mathematics

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Best Approximation of Fixed-Point Results for Branciari Contraction of Integral Type on Generalized Modular Metric Space

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Abstract: In the realm of generalized modular metric spaces, we substantiate the validity of fixed-point theorems with Branciari contractions. This paper expands and broadens the original theorems in this context. Subsequently, by building upon this foundation, we explore various integral contractions to identify and characterize fixed points within this context. To highlight the practical implications of our work, we introduce the concept of the best proximity pair, thereby culminating in the best approximation theorem. We apply this theoretical construct to a specific example—one that is guided by the best approximation method described in prior research.

Keywords: fixed-point theorem; generalized metric space; modular metric space; integral-type contraction; best approximation

MSC: 54H25; 47H10

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Citation: Manav Tatar, N.; Agarwal, R.P. Best Approximation of Fixed-Point Results for Branciari Contraction of Integral Type on Generalized Modular Metric Space. *Mathematics* **2023**, *11*, 4455. https://doi.org/10.3390/math11214455

Academic Editor: Salvatore Sessa

Received: 23 September 2023
Revised: 24 October 2023
Accepted: 26 October 2023
Published: 27 October 2023

1. Introduction and Preliminaries

Banach's classical contraction mapping principle asserts that, in a complete metric space (X, d), if $T : X \to X$ is a contraction mapping, this implies the existence of a unique fixed point. This foundational principle has garnered substantial interest from various scholars, thereby motivating them to explore and enhance the theory of fixed points in the metric context.

In 1970, Takahashi [1] offered models of convex structures in metric spaces that were not inserted in a Banach space, a contribution that continues to inspire ongoing research in this field. He proved the existence of fixed-point theorems if there were mappings with invariant properties.

In 2002, Branciari [2] discovered a fixed-point theorem for a single-valued mapping that fulfills a correspondence of the Banach contraction for this inequality. One such exploration is exemplified by an illustrative graphical example in [3], which emphasizes the practical utility of integral-type rational contractions, as well as offering one of many instances of their application.

To pursue this objective, researchers have embarked on a journey to broaden the horizons of metric fixed-point theory by exploring various abstract spaces. One notable endeavor involves the examination of modular metric spaces, an extension initially introduced by Chistyakov in [4–6] as a generalized concept derived from modular spaces.

In 2013, Azadifar et al. [7] established an important part of a provided fixed point of integral-type compatible mappings, as well as the uniqueness of their existence in modular metric spaces.

In 2021, the authors of [8] developed the approach of Gupta et al. [9] to a common fixed-point theorem for the contraction of integral types to investigate the breadth of the Banach fixed-point theorem in modular metric spaces, and this was achieved by studying the contractive conditions that contain integral types. They provided numerous theorems that illustrated the presence and singularity of a common fixed point among the self-mappings that satisfy integral variety contraction requirements. They also gave various corollaries and instances to show the validity of their results.

On the other hand, a fundamental classical best-approximation theorem attributed to Fan [10] provides assurances regarding the existence of an element belonging to a set.

O'Regan and Shahzad [11] examined Fan's approximation result in a multivalued form in 2003. In 2005, Hussain et al. [12] examined the example of non-star-shaped input values and common fixed-point findings for operators on non-star-shaped domains, as well as offering Brosowski–Meinardus-type approximation theorems as applications.

In 2021, Malih [13] explained the best approximation concept in metric space for contractions of an integral type via several methods. The investigation of integral-type contractions, along with the approach to best proximity points, was also generalized to encompass the study of metric spaces, as exemplified in the scholarly work presented in the studies of [14,15].

Reich's [16] study involved combining the results obtained in earlier sections, thereby ultimately leading to the formulation of novel fixed-point theorems. Furthermore, he introduced a distinctive version of Fan's [10] fixed-point theorem by integrating the principles of approximation theory into the realm of fixed-point theory.

The authors of [17] presented a theorem concerning proximity pairs, in which the essential conditions for verifying the presence of a specific element were outlined. Moreover, the authors established the correlations between the best approximation theorem and the theorem on proximity pairs.

Furthermore, Asadi et al., as documented in [18], have played an instrumental role in the development of fixed-point theory within the realm of modular metric spaces. They introduced the concept of ω-proximal quasi-contraction mappings and the notion of the best ω-proximity point within the context of modular metric spaces. Notably, their pioneering work has not only advanced modular metric spaces, but has also paved the way for the applicability of best proximity point results in the broader context of generalized modular metric spaces.

The authors of one study formulated a generalization of the modular metric notion that can be applied to any abstract set, whereby a more dynamic and varied application of this mathematical framework was substituted [19]. They also introduced certain important definitions that supplement and improve upon our understanding of the expanded modular metric notion. This section outlines these topics and provides comprehensive explanations of them.

Definition 1 ([19])**.** *Let M be a set, and $U : (0,\infty) \times M \times M \to [0,\infty]$ be a function with the following properties:*

- (U_1) *If $U_p(\xi,\varrho) = 0$ for all $p > 0$, then $\xi = \varrho$ for all $\xi,\varrho \in M$.*
- (U_2) *$U_p(\xi,\varrho) = U_p(\varrho,\xi)$ for all $p > 0$ and all $\xi,\varrho \in M$.*
- (U_3) *There exists a constant $K > 0$ such that if $(\xi,\varrho) \in M \times M$ and $\{\xi_n\} \subset M$ with $\lim_{n\to\infty} U_p(\xi_n,\xi) = 0$ for some $p > 0$, then:*

$$U_p(\xi,\varrho) \le K \limsup_{n\to\infty} U_p(\xi_n,\varrho).$$

Then, the pair (M,U) forms a generalized modular metric space.

If there exist ξ and ϱ in M such that $\{\xi_n\} \subset M$ and $\lim_{n\to\infty} U_p(\xi_n,\xi) = 0$ for some $p > 0$ and $U_p(\xi,\varrho) < \infty$, then we must have $K \ge 1$.

Let us fix $\xi_0 \in M$, while U_p (abbreviated as U) is a generalized modular metric on M. Consider the set

$$M_U = M_U(\xi_0) = \{\xi \in M : U_p(\xi,\xi_0) \to 0 \text{ as } p \to \infty\}.$$

This set is known as a generalized modular set. We provide additional useful definitions below.

Definition 2 ([19]). *Let (M_U, U) be a generalized modular metric space. We define the following concepts:*

(1) *A sequence $\{\xi_n\}_{n\in\mathbb{N}}$ in M_U is called U-convergent to $\xi \in M_U$ if and only if $U_p(\xi_n, \xi) \to 0$, as $n \to \infty$, when some $p > 0$.*
(2) *A sequence $\{\xi_n\}_{n\in\mathbb{N}}$ in M_U is said to be U-Cauchy if $U_p(\xi_m, \xi_n) \to 0$ for any m, and $n \to \infty$ when $p > 0$.*
(3) *A subset A of M_U is called U-closed if, for any $\{\xi_n\}$ from A that U-converges to ξ (where $\xi \in A$), we denote the set of all nonempty U-closed subsets of M_U as $\mathcal{V}(M_U)$.*
(4) *A subset A of M_U is called U-complete if there exists a point $\xi \in A$ such that $\lim_{n\to\infty} U_p(\xi_n, \xi) = 0$ for any $\{\xi_n\}$ that forms a U-Cauchy sequence, and $\lim_{n,m\to\infty} U_p(\xi_n, \xi_m) = 0$ for some p.*

The definition of ω-compatible mapping is as follows: if two self-mappings, such as K and L on M_ω space, exist with the property $\omega_p(KL(\xi_n), LK(\xi_n)) \to 0$ while $\{\xi_n\}_{n=1}^{\infty}$ is a sequence, and when $K\xi_n \to \varrho$, $L\xi_n \to \varrho$, then the following is the case for some $\varrho \in M_\omega$ when $p > 0$.

Theorem 1 ([7]). *Let M_ω be a complete modular metric space, and let $l, m \in \mathbb{R}$ be $l > m$. Consider the self-compatible mappings L and $K : M_\omega \to M_\omega$ such that they satisfy the inclusion condition as follows:*

$$L(M_\omega) \subseteq K(M_\omega).$$

Additionally, if for some $\aleph \in (0,1)$ and for all $\xi, \varrho \in M_\omega$ and $p > 0$, then the following inequality holds:

$$\int_0^{\omega_{p/l}(L\xi, L\varrho)} \kappa(\vartheta)d\vartheta \le \aleph \int_0^{\omega_{p/m}(K\xi, K\varrho)} \kappa(\vartheta)d\vartheta, \tag{1}$$

where $\kappa : \mathbb{R}^+ \to \mathbb{R}^+$ is a non-negative Lebesgue integrable and summable function with $\int_0^{\epsilon} \kappa(\vartheta)d\vartheta > 0$ for every time $\epsilon > 0$. Furthermore, if either K or L are continuous, there exists a unique fixed point for both mappings.

In 2021, Kerim et al. [8] proved some common fixed-point theorems in modular metric spaces, as well as providing certain examples about integral-type contractions.

Theorem 2 ([8]). *Let M_ω be a complete modular metric space with modular metric ω and let $L, K : M_\omega \to M_\omega$ be self-compatible mappings that satisfy*

$$L(M_\omega) \subseteq K(M_\omega)$$

$$\int_0^{\omega_p(L\xi, L\varrho)} \kappa(\vartheta)d\vartheta \le \beta(\omega_p(K\xi, K\varrho)) \int_0^{\omega_p(K\xi, K\varrho)} \kappa(\vartheta)d\vartheta - \psi(\int_0^{\omega_p(K\xi, K\varrho)} \kappa(t)dt) \tag{2}$$

for all $\xi, \varrho \in M_\omega$, where $(\kappa, \psi) \in (\Phi_1, \Phi_2)$ and $\beta : \mathbb{R} \to [0,1)$ is a function with $\limsup_{s\to\vartheta} \beta(s) < 1$, for all $\vartheta > 0$. Then, L and K have a unique fixed point $u \in M_\omega$.

In the article, Φ_1 refers to the family of all functions $\kappa : [0,\infty) \to [0,\infty)$ such that:

(1) κ is continuous and non-decreasing;
(2) $\kappa(\vartheta) = 0$ if and only if $\vartheta = 0$. Φ_2 is denoted to the set of all functions $\kappa : [0,\infty) \to [0,\infty)$ such that ϕ is a Lebesgue integrable function with $\int_0^{\nu} \kappa(\vartheta)d\vartheta > 0$ for all $\nu > 0$, which are summable and non-negative.

Then, $\psi : [0,\infty) \to [0,\infty)$ while $\psi(0) = 0$ is the collection of all functions, which is Φ_3.

In 1976, Jungck [20] subsequently developed a more generalized commutativity, known as compatibility, which is more broad than weak commutativity. However, in this study, we establish a new definition for generalized, modular metric space [21].

Definition 3. *Let (M, u) be a metric space. Then, the self-mappings K and L are said to be compatible if* $\lim_{n\to\infty} u(KL(\xi_n), LK(\xi_n)) = 0$ *whenever* $\{\xi_n\}_{n=1}^{\infty}$ *is a sequence in M such that* $\lim_{n\to\infty} L(\xi_n) = \lim_{n\to\infty} K(\xi_n) = z$ *for some* $z \in M$ *and when* $p > 0$.

Jungck [20] presented a fixed-point theorem for commuting maps that generalized Banach's principle. Various writers subsequently generalized and expanded his findings in numerous ways, whereas Sessa [22] specified weak commutativity in metric spaces.

Definition 4 ([22])**.** *Let (M, u) be a metric space. Then, the self-mappings K and L are said to be weakly commuting if:*

$$u(KL(\xi), LK(\xi)) \leq u(L(\xi), K(\xi))$$

for all $\xi \in M$ and when $p > 0$.

In 2002, the authors of [23] gave the definition of the E.A. property, and they proved that if weakly-compatible self-mappings have this property, then they have common fixed points.

Definition 5 ([23])**.** *Let K and L be two self-mappings of a metric space (M, u). If there exists a sequence (ξ_n) such that*

$$\lim_{n\to\infty} K\xi_n = \lim_{n\to\infty} L\xi_n = \vartheta \tag{3}$$

for some $\vartheta \in M$, then K and L satisfy the E.A. property.

Theorem 3 ([23])**.** *Let K and L be two weakly-compatible self-mappings of a metric space (M, u) such that:*

(i) *K and L satisfy the E.A. property;*
(ii) $u(K\xi, K\varrho) < \max\{u(L\xi, L\varrho), [u(K\xi, Lx) + u(K\varrho, L\varrho)]/2, [u(K\varrho, L\xi) + u(K\xi, L\varrho)]/2\}$, $\xi \neq \varrho \in M$;
(iii) $KM \subseteq LM$.

If KM or LM are complete subspaces of M, then K and L have a unique fixed point.

Following this, Malih [13] revealed weakly-compatible self-mappings that satisfy the condition of integral-type contractions; in addition, it was also shown that if they satisfy as such, then they have a unique fixed point.

Theorem 4 ([13])**.** *Let K and L be weakly-compatible self-mappings of a subset M of metric space (A, u) satisfying the following condition: $clK(M) \subseteq L(M)$, $cl(K(M))$. Or, if $L(M)$ is complete, we have the following:*

$$\int_0^{u(K(\xi),K(\varrho))} \kappa(\vartheta)d\vartheta \leq \aleph \int_0^{u(L(\xi),L(\varrho))} \kappa(\vartheta)d\vartheta \tag{4}$$

for each $\xi, \varrho \in M$, $\aleph \in [0, 1)$, where $\kappa : [0, \infty) \to [0, \infty)$ is a Lebesgue-integrable function, which is non-negative and summable on each compact subset $[0, \infty)$ such that for each $\epsilon > 0$, $\int_0^{\epsilon} \kappa(\vartheta)d\vartheta > 0$. Then, K and L have a unique fixed point.

Khan and Khan [24] gave the following definitions:

Definition 6 ([24])**.** *Let (M, u) be a given metric space, $V \subseteq M$ be a nonempty subset, and $\xi \in M$. We define the following concepts:*

(i) *The distance from ξ to V is given by $u(\xi, V) = \inf\{u(\xi, \varrho); \varrho \in V\}$.*
(ii) *If $\varrho \in V$ and $\xi \in M$, ξ is called a ϱ-approximation to ξ if $u(\xi, \varrho) = u(\xi, V) = \inf\{u(\xi, z); z \in V\}$, which is according to V.*
(iii) *The best approximation of ξ is defined as $P_V(\xi) = \{\varrho \in V; u(\xi, \varrho) = u(\xi, V)\}$ from V.*
(iv) *When $P_V(\xi)$ is nonempty for all $\xi \in M$, V is called proximinal.*
(v) *Proximinality is defined as when $P_V(\xi)$ is bounded and V is closed; as such, $P_V(\xi)$ is closed.*
(vi) *If $K : M \to M$ is a mapping with $K(V) \subseteq V$, then V is called a K-invariant subset of M.*

Additionally, Khan and Khan [24] also demonstrated an example of invariant approximation with the following theorem in metric spaces.

Theorem 5 ([24])**.** *Let (M, u) be a metric space with a $K : M \to M$ non-expansive mapping, and let $\Gamma \in M$ be a fixed point of K. If V is a closed K-invariant subset of M, and if $K : M \to M$ has the restriction $K|V : V \to M$ and this mapping is a compact mapping, then the best approximation set $P_V(\Gamma)$ is not empty.*

In his work, Malih [13] merged Theorem 1.4 and Theorem 1.5 from metric spaces to create what is now known as Theorem 2.2. Similarly, we undertake a comparable approach to theorems in this paper.

2. Main Results

In this section, we rigorously present and prove theorems within the framework of generalized modular metric spaces from Branciari's perspective. Theorem 2.1, along with several subsequent theorems, was originally introduced in [25] without accompanying proof. However, in this paper, we provide a comprehensive and rigorous proof for Theorem 2.1 and the following theorems, contributing to a deeper understanding of their underlying principles.

Theorem 6. *Let (M, U_p) be a U-complete generalized modular metric space, and let $\aleph \in (0, 1)$ and $K : M \to M$ be a Branciari contraction (BC) such that for all $\xi, \varrho \in M$,*

$$\int_0^{U_p(K(\xi),K(\varrho))} \kappa(\vartheta) d\vartheta \leq \aleph \int_0^{U_p(\xi,\varrho)} \kappa(\vartheta) d\vartheta. \tag{5}$$

Let there be a Lebesgue-integrable mapping summable (LIMS) (i.e., with a finite integral) on an all-compact subset of $[0, \infty)$ that is non-negative when all $\nu > 0$, $\int_0^{\nu} \kappa(\vartheta) d\vartheta > 0$ is $\kappa : \mathbb{R}^+ \to \mathbb{R}^+$. Then, K has a unique fixed point $a \in M$ such that there is for all $\xi \in M$, $\lim_{n\to\infty} K^n(\xi) = a$ when all $p > 0$.

Proof. Let us take

$$\int_0^{U_p(K^n(\xi),K^{n+1}(\xi))} \kappa(\vartheta) d\vartheta \leq \aleph \int_0^{U_p(K^{n-1}(\xi),K^n(\xi))} \kappa(\vartheta) d\vartheta \leq \ldots \leq \aleph^n \int_0^{U_p(\xi,K(\xi))} \kappa(t) dt. \tag{6}$$

There is an n-times iteration of (5), as follows:

$$\int_0^{U_p(K^n(\xi),K^{n+1}(\xi))} \kappa(\vartheta) d\vartheta \leq \aleph^n \int_0^{U_p(\xi,K(\xi))} \kappa(\vartheta) d\vartheta. \tag{7}$$

Since $\aleph \in (0, 1)$, we have:

$$\int_0^{U_p(K^n(\xi),K^{n+1}(\xi))} \kappa(\vartheta) d\vartheta \to 0_+ \text{ as } n \to \infty. \tag{8}$$

Then, we have $U_p(K^n(\xi), K^{n+1}(\xi)) \to 0$ as $n \to \infty$. Thus, we select the following:

$$\lim_{n\to\infty} \sup(U_p(K^n(\xi), K^{n+1}(\xi))) = \nu > 0, \tag{9}$$

As such, there then exists $Q_\nu \in \mathbb{N}$ and a sequence $\{K^{n_Q}(\xi)\}_{Q \geq Q_\nu}$ such that $(U_p(K^{n_Q}((\xi), K^{n_Q+1}(\xi))) \to \nu > 0$ as $Q \to \infty$ and $(U_p(K^{n_Q}((\xi), K^{n_Q+1}(\xi))) \to \nu \geq \frac{\nu}{2}$ when all $Q \geq Q_\nu$. Therefore, we have the following:

$$0 < \int_0^{\frac{\nu}{2}} \kappa(\vartheta)d\vartheta \leq \lim_{Q\to\infty} \int_0^{U_p(K^{n_Q}((\xi), K^{n_Q+1}(\xi))} \kappa(\vartheta)d\vartheta = 0, \tag{10}$$

which is a contradiction.

For all $\xi \in M$ and $\{K^n(\xi)\}_{n\in\mathbb{N}}$ as a U-Cauchy sequence, we have the following: $\forall m, n \in \mathbb{N}, Q_\nu < m < n : U_p(K^n(\xi), K^m(\xi)) < \nu$ when all $\nu > 0$ and when there is at least one $Q_\nu \in \mathbb{N}$. Now, we assume that there exists an $\nu > 0$ such that for each $Q \in \mathbb{N}$ there are $n_Q, m_Q \in \mathbb{N}$ as $Q < m_Q < n_Q$, which means that $\nu \leq U_p(K^{n_Q}(\xi), K^{m_Q}(\xi))$. Then, we assume the minimal sense of $m_Q, Q \in \mathbb{N}$ such that $\nu \leq U_p(K^{n_Q}(\xi), K^{m_Q}(\xi))$ but $U_p(K^r(\xi), K^{m_Q}(\xi)) < \nu$ for each $r \in \{m_Q + 1, \ldots, n_Q - 1\}$ and for the sequences $\{n_Q\}_{Q>0}$ and $\{m_Q\}_{Q>0}$ as in $\mathbb{N}$.

Now, we investigate the character of $U_p(K^{n_Q}(\xi), K^{m_Q}(\xi))$ and $U_p(K^{n_Q+1}(\xi), K^{m_Q+1}(\xi))$. Primarily, we have $U_p(K^{n_Q}(\xi), K^{m_Q}(\xi)) \to \nu$ while $Q \to \infty$. This is according to (U_3) and (6), as well as (7), as follows:

$$\nu \leq U_p(K^{n_Q}(\xi), K^{m_Q}(\xi)) \leq U_p(K^{n_Q-1}(\xi), K^{n_Q}(\xi)) \tag{11}$$

$$\nu \leq U_p(K^{n_Q}(\xi), K^{m_Q}(\xi)) \leq U_p(K^{n_Q-1}(\xi), K^{m_Q}(\xi)) \tag{12}$$

which is as a result of the below equations:

$$U_p(K^{n_Q-1}(\xi), K^{n_Q}(\xi)) + \nu \to \nu_+, \tag{13}$$

while $Q \to \infty$, and thus there exists $\varrho \in \mathbb{N}$, for each $\varrho < Q$. Indeed, one of them has $U_p(K^{n_Q+1}(\xi),\, K^{m_Q+1}(\xi)) < \nu$; thus, if there exists a subsequence $(Q_\varrho)_{\varrho\in\mathbb{N}} \subseteq \mathbb{N}$ such that $\nu \leq U_p(K^{n_{Q_\varrho}+1}(\xi),\, K^{m_{Q_\varrho}+1}(\xi))$, then from (1), (9), and (10) we have the following:

$$\nu \leq U_p(K^{n_{Q_\varrho}+1}(\xi), K^{m_{Q_\varrho}+1}(\xi)) \to \nu, \tag{14}$$

while we have $\varrho \to \infty$, and then:

$$\int_0^{U_p(K^{n_{Q_\varrho}+1}(\xi), K^{m_{Q_\varrho}+1}(\xi))} \kappa(\vartheta)d\vartheta \leq \aleph \int_0^{U_p(K^{n_Q}(\xi), K^{m_Q}(\xi))} \kappa(\vartheta)d\vartheta \tag{15}$$

If we consent to $\varrho \to \infty$ on both sides of (12), then we have for $\nu > 0, \int_0^\nu \kappa(\vartheta)d\vartheta \leq \aleph \int_0^\nu \kappa(\vartheta)d\vartheta$, which is a contradiction of $\aleph \in (0,1)$, which is where the integral is positive. Accordingly, for a fixed $\varrho \in \mathbb{N}$, we have $U_p(K^{n_{Q_\varrho}+1}(\xi), K^{m_{Q_\varrho}+1}(\xi)) < \nu$ when all $\varrho < Q$. Lastly, we justify the robust characteristic that there exists a $\beta_\nu \in (0,\nu)$ and a $Q_\nu \in \mathbb{N}$ such that for each $Q_\nu < Q, Q \in \mathbb{N}$ we have $U_p(K^{n_{Q_\varrho}+1}(\xi), K^{m_{Q_\varrho}+1}(\xi)) \leq \nu - \beta_\nu$, which helps estimate the existence of a subsequence $(Q_\varrho)_{\varrho\in\mathbb{N}} \subseteq \mathbb{N}$ such that

$$U_p(K^{n_{Q_\varrho}+1}(\xi), K^{m_{Q_\varrho}+1}(\xi)) \to \nu_-$$

as $\varrho \to \infty$. Then, starting with

$$\int_0^{U_p(K^{n_{Q_\varrho}+1}(\xi), K^{m_{Q_\varrho}+1}(\xi))} \kappa(\vartheta)d\vartheta \leq k \int_0^{U_p(K^{n_{Q_\varrho}}(\xi), K^{m_{Q_\varrho}}(\xi))} \kappa(\vartheta)d\vartheta, \tag{16}$$

we obtain $\varrho \to \infty$, which we have once more when the contradiction is such that $\int_0^\nu \kappa(\vartheta)d\vartheta \leq \aleph \int_0^\nu \kappa(t)dt$. Eventually, at the end of this step, we are able to show the Cauchy frame of

$\{K^n(\xi)\}, \xi \in M$. Genuinely, for each $Q_\nu \in \mathbb{N}$, we have the following for each $Q_\nu < Q, Q \in \mathbb{N}$:

$$\nu \leq U_p(K^{n_Q}(\xi), K^{m_Q}(\xi)) \leq U_p(K^{n_Q+1}(\xi), K^{n_Q}(\xi)) \tag{17}$$

$$\nu \leq U_p(K^{n_Q}(\xi), K^{m_Q}(\xi)) \leq U_p(K^{m_Q+1}(\xi), K^{m_Q}(\xi)) \tag{18}$$

$$\nu \leq U_p(K^{n_Q}(\xi), K^{m_Q}(\xi)) \leq U_p(K^{n_Q+1}(\xi), K^{m_Q+1}(\xi)) \tag{19}$$

with which, as a result of (14)–(16), we have

$$U_p(K^{m_Q+1}(\xi), K^{m_Q}(\xi)) + U_p(K^{n_Q+1}(\xi), K^{n_Q}(\xi)) + \nu - \beta_\nu < \nu - \beta_\nu \tag{20}$$

Thus, while we have $Q \to \infty$, there is $\nu \leq \nu - \beta_\nu$, which is a contradiction. This step is proven. Now, we show the existence of a fixed point.

As M is a U-complete generalized modular metric space, a point of a in M can be established such that a serves as a fixed point. To illustrate this, consider $a = \lim_{n\to\infty} K^n(\xi)$. In fact, if $0 < U_p(a, K(a))$, then:

$$0 < U_p(a, K(a)) \leq \aleph \limsup_{n\to\infty} U_p(K^{n+1}(\xi), a) \to 0 \tag{21}$$

$$0 < U_p(a, K(a)) \leq \aleph \limsup_{n\to\infty} U_p(K^{n+1}(\xi), K(a)) \to 0 \tag{22}$$

Meanwhile, $n \to \infty$ is such that we must respectfully have both $U_p(K^{n+1}(\xi), K(a))$ and $U_p(K^{n+1}(\xi), a)$ converge to 0. As such, we have

$$\int_0^{U_p(K^{n+1}(\xi),K(a))} \kappa(\vartheta)d\vartheta \leq \aleph \int_0^{U_p(K^{n+1}(\xi),a)} \kappa(\vartheta)d\vartheta \to 0 \tag{23}$$

while $n \to \infty$ is for the proof of (19). Now, if $U_p(K^{n+1}(\xi), K(a))$ does not converge when there is $n \to \infty$, there exists a subsequence $K^{n_Q+1}(\xi)_{Q\in\mathbb{N}} \subseteq K^{n+1}(\xi)_{n\in\mathbb{N}}$ such that $\nu \leq U_p(K^{n_{Q+1}}(\xi), K(a))$ when there is a fixed $\nu > 0$, which is as a result of the next contradiction:

$$0 < \int_0^{\nu} \kappa(t)dt \leq k \int_0^{U_p(K^{n_{Q+1}}(\xi),K(a))} \kappa(\vartheta)d\vartheta \to 0 \tag{24}$$

while $Q \to \infty$.

Now, we show the uniqueness of the fixed point. Assume that there are two different points $a_1, a_2 \in M$ such that $K(a_1) = a_1$ and $K(a_2) = a_2$. Then, according to (1), we have the contradiction

$$0 < \int_0^{U_p(a_1,a_2)} \kappa(\vartheta)d\vartheta = \int_0^{U_p(K(a_1),K(a_2))} \kappa(\vartheta)d\vartheta \leq \aleph \int_0^{U_p(a_1,a_2)} \kappa(\vartheta)d\vartheta < \int_0^{U_p(a_1,a_2)} \kappa(\vartheta)d\vartheta, \tag{25}$$

which converges to 0. Moreover, the last stage proves that, for each $\xi \in M$, there is $a = K(a) = \lim_{n\to\infty} K^n(\xi)$. The proof is thus complete. □

In this paper, we argue that Theorem 6 is no longer valid if we concede that the 0 value is near zero practically everywhere for the mapping κ; thus, we establish this with the following example (please note that we cannot display a negative value for κ):

Example 1. *Let $K : \mathbb{N} \to \mathbb{N}$ and $\kappa : \mathbb{R}^+ \to \mathbb{R}^+$ be defined by*

$$K(\xi) = \begin{cases} 1 & \text{if } \xi \neq 1, \\ 2 & \text{if } \xi = 1, \end{cases} , \qquad \kappa(\xi) = \begin{cases} e^{\frac{1}{1-\vartheta}} & \text{if } \vartheta > 1, \\ 0 & \text{if } \vartheta \in [0,1]. \end{cases}$$

Thus, we choose $U : (0,\infty) \times \mathbb{N} \times \mathbb{N} \to [0,\infty]$, which is the U-complete generalized modular metric space as per the following: $\limsup_{n\to\infty} U_p(K(\xi), K(\varrho)) \leq 1$, *which is for each $\xi, \varrho \in \mathbb{N}$ when all $p > 0$ and there is the $k \in (0,1)$.*

$$0 < \int_0^{U_p(K(\xi),K(\varrho))} \kappa(\vartheta)d\vartheta \leq \int_0^1 \kappa(\vartheta)d\vartheta = 0 \leq \aleph \int_0^{U_p(\xi,\varrho)} \kappa(\vartheta)d\vartheta; \tag{26}$$

Thus, Theorem 6 is satisfied for all $\aleph \in (0,1)$, but K has no fixed point. In a similar manner, if we choose $\kappa \equiv 1$, then when all $p > 0$ and there is a $\aleph \in (0,1)$, we have the following:

$$0 < \int_0^{U_p(K(\xi),K(\varrho))} \kappa(\vartheta)d\vartheta = U_p(K(\xi), K(\varrho)) \leq 1 \leq \aleph\, U_p(\xi,\varrho) = \aleph \int_0^{U_p(\xi,\varrho)} \kappa(\vartheta)d\vartheta; \tag{27}$$

but, again, K has no fixed points.

Other results are also applicable for generalized, modular metric space [7,8].

Theorem 7. *Let (M, U_p) be a complete generalized modular metric space, and let $a, b, c \in \mathbb{R}, a > c$ and $L, K : M \to M$ be self-compatible mappings that satisfy*

$$L(M) \subseteq K(M)$$

$$\int_0^{U_{p/a}(L\xi,L\varrho)} \kappa(\vartheta)d\vartheta \leq \aleph \int_0^{U_{p/c}(K\xi,K\varrho)} \kappa(\vartheta)d\vartheta \tag{28}$$

for some $\aleph \in (0,1)$ and when $p > 0$, then all $\xi, \varrho \in M$ are LIMSs with $\int_0^{\epsilon} \kappa(\vartheta)d\vartheta > 0$ when all $\epsilon > 0$, which is $\kappa : \mathbb{R}^+ \to \mathbb{R}^+$. Also, whether K or L is continuous, then there exists a unique fixed point of both.

Theorem 8. *Let (M, U_p) be a complete generalized modular metric space, and let $L, K : M \to M$ be a self-compatible mapping that satisfies*

$$L(M) \subseteq K(M)$$

$$\int_0^{U_p(L\xi,L\varrho)} \kappa(\vartheta)d\vartheta \leq \beta(U_p(K\xi, K\varrho)) \int_0^{U_p(K\xi,K\varrho)} \kappa(\vartheta)d\vartheta - \psi\Big(\int_0^{U_p(K\xi,K\varrho)} \kappa(\vartheta)d\vartheta\Big) \tag{29}$$

for all $\xi, \varrho \in M$, where $(\kappa, \psi) \in (\Phi_1, \Phi_2)$ and $\beta : \mathbb{R} \to [0,1)$ are a function with $\limsup_{s\to\vartheta} \beta(s) < 1$; then, we have the following for all: $\vartheta > 0$. As such, L and K have a unique fixed point, $u \in M$.

In 1976, Jungck [20] presented a fixed-point theorem for commuting maps that generalized Banach's Principle, and many writers have generalized and expanded his findings in numerous ways. However, Sessa [22] has defined weak commutativity in their work. Thus, we define a new version of it for generalized modular metric space as follows.

Definition 7. *Let (M, U_p) be a generalized modular metric space, and then the self-mappings K and L are said to be U-weakly commuting if*

$$U_p(KL(\xi), LK(\xi)) \leq U_p(L(\xi), K(\xi))$$

for all $\xi \in M$ and $p > 0$.

Jungck [21] subsequently developed a more generalized commutativity, known as compatibility, which covers a broader range of conditions compared to weak commutativity. Consequently, we are now introducing a novel conceptualization of generalized modular metric spaces.

Definition 8. *Let (M, U_p) be a generalized modular metric space. Then, the self-mappings K and L are said to be U-compatible if $\lim_{n\to\infty} U_p(KL(\xi_n), LK(\xi_n)) = 0$ whenever $\{\xi_n\}_{n=1}^{\infty}$ is a sequence in M such that $\lim_{n\to\infty} L(\xi_n) = \lim_{n\to\infty} K(\xi_n) = z$ is for some $z \in M$ and $p > 0$.*

Demonstrating that weakly commuting mappings are compatible is straightforward; however, it is important to note that neither implication holds in reverse.

Vijayaraju et al. [26] examined the existence of a unique fixed-point theorem for a set of maps satisfying a generic contractive condition of the integral type. Razani and Moradi [27] have illustrated the common fixed-point theorem within modular spaces of this category. Our work builds upon Jungck's [20] objectives and extends Branciari's [2] conclusions into the realm of generalized modular metric spaces, specifically focusing on compatible mappings. This section explores the presence of a common fixed point for U-compatible mappings when they satisfy a contractive condition of this nature in modular metric spaces.

Vijayaraju et al. [26] addressed common fixed-point theorem for a set of maps that satisfy a generic contractive circumstance of the integral type. The authors [27] have demonstrated the common fixed-point theorem of this type in modular spaces. We expand and enhance Jungck's [20] objectives and Branciari's [2] conclusion in generalized modular metric space to compatible mappings. This section is dedicated to the exploration of the existence of a common fixed point among recently-introduced compatible mappings, subject to the constraints of a contractive condition of this specific nature within modular metric spaces.

Now, we give a proof of the theorem given in their work [25].

Theorem 9 ([25]). *Let M be a U-complete generalized modular metric space, and $\aleph \in \mathbb{R}^+$ and $K, L : M \to M$ be two U-compatible mappings such that $K(M) \subseteq L(M)$,*

$$\int_0^{U_p(K(\xi),K(\varrho))} \kappa(\vartheta)d\vartheta \le \aleph \int_0^{U_p(L(\xi),L(\varrho))} \kappa(\vartheta)d\vartheta. \tag{30}$$

Thus, the LIMS (i.e., with a finite integral) on the all-compact subset of $[0, \infty)$ is non-negative, and this is such that we have $\kappa : \mathbb{R}^+ \to \mathbb{R}^+$ $\nu > 0$, $\int_0^{\nu} \kappa(\vartheta)d\vartheta > 0$. If one of K or L is continuous, then there exists a unique fixed point of K and L.

Proof. Let ξ be an arbitrary point of M and produce the sequence $\{K(\xi_n)\}_{n=1}^{\infty}$ such as $K(\xi_n) = L(\xi_{n+1})$ for each $n \in \mathbb{N}, \xi_1 = \xi$. It is thus possible that $K(M) \subseteq L(M)$. This is such when $n \ge 1$ and $p > 0$. As such, from (25) we have the following:

$$\begin{aligned} \int_0^{U_p(K(\xi_{n+1}),K(\xi_n))} \kappa(\vartheta)d\vartheta &\le \aleph \int_0^{U_p(L(\xi_{n+1}),L(\xi_n))} \kappa(\vartheta)d\vartheta \\ &\le \aleph \int_0^{U_p(K(\xi_n),K(\xi_{n-1}))} \kappa(\vartheta)d\vartheta \\ &\le \aleph^2 \int_0^{U_p(L(\xi_n),L(\xi_{n-1}))} \kappa(\vartheta)d\vartheta, \end{aligned}$$

by installing this formula, we have

$$\int_0^{U_p(K(\xi_{n+1}),K(\xi_n))} \kappa(\vartheta)d\vartheta \le \aleph \int_0^{U_p(K(\xi),\xi)} \kappa(\vartheta)d\vartheta, \tag{31}$$

and by taking the limit while $n \to \infty$, we thus take

$$\lim_{n\to\infty} \int_0^{U_p(K(\xi_{n+1}),K(\xi_n))} \kappa(\vartheta)d\vartheta \le 0. \tag{32}$$

Meanwhile, when each $\nu > 0$, *and* $\int_0^{\nu} \kappa(\vartheta)d\vartheta > 0$, we have the following remarks:

$$\lim_{n\to\infty} U_p(K(\xi_{n+1}), K(\xi_n)) = 0. \tag{33}$$

We display $\{K(\xi_n)\}_{n=1}^{\infty}$ as a U-Cauchy sequence. When all $\nu > 0$, there exists $n_\nu \in \mathbb{N}$ such that $U_p(K(\xi_{n+1}), K(\xi_n)) < \nu$ for all $n \in \mathbb{N}$ while $n \geq n_\nu$ and $0 < p$. When used in similar manner, let us assume that $n_1, n \in \mathbb{N}$ and $n < n_1$; we then have $U_p(K(\xi_{n+1}), K(\xi_n)) < \frac{\nu}{n_1 - n}$ for all $n < n_1$. Now, we produce

$$\begin{aligned}
&U_p(K(\xi_{n_1}), K(\xi_n)) \\
&\leq U_p(K(\xi_n), K(\xi_{n+1})) + U_p(K(\xi_{n+1}), K(\xi_{n+2})) + ... + U_p(K(\xi_{n_1-1}), K(\xi_{n_1})) \\
&< \tfrac{\nu}{n_1-n} + \tfrac{\nu}{n_1-n} + ... + \tfrac{\nu}{n_1-n} \\
&= \nu
\end{aligned}$$

when all $n < n_1$. Then, $\{K(\xi_n)\}_{n=1}^{\infty}$ is a U-Cauchy sequence. Since M is U-complete, there exists $a \in M$ such that $U_p(K(\xi_n), a) \to 0$ is as $n \to \infty$. If K is continuous, then $K^2(\xi_n) \to K(a)$ and $KL(\xi_n) \to K(a)$. From the U-compatibility of M, we have $U_p(LK(\xi_n), KL(\xi_n)) \to 0$ as $n \to \infty$ for $0 < p$. In addition, when $LK(\xi_n) \to K(a)$ considers that $U_p(LK(\xi_n), K(a)) \leq U_p(LK(\xi_n), KL(\xi_n))$ and $U_p(LK(\xi_n), K(a)) \leq U_p(KL(\xi_n), K(a))$, then, in the latter part, we show that a is a common fixed point of K and L. From (25), it can be that:

$$\int_0^{U_p(K^2(\xi_n), K(\xi_n))} \kappa(\vartheta)d\vartheta \leq \aleph \int_0^{U_p(LK(\xi_n), L(\xi_n))} \kappa(\vartheta)d\vartheta. \tag{34}$$

If we take the limit $n \to \infty$, we have:

$$\int_0^{U_p(K(a), a)} \kappa(\vartheta)d\vartheta \leq \aleph \int_0^{U_p((K(a), a))} \kappa(\vartheta)d\vartheta, \tag{35}$$

which answers the expression that $U_p((K(a), a)) = 0$ when $0 < p$. As a result of $K(a) = a$, then, from $K(M) \subseteq L(M)$, there exists a point a_1 such that $K(a) = a = a_1$. In addition, from (25) we have

$$\int_0^{U_p(K^2(\xi_n), K(a_1))} \kappa(\vartheta)d\vartheta \leq \aleph \int_0^{U_p(LK(\xi_n), L(a_1))} \kappa(\vartheta)d\vartheta, \tag{36}$$

and if we take the limit while $n \to \infty$, we have

$$\int_0^{U_p(K(a), K(a_1))} \kappa(\vartheta)d\vartheta \leq \aleph \int_0^{U_p(K(a), L(a_1))} \kappa(\vartheta)d\vartheta \tag{37}$$

and

$$\int_0^{U_p(a, K(a_1))} \kappa(\vartheta)d\vartheta \leq \aleph \int_0^{U_p(a, L(a_1))} \kappa(\vartheta)d\vartheta \leq \aleph \int_0^{U_p(a,a)} \kappa(\vartheta)d\vartheta \tag{38}$$

Hence, $a = K(a_1) = L(a_1)$ and $L(a) = LK(a_1) = KL(a_1) = K(a) = a$. Likewise, if we take L to be continuous (in place of K), then we can show $L(a) = K(a) = a$ analogously. Eventually, if we assume that a and b are two common fixed points of K and L, then

$$\begin{aligned}
\int_0^{U_p(a,b)} \kappa(\vartheta)d\vartheta &\leq \aleph \int_0^{U_p(K(a),K(b))} \kappa(\vartheta)d\vartheta \\
&\leq \aleph \int_0^{U_p(L(a),L(b))} \kappa(\vartheta)d\vartheta \\
&\leq \aleph \int_0^{U_p(a,b)} \kappa(\vartheta)d\vartheta,
\end{aligned}$$

which asserts that $U_p(a, b) = 0$ when $0 < p$; thus, as a result, we have $a = b$. $\square$

The next theorem is a supplementary revision of Theorem 6, which is when we take a condition such as $K, L : M_1 \to M_1$, where M_1 is a U-closed and a U-bounded subset of M.

Theorem 10 ([25]). *Let M be a U-complete generalized modular metric space and M_1 be a U-closed and U-bounded subset of M, $\aleph \in \mathbb{R}^+$, and $K, L : M \to M$. Thus, we have two U-compatible mappings such that $K(M) \subseteq L(M)$,*

$$\int_0^{U_p(K(\xi),K(\varrho))} \kappa(\vartheta)d\vartheta \leq \aleph \int_0^{U_p(L(\xi),L(\varrho))} \kappa(\vartheta)d\vartheta \tag{39}$$

when all $\xi, \varrho \in M_1$ and $0 < p$. Thus, the LIMSs (i.e., with a finite integral) on the all-compact subset of $[0, \infty)$ is non-negative, which is such that we have the following for all: $\kappa : \mathbb{R}^+ \to \mathbb{R}^+$ $\nu > 0$, $\int_0^\nu \kappa(\vartheta)d\vartheta > 0$. If one of K or L is continuous, then there exists a unique fixed point of K and L.

Rhoades [28] demonstrated the existence of two fixed-point theorems for mappings that satisfy a generic contractive inequality of the integral type in the context of a version of generalized modular metric space [25]. These results follow Branciari's [2] established theorem for contraction mapping:

Theorem 11 ([25]). *Let K be a mapping from complete metric space (M, U_p) that is into itself and satisfies for all $\xi, \varrho \in M$*

$$\int_0^{U_p(K(\xi),K(\varrho))} \kappa(\vartheta)d\vartheta \leq \aleph \int_0^{m(\xi,\varrho)} \kappa(\vartheta)d\vartheta, \tag{40}$$

where

$$m(\xi, \varrho) = \max\left\{U_p(\xi, \varrho), U_p(\xi, K(\xi)), U_p(\varrho, K(\varrho)), \frac{U_p(\xi, K(\varrho)) + U_p(\varrho, K(\xi))}{2}\right\} \tag{41}$$

and for all $\xi, \varrho \in M$,

$$\int_0^{U_p(K(\xi),K(\varrho))} \kappa(\vartheta)d\vartheta \leq \aleph \int_0^{M(\xi,\varrho)} \kappa(\vartheta)d\vartheta, \tag{42}$$

where

$$M(\xi, \varrho) = \max\{U_p(\xi, \varrho), U_p(\xi, K(\xi)), U_p(\varrho, K(\varrho)), U_p(\xi, K(\varrho)), U_p(\varrho, K(\xi))\} \tag{43}$$

for $\aleph \in [0, 1)$ and $\kappa : \mathbb{R}^+ \to \mathbb{R}^+$, which are given in Theorem 6. Then, K has a unique fixed point $a \in M$ such that for all, we have $\xi \in M, \lim_{n\to\infty} K^n(\xi) = a$.

In 2004, Berinde [29] provided a definition that serves as an inspiration for the following:

Definition 9 ([29]). *Let (M, u) be a metric space. A self operator $K : M \to M$ is said to be a weak ψ-contraction or (V, ψ)-weak contraction, provided that there exist a comparison function K and some $V \geq 0$ such that*

$$u(K(\xi), K(\varrho)) \leq \psi u(\xi, \varrho) + V(u(\varrho, K(\xi)))$$

for all $\xi, \varrho \in M$.

Definition 10. *In the context of a generalized modular metric space represented as M, a mapping denoted as K qualifies as a weak contraction, or more specifically, a (V, ψ)-weak contraction. This categorization is applicable under the condition that there exist two constants, $V \geq 0$ and $\psi \in [0, 1)$, which satisfy the subsequent inequality for all $\xi, \varrho \in M$:*

$$U_p(K(\xi), K(\varrho)) \leq \psi U_p(\xi, \varrho) + V(U_p(\varrho, K(\xi)))$$

In 2010, Olatinwo [30] generalized Branciari's conclusion using Definition 8, and they used the same approach as Berinde and Berinde [31] in their work. We thus established the following fixed-point theorem for generalized modular metric space through Olatinwo's step.

Definition 11. *A function $V : \mathbb{R}^+ \to \mathbb{R}^+$ is called a comparison function when it captures the subsequent situations:*

(i) *If $V(\vartheta) < \vartheta$ for some $\vartheta > 0$, then ψ is monotone increasing.*
(ii) $V(0) = 0$.
(iii) $\lim_{n\to\infty} V^n(\vartheta) = 0, \forall\vartheta \geq 0$.

Definition 12. *If and only if the following two properties encounter one another, then mapping $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ is called an altering distance function.*

(i) *Ψ is continuous and non-decreasing.*
(ii) *$\Psi(\vartheta) = 0$ if and only if $\vartheta = 0$.*

Here, we define the similar aspect for a generalized modular metric.

Theorem 12. *Let (M, U) be a complete generalized modular metric space, and let $K : M \to M$ satisfy a (V, ψ)-weak contraction of the integral type, which is defined as follows:*

$$\int_0^{U_p(K(\xi),K(\varrho))} \kappa(\vartheta)d\vartheta \leq V\left(\int_0^{U_p(\xi,K(\xi))} \kappa(\vartheta)d\vartheta\right)^y \left(\int_0^{U_p(\varrho,K(\xi))} \kappa(\vartheta)d\vartheta\right) + \psi\left(\int_0^{U_p(\xi,\varrho)} \kappa(\vartheta)d\vartheta\right),$$

for all $\xi, \varrho \in M$, where there is $V \geq 0$ and $y \geq 0$. We thus assume the following:

(i) *$v : \mathbb{R}^+ \to \mathbb{R}^+$ is a monotonically increasing function, and $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous comparison function.*
(ii) *The Lebesgue-integrable mapping summable (LIMS)-Stieltjes function $\kappa : \mathbb{R}^+ \to \mathbb{R}^+$ is non-negative, and this applies when all $\nu > 0$, $\int_0^\nu \kappa(\vartheta)d\vartheta > 0$.*

Then, K possesses a unique fixed point $u \in M$ such that for all $\xi \in M$, we have $\lim_{n\to\infty} K^n(\xi) = u$.

For a generalized modular metric, they provided a weak contraction theorem of a version of the integral-type [25].

Theorem 13. *Let M be a U-complete generalized modular metric space and let $K : M \to M$ satisfy a (ξ_i, ψ)-weak contraction of the integral type as defined below:*

$$\int_0^{U_p(K(\xi),K(\varrho))} \kappa(\vartheta)d\vartheta \leq \xi_i\left(\int_0^{U_p(\xi,K(\xi))} \kappa(\vartheta)d\vartheta\right) \left(\int_0^{U_p(\varrho,K(\xi))} \kappa(\vartheta)d\vartheta\right) + \psi\left(\int_0^{U_p(\xi,\varrho)} \kappa(\vartheta)d\vartheta\right),$$

for all $\xi, \varrho \in M$. We thus assume the following conditions:

(i) *$\xi_i : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous comparison function.*
(ii) *ψ is continuous with $\psi(0) = 0$, and both ψ and $v : \mathbb{R}^+ \to \mathbb{R}^+$ are monotone increasing functions.*
(iii) *The Lebesgue-integrable mapping summable (LIMS)-Stieltjes function $\kappa : \mathbb{R}^+ \to \mathbb{R}^+$ is non-negative, and this applies when all are $\nu > 0$, $\int_0^\nu \kappa(\vartheta)d\vartheta > 0$. Additionally, $v : \mathbb{R}^+ \to \mathbb{R}^+$ is an increasing function.*

Under these conditions, K possesses a unique fixed point $u \in M$ such that for all $\xi \in M$, there is $\lim_{n\to\infty} K^n(\xi) = u$.

Aydi [32] proposed the formulation and fixed-point theory for the contractive conditions of an integral type with changing distances. They [25] used the same approach for a generalized modular metric.

Theorem 14. *$V : [0,\infty) \to [0,\infty)$ is subadditive on all $[m,n] \subset [0,\infty)$ if*

$$\int_0^{m+n} V(\vartheta)d\vartheta \leq \int_0^{m} V(\vartheta)d\vartheta + \int_0^{n} V(\vartheta)d\vartheta. \quad (44)$$

Theorem 15. *Let M be a U-complete generalized modular metric space and $K : M \to M$ such that*

$$\psi(\int_0^{U_p(K(\xi),K(\varrho))} \kappa(\vartheta)d\vartheta) \leq \psi(\theta(\xi,\varrho)) - \Phi(\theta(\xi,\varrho)), \quad (45)$$

when all $\xi, \varrho \in M$ with non-negative real numbers θ, γ, ϱ such that $2\theta + \gamma + 2\varrho < 1$, where ψ, Φ are altering distances. Thus, we have:

$$\theta(\xi,\varrho) = \theta \int_0^{U_p(\xi,K(\xi))+U_p(\varrho,K(\varrho))} \kappa(\vartheta)d\vartheta + \gamma \int_0^{U_p(\xi,\varrho)} \kappa(\vartheta)d\vartheta + \varrho \int_0^{\max\{U_p(\xi,K(\varrho)),U_p(\varrho,K(\xi))\}} \kappa(\vartheta)d\vartheta$$

The LIMSs for $\kappa(\vartheta) : [0,\infty) \to [0,\infty)$ is sub-additive on all of the subsets of $\mathbb{R}^+$, which are non-negative when all $\nu > 0$,

$$\int_0^{\nu} \kappa(\vartheta)d\vartheta > 0.$$

Then, K has a unique fixed point in M.

3. Best Approximation Results with Application

In this paper, we provide fixed-point results for a best approximation of generalized modular metric space via Malih's angle [13], which was based on the structure of Theorems 6 and 7.

Theorem 16. *Let (A, U_p) be a subset of generalized modular metric space M with the $K, L : M \to M$ weakly-compatible self-mappings that satisfy $clK(M) \subseteq L(M)$, $cl(K(M))$ or that let $L(M)$ be complete. As such, we obtain*

$$\int_0^{U_p(K(\xi),K(\varrho))} \kappa(\vartheta)d\vartheta \leq \aleph \int_0^{U_p(L(\xi),L(\varrho))} \kappa(\vartheta)d\vartheta, \quad (46)$$

LIMS for $\xi, \varrho \in M, \aleph \in [0,1)$ and $\kappa : \mathbb{R}^+ \to \mathbb{R}^+$, which is a non-negative function with $\int_0^{\epsilon} \kappa(\vartheta)d\vartheta > 0$, which is when all $\epsilon > 0$. Then, K and L have a unique fixed point.

An example of a better understanding of the theorem is given below:

Example 2. *$M = \{1,2,3,4,5\}$ and $U_p(1,2) = U_p(2,1) = 5$, $U_p(2,3) = U_p(3,2) = U_p(1,3) = U_p(3,1) = 1$, $U_p(1,4) = U_p(4,1) = U_p(2,4) = U_p(4,2) = U_p(3,4) = U_p(4,3) = 4$, $U_p(1,5) = U_p(5,1) = U_p(2,5) = U_p(5,2) = U_p(3,5) = U_p(5,3) = U_p(4,5) = U_p(5,4) = 2$ $K(5) = 1$ and 2 for other values. In addition, when $L(\xi) = \xi$, then $\int_0^{\epsilon} \kappa(\vartheta)d\vartheta > 0$ is for $\kappa(\vartheta) = 2\vartheta$ and when all $\epsilon > 0$. When K and L are weakly compatible, then 2 is a fixed point for all $\aleph$.*

Theorem 17. *Let M be generalized modular metric space and $K : M \to M$ satisfy a fixed point $z \in M$; thus, the following is satisfied:*

$$\int_0^{U_p(K(\xi),K(\varrho))} \kappa(\vartheta)d\vartheta \leq \aleph \int_0^{U_p(\xi,\varrho)} \kappa(\vartheta)d\vartheta, \quad (47)$$

LIMS for $\xi, \varrho \in M, \aleph \in [0,1)$ and $\kappa : \mathbb{R}^+ \to \mathbb{R}^+$, which is a non-negative function with $\int_0^{\epsilon} \kappa(\vartheta)d\vartheta > 0$ when all $\epsilon > 0$. If V is a closed K-invariant subset of M and $K|V$, as well as a compact mapping, then the set $P_V(z)$ is the best approximation, which is not empty.

Samet's [33] work is inspirational for this following example:

Example 3. *$M = \{1,2,3,4,5\}$ and $U_p(1,2) = U_p(2,1) = 5/p$, $U_p(2,3) = U_p(3,2) = U_p(1,3) = U_p(3,1) = 1/p$, $U_p(1,4) = U_p(4,1) = U_p(2,4) = U_p(4,2) = U_p(3,4) = U_p(4,3) = 4/p$, $U_p(1,5) = U_p(5,1) = U_p(2,5) = U_p(5,2) = U_p(3,5) = U_p(5,3) = U_p(4,5) = U_p(5,4) = 2/p$ $U_{2p}(1,2) \leq U_p(1,3) + U_p(3,2)$, which shows U_p is not even a modular expression when $K(5) = 1$ and 2 are present for the other values. This applies when $\int_0^{\epsilon} \kappa(\vartheta)d\vartheta > 0$ is for $\kappa(\vartheta) = e^{\vartheta}$ when all $\epsilon > 0$. When $K = \{1,2\}$ and V are a closed K-invariant subset of M, and when $K|V$ is a compact mapping, then the set $P_V(2) = 0$ applies.*

Further extensions of our results are applicable from two to four finite self-mappings for common fixed points such as the symmetric spaces given in [34]. For the set-valued mappings, similar outcomes are useful for optimization in other areas; these serve as guiding lights. In addition, for those who want to explore more in this field, we recommend the work of [35]; for more work on the applications of systems of initial value problems, we recommend [36]; regarding graphical extended metric spaces applications, we recommend [37]; and for those seeking more on (α, β)-generalized contractions and their applications in matrix equations, there is the work of [38].

Author Contributions: Correspondence, N.M.T. Writing—original draft, N.M.T.; Writing—review & editing, R.P.A. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: No data were used in the study as it was primarily conceptual and did not involve the collection or analysis of specific datasets.

Conflicts of Interest: The authors declare no conflict of interest.

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