mathematics

Article

On the Solution of the Dirichlet Problem for Second-Order Elliptic Systems in the Unit Disk

Astamur Bagapsh [1,2] and Alexandre Soldatov [1,2,3,*]

[1] Federal Research Center "Computer Science and Control" of the Russian Academy of Sciences, 40 Vavilova St., Moscow 119333, Russia; a.bagapsh@gmail.com
[2] Moscow Center for Fundamental and Applied Mathematics, Lomonosov Moscow State University, GSP-1, Leninskie Gory, Moscow 119991, Russia
[3] Department of Hiah Mathematics, Moscow Power Engineering Institute, National Research University, 14 Krasnokazarmennaya St., Moscow 111250, Russia
* Correspondence: soldatov48@gmail.com

Abstract: The role played by explicit formulas for solving boundary value problems for elliptic equations and systems is well known. In this paper, explicit formulas for a general solution of the Dirichlet problem for second-order elliptic systems in the unit disk are given. In addition, an iterative method for solving this problem for systems with respect to two unknown functions is described, and an integral representation of the Poisson type is obtained by applying this method.

Keywords: Dirichlet problem; second-order elliptic system; integral representation; Poisson integral; unit disk; Fredholm property

MSC: 45B05; 35J47; 35J57

1. Introduction

The classical method of studying boundary value problems for elliptic equations and systems on a plane is based on representing the solutions of elliptic equations in terms of analytic functions, which allows one to reduce the matter to the study of the boundary value problems of function theory. For elliptic systems with constant leading coefficients, a method was developed by A. V. Bitsadze [1] (see also [2,3]). In the Bitsadze representation of solutions to elliptic systems, along with analytical functions, its derivatives up to a certain order are also involved. This representation is significantly simplified [4,5] if the analytical functions are replaced by the solutions of canonical elliptic systems of the first order:

$$\frac{\partial\phi}{\partial y} - J\frac{\partial\phi}{\partial x} = 0$$

where J is a constant $l \times l-$matrix with the eigenvalues ν located in the upper half-plane $\operatorname{Im} z > 0$. As was shown by A. Douglis [6], all of the elements of the theory of analytic functions extend to solutions of the last system, which, following G. Hile [7], we call J-analytic functions. They are studied in more detail in [8].

In the case where $l = 2$, the theory of the first-order elliptic systems was constructed in the works by I. N. Vekua [9] and L. Bers [10] and is known as the theory of generalized analytic functions. This theory was later extended to the case where $l > 2$ in the works by B.V. Boyarskiy [11], R. Gilbert [12], R. Gilbert, G. Hile [13], and others.

The important results for general elliptic problems on a plane were obtained by A.I. Volpert [14] and M.M. Sirazhudinov [15] using methods that are analogous to the ones of function theory.

In this paper, within the framework of the general functional-theoretic approach developed in [16], necessary and sufficient conditions for the unique solvability of the

check for updates

Citation: Bagapsh, A.; Soldatov, A. On the Solution of the Dirichlet Problem for Second-Order Elliptic Systems in the Unit Disk. *Mathematics* **2023**, *11*, 4360. https://doi.org/10.3390/math11204360

Academic Editor: Juan Eduardo Nápoles Valdes

Received: 16 September 2023
Revised: 16 October 2023
Accepted: 18 October 2023
Published: 20 October 2023

Dirichlet problem for a second-order elliptic system with constant leading coefficients in the unit disk are given. In addition, an iterative method for solving this problem for systems with respect to two unknown functions is described. In addition, the integral representation of the Poisson type is obtained by virtue of the specified method application.

2. Main Results

Criterion for unique solvability: Let us consider the elliptic system of the second order

$$a_0\frac{\partial^2 u}{\partial x^2}+a_1\frac{\partial^2 u}{\partial x\partial y}+a_2\frac{\partial^2 u}{\partial y^2}=0 \tag{1}$$

on a plane with constant coefficients $a_j\in\mathbb{R}^{l\times l}$. As it is known, the ellipticity condition can be described as follows: the matrix a_2 is invertible, and the characteristic polynomial

$$\chi(z)=\det p(z),\quad p(z)=a_0+a_1z+a_2z^2, \tag{2}$$

does not have real roots.

Let D be a simply connected domain on a plane bounded by the Lyapunov contour Γ. Let us consider the Dirichlet problem

$$u\big|_\Gamma=f \tag{3}$$

for the system (1) within the class $C(\overline{D})\cap C^2(D)$.

In 1948, A.V. Bitsadze [17] constructed an example of an elliptic 2×2 system with the coefficients

$$a_0=-a_2=1,\ \ a_1=\pm\begin{pmatrix}0&2\\-2&0\end{pmatrix},$$

for which the homogeneous Dirichlet problem in the unit disk has an infinite number of linearly independent solutions.

After the appearance of this example, the question arose as to how to describe systems for which the Dirichlet problem is well-posed. A class of such systems was introduced by M.I. Vishik [18], who called them strongly elliptic. They are determined by the positive definiteness condition for the matrix $p(t)$ given in (2) for an arbitrary $t\in\mathbb{R}$. Note that this condition is equivalent [19] to

$$\det(a_0\alpha+2a_1\beta+a_2\gamma)\neq 0\quad\text{for}\quad \beta^2-\alpha\gamma<0. \tag{4}$$

Later, A.V. Bitsadze [1] described a class of elliptic systems, called weakly connected, for which the Dirichlet problem is Fredholm. This class can be defined [20] as follows. Using the coefficients a_i of the system (1), we compose the block $2l\times 2l$ matrix

$$A=\begin{pmatrix}0&1\\-a_2^{-1}a_0&-a_2^{-1}a_1\end{pmatrix},$$

with the eigenvalues being coincided with the roots of the characteristic polynomial (2). It reduces to the Jordan form

$$B^{-1}AB=\widetilde{J},\quad \widetilde{J}=\begin{pmatrix}J&0\\0&\overline{J}\end{pmatrix},$$

where J consists of Jordan cells with eigenvalues belonging to the upper half-plane. Moreover, the matrix B can be chosen of the block form

$$B=\begin{pmatrix}b&\overline{b}\\bJ&\overline{bJ}\end{pmatrix}. \tag{5}$$

The equivalent construction is that the matrix $b \in \mathbb{C}^{l\times l}$ satisfies the matrix relation $a_0 b + a_1 bJ + a_2 bJ^2 = 0$, and the block matrix B must be invertible. Moreover, any two matrices b and b_1 are related by $b_1 = bd$ to some invertible matrix d commuting with J.

In this notation, the class of weakly connected systems is defined by the condition $\det b \neq 0$, which does not depend on the choice of the matrix b.

From the point of view of the general elliptic theory [21] the Fredholm property of the problem (1), (3) is described by the complementarity condition. It can be shown [20] that it is equivalent to the weak connection of the system (1). In the paper [22], the author also states that, in the notation of (2), this condition is equivalent to the invertibility of the matrix

$$\Delta = \int_{\mathbb{R}} p^{-1}(t)dt.$$

The Fredholm property of the Dirichlet problem in the class of functions satisfying the Hölder condition in the closed domain $\overline{D}$ was established by A.V. Bitsadze [1] and N.E. Tovmasyan [23]. Note that M.I. Vishik showed [18] that the Dirichlet problem is uniquely solvable for a strongly elliptic system (1). The paper [20] introduces the notion of a generalized double layer potential, which makes it possible to reduce the Dirichlet problem to the equivalent Fredholm equation on the boundary contour Γ. Firstly, let us illustrate this approach on the classical potential of the double layer for the Laplace equation, which can be defined by the equality

$$(P_0\varphi)(z) = \frac{1}{\pi}\int_\Gamma p(t, t-z)\varphi(t)d_1t, \quad p(t,\xi) = \frac{n_1(t)\xi_1 + n_2(t)\xi_2}{|\xi|^2}, \tag{6}$$

where d_1t is the arc length differential, $z = x + iy$, and $n(t) = n_1(t) + in_2t)$ is the unit outward normal. It is a well-known fact that the operator P_0 is bounded $C(\Gamma) \to C(\overline{D})$, and the boundary value u^+ on Γ satisfies the formula $(P_0\varphi)^+ = \varphi + K_0\varphi$, where the operator K_0 is defined similarly to (6) by replacing $z \in D$ with $t_0 \in \Gamma$ in the corresponding kernel $p(t, t-t_0)$. Since Γ is a Lyapunov contour, this kernel has a weak singularity, which means that the operator K_0 is compact in the space $C(\Gamma)$. The procedure for studying the Dirichlet problem with the help of these potentials is described in the classical monographs [24] on mathematical physics. It is based on the involvement of the Dirichlet problem in the outer domain and can be replaced with the following property. Any harmonic function $u \in C(\overline{D})$ in the simply connected domain D can be uniquely represented in the form $u = P_0\varphi$ with some real density $\varphi \in C(\Gamma)$. This immediately implies that the Dirichlet problem (3) for the Laplace equation is reduced to the equivalent Fredholm equation $\varphi + K_0\varphi = f$.

Exactly the same procedure holds for the weakly coupled elliptic system (1) with respect to the integral (6)

$$(P\varphi)(z) = \frac{1}{\pi}\int_\Gamma p(t, t-z)H(t-z)\varphi(t)d_1t, \tag{7}$$

where a vector function is defined as $\varphi = (\varphi_1, \varphi_2) \in C(\Gamma)$, and

$$H(\xi) = \mathrm{Im}[b(\xi_1 1 + \xi_2 J)^{-1}(-\xi_2 1 + \xi_1 J)b^{-1}],$$

is the homogeneous matrix function of degree zero. Recall that the complex $l \times l-$matrices b, J appear in (5), and one is the identity matrix. This integral in the domain D defines solution of the system (1), and the operator P is bounded $C(\Gamma) \to C(\overline{D})$. As above, the boundary value of the function $u = P\varphi$ satisfies the formula $u^+ = \varphi + K\varphi$ with the operator K acting similarly to (7) with the replacement $z \in D$ to $t_0 \in \Gamma$. As in the classical situation, the kernel of this operator has a weak singularity, which means that it is compact in the space $C(\Gamma)$ of vector functions $\varphi = (\varphi_1, \varphi_2)$. The main result established in [20] (in the more general situation of multiplying connected domains) is that in the considered case of a simply connected domain, any solution of Equation (1) in the domain D from class $u \in C(\overline{D})$ can be uniquely represented in the form given by (7) with some real density

$\varphi = (\varphi_1, \varphi_2) \in C(\Gamma)$. This immediately implies that the Dirichlet problem (3) is reduced to the equivalent Fredholm equation $\varphi + K\varphi = f$.

In this paper, we consider the Dirichlet problem for the Equation (1) in the unit circle $\mathbb{D} = \{|z| < 1\}$, and its boundary circle will be denoted by $\mathbb{T}$. It is convenient to use the notation $\varphi \in C(\mathbb{T})$ for $2\pi-$ periodic functions $\varphi(\theta) = \varphi(e^{i\theta})$. Using Fourier coefficients

$$\widehat{\varphi}_n = \frac{1}{2\pi}\int_0^{2\pi} \varphi(\theta)e^{-in\theta}d\theta$$

this function is restored by the formula

$$\varphi(\theta) = \widehat{\varphi}_0 + 2\mathrm{Re}\sum\nolimits_{n=1}^{\infty} \widehat{\varphi}_n e^{in\theta}.$$

As the function

$$H(\theta) = \mathrm{Im}[b(1\cos\theta + J\sin\theta)^{-1}(-1\sin\theta + J\cos\theta)b^{-1}] \in C^\infty$$

is $\pi-$ periodic, the equality

$$k(\theta) = H(\theta/2) \tag{9}$$

defines a 2π-periodic matrix function; thus,

$$\widehat{k}_n = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} H(\theta)e^{-2in\theta}d\theta.$$

Let us suppose that

$$\det(1 + \widehat{k}_n) \neq 0, \quad n = 0, 1, \ldots. \tag{10}$$

Under this assumption, the subsequence

$$\widehat{r}_n = (1 + \widehat{k}_n)^{-1} - 1 \to 0 \tag{11}$$

is faster than any power of n and serves to provide the Fourier coefficients of the corresponding matrix function $r(\theta) \in C^\infty$.

The above considerations make it possible to explicitly solve the question of the solvability of the Dirichlet problem in the unit disk.

Theorem 1. *The Dirichlet problem for a weakly connected system (1) is uniquely solvable in the unit disk if and only if condition (10) is satisfied. Under this assumption, a solution to the problem is given by the formula*

$$u = P(1 + R)f, \tag{12}$$

where the operator R acts in the class of 2π-periodic functions according to the rule

$$(Rf)(\theta_0) = \frac{1}{2\pi}\int_0^{2\pi} r(\theta - \theta_0)\varphi(\theta)d\theta \tag{13}$$

with the matrix-value function $r(\theta)$ defined above.

Proof. The unit normal is $n(t) = t$ on $\mathbb{T}$, and, for the points $t = e^{i\theta},\ t_0 = e^{i\theta_0}$, we have the expressions

$$p(t, t - t_0) = \mathrm{Re}\left(\frac{t}{t - t_0}\right) = \frac{1}{2}, \quad t - t_0 = 2\sin\left(\frac{\theta + \theta_0}{2}\right)e^{i(\theta - \theta_0)/2}.$$

Therefore, the operator

$$(K\varphi)(t_0) = \frac{1}{\pi}\int_\Gamma p(t, t - t_0)H(t - t_0)\varphi(t)d_1 t, \quad t_0 \in \mathbb{T},$$

acts according to Formula (13), where the role of r is played by function (9). Thus,

$$(K\varphi)_n^\wedge = \widehat{k}_{-n}\widehat{\varphi}_n,$$

which means condition (10) is necessary for the invertibility of the operator $1+K$ and, consequently, for the solvability of the Dirichlet problem. Let it be done. It is easy to see that the operator $S = RK$ acts similarly to (13) with respect to the function

$$s(\theta_0) = \frac{1}{2\pi}\int_0^{2\pi} r(\theta_0-\theta)k(\theta)d\theta.$$

Taking into account (11) and defining $\psi = (1+R)(1+K)\varphi$, we have

$$\widehat{\psi}_n = (1+\widehat{r}_{-n}+\widehat{k}_{-n}+\widehat{r}_{-n}\widehat{k}_{-n})\widehat{\varphi}_n = \widehat{\varphi}_n.$$

Consequently, the operator $1+K$ is invertible, and $(1+K)^{-1} = 1+R$, which leads to formula (12) for solving the Dirichlet problem. □

Let us make sure that, in the case of the Laplace equation, this formula goes over into the Poisson integral. To this end, note that, according to (6),

$$p(t, t-z) = \mathrm{Re}\left(\frac{t}{t-z}\right)$$

and, therefore,

$$(P_0\varphi)(z) = \mathrm{Re}\left[\frac{1}{\pi i}\int_{\mathbb{T}}\frac{\varphi(t)dt}{t-z}\right], \tag{14}$$

where it is taken into account that the complex differential $dt = itd_1t$.

For the Laplace equation in expression (5), we can set $b=1$ and $J=i$ so that the function k in (9) is identically equal to one. Accordingly, the operator K_0 transforms the function φ into a constant

$$\widehat{\varphi}_0 = \frac{1}{2\pi}\int_0^{2\pi}\varphi(\theta)d\theta = \frac{1}{2\pi i}\int_{\mathbb{T}}\frac{\varphi(t)dt}{t}.$$

Therefore, the equation $\varphi + K_0\varphi = f$ is uniquely solvable with $\varphi = f - \widehat{f}_0/2$. Accordingly, (12) becomes equal to

$$u = P_0[f - \widehat{f}_0/2].$$

Expression (14) shows that $P_0 1 = 2$, and, hence, $P_0[f-\widehat{f}_0/2] = P_0 f - \widehat{f}_0$, which in its expanded form is

$$u(z) = \mathrm{Re}\left[\frac{1}{\pi i}\int_{\mathbb{T}}\frac{\varphi(t)dt}{t-z} - \frac{1}{2\pi i}\int_{\mathbb{T}}\frac{\varphi(t)dt}{t}\right] = \frac{1}{2\pi}\int_{\mathbb{T}}\mathrm{Re}\left[\frac{t+z}{t-z}\right]\varphi(t)d_1t.$$

As a result, we arrive at the Poisson formula

$$u(z) = \frac{1}{2\pi}\int_{\mathbb{T}}\frac{1-|z|^2}{|t-z|^2}\varphi(t)d_1t.$$

For this reason, in the general case, equality (12) is naturally called the generalized Poisson formula. It gives a solution to the problem (3) for weakly connected elliptic system (1) in the unit disk.

Iterative method of solution. Consider a strongly elliptic system of the form (1) with dimension $l=2$, which is defined with respect to the vector function $u=(u_1,u_2)$ of the two real variables x and y. It is convenient to introduce a complex variable $z = x+iy$ and a complex-valued function, which will be denoted $u = u_1 + iu_2$. With the help of suitable linear transformations of variables (x, y) and of unknowns (u_1, u_2) this system can be reduced to the form

$$\mathcal{L}_{\tau,\sigma}u(z) = (\partial\bar{\partial} + \tau\partial^2)u(z) + \sigma(\tau\partial\bar{\partial} + \partial^2)\overline{u(z)} = 0 \tag{15}$$

with just two parameters $\tau, \sigma \in [0, 1)$; see [25,26]. Here, the symbols ∂ and $\bar{\partial}$ denote Cauchy–Riemann operators in the new variable z, and the bar over $u(z)$ means, as usual, complex conjugation.

Let us list some well-known particular cases of Equation (15). First of all, for $\tau = \sigma = 0$, we have the complex Laplace equation $\Delta u = 4\partial\bar{\partial}u = 0$. Furthermore, for $\tau = 0$ and $\sigma \neq 0$, we have the isotropic Lame equation from the plane theory of elasticity [27], and the parameter σ is related to the Poisson's ratio p of the elastic body, $\sigma = 1/(3 - 4p)$; since $p \in (0, 1/2)$ is known, then $\sigma \in (1/3, 1)$. If $\sigma = 0$ and $\tau \neq 0$, then we have a skew-symmetric system that can be written as the equation $au_{xx} + 2bu_{xy} + cu_{yy} = 0$ with complex coefficients a, b, and c. Finally, for $\sigma > \tau$, we obtain an anisotropic Lame equation.

Let us arrange one more transformation over Equation (15), having previously rewritten it in the form

$$\partial\bar{\partial}(\mathcal{T}_{1,\sigma\tau}u) + \partial^2(\mathcal{T}_{\tau,\sigma}u) = 0,$$

where

$$\mathcal{T}_{\alpha,\beta}w = \alpha w + \beta\overline{w}, \qquad \alpha, \beta \in \mathbb{C}, \quad |\alpha| \neq |\beta|,$$

is the operator of a nondegenerate affine transformation of a plane of the complex variable w that preserves the origin. Now, let us replace the function u on

$$v = \mathcal{T}_{1,\sigma\tau}u; \tag{16}$$

then,

$$\partial\bar{\partial}v + \partial^2(Tv) = 0,$$

where

$$Tw = \mathcal{T}_{\tau,\sigma}\mathcal{T}^{-1}_{1,\sigma\tau}w = \alpha_0 w + \beta_0\overline{w}, \qquad \alpha_0 = \frac{\tau(1-\sigma^2)}{1-\sigma^2\tau^2}, \quad \beta_0 = \frac{\sigma(1-\tau^2)}{1-\sigma^2\tau^2},$$

and the operator $\mathbb{C} \to \mathbb{C}$ has the norm

$$|T| = \alpha_0 + \beta_0 = \frac{\tau + \sigma}{1 + \sigma\tau} < 1.$$

The resulting equation can be rewritten as

$$\partial\bar{\partial}v + |T|\partial^2(T_0 v) = 0, \tag{17}$$

where the operator $T_0 = |T|^{-1}T$ has a unit norm.

The solution of the Dirichlet problem for the complex Equation (17) in the unit disk $\mathbb{D}$ with boundary data $v|_{\mathbb{T}} = \varphi \in C(\mathbb{T})$ will be sought in the form of a functional series

$$v(z) = \sum_{n=0}^{\infty} |T|^n v_n(z) \tag{18}$$

in powers of a small parameter $|T| < 1$. Substitute (18) into (17), and set the factors equal to zero for all powers of $|T|$; moreover, we set $v_0 = \varphi$ and $v_n = 0$ for $n \geqslant 1$ on the boundary $\mathbb{T}$ of the disk $\mathbb{D}$. As a result, we obtain the following sequence of Dirichlet problems for the Laplace equation

$$\partial\bar{\partial}v_0 = 0 \quad \text{on} \quad \mathbb{D}, \qquad v_0|_{\mathbb{T}} = \varphi$$

and the Poisson equation

$$\partial\overline{\partial}v_n = -\partial^2(T_0 v_{n-1}) \quad \text{on} \quad \mathbb{D}, \qquad v_n|_{\mathbb{T}} = 0, \qquad n \geqslant 1.$$

Let the boundary function of the Dirichlet problem for Equation (17) in the unit disk $\mathbb{D}$ have the Fourier expansion

$$f(z) = \sum_{m=0}^{\infty} a_m z^m + \sum_{m=1}^{\infty} b_m \overline{z}^m,$$

which means that

$$a_m = \frac{1}{2\pi}\int_{\mathbb{T}} f(\zeta)\overline{\zeta}^m d_1\zeta, \qquad b_m = \frac{1}{2\pi}\int_{\mathbb{T}} f(\zeta)\zeta^m d_1\zeta.$$

Using the linearity of Equation (17), one can search for the solution to the Dirichlet problem for it as a sum of solutions to the Dirichlet problems for the functions $a_m z^m$ and $b_m\overline{z}^m$. Each such problem can be solved using the described iterative method. To satisfy the boundary conditions on the function $v_n(z)$, we use the fact that $z\overline{z} = 1$ for $z \in \mathbb{T}$.

Thus, for the boundary function $f(z) = a_m z^m$, $m \geqslant 2$, we have

$$v_0(z) = a_m z^m, \qquad v_1(z) = -\alpha_0 m a_m \overline{z} z^{m-1} + \alpha_0 m a_m z^{m-2}, \ldots,$$

and for $f(z) = b_m \overline{z}^m$, $m \geqslant 2$, we find

$$v_0(z) = b_m\overline{z}^m, \qquad v_1(z) = -\beta_0 m\overline{b_m}\overline{z}z^{m-1} + \beta_0 m\overline{b_m}z^{m-2}, \ldots$$

The summation of the solutions leads to a Poisson-type integral representation, which we present (see also [25,28,29]) for the solution $u(z)$ of Equation (15), which is related to the solution $v(z)$ of Equation (17) by way of relation (16).

Theorem 2. *Let $u \in C(\overline{\mathbb{D}})$ and $\mathcal{L}_{\tau,\sigma}u = 0$, with $\tau, \sigma \in [0,1)$ in the unit disk $\mathbb{D}$. Then, for $z \in \mathbb{D}$, there is a Poisson-type integral representation*

$$u(z) = \frac{1}{2\pi}\int_{\mathbb{T}} \frac{1-|z|^2}{|\zeta - z|^2}\varphi(\zeta)d_1\zeta +$$
$$+\frac{1}{2\pi}\int_{\mathbb{T}}(1-|z|^2)\sum_{m=0}^{\infty}\Bigg(\frac{\sigma^{2m}\tau^{2m}(2\zeta - \tau^{2m}z_\tau)(\tau\varphi(\zeta) + \sigma\overline{\varphi(\zeta)})}{(\zeta_{\tau^{4m+1}} - \tau^{2m}z_\tau)(\zeta - \tau^{2m}z)(\zeta + \tau^{2m+1}\overline{z})} -$$
$$-\frac{\sigma^{2m+1}\tau^{2m+1}(2\overline{\zeta} + \tau^{2m}z_\tau)(\tau\overline{\varphi(\zeta)} + \sigma\varphi(\zeta))}{(\overline{\zeta}_{\tau^{4m+2}} + \tau^{2m+1}z_\tau)(\overline{\zeta} + \tau^{2m+1}z)(\overline{\zeta} - \tau^{2m+2}\overline{z})}\Bigg)d_1\zeta, \tag{19}$$

where $z_\tau = z - \tau\overline{z}$.

Note that this Poisson-type formula becomes much simpler when one of the parameters τ or σ vanishes. In these cases, we have

$$u(z) = \frac{1}{2\pi}\int_{\mathbb{T}}\frac{1-|z|^2}{|\zeta - z|^2}\varphi(\zeta)d_1\zeta + \frac{\sigma}{2\pi}\int_{\mathbb{T}}\frac{1-|z|^2}{(\zeta - z)^2}(2-\overline{\zeta}z)\overline{\varphi(\zeta)}d_1\zeta$$

for $\tau = 0$ (isotropic Lame equation) and

$$u(z) = \frac{1}{2\pi}\int_{\mathbb{T}}\frac{(1-|z|^2)(\zeta + \tau\overline{\zeta})\varphi(\zeta)d_1\zeta}{(\zeta - z)_\tau(\overline{\zeta} - \overline{z})(\zeta + \tau\overline{z})}$$

for $\sigma = 0$ (skew-symmetric system).

3. Conclusions

We emphasize that in this paper the main attention has been paid to explicit formulas for a general solution of the Dirichlet problem for second-order elliptic systems on the unit disk (especially for systems of two equations). Thus, we have equality (12), which is naturally called the generalized Poisson formula. It gives a solution to the problem (3) for a weakly connected elliptic system (1) in the unit disk. Another formula (19) is the generalized Poisson formula for a strongly elliptic system in the particular case where $l = 2$. As for the general questions of the solvability of this problem, numerous studies have been devoted to them. See, for example, A.H. Babayan [30–32], V.P. Burskii [33], etc.

Author Contributions: Conceptualization, A.B. and A.S.; metodology, A.B. and A.S.; validation, A.B. and A.S.; writing—original draft preparation, A.B. and A.S.; writing—review and editing, A.B. and A.S. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Bitsadze, A.V. *Boundary Value Problem for Second Order Elliptic Equations*; North-Holland Publishing Company Amsterdam: Amsterdam, The Netherlands, 1968.
2. Tovmasyan, N.E. General boundary value problem for second-order elliptic systems with constant coefficients. *Differ. Uravn.* **1966**, *2*, 163–171. (In Russian)
3. Saks, R.S. *Boundary Value Problems for Elliptic Systems of Differential Equations*; Izd-vo NGU: Novosibirsk, Russia, 1975. (In Russian)
4. Soldatov, A.P. Elliptic systems of high order. *Differ. Uravn.* **1989**, *25*, 136–144. (In Russian)
5. Soldatov, A.P. To elliptic theory for domains with piecewise smooth boundary on the plane. In Proceedings of the Congress ISAAC'97, Newark, NJ, USA, 3–7 June 1997; pp. 177–186.
6. Douglis, A.A. A function-theoretic approach to elliptic systems of equations in two variables. *Comm. Pure Appl. Math.* **1953**, *6*, 259–289. [CrossRef]
7. Hile, G.N. Elliptic systems in the plane with order terms and coustant coefficients. *Comm. Part Diff. Equ.* **1978**, *3*, 949–977. [CrossRef]
8. Soldatov, A.P. Hyperanalytic functions. *J. Math. Sci.* **2006**, *17*, 827–882. [CrossRef]
9. Vekua, I.N. *Generalized Analytic Functions*; Pergamon: Oxford, UK, 1962.
10. Bers, L. *Theory of Pseudo-Analytic Functions*; Lecture Notes; New York University: New York, NY, USA, 1953.
11. Boyarskii, B.V. Theory of generalized analytic vector. *Ann. Pol. Math.* **1966**, *17*, 281–320.
12. Gilbert, R.P. *Constructive Methods for Elliptic Equations;* Springer Lecture Notes; Springer: Berlin/Heidelberg, Germany, 1974; Volume 365.
13. Gilbert, R.P.; Hile, G.N. Generalized hypcrcomplex function theory. *Trans. Amer Math. Soc.* **1974**, *195*, 1–29. [CrossRef]
14. Volpert, I. On the index and normal solvability for elliptic systems of differential equations on the plane. *Tr. Mosk. Mat. Obs.* **1961**, *10*, 41–87. (In Russian)
15. Sirazhudinov, M.M. Boundary value problems for general elliptic systems on a plane. *Math. USSR Izv.* **1997**, *61*, 137–176. (In Russian) [CrossRef]
16. Soldatov, A.P. A function theory method in boundary value problems 1n the plane. I: The smooth case. *Math. USSR Izv.* **1991**, *55*, 1070–1100. (In Russian)
17. Bitsadze, A.V. On the uniqueness of the solution of the Dirichlet problem for elliptic partial differential equations. *Uspekhi Mat. Nauk.* **1948**, *3*, 211–212. (In Russian)
18. Vishik, M.I. On strongly elliptic systems of differential equations. *Mat. Sb.* **1951**, *29*, 615–676. (In Russian)
19. Hua, L.K.; Lin, W.; Wu, C.Q. On the uniqueness of the solution of the Dirichlet problem of the elliptic system of differential equations. *Acta Math. Sin.* **1965**, *15*, 174–187.
20. Soldatov, A.P. The Dirichlet problem for weakly connected elliptic systems in the plane. *Diff. Equ.* **2013**, *49*, 734–745.
21. Nazarov, S.A.; Plamanevskiy, B.A. *Elliptic Problems in Domains with Piecewise Smooth Boundary*; Nauka: Moscow, Russia, 1991. (In Russian)
22. Soldatov, A.P. On the first and second boundary value problems for elliptic systems in the plane. *Diff. Equ.* **2003**, *39*, 674–686. (In Russian)
23. Tovmasyan, N.E. Dirichlet problem for elliptic systems of second-order differential equations. *Dokl. Akad. Nauk.* **1964**, *159*, 995–997. (In Russian)
24. Bitsadze, A.V. *Equations of Mathematical Physics*; Nauka: Moscow, Russia, 1976. (In Russian)
25. Hua, L.K.; Lin, W.; Wu, C.-Q. *Second-Order Systems of Partial Differential Equations in the Plane*; Pitman Advanced Publishing Program: Boston, MA, USA; London, UK; Melbourne, Australia, 1985.

26. Bagapsh, A.O.; Fedorovskiy, K.Y. On energy functionals for second-order elliptic systems with constant coefficients. *Ufa Math. J.* **2022**, *14*, 14–25.
27. Landau, L.D.; Lifshitz, E.M. *Theory of Elasticity*; Butterworth-Heinemann: Oxford, UK, 1986.
28. Zaitsev, A.B. On univalence of solutions of second-order elliptic equations in the unit disk on the plane. *J. Math. Sci.* **2015**, *215*, 601–607. [CrossRef]
29. Bagapsh, A.O. The Poisson Integral and Green's Function for One Strongly Elliptic System of Equation in a Circle and an Ellipse. *Comput. Math. Math. Phys.* **2016**, *56*, 2035–2042. [CrossRef]
30. Babayan, A.H. On the Dirichlet problem for properly elliptic equation in the unit disc. *J. Contemp. Math. Anal.* **2003**, *38*, 16–26.
31. Babayan, A.H. On a Dirichlet problem for fourth order improperly elliptic equation. In *Neklassicheskije Uravnenija Matematicheskoj Fziki*; Izd. Mathem. Inst.: Novosibirsk, Russia, 2007; pp. 56–69. (In Russian)
32. Babayan, A.H. On a Dirichlet problem for fourth order partial differential equation in the case of double roots of characteristic equation. *Math. Montisnigri* **2015**, *32*, 66–80. (In Russian)
33. Burskii, V.P. *Investigations Methods of Boundary Value Problems for General Differential Equations*; Naukova Dumka: Kiev, Ukraine, 2002; 315p. (In Russian)