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Modulations of Collapsing Stochastic Modified NLSE Structures

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Abstract: The exact solutions of the nonlinear Schrödinger equation (NLSE) predict consistent novel applicable existences such as solitonic localized structures, rouge forms, and shocks that rely on physical phenomena to propagate. Theoretical explanations of randomly nonlinear new extension NLSE structure solutions have undergone stochastic mode examination. This equation enables accurate and efficient solutions capable of simulating developed solitonic structures with dynamic features. The generated random waves are a dynamically regulated system that are influenced by random water currents behaviour. It has been noticed that the stochastic parameter modulates the wave force and supplies the wave collapsing energy with related medium turbulence. It has been observed that noise effects can alter wave characteristics, which may lead to innovative astrophysics, physical density, and ocean waves.

Keywords: nonlinear stochastic behaviour; new solver method; stochastic collapsing wave

MSC: 35C07; 35R60; 60H40; 35Q40

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1. Introduction

The investigation of nonlinear stochastic partial differential equations (NSPDEs) is a vital topic that is employed in a variety of applications, such as new physics, biology, superfluids, image processing, optical fiber communications, plasma physics, and finance [1–5]. As a result, addressing NSPDEs is a exciting and current area of research. A common stochastic process that is both a martingale and a Markov process is the Wiener process, also known as Brownian motion [6]. The Wiener process is the foundation of stochastic calculus and, as such, is essential for modelling stochastic processes. It is a continuous process, the increments of which are chosen from a normal distribution for any time scale. The Wiener process is a frequently used stochastic process in dispersive situations [7,8]. Further, there is a crucial link between PDEs and stochastic processes. On the other hand, fractional Brownian dynamics via the stochastic anomalous diffusion methods play an important role in describing the considerable experimental observations of non-Brownian nonlinear diffusion for various length and time scales, from nano to interstellar spaces [9–11]. Using fractional Gaussian noise, Cherstvy et al. reported that the behaviors of ergodicity breaking for underdamped massive Brownian fractional motion for changing particle mass and trace length are in perfect accordance with the findings of stochastic computer simulations. The experimental community, that are using different single-particle tracking techniques and attempting to determine the level of nonergodicity for the recorded time series, may find the present results of interest [12].

The nonlinear models that are used the most frequently in the field of applied science are the nonlinear Schrodinger's equations (NLSEs), due to their extensive range of applications [13–16]. An investigation of their soliton solutions is critical in nonlinear science studies because they aid in describing the physical mechanism of a complicated natural phenomenon, and this subject has become one of the most intriguing and incredibly active areas of research [17–20]. Recently, various new types of solitary solutions were produced through innovative applications of nonlinear equation models [21–26]. Studies on N-soliton solutions, which may result in lump and rogue wave solutions, have been conducted for modified Korteweg–De-Vries-type integrable equations and reduced integrable nonlinear Schrödinger-type equations. The appearance of NLSEs in optical solitons with nonlinearities might be considered a growing subject of research in nonlinear photonics [27,28]. In recent years, several different types of nonlinearities have been studied, including parabolic law, Kerr law, power law, polynomial law, and saturable law [29]. Islam et al. explain the parameters of wave dispersion and nonlinearity impacts on the solitonic KMNE properties. It was noted that the optical wave propagations are expressed by the bell, bright, dark, periodic, kink, and singular with dynamical features depending on the dispersion parameters [30]. The important solitonic applications of a GP equation in water wave and plasma physics, as a nonlinear unidirectional propagating wave model, have been theoretically investigated [31]. It was reported that the soliton nature is affected by free parameter and dispersion coefficients.

The paper will provide an overview of recent advances in statistical models based on NSPDEs. In terms of the Wiener process, we will pay particular attention to the new extension NLSE and discuss when and why such models are helpful. As motivating applications, the effect of the noise term on the behaviour of the solution will be considered. The proposed deterministic model is given in references [32–35] and we presented the stochastic form as follows:

$$i(\phi_t - \delta\,\phi(x,t)\,\Gamma_t) + a\,\phi_{xy} + i\,b\,\phi(\phi\,\phi_x^* - \phi^*\,\phi_x) = 0\,, \quad i = \sqrt{-1}, \tag{1}$$

$\phi(x,y,t)$ symbolizes the nonlinear wave envelope, and $*$ represents complex conjugate. The first and second terms symbolize the temporal evolution of the wave and the disturbance of the dispersion that is given by the coefficient of a. The parameter b is distinct from the conventional Kerr law nonlinearity. The noise Γ_t is a Brownian times derivative of $\Gamma(t)$ and δ identifies noise amplitude [36]. Equation (1) depicts the bending of light beams, hole waves, oceanic rogue waves, and erbium atoms [34,35].

This research analyzes many aspects of noise's influence on the new extension NLSE using Itô sense via the Wiener process. This is a vast and fascinating field, with active research in a variety of approaches. One of the topic's fascinating features is its capacity to combine techniques from both classical and stochastic analysis [37]. We apply the unified technique [38] to produce some new stochastic solutions for the new extension NLSE. Compared to most existing methods, the recommended strategy has a number of benefits, including the avoidance of tedious computations and the generation of vital families of solutions. It is straightforward, dependable, and efficient. The proposed technique can be used as a box-solver for a number of natural science systems. Also, this method contains the rational solution which is important to describe the wave at critical points. The presented stochastic solutions for Equation (1) show a variety of crucial physical aspects, including erbium atoms, fiber communications, oceanic rogue waves, and the bending of light beams.

This work is arranged as follows. Section 2 gives the new extension NLSE via the Wiener process and its corresponding potential. Section 3 introduces the stochastic solutions utilizing a robust technique. The physical interpretation of the new extension NLSE equation's solutions is provided in Section 4. The findings are then presented in Section 5.

2. Mathematical Analysis

Using the traveling wave solution [34]:

$$\phi(x,y,t) = e^{i(-r_1x - r_2y + \Omega t + \vartheta) + \delta\Gamma(t) - \delta^2 t} U(\zeta), \quad \zeta = p_1 x + p_2 y - v t. \tag{2}$$

Here, r_1 and r_2 represent the wave numbers in x and y directions, ϑ is phase constant of soliton, δ represents the noise amplitude, and Ω identifies wave speed, whereas p_1 and p_2 denote the inverse width of the soliton along the x- and y-directions and v denotes the soliton velocity. Equation (1) becomes

$$-ar_1r_2U(\zeta) - 2br_1U^3(\zeta)e^{2\delta\Gamma(t) - 2\delta^2 t} + a\, p_1p_2U''(\zeta) - \Omega U(\zeta) = 0 \tag{3}$$

from the real part. Taking expectations on both sides gives

$$-a\, r_1r_2U(\zeta) - 2b\, r_1U^3(\zeta)e^{-2\delta^2 t}E(e^{2\delta\Gamma(t)}) + ap_1p_2U''(\zeta) - \Omega U(\zeta) = 0. \tag{4}$$

Indeed, $E(e^{2\delta\Gamma(t)}) = e^{2\delta^2 t}$, then Equation (4) becomes

$$ap_1p_2U''(\zeta) - 2br_1U^3(\zeta) - (ar_1r_2 + \Omega)U(\zeta) = 0. \tag{5}$$

On the other hand, the imaginary part gives

$$ar_2p_1U'(\zeta) + ar_1p_2U'(\zeta) + vU'(\zeta) + \delta^2U(\zeta) = 0, \tag{6}$$

with a dispersion constraint

$$-r_1\left(ar_2 + 2be^{\frac{2\delta^2\zeta}{ar_2p_1 + ar_1p_2 + v}}\right) + \frac{a\delta^4 p_1p_2}{(ar_2p_1 + ar_1p_2 + v)^2} - \Omega = 0. \tag{7}$$

Equation (5) depicts an energy equation with potential

$$V = -\frac{1}{2ap_1p_2}ar_1r_2U^2(\zeta) - \frac{1}{2ap_1p_2}br_1U^4(\zeta) - \frac{1}{2ap_1p_2}\Omega U^2(\zeta). \tag{8}$$

The model has an exact solution

$$U(\zeta) = \frac{2(ar_1r_2 + \Omega)e^{\frac{\zeta\sqrt{ar_1r_2+\Omega}}{\sqrt{a}\sqrt{p_1}\sqrt{p_2}}}}{\sqrt{-br_1(ar_1r_2 + \Omega)}\left(e^{\frac{2\zeta\sqrt{ar_1r_2+\Omega}}{\sqrt{a}\sqrt{p_1}\sqrt{p_2}}} + 1\right)},$$

$$\phi(x,y,t) = \frac{2(ar_1r_2 + \Omega)e^{\frac{\sqrt{ar_1r_2+\Omega}\,(p_1\,x + p_2\,y - v\,t)}{\sqrt{a}\sqrt{p_1}\sqrt{p_2}}}\, e^{i(-r_1x - r_2y + \Omega t + \vartheta) + \delta\Gamma(t) - \delta^2 t}}{\sqrt{-bb_1(ar_1r_2 + \Omega)}\left(e^{\frac{2\sqrt{ar_1r_2+\Omega}\,(p_1\,x + p_2\,y - v\,t)}{\sqrt{a}\sqrt{p_1}\sqrt{p_2}}} + 1\right)}. \tag{9}$$

3. The New Stochastic Solutions

We produce new stochastic solutions to Equation (1). According to the unified technique [38], the stochastic solutions of Equation (1) are as follows:

Family I:

$$U_{1,2}(\zeta) = \left(\mp\sqrt{\frac{2br_1}{2ap_1p_2}}(\zeta + \varsigma)\right)^{-1}. \tag{10}$$

Thus, the solutions for Equation (1) are

$$\phi_{1,2}(x,y,t) = \left(\mp\sqrt{\frac{2br_1}{2ap_1p_2}}(p_1\,x + p_2\,y - v\,t + \varsigma)\right)^{-1} e^{i(-r_1x-r_2y+\Omega\,t+\vartheta)+\delta\Gamma(t)-\delta^2t}, \quad (11)$$

where ς is an arbitrary constant.

Family II:

$$U_{3,4}(\zeta) = \pm\sqrt{\frac{ar_1r_2+\Omega}{2br_1}}\,tan\left(\sqrt{\frac{ar_1r_2+\Omega}{2ap_1p_2}}(\zeta+\varsigma)\right) \quad (12)$$

and

$$U_{5,6}(\zeta) = \pm\sqrt{\frac{ar_1r_2+\Omega}{2br_1}}\,cot\left(\sqrt{\frac{ar_1r_2+\Omega}{2ap_1p_2}}(\zeta+\varsigma)\right). \quad (13)$$

Thus, the solutions for Equation (1) are

$$\phi_{3,4}(x,y,t) = \pm\sqrt{\frac{ar_1r_2+\Omega}{2br_1}}\,e^{i(-r_1x-r_2y+\Omega\,t+\vartheta)+\delta\Gamma(t)-\delta^2t}\,tan\left(\sqrt{\frac{ar_1r_2+\Omega}{2ap_1p_2}}(p_1\,x+p_2\,y-v\,t+\varsigma)\right) \quad (14)$$

and

$$\phi_{5,6}(x,y,t) = \pm\sqrt{\frac{ar_1r_2+\Omega}{2br_1}}\,e^{i(-r_1x-r_2y+\Omega\,t+\vartheta)+\delta\Gamma(t)-\delta^2t}\,cot\left(\sqrt{\frac{ar_1r_2+\Omega}{2ap_1p_2}}(p_1\,x+p_2\,y-v\,t+\varsigma)\right). \quad (15)$$

Family III:

$$U_{7,8}(\zeta) = \pm\sqrt{\frac{-(ar_1r_2+\Omega)}{2br_1}}\,tanh\left(\sqrt{\frac{-(ar_1r_2+\Omega)}{2ap_1p_2}}(\zeta+\varsigma)\right) \quad (16)$$

and

$$U_{9,10}(\zeta) = \pm\sqrt{\frac{-(ar_1r_2+\Omega)}{2br_1}}\,coth\left(\sqrt{\frac{-(ar_1r_2+\Omega)}{2ap_1p_2}}(\zeta+\varsigma)\right). \quad (17)$$

Thus the solutions for Equation (1) are

$$\phi_{7,8}(x,y,t)=\pm\sqrt{\frac{-(ar_1r_2+\Omega)}{2br_1}}e^{i(-r_1x-r_2y+\Omega t+\vartheta)+\delta\Gamma(t)-\delta^2t}tanh\left(\sqrt{\frac{-(ar_1r_2+\Omega)}{2ap_1p_2}}(p_1x+p_2y-vt+\varsigma)\right) \quad (18)$$

and

$$\phi_{9,10}(x,y,t)=\pm\sqrt{\frac{-(ar_1r_2+\Omega)}{2br_1}}e^{i(-r_1x-r_2y+\Omega t+\vartheta)+\delta\Gamma(t)-\delta^2t}coth\left(\sqrt{\frac{-(ar_1r_2+\Omega)}{2ap_1p_2}}(p_1x+p_2y-vt+\varsigma)\right). \quad (19)$$

4. Physical Interpretation

Here we present the mathematical analysis of our model (1), which characterized oceanic and hole waves, and produced accurate solitons, oscillatory disturbances, super solitons, and breathers formations. The model under investigation reduced to Equation (5). The expectations of (3) by $E(e^{2\delta\Gamma(t)}) = e^{2\delta^2t}$ transforms the model into a differential equation which was solved using a mathematical solver to give many important solitary solutions. We introduce some 2D and 3D graphs for some chosen solutions of Equation (1) for suitable parametric choices using Matlab release 18 and Mathematica release 13. Equation (9) represents a group of randomly generated solitons, as seen in Figures 1–3. The mathematical method produces different effective solitary wave generations as Equations (11), (14), (15), (18), and (19). Equation (11) is a rational growing rapid explosive wave. Solutions (14) and (15) are periodic blow-up structures. Furthermore, solutions (18) and (19) are dissipative shock wave formations.

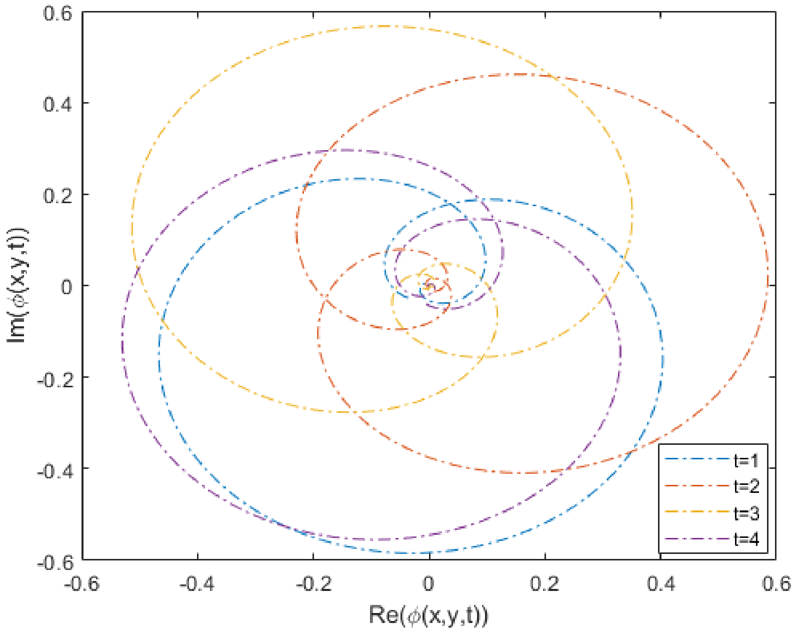

Figure 1. Trajectory of $\phi(x,t)$ for $r_1 = 0.5$, $r_2 = 0.5$, $p_1 = 0.5$, $p_2 = 0.5$, $\Omega = 1.5$, $\delta = 0$, $\vartheta = 0.05$, $a = 2$, $b = -1.7$.

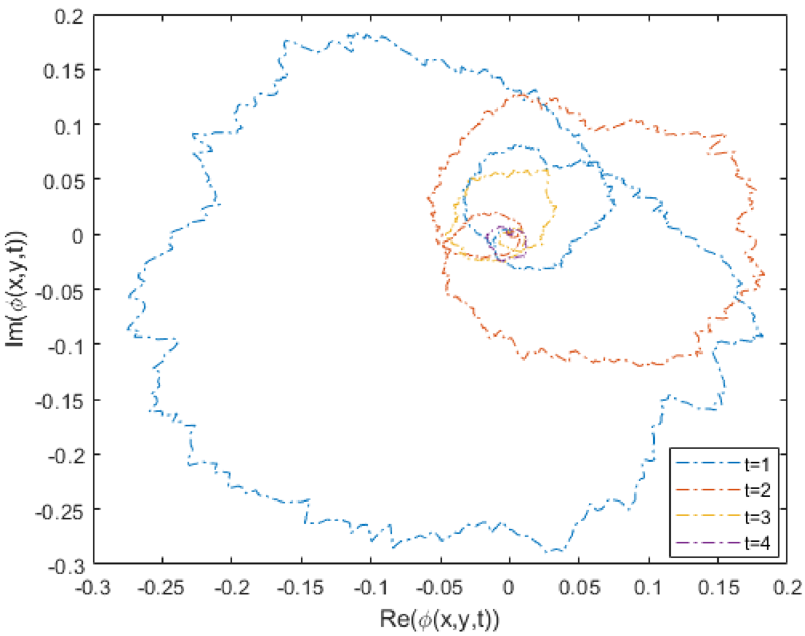

Figure 2. Trajectory of $\phi(x,t)$ for $r_1 = 0.5$, $r_2 = 0.5$, $p_1 = 0.5$, $p_2 = 0.5$, $\Omega = 1.5$, $\delta = 1$, $\vartheta = 0.05$, $a = 2$, $b = -1.7$.

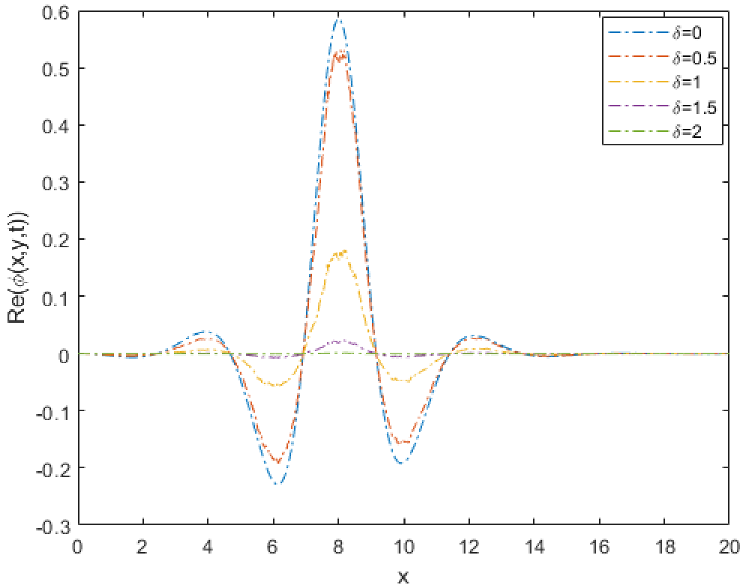

Figure 3. Plot of $\phi(x,t)$ with x,δ for r_1 = 0.5, r_2 = 0.5, p_1 = 0.5, p_2 = 0.5, Ω = 1.5, ϑ = 0.05, a = 2, b = -1.7.

The rigorous randomness factor influences on structure, amplitude, band width, and soliton energy are given in Figures 1 and 2. The ability of the abrupt wave collapse, which depends mainly on the impact of randomness, grows with increasing time t, as seen in Figure 2. At time $t = 4$, it was determined that the system almost totally collapses. When δ increases, we observe that the wave's amplitude and width both shrink, and the wave starts to collapse, which is complete at $\delta = 1$, as shown in Figure 3. Also, the dark solution (18), which performs the dissipative pictures, was identified to be impacted by time t and the random variable δ, as illustrated in Figures 4–6. The rate of collapse of the dissipative wave rises when t is increased, as shown in Figure 5. Additionally, as seen in Figure 6, the parameter δ induces the wave to collapse and convert into a distorted waveform with a limited amplitude.

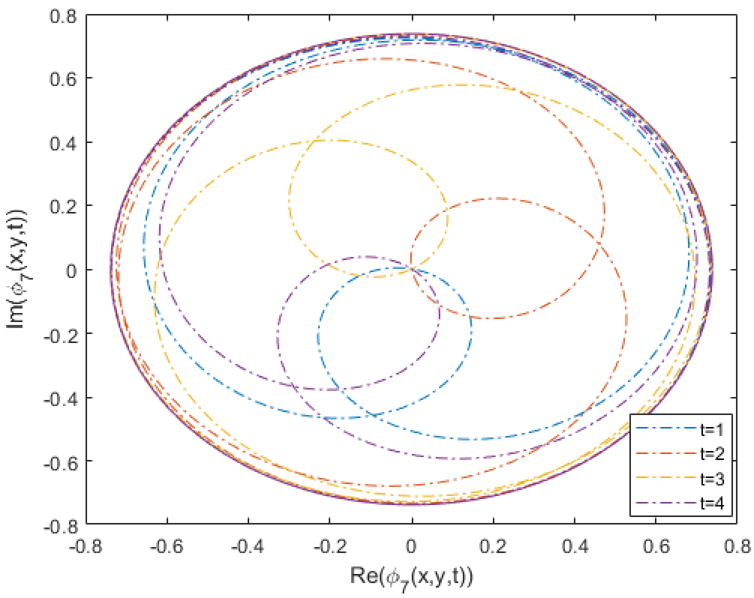

Figure 4. Trajectory of $\phi_7(x,t)$ for r_1 = 0.5, r_2 = 0.5, p_1 = 0.5, p_2 = 0.5, Ω = 1.5, δ = 0, ϑ = 0.05, a = -2, b = 1.7.

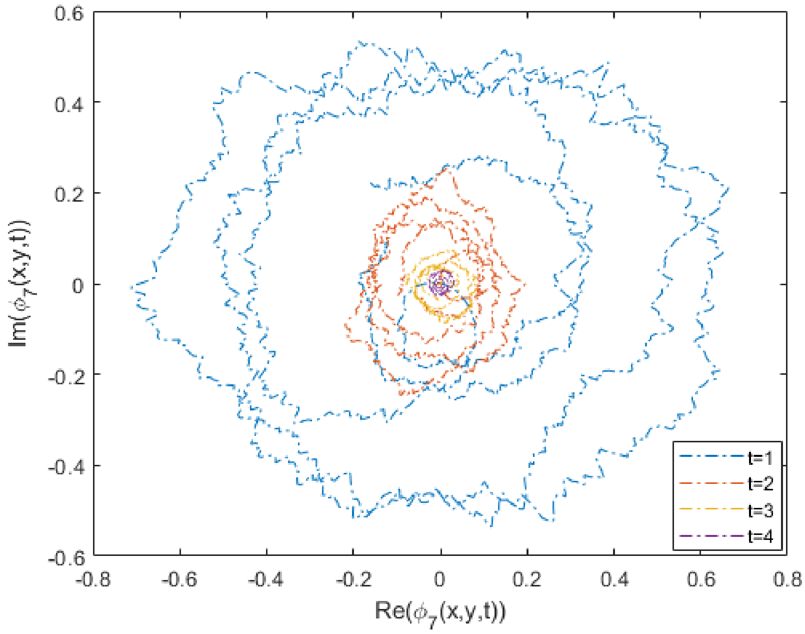

Figure 5. Trajectory of $\phi_7(x,t)$ for $r_1 = 0.5$, $r_2 = 0.5$, $p_1 = 0.5$, $p_2 = 0.5$, $\Omega = 1.5$, $\delta = 1$, $\vartheta = 0.05$, $a = -2$, $b = 1.7$.

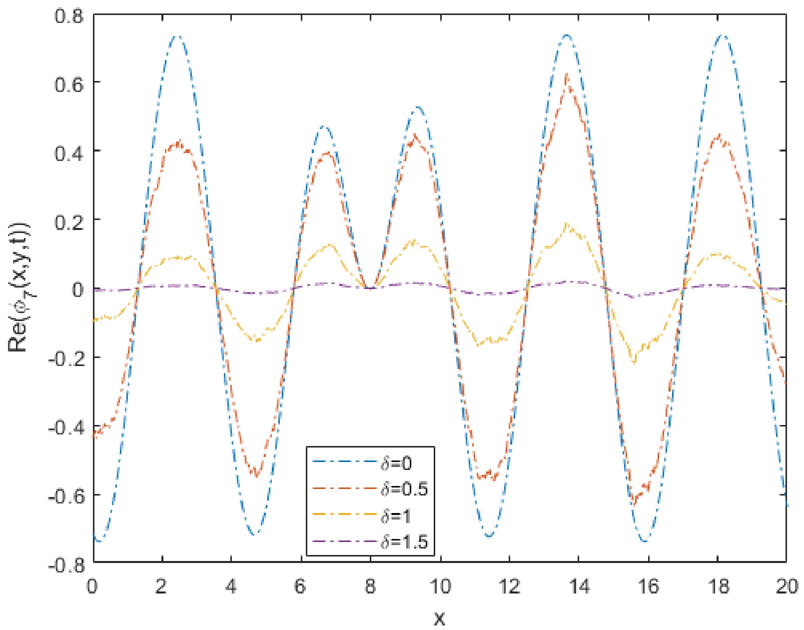

Figure 6. Plot of $\phi_7(x,t)$ with x, δ for $r_1 = 0.5$, $r_2 = 0.5$, $p_1 = 0.5$, $p_2 = 0.5$, $\Omega = 1.5$, $\vartheta = 0.05$, $a = -2$, $b = 1.7$.

In being devoid of random impacts, the enormous significance of the numerous solitary properties of the investigated solutions must be examined. For example, Equation (9) provides breathers structures, and stationary and super solitons, as shown in Figures 7 and 8. The solution (11) produces two wave structure types; the first type is a bright explosive envelope wave and the second is an explosive solitonic form, as depicted in Figures 9 and 10.

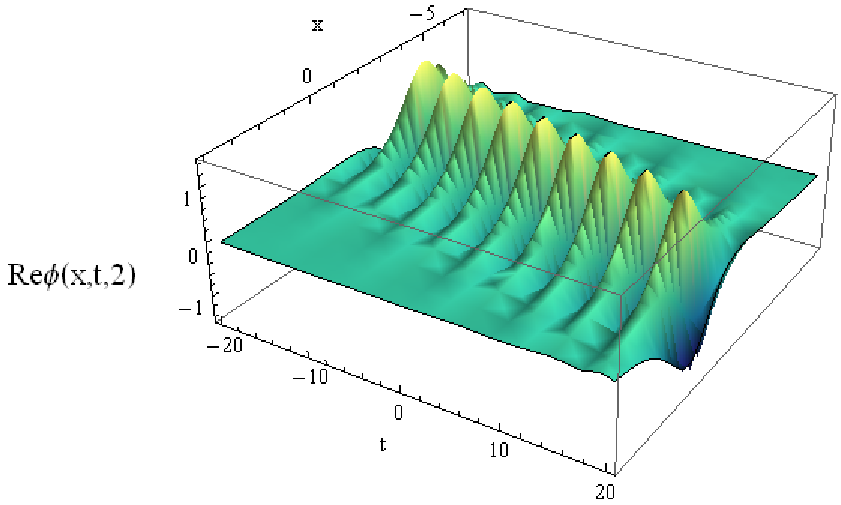

Figure 7. Plot of $\phi(x,t)$ with x,t for $r_1 = 0.5$, $r_2 = 0.5$, $p_1 = 0.5$, $p_2 = 0.5$, $\Omega = 1.5$, $\delta = 0$, $\vartheta = 0.05$, $a = 2$, $b = -1.7$.

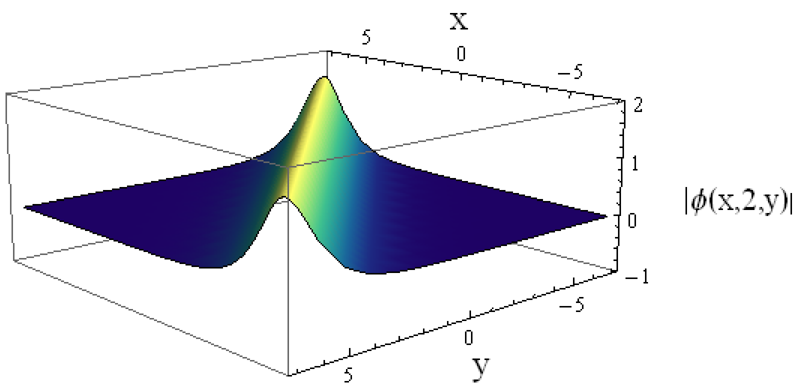

Figure 8. Plot of $|\phi(x,t)|$ with x,y for $r_1 = 0.5$, $r_2 = 0.5$, $p_1 = 0.5$, $p_2 = 0.5$, $\Omega = 1.5$, $\delta = 0$, $\vartheta = 0.05$, $a = 2$, $b = -1.7$.

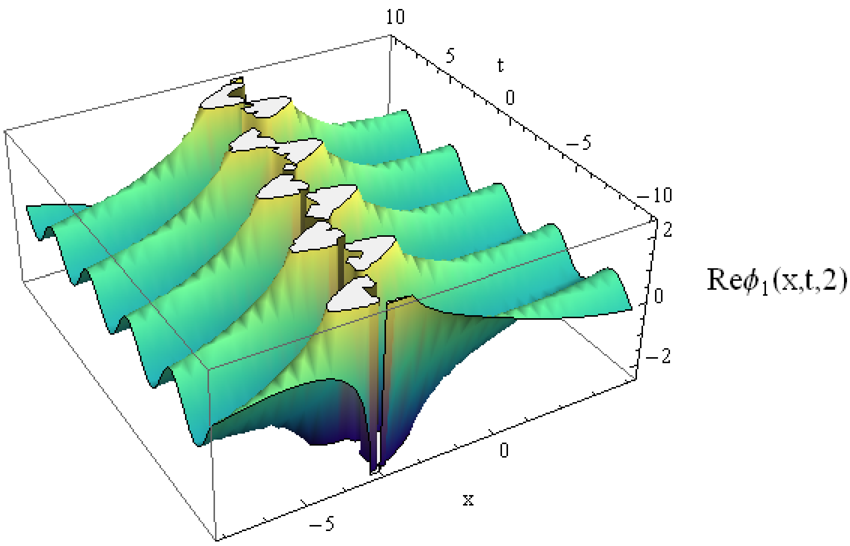

Figure 9. Plot of $\phi_1(x,t)$ with x,y for $r_1 = 0.5$, $r_2 = 0.5$, $p_1 = 0.5$, $p_2 = 0.5$, $\Omega = 1.5$, $\delta = 0$, $\vartheta = 0.05$, $a = 2$, $b = 1.7$.

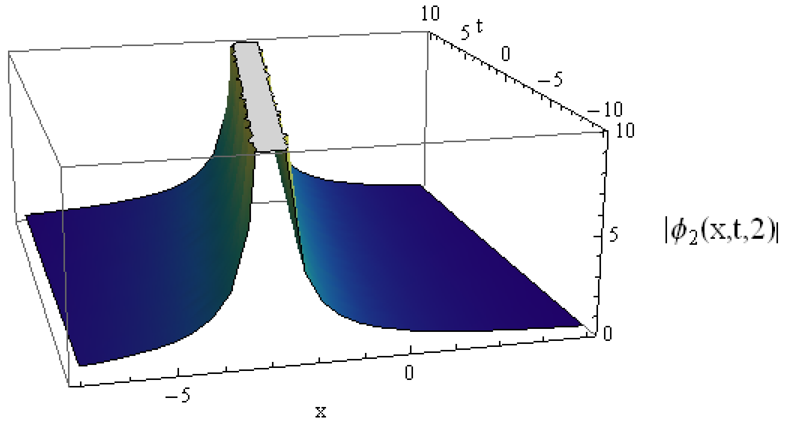

Figure 10. Plot of Re$\phi_2(x,t)$ with x,t for $r_1 = 0.5, r_2 = 0.5$, $p_1 = 0.5$, $p_2 = 0.5$, $\Omega = 1.5$, $\delta = 0$, $\vartheta = 0.05$, $a = 2$, $b = 1.7$.

In contrast, the form (15) is regarded as one of the important physical aspects in the investigation of super-explosive forms and dissipative blow-up structures. Figures 11 and 12 demonstrate the generation of the dissipative blow-up waveforms and rational super-explosive structures. Finally, Figure 13 describes the explosive envelopes of Equation (18) in x and t directions.

In summary, the characteristics of the stochastic nonlinear solitonic structures of the studied model with a stochastic noise term provoked the dynamical energy advantages of the obtained solitary envelopes and dissipative–dispersive waves.

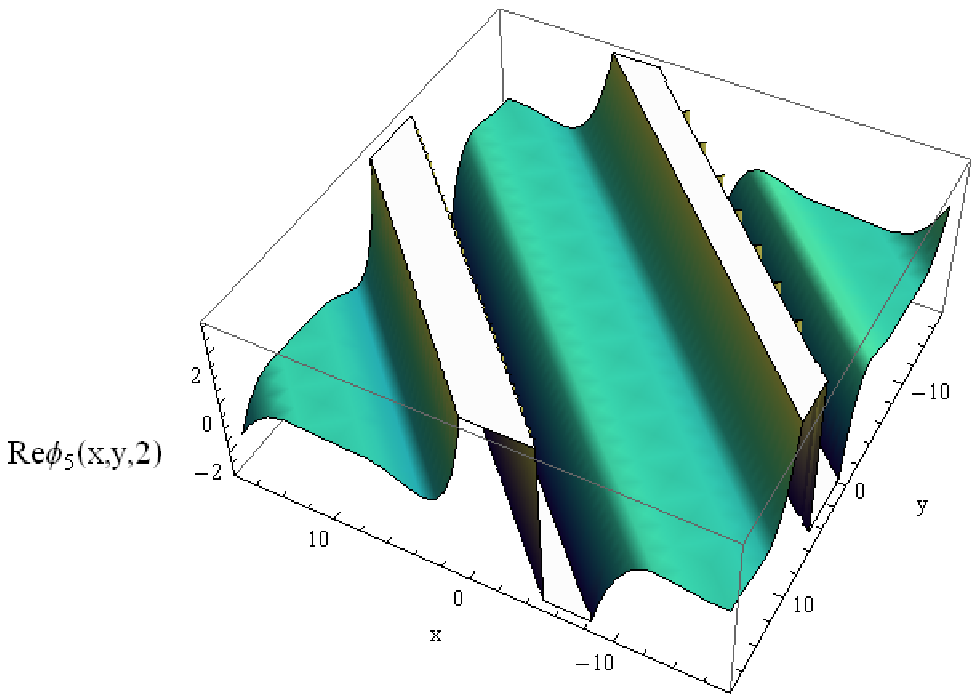

Figure 11. Plot of Im$\phi_5(x,t)$ with x,t for $\vartheta = 0.1$, $r_1 = 0.5$, $r_2 = 0.5$, $p_1 = 0.5$, $p_2 = 0.5$, $\Omega = 1.5$, $\delta = 0$, $\vartheta = 0.05$, $a = 2$, $b = 1.7$.

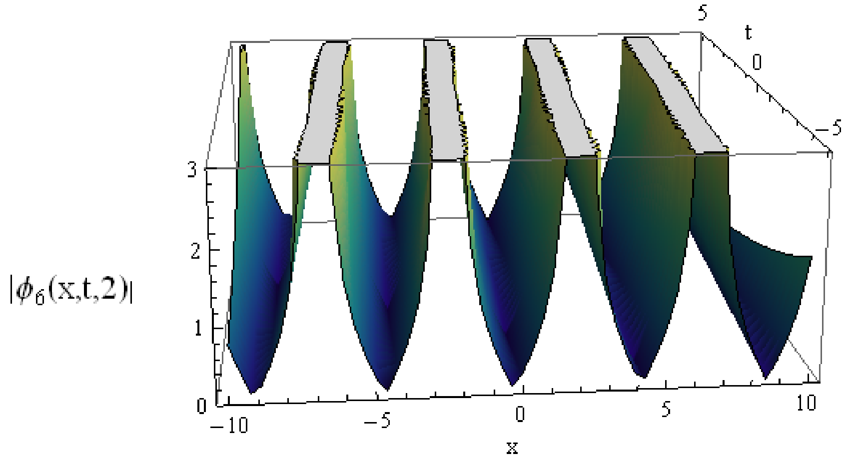

Figure 12. Plot of $\mathrm{Re}\phi_6(x,t)$ with x,t for $r_1 = 0.5$, $r_2 = 0.5$, $p_1 = 0.5$, $p_2 = 0.5$, $\Omega = 1.5$, $\delta = 0$, $\vartheta = 0.05$, $a = 2$, $b = 1.7$.

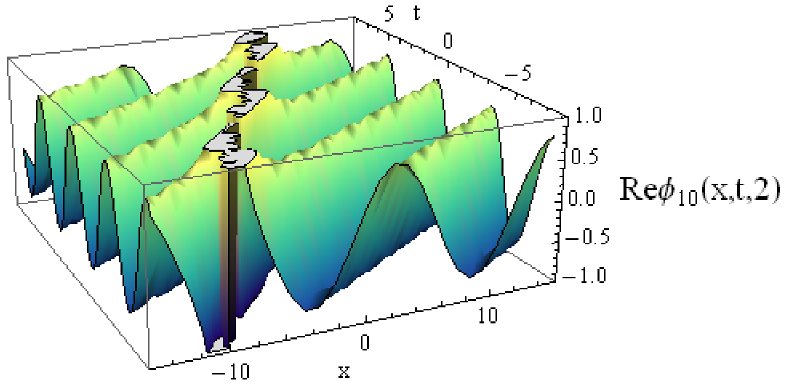

Figure 13. Plot of $|\phi_{10}(x,t)|$ with x,t for $r_1 = 0.5$, $r_2 = 0.5$, $p_1 = 0.5$, $p_2 = 0.5$, $\Omega = 1.5$, $\delta = 0$, $\vartheta = 0.05$, $a = -2$, $b = 1.7$.

5. Conclusions

The new extension NLSE model has been used to analyze the fundamental wave properties for exact stochastic solitary, blow-up dispersive–dissipative and super explosive shocks, and breather and explosive super structures. The Kerr nonlinearity coefficient affected the characteristic properties of the obtained solitary structures. The alterations of the stochastic noise in the obtained waveform amplitudes and energies have been examined. It was reported that the random influences can be demonstrated by some modulations in collapsing dissipative and dispersive explosive water structures. The noise stochastic parameter modulates and fluctuates the resulting wave, producing collapsing solitonic tails. The applications of these mathematical discussions might be utilized in sea-ocean wave applications.

Author Contributions: M.A.E.A.: Conceptualization, Software, Formal analysis, Writing—review editing. E.K.E.-S.: Conceptualization, Software, Formal analysis, Writing—review editing. Y.O.: Conceptualization, Data curation, Writing—original draft. N.F.A.: Conceptualization, Data curation, Writing—original draft. All authors have read and agreed to the published version of the manuscript.

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Data Availability Statement: The data used to support the findings of this study are included within the article.

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