infty}^{+\infty} (\xi^2+\eta)|\Phi(\xi,s)|^2 d\xi d\rho ds.
\end{aligned} \tag{53}
$$

Since $p > 2$, we can write

$$
\begin{aligned}
J''(t) \geq &(p+2)\|v\|_2^2 - 2pE(0) + 2p\int_0^t \|w_s\|_2^2 ds \\
&+ 2pb_1 \int_0^t \int_{\Gamma_0} \int_{-\infty}^{+\infty} (\xi^2+\eta)|\Phi(\xi,s)|^2 d\xi d\rho ds.
\end{aligned} \tag{54}
$$

Since b_1 and η are non-negative, one can estimate (54) by

$$
J''(t) \geq p\Big(\|v\|_2^2 - 2E(0) + \int_0^t \|w_s\|_2^2 ds + b_1 \int_0^t \int_{\Gamma_0} \int_{-\infty}^{+\infty} (\xi^2+\eta)|\Phi(\xi,s)|^2 d\xi d\rho ds\Big).
$$

□

Lemma 10. *For $2 < p$, if the initial energy is non-negative, then $J'(t) > \|w_0\|_2^2$, with $t > \max\Big\{0, \frac{J'(0)-\|w_0\|_2^2}{2pE(0)}\Big\}$.*

Proof. According to the Lemma 9, we obtain

$$J''(t) \geq -2pE(0).$$

Integrating the above over $(0,t)$, we have

$$J'(t) \geq J'(0) - 2pE(0)t. \tag{55}$$

After that, we have

$$J'(t) - \|w_0\|_2^2 \geq J'(0) - \|w_0\|_2^2 - 2pE(0)t.$$

Consequently, $J'(t) > \|w_0\|_2^2, \ \forall t > \max\left\{0, \frac{J'(0)-\|w_0\|_2^2}{2pE(0)}\right\}$. □

Theorem 3. *Assume $p > 2$ and that $E(0) < 0$. Then, according to Definition 1, the solution (w, Φ) blows up at T^*, with*

$$T^* \leq t_0 - \frac{\varphi(t_0)}{\varphi'(t_0)}, \tag{56}$$

where T^* is finite time and

$$\varphi(t) = \left(J(t) + (T-t)\|w_0\|_2^2\right)^{-\gamma_1}. \tag{57}$$

Moreover, if $\varphi(t_0) < min\left\{1, \sqrt{\frac{\sigma}{-m}}\right\}$, we have

$$T^* \leq t_0 + \frac{1}{\sqrt{-m}} ln \frac{\sqrt{-\sigma/m}}{\sqrt{-\sigma/m} - \varphi(t_0)}, \tag{58}$$

where m and σ are two constants to be determined later.

Proof. From $\varphi(t)$, we have the following:

$$\begin{aligned} \varphi'(t) &= -\gamma_1\left(J'(t) - \|w_0\|_2^2\right)\left(J(t) + (T-t)\|w_0\|_2^2\right)^{-\gamma_1 - 1} \\ &= -\gamma_1\left(J'(t) - \|w_0\|_2^2\right)(\varphi(t))^{1+(1/\gamma_1)}. \end{aligned} \tag{59}$$

Then, $\varphi'(t)$ implies that

$$\varphi''(t) = -\gamma_1 \varphi^{1+2/\gamma_1}(t)\left(J''(t)\left(J(t) + (T-t)\|w_0\|_2^2\right) - (1+\gamma_1)\left(J'(t) - \|w_0\|_2^2\right)^2\right), \tag{60}$$

and set

$$H(t) = J''(t)\left(J(t) + (T-t)\|w_0\|_2^2\right) - (1+\gamma_1)\left(J'(t) - \|w_0\|_2^2\right)^2.$$

By applying the Lemma 9 and (47), we obtain

$$H(t) \geq -p\left(b^2 - ac + 2E(0)\varphi^{-1/\gamma_1}\right), \tag{61}$$

where

$$a = \left(\|w\|_2^2 + \int_0^t \|w\|_2^2 ds + b_1 \int_0^t \int_{\Gamma_0} \int_{-\infty}^{+\infty} (\xi^2 + \eta)\left(\int_0^s |\Phi(\xi, z)dz\right)^2 d\xi d\rho ds\right),$$

$$c = \left(\|v\|_2^2 + \int_0^t \|w_s\|_2^2 ds + b_1 \int_0^t \int_{\Gamma_0} \int_{-\infty}^{+\infty} (\xi^2 + \eta)|\Phi(\xi, s)|^2 d\xi d\rho ds\right),$$

and

$$b^2 = \left(\int_\Omega wvdy + \int_0^t \int_\Omega w_s wdyds + b_1 \int_0^t \int_{\Gamma_0} \int_{-\infty}^{+\infty} (\xi^2+\eta)\Phi(\xi,s) \int_0^s \Phi(\xi,z)dzd\xi d\rho ds\right)^2.$$

$$\forall y \in \mathbb{R},$$

$$\begin{aligned} ax^2 + 2bx + c = {} & \|w\|_2^2 x^2 + 2\left(\int_\Omega wvdx\right)x + \|v\|_2^2 \\ & + \int_0^t \left(\|w\|_2^2 x^2 + 2\left(\int_\Omega w_s wdy\right)x + \|w_s\|_2^2\right)ds \\ & + b_1 \int_0^t \int_{\Gamma_0} \int_{-\infty}^{+\infty} (\xi^2+\eta)\left[\left(\int_0^s \Phi(\xi,z)dz\right)^2 x^2 \right. \\ & \left. + 2\left(\Phi(\xi,s)\int_0^s \Phi(\xi,z)dz\right)x + |\Phi(\xi,s)|^2\right] d\xi d\rho ds \geq 0. \end{aligned} \tag{62}$$

It is clear that $b^2 - ac$ is negative. Consequently,

$$H(t) \geq -2pE(0)\varphi^{-1/\gamma_1}, \;\; t \geq t_0. \tag{63}$$

From (60) and (63), we also obtain

$$\varphi''(t) \leq 2p\gamma_1 E(0)\varphi^{1+1/\gamma_1}, \tag{64}$$

since $\varphi'(t) < 0$. Furthermore, multiplying the equation in (64) by $\varphi'(t)$ and integrating over (t^*, t), the following is obtained:

$$(\varphi'(t))^2 \geq (\varphi'(t^*))^2 + \frac{4p\gamma_1^2}{2\gamma_1+1}E(0)(\varphi(t))^{2+1/\gamma_1} - \frac{4p\gamma_1^2}{2\gamma_1+1}E(0)(\varphi(t^*))^{2+1/\gamma_1}) \tag{65}$$

$$(\varphi'(t))^2 \geq m(\varphi(t))^{2+1/\gamma_1} + \sigma, \tag{66}$$

where $m = \frac{4p\gamma_1^2}{2\gamma_1+1}E(0) < 0$ and $\sigma = (\varphi'(t^*))^2 - \frac{4p\gamma_1^2}{2\gamma_1+1}E(0)(\varphi(t^*))^{2+1/\gamma_1}\ \sigma > 0$. By using Lemma 3, one can find a T^* in a way that

$$\lim_{t \to T^{*-}} \varphi(t) = 0.$$

Thus, we have

$$\lim_{t \to T^{*-}} (J(t))^{-1} = 0, \tag{67}$$

that is to say

$$\lim_{t \to T^{*-}} J(t) = +\infty. \tag{68}$$

Applying Lemma 8, the definition of $J(t)$, and (68), one can find a T in a manner that

$$\|\nabla u\|_2^2 \to +\infty \;\; as\ t \to T^-.$$

□

5. Conclusions

In this paper, we use a semigroup theory approach to offer a comprehensive solution to an initial boundary value problem associated with a wave equation featuring logarithmic nonlinear source terms and fractional boundary dissipation. Through our analysis, we not only provide a global solution but also establish a noteworthy blow-up result for the solution. The identification of a blow-up phenomenon under the condition of non-positive initial energy adds a significant dimension to our understanding of the system's behavior, shedding light on the intricate interplay of nonlinearities and fractional dissipation in wave dynamics. This work contributes to the broader exploration of complex mathematical models, offering insights that advance our comprehension of wave equations with unique nonlinear and dissipative characteristics.

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