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Regularized Asymptotics of the Solution of a Singularly Perturbed Mixed Problem on the Semiaxis for the Schrödinger Equation with the Potential $Q = X^2$ [†]

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Abstract: In this paper, we study the solution of a singularly perturbed inhomogeneous mixed problem on the half-axis for the Schrödinger equation in the presence of a “strong” turning point for the limit operator on time interval that do not contain focal points. Based on the ideas of the regularization method for asymptotic integration of problems with an unstable spectrum, it is shown how regularizing functions should be constructed for this type of singularity. The paper describes in detail the formalism of the regularization method, justifies the algorithm, constructs an asymptotic solution of any order in a small parameter, and proves a theorem on the asymptotic convergence of the resulting series.

Keywords: singularly perturbed problem; asymptotic solution; regularization method; turning point

MSC: 35J10



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1. Introduction

At present, a huge number of works are devoted to various methods of asymptotic integration of singularly perturbed problems. There are so many of them that it is not possible to give a complete review in a paper of limited volume. We refer the reader to the monographs [1,2], where detailed bibliographies on existing approaches in the theory of singular perturbations are given, and a review is made of the current state of the S.A. Lomov regularization method, the main principles of which, according to the author himself, are in the monograph [1], and were laid down in the late fifties and early sixties of the last century in the series [3–8]. The main problem that the researcher faces when applying the regularization method is related to the search and description of regularizing functions that contain a non-uniform singular dependence of the solution of the desired problem, highlighting that you can search for the rest of the solution in the form of power series in a small parameter. The development of the regularization method led to the understanding that this search is closely related to the spectral characteristics of the limit operator. In particular, it is established how the singular dependence of the asymptotic solution on a small parameter should be described under the condition that the spectrum is stable (see [1]). When stability conditions are violated, things are much more complicated. Moreover, there is still no complete mathematical theory for singularly perturbed problems with an unstable spectrum, although they began to be studied from a general mathematical standpoint about 50 years ago. Of particular interest among such problems are those in which the spectral features are expressed in the form of point instability (see, for example, [9–12]). In works devoted to singularly perturbed problems, some of the features of this type are called turning points, and their classification is as follows:

- (1) Simple turning point: the eigenvalues of the limit operator are isolated from each other and one eigenvalue vanishes at separate points;
- (2) Weak turning point: at least a pair of eigenvalues intersect at separate points, but the limit operator retains the diagonal structure up to the intersection points, and the basis of eigenvectors retains smoothness;
- (3) Strong turning point: at least a pair of eigenvalues intersect at separate points, but in this case, the limit operator changes the diagonal structure to Jordan at the intersection points, and the basis of eigenvectors loses smoothness.

Here, we give links to several recent studies in the framework of the method of regularization of singularly perturbed problems with singularities in the spectrum of the limit operator of the indicated form: for a simple turning point, see papers [9,10], for a weak turning point, see [11], and for a strong turning point, see [12,13].

Typical physical examples of singularly perturbed problems are the Navier–Stokes equation with low viscosity and the Schrödinger equation, if the Planck constant \hbar is considered a small quantity. Strictly speaking, the Planck constant \hbar is a dimensional quantity and has a very specific value, and the assertion that \hbar is small should be understood in the sense that it is always possible to single out a dimensionless combination of parameters that contains \hbar to some extent, which is small compared to other dimensionless parameters that do not contain \hbar . The formal passage to the limit $\hbar \rightarrow 0$ in the relations of quantum theory makes the transition from quantum to classical mechanics (see, for example, [14], §6); therefore, in cases where it is expedient to look for \hbar solutions of the Schrödinger equation, speak of a semiclassical approximation (see [14], Chap. 7). The described semiclassical transition in the nonstationary Schrödinger equation in the coordinate representation on the semiaxis with the Hamiltonian $\hat{H}(p, x) = \hat{p}^2 + \hat{x}^2$ generates a singularly perturbed problem whose asymptotic integration dedicated to the present work. It should be immediately noted that the problem we are considering is considered on a time interval in which no focal points arise, but only a turning point $x = 0$ is present. In addition, it contains an inhomogeneous Schrödinger equation, which, as will become clear in the main text of the article, significantly complicates the process of constructing a regularized asymptotic series. In many ways, our studies on the asymptotic integration of a mixed problem on a semiaxis for a nonstationary and inhomogeneous Schrödinger equation with the above-mentioned Hamiltonian at $\hbar \rightarrow 0$ represent the development of ideas from [12,13], where the Cauchy problem for a parabolic equation with “strong” turning point.

2. Nomenclature

All quantities in the article are dimensionless:

1. x, t, τ variables;
2. $x \in [0, +\infty)$, $\tau, t \in [0, T]$ and $0 \leq \tau \leq t \leq T$;
3. ε is a small parameter varying within $0 < \varepsilon < \varepsilon_0$;
4. $f(x, t)$, $u(x, t, \varepsilon)$, $\varphi(x, t)$, $\psi(t)$, $v(x, t, \varepsilon)$, $v_k(x, t, \varepsilon)$, $y_i(t, \varepsilon)$, $z_i(t, \varepsilon)$, $w(x, t, \varepsilon)$, $W(x, t, \varepsilon)_k$, $H(x, t, \varepsilon)$ functions;
5. $G, F, T_\varepsilon, \sigma_i$ operators;
6. M, k, m, C, C_1, C_2 constants;
7. $Q_T = [0, +\infty) \times [0, T]$ problem solution area.

3. Formulation of the Problem

Let the task be given

$$\begin{cases} i\varepsilon \frac{\partial u}{\partial t} = -\varepsilon^2 \frac{\partial^2 u}{\partial x^2} + x^2 u + h(x, t), \\ u(x, 0) = f(x), \\ u(x, t) = \psi(t), f(0) = \psi(0), \quad 0 \leq x < +\infty, \\ t \in [0, T], 0 \leq T < \frac{\pi}{4} (\text{no focal point condition}). \end{cases} \quad (1)$$

and the following conditions are met:

- (1) $f(x) \in C^\infty(0, +\infty)$;
- (2) $h(x, t) \in C^\infty(0, +\infty) \times [0, T]$;
- (3) $\int_0^{+\infty} x^2 |f(x)| dx < \infty$ and $\int_0^{+\infty} x^2 |h(x, t)| dx < \infty$ converge uniformly in t (sufficient conditions for the existence of a classical solution to the problem);
- (4) $\forall k, m, n \in \mathbb{N}$: $\int_0^{+\infty} x^m |f_0^{(k)}(x)| dx < \infty$, $\int_0^{+\infty} x^m |h^{(k,n)}(x, t)| dx < \infty$ converge uniformly in t (which are sufficient conditions for constructing an asymptotic series).

A classical solution to the problem (1) is a function $u(x, t, \varepsilon)$ continuous in $\overline{Q_T} = [0, +\infty) \times [0, T]$, having continuous $\frac{\partial u}{\partial t}$, $\frac{\partial u}{\partial x}$, $\frac{\partial^2 u}{\partial x^2}$ in Q_T , which satisfies Equation (1) at all points of Q_T , and continuously adjoins the initial conditions $f(x)$ and edge $\psi(t)$. The following theorem is true.

Theorem 1. *The classical solution of the problem (1) under conditions (1)–(3) exists and is unique.*

Proof. See Appendix A. \square

For a visual representation of the form of the spectral feature in the problem posed, one should switch to the matrix form of notation:

$$\varepsilon \frac{\partial}{\partial x} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ x^2 & 0 \end{pmatrix} \cdot \begin{pmatrix} u \\ v \end{pmatrix} - i\varepsilon \begin{pmatrix} 0 & 0 \\ \frac{\partial}{\partial t} & 0 \end{pmatrix} \cdot \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ h \end{pmatrix},$$

here, the replacement $\varepsilon \partial u / \partial x = v$ is introduced. Then, the matrix of the limit operator has the form:

$$A(x) = \begin{pmatrix} 0 & 1 \\ x^2 & 0 \end{pmatrix}.$$

Now, it is easy to see that the matrix $A(x)$ is diagonalizable and has a smooth basis of eigenvectors at $x \neq 0$, and at the intersection point of the eigenvalues (that is, at $x = 0$) the corresponding limit operator changes diagonal structure onto a Jordan structure and the basis of eigenvectors loses smoothness in x . According to the classification given in the introduction, such a spectral feature is a strong turning point.

In the general case, regularizing functions must be constructed based on the canonical form of the limit operator, which can be reduced by smooth transformations (see, for example, [15]), and the corresponding basis, but in the proposed problem, the operator already has the canonical form and there is no need for corresponding constructions. Moreover, it is necessary to regularize the right side of $h(x, t)$, due to the fact that the limit operator with matrix $A(x)$ at the point $x = 0$ is not invertible.

4. Formalism of the Regularization Method

4.1. Regularizing Function

The regularizing function of the problem (1) will be sought in the standard form $e^{-i\varphi(x,t)/\varepsilon}$. For solutions of linear homogeneous equations, such singularities were highlighted by J. Liouville in [16]. So, substituting $u(x, t) = v(x, t)e^{-i\varphi(x,t)/\varepsilon}$ into the corresponding homogeneous equation of (1) and collecting the terms for the same powers of ε , we obtain:

$$\left(\frac{\partial \varphi}{\partial t} - \left(\frac{\partial \varphi}{\partial x} \right)^2 - x^2 \right) v + i\varepsilon \left(\frac{\partial v}{\partial t} - 2 \frac{\partial \varphi}{\partial x} \frac{\partial v}{\partial x} - \frac{\partial^2 \varphi}{\partial x^2} v \right) + \varepsilon^2 \frac{\partial^2 v}{\partial x^2} = 0. \quad (2)$$

An analysis of the last expression allows us to state that to search for $u(x, t)$ as a regular series in ε ; we need to take the solution of the following problem as $\varphi(x, t)$:

$$\frac{\partial \varphi}{\partial t} - \left(\frac{\partial \varphi}{\partial x} \right)^2 = x^2, \varphi(x, 0) = 0. \quad (3)$$

The choice of the initial condition for $\varphi(x, t)$ is due to the fact that, in what follows, the initial condition for $v(x, t)$ does not contain a singular dependence on ε . Moreover, with such a choice, the initial condition on $v(x, t)$ inherits the initial condition of the (1) problem.

The problem (3) is a problem for a nonlinear partial differential equation of the first order, which we will solve by the method of characteristics (see [17], Ch. 5, §4, pp. 268–272). Denoting $p = \partial \varphi / \partial t$ and $q = \partial \varphi / \partial x$, we obtain the following characteristic system for the equation of problem (3):

$$\begin{aligned} \frac{dt}{1} &= \frac{dx}{-2q} = \frac{dp}{0} = \frac{dq}{2x} = \frac{d\varphi}{p - 2q^2} = d\tau, \\ \text{N.W.: } t &= 0, x = s, \varphi = 0, q = 0, p = s^2. \end{aligned} \quad (4)$$

The initial conditions in the last system are obtained by parametrization (s —parameter) of the initial condition of problem (3).

Integrating system (4), we obtain the desired surface in a parametric form:

$$t = \tau, x = s \cos(2\tau), \varphi = s^2 \frac{\sin(4\tau)}{4}.$$

Then, finally, for the function $\varphi(x, t)$ we explicitly have:

$$\varphi(x, t) = \frac{x^2 \tan(2t)}{2}. \quad (5)$$

Thus, the regularizing function has the form $e^{-i \frac{x^2 \tan(2t)}{2\varepsilon}}$.

4.2. Regularizing Singular Operators

Additional regularizing singular operators, related to the pointwise irreversibility of the limit operator $A(x)$, are constructed using the fundamental solution of problem (1) on the entire line (see item 8). We write here only the final form of the fundamental solution:

$$K(x, \xi, t) = \frac{1-i}{2\sqrt{\pi\varepsilon \sin(2t)}} \exp \left[i \left(\cot(2t) \frac{x^2 + \xi^2}{2\varepsilon} - \frac{x\xi}{\varepsilon \sin(2t)} \right) \right]$$

$K(x, \xi, t)$ has the property $K(x, \xi, 0) = \delta(x - \xi)$.

Additional singular integral operators for the regularization of the right-hand sides of iterative problems are obtained by integrating $K(x, \xi, t)$ over the variable ξ (see Section 5) and dividing by i for convenience. Then, we obtain:

$$\begin{aligned} \sigma_0(x, t, \varepsilon)(\cdot) &= -i \int_0^t (\cdot) d\tau \int_{-\infty}^{+\infty} K(x, \xi, t - \tau) d\xi = \\ &= -i \int_0^t (\cdot) \frac{1}{\sqrt{\cos(2(t - \tau))}} e^{-\frac{ix^2 \tan(2(t - \tau))}{2\varepsilon}} d\tau, \\ \sigma_1(x, t, \varepsilon)(\cdot) &= -ix \int_0^t (\cdot) d\tau \int_{-\infty}^{+\infty} \xi K(x, \xi, t - \tau) d\xi = \\ &= -ix \int_0^t (\cdot) \frac{1}{\sqrt{\cos 2(t - \tau)}} e^{-\frac{ix^2 \tan(2(t - \tau))}{2\varepsilon}} d\tau. \end{aligned}$$

In fact, the singular operators $\sigma_0(x, t, \varepsilon)(\cdot)$, $\sigma_1(x, t, \varepsilon)(\cdot)$ are solutions of the Schrödinger equation with right-hand sides $-i\varepsilon$, $-i\varepsilon x$. The actions of operators on a function will be written as:

$$\begin{aligned}\sigma_0(f(t)) &= -i \int_0^t \frac{f(\tau)}{\sqrt{\cos 2(t-\tau)}} e^{-\frac{ix^2 \tan(2(t-\tau))}{2\varepsilon}} d\tau = -if(t) * \frac{e^{-\frac{ix^2 \tan(2t)}{2\varepsilon}}}{\sqrt{\cos(2t)}}, \\ \sigma_1(f(t)) &= -ix \int_0^t \frac{f(\tau)}{\sqrt{\cos 2(t-\tau)}^3} e^{-\frac{ix^2 \tan(2(t-\tau))}{2\varepsilon}} d\tau = -ixf(t) * \frac{e^{-\frac{ix^2 \tan(2t)}{2\varepsilon}}}{\sqrt{\cos 2t}^3}.\end{aligned}\quad (6)$$

4.3. Singular Integral Operator for Describing the “Boundary Layer” in the Vicinity of the Point $x = 0$

This operator has the form:

$$G(\Psi) = \int_0^t \Psi(\tau) W(x, t - \tau, \varepsilon) d\tau = (1+i) \sqrt{\frac{2}{\pi}} \int_{\frac{x}{\sqrt{2\varepsilon \tan(2t)}}}^{\infty} \frac{\psi\left(t - \frac{1}{2} \arctan(b)\right) e^{-z^2}}{\sqrt[4]{1+b^2}} dz,$$

where $W(x, t, \varepsilon) = \frac{(1+i)x}{\sqrt{\varepsilon\pi}} \frac{e^{-\frac{ix^2 \cot(2t)}{2\varepsilon}}}{\sqrt{\sin(2t)}^3}$, $b = \frac{x^2}{2\varepsilon z^2}$. We note the properties of the operator G :

$$T_\varepsilon G(\Psi) = 0, \quad G(\Psi)|_{x=0} = \Psi(t).$$

5. Construction of a Regularized Asymptotic Series: Iterative Problems

The regularizing function $e^{-i\varphi(x,t)/\varepsilon}$ introduced in the previous section, and the additional regularizing operators $\sigma_0(x, t, \varepsilon)$, $\sigma_1(x, t, \varepsilon)$, allow us to expect that the rest of the solution can be sought in the form of power series in ε . The regularized solution of the problem (1) is sought in the form

$$u(x, t, \varepsilon) = v(x, t, \varepsilon) e^{-\frac{ix^2 \tan(2t)}{2\varepsilon}} + G(\Psi(t, \varepsilon)) + \sigma_0(y(t, \varepsilon)) + \sigma_1(z(t, \varepsilon)) + w(x, t, \varepsilon). \quad (7)$$

Substituting (7) into (1) and extracting the terms of the regularizing functions, we obtain the problem:

$$\begin{aligned}\frac{\partial v}{\partial t} - 2x \tan(2t) \frac{\partial v}{\partial x} - \tan(2t)v &= i\varepsilon \frac{\partial^2 v}{\partial x^2}, \\ \Psi(t, \varepsilon) * T_\varepsilon W(t, \varepsilon) &= 0, \\ y(t, \varepsilon) * T_\varepsilon \left(\frac{e^{-\frac{ix^2 \tan(2t)}{2\varepsilon}}}{\sqrt{\cos(2t)}} \right) &= 0, \\ z(t, \varepsilon) * T_\varepsilon \left(x \frac{e^{-\frac{ix^2 \tan(2t)}{2\varepsilon}}}{\sqrt{\cos(2t)}^3} \right) &= 0, \\ x^2 w &= -h(x, t) + i\varepsilon \frac{\partial w}{\partial t} + \varepsilon^2 \frac{\partial^2 w}{\partial x^2} + \varepsilon y(t, \varepsilon) + \varepsilon x z(t, \varepsilon), \\ v(x, 0) + w(x, 0) &= f(x), \\ v(0, t, \varepsilon) + \Psi(t, \varepsilon) + \int_0^t \frac{y(\tau, \varepsilon)}{\sqrt{\cos(2(t-\tau))}} d\tau + w(0, t, \varepsilon) &= \psi(t).\end{aligned}\quad (8)$$

From (8), we obtain a series of iterative problems:

$$\begin{aligned}
 \frac{\partial v_k}{\partial t} - 2x \tan(2t) \frac{\partial v_k}{\partial x} - \tan(2t) v_k &= \frac{\partial^2 v_{k-1}}{\partial x^2}, \\
 \Psi_k(t) * T_\varepsilon W(t, \varepsilon) &= 0, \\
 y_k(t) * T_\varepsilon \left(\frac{e^{-\frac{ix^2 \tan(2t)}{2\varepsilon}}}{\sqrt{\cos(2t)}} \right) &= 0, \\
 z_k(t) * T_\varepsilon \left(x \frac{e^{-\frac{ix^2 \tan(2t)}{2\varepsilon}}}{\sqrt{\cos(2t)}^3} \right) &= 0, \\
 x^2 w_k &= -h(x, t) \delta_0^k + i \frac{\partial w_{k-1}}{\partial t} + \frac{\partial^2 w_{k-2}}{\partial x^2} + y_{k-1}(t) + x z_{k-1}(t), \\
 v_k(x, 0) + w_k(x, 0) &= f(x) \delta_k^0, \\
 v_k(0, t) + \Psi_k(t) + \int_0^t \frac{y_k(\tau)}{\sqrt{\cos 2(t-\tau)}} d\tau + w_k(0, t) &= \delta_k^0 \psi(t), \quad k = \overline{-1, \infty}.
 \end{aligned} \tag{9}$$

Here, δ_k^0 is the Kronecker symbol: $\delta_0^0 = 1$, $\delta_k^0 = 0$ for $k \neq 0$. The system at step $k = -1$ has the form

$$\begin{aligned}
 \frac{\partial v_{-1}}{\partial t} + 2x \tan(2t) \frac{\partial v_{-1}}{\partial x} + \tan(2t) v_{-1} &= 0, \\
 \Psi_{-1}(t) * T_\varepsilon W(t, \varepsilon) &= 0, \\
 y_{-1}(t) * T_\varepsilon \left(\frac{e^{-\frac{ix^2 \tan(2t)}{2\varepsilon}}}{\sqrt{\cos(2t)}} \right) &= 0, \\
 z_{-1}(t) * T_\varepsilon \left(x \frac{e^{-\frac{ix^2 \tan(2t)}{2\varepsilon}}}{\sqrt{\cos(2t)}^3} \right) &= 0, \\
 x^2 w_{-1} &= 0, \\
 v_{-1}(x, 0) + w_{-1}(x, 0) &= 0, \\
 v_{-1}(0, t) + \Psi_{-1}(t) + \int_0^t \frac{y_{-1}(\tau)}{\sqrt{\cos(2(t-\tau))}} d\tau + w_{-1}(0, t) &= 0.
 \end{aligned}$$

Solutions at the iterative step $k = -1$ will be $v_{-1}(x, t) \equiv 0$, $w_{-1}(x, t) \equiv 0$, and $\Psi_{-1}(t)$, $y_{-1}(t)$, $z_{-1}(t)$ are arbitrary functions. To determine them, consider the iteration problem at the zero iteration step $k = 0$:

$$\begin{aligned}
 \frac{\partial v_0}{\partial t} + 2x \tan(2t) \frac{\partial v_0}{\partial x} + \tan(2t) v_0 &= 0, \\
 \Psi_0(t) * T_\varepsilon W(t, \varepsilon) &= 0, \\
 y_0(t) * T_\varepsilon \left(\frac{e^{-\frac{ix^2 \tan(2t)}{2\varepsilon}}}{\sqrt{\cos(2t)}} \right) &= 0, \\
 z_0(t) * T_\varepsilon \left(x \frac{e^{-\frac{ix^2 \tan(2t)}{2\varepsilon}}}{\sqrt{\cos(2t)}^3} \right) &= 0, \\
 x^2 w_0 &= -h(x, t) + y_{-1}(t) + x z_{-1}(t), \\
 v_0(x, 0) + w_0(x, 0) &= f(x), \\
 v_0(0, t) + \Psi_0(t) + \int_0^t \frac{y_0(\tau)}{\sqrt{\cos 2(t-\tau)}} d\tau + w_0(0, t) &= \psi(t).
 \end{aligned} \tag{10}$$

The functions $y_0(t)$, $z_0(t)$ are arbitrary at this step. For the equation to be solvable with respect to $w_0(x, t)$, it is necessary and sufficient that the relations $y_{-1}(t) = h(0, t)$, $z_{-1}(t) = \frac{\partial h}{\partial x}(0, t)$. From here,

$$w_0(x, t) = \frac{h(x, t) - h(0, t) - x \frac{\partial h}{\partial x}(0, t)}{x^2} = h_0(x, t).$$

where $h_0(x, t)$ is a smooth function. Having determined $y_{-1}(t)$, we find the function $\Psi_{-1}(t)$ from the boundary condition:

$$\Psi_{-1}(t) = - \int_0^t \frac{h(0, \tau)}{\sqrt{\cos(2(t-\tau))}} d\tau.$$

Now, we can write the solution at step $k = -1$:

$$u_{-1}(x, t) = G(\Psi_{-1}(t)) + \int_0^t \frac{h(0, \tau)}{\sqrt{\cos(2(t-\tau))}} e^{-\frac{ix^2 \tan(2(t-\tau))}{2\varepsilon}} d\tau + \\ + x \int_0^t \frac{\frac{\partial h(0, \tau)}{\partial x}}{\sqrt{\cos(2(t-\tau))}^3} e^{-\frac{ix^2 \tan(2(t-\tau))}{2\varepsilon}} d\tau.$$

To solve the equation for $v_0(x, t)$, we make the change $v_0(x, t) = \frac{\alpha(x, t)}{\sqrt{\cos(2t)}}$. Then, we obtain the equation

$$\frac{\partial \alpha}{\partial t} + 2x \tan(2t) \frac{\partial \alpha(x, t)}{\partial x} = 0.$$

Let us write the equation of the characteristics:

$$dt = \frac{dx}{2x \tan(2t)} = \frac{d\alpha}{0}.$$

The first integral is, respectively, equal to:

$$\frac{x}{\cos(2t)} = c_1.$$

From this, we obtain the general solution

$$\alpha(x, t) = g_0\left(\frac{x}{\cos(2t)}\right).$$

where the function $g_0(x, t)$ is determined from the initial conditions. Thus, the general solution $v_0(x, t)$ has the form:

$$v_0(x, t) = \frac{g_0\left(\frac{x}{\cos(2t)}\right)}{\sqrt{\cos(2t)}}.$$

From the initial condition, we define an arbitrary function $g_0(x, t)$. For $t = 0$, we have

$$g_0(x) + h_0(x, 0) = f(x).$$

Hence, $g_0(x) = f(x) - h_0(x, 0)$ (here, it is taken into account that $\Psi_0(0) = 0$). Or, expanded,

$$g_0(x, t) = f\left(\frac{x}{\cos(2t)}\right) - h_0\left(\frac{x}{\cos(2t)}, 0\right) = \\ = f\left(\frac{x}{\cos(2t)}\right) - \frac{h\left(\frac{x}{\cos(2t)}, 0\right) - h(0, 0) - \frac{x}{\cos(2t)} \frac{\partial h}{\partial x}(0, 0)}{\left(\frac{x}{\cos(2t)}\right)^2}.$$

To determine arbitrary functions $y_0(t)$, $z_0(t)$, consider the problem at the ε step:

$$\begin{aligned} \frac{\partial v_1}{\partial t} + 2x \tan(2t) \frac{\partial v_1}{\partial x} + \tan(2t) v_1 &= \frac{\partial^2 v_0}{\partial x^2}, \\ \Psi_1(t) * T_\varepsilon W(t, \varepsilon) &= 0, \\ y_1(t) * T_\varepsilon \frac{e^{-\frac{ix^2 \tan(2t)}{2\varepsilon}}}{\sqrt{\cos(2t)}} &= 0, \\ z_1(t) * T_\varepsilon \left(x \frac{e^{-\frac{ix^2 \tan(2t)}{2\varepsilon}}}{\sqrt{\cos(2t)}}\right)^3 &= 0, \\ x^2 w_1 &= -\frac{\partial h_0}{\partial t}(x, t) - y_0(t) - x z_0(t), \\ v_1(x, 0) + w_1(x, 0) &= 0, \\ v_1(0, t) + \Psi_1(t) + \int_0^t \frac{y_1(\tau)}{\sqrt{\cos(2(t-\tau))}} d\tau + w_1(0, t) &= 0. \end{aligned} \quad (11)$$

To define $w_1(x, t)$ it is necessary and sufficient that

$$y_0(t) = -\frac{\partial h_0}{\partial t}(0, t), \quad z_0(t) = -\frac{\partial^2 h_0}{\partial x \partial t}(0, t),$$

Now, from the boundary condition of problem (5), we define the function $\Psi_0(t)$. To do this, consider the boundary condition for $x = 0$:

$$\Psi_0(t) = \psi(t) - v_0(0, t) + \int_0^t \frac{\frac{\partial h_0}{\partial \tau}(0, \tau) d\tau}{\sqrt{\cos(2(t-\tau))}} - w_0(0, t).$$

Thus, at this step, the term at the zero step is found. It can be written as:

$$\begin{aligned} u_0(x, t) &= \frac{1}{\sqrt{\cos(2t)}} \left[f\left(\frac{x}{\cos(2t)}\right) - h_0\left(\frac{x}{\cos(2t)}, 0\right) \right] e^{-\frac{ix^2 \tan(2t)}{2\varepsilon}} + G(\Psi_0(t)) - \\ &- \int_0^t \frac{\frac{\partial h_0(0, \tau)}{\partial \tau}}{\sqrt{\cos(2(t-\tau))}} e^{-\frac{ix^2 \tan(2(t-\tau))}{2\varepsilon}} d\tau - x \int_0^t \frac{\frac{\partial^2 h_0}{\partial \tau \partial x}(0, \tau)}{\sqrt{\cos(2(t-\tau))}^3} e^{-\frac{ix^2 \tan(2(t-\tau))}{2\varepsilon}} d\tau + \\ &+ \frac{h(x, t) - h(0, t) - x \frac{\partial h}{\partial x}(0, t)}{x^2}. \end{aligned}$$

Now, we can write the leading term of the asymptotics:

$$\begin{aligned}
u_{gl}(x, t) = & \frac{1}{\varepsilon} \left[G(\Psi_{-1}(t)) + \int_0^t \frac{h(0, \tau)}{\sqrt{\cos(2(t-\tau))}} e^{-\frac{ix^2 \tan(2(t-\tau))}{2\varepsilon}} d\tau + \right. \\
& + x \int_0^t \frac{\frac{\partial h(0, \tau)}{\partial x}}{\sqrt{\cos(2(t-\tau))}^3} e^{-\frac{ix^2 \tan(2(t-\tau))}{2\varepsilon}} d\tau \left. + \frac{1}{\sqrt{\cos(2t)}} \left[f\left(\frac{x}{\cos(2t)}\right) - \right. \right. \\
& - h_0\left(\frac{x}{\cos(2t)}, 0\right) \left. \right] e^{-\frac{ix^2 \tan(2t)}{2\varepsilon}} + G(\Psi_0(t)) - \int_0^t \frac{\frac{\partial h_0(0, \tau)}{\partial \tau}}{\sqrt{\cos(2(t-\tau))}} e^{-\frac{ix^2 \tan(2(t-\tau))}{2\varepsilon}} d\tau - \\
& - x \int_0^t \frac{\frac{\partial^2 h_0}{\partial \tau \partial x}(0, \tau)}{\sqrt{\cos(2(t-\tau))}^3} e^{-\frac{ix^2 \tan(2(t-\tau))}{2\varepsilon}} d\tau + \frac{h(x, t) - h(0, t) - x \frac{\partial h}{\partial x}(0, t)}{x^2}. \quad (12)
\end{aligned}$$

Now, we can write the solution $w_1(x, t)$ of system (11):

$$w_1(x, t) = -\frac{\frac{\partial h_0}{\partial t}(x, t) - \frac{\partial h_0}{\partial t}(0, t) - x \frac{\partial^2 h_0}{\partial t \partial x}(0, t)}{x^2} = h_1(x, t).$$

where $h_1(x, t)$ is a smooth function.

Let us solve the inhomogeneous equation with respect to $v_1(x, t)$

$$\frac{\partial v_1}{\partial t} + 2x \tan(2t) \frac{\partial v_1}{\partial x} + \tan(2t) v_1 = \frac{\partial^2 v_0}{\partial x^2}$$

To solve the equation for $v_1(x, t)$, make the change $v_1(x, t) = \frac{\alpha(x, t)}{\sqrt{\cos(2t)}}$ and compute $\frac{\partial^2 v_0}{\partial x^2}$. We obtain the equation

$$\frac{\partial \alpha}{\partial t} + 2x \tan(2t) \frac{\partial \alpha(x, t)}{\partial x} = \frac{g_0''\left(\frac{x}{\cos(2t)}\right)}{\cos^2(2t)}.$$

Let us write the equation of the characteristics:

$$dt = \frac{dx}{2x \tan(2t)} = \frac{\cos^2(2t) d\alpha}{g_0''\left(\frac{x}{\cos(2t)}\right)}.$$

The first integrals are, respectively, equal:

$$\frac{x}{\cos(2t)} = c_1, \quad \alpha(x, t) - \frac{1}{2} \tan(2t) \cdot g_0''(c_1) = c_2.$$

From this, we obtain the general solution

$$\alpha(x, t) = \frac{1}{2} \tan(2t) \cdot g_0''\left(\frac{x}{\cos(2t)}\right) + g_1\left(\frac{x}{\cos(2t)}\right),$$

where the function g_1 is determined from the initial conditions. Thus, the solution $v_1(x, t)$ has the form:

$$v_1(x, t) = \frac{1}{\sqrt{\cos(2t)}} \left[\frac{\tan(2t)}{2} g_0''\left(\frac{x}{\cos(2t)}\right) + g_1\left(\frac{x}{\cos(2t)}\right) \right].$$

We define the function $g_1\left(\frac{x}{\cos(2t)}\right)$. Let us use the initial condition $g_1(x) = -h_1(x, 0)$. Hence,

$$v_1(x, t) = \frac{1}{\sqrt{\cos(2t)}} \left[\frac{\tan(2t)}{2} g_0''\left(\frac{x}{\cos(2t)}\right) - h_1\left(\frac{x}{\cos(2t)}, 0\right) \right].$$

The functions $y_1(t)$, $z_1(t)$ are at the next iterative step. They are found from the condition of solvability of the equation with respect to $w_2(x, t)$:

$$y_1(t) = -\frac{\partial h_1}{\partial t}(0, t) + \frac{\partial^2 h_0}{\partial x^2}(0, t),$$

$$z_1(t) = -\frac{\partial^2 h_1}{\partial t \partial x}(0, t) + \frac{\partial^3 h_0}{\partial x^3}(0, t).$$

Thus, at this step, the term at the $k = 1$ step is found. It can be written as follows:

$$\begin{aligned} u_1(x, t) = & \frac{1}{\sqrt{\cos(2t)}} \left[\frac{\tan(2t)}{2} g_0''\left(\frac{x}{\cos(2t)}\right) + g_1\left(\frac{x}{\cos(2t)}\right) \right] e^{-\frac{ix^2 \tan(2t)}{2\varepsilon}} + \\ & + G(\Psi_1(t)) - \int_0^t \frac{\frac{\partial h_1}{\partial \tau}(0, \tau) - \frac{\partial^2 h_0}{\partial x^2}(0, \tau)}{\sqrt{\cos(2(t-\tau))}} e^{-\frac{ix^2 \tan(2t)(t-\tau)}{2\varepsilon}} d\tau - \\ & - x \int_0^t \frac{\frac{\partial^2 h_1}{\partial \tau \partial x}(0, \tau) - \frac{\partial^3 h_0}{\partial x^3}(0, \tau)}{\sqrt{\cos(2(t-\tau))}^3} e^{-\frac{ix^2 \tan(2t)(t-\tau)}{2\varepsilon}} d\tau - \frac{\frac{\partial h_0}{\partial t}(x, t) - \frac{\partial h_0}{\partial t}(0, t) - x \frac{\partial^2 h_0}{\partial t \partial x}(0, t)}{x^2}. \end{aligned}$$

Using this scheme, the following terms of the asymptotic series are found by induction.

6. Estimation of the Remainder Term

Let $(N + 1)$ iteration problems be solved. Then, the solution of the problem can be represented as

$$u(x, t, \varepsilon) = \sum_{k=-1}^N \varepsilon^k u_k(x, t) + \varepsilon^{N+1} R_N(x, t, \varepsilon), \quad (13)$$

where $R_N(x, t, \varepsilon)$ is the remainder.

$$u_k = v_k(x, t) e^{-\frac{i\varphi(x, t)}{\varepsilon}} + G(\Psi_k(t)) + \sigma_0(y_k(t)) + \sigma_1(z_k(t)) + w_k(x, t).$$

Substituting Equation (8) into (1), we obtain the problem for the remainder $R_N(x, t, \varepsilon)$:

$$\begin{aligned} i\varepsilon \frac{\partial R_N}{\partial t} + \varepsilon^2 \frac{\partial^2 R_N}{\partial x^2} - x^2 R_N &= H(x, t, \varepsilon), \\ R_N(0, t, \varepsilon) &= 0, \\ R_N(x, 0, \varepsilon) &= 0, \end{aligned} \quad (14)$$

where $H(x, t, \varepsilon) = x^2 w_{N+1}(x, t) + \varepsilon \left(\frac{\partial^2 v_N(x, t)}{\partial x^2} e^{-\frac{i\varphi(x, t)}{\varepsilon}} + \frac{\partial^2 w_N(x, t)}{\partial x^2} \right)$. Note that since the iterative problems are solved up to ε^{N+1} , the term $x^2 w_{N+1}(x, t) = O(1)$.

A classical solution problem (14) is a function $R(x, t, \varepsilon)$ continuous in $\overline{Q}_T = (0, +\infty) \times [0, T]$ with continuous $\frac{\partial R}{\partial t}$, $\frac{\partial R}{\partial x}$, $\frac{\partial^2 R}{\partial x^2}$ in Q_T and satisfies Equation (14) at all points of Q_T and the initial conditions for $t = 0$.

Theorem 2 (Evaluation of the remainder term). *Let the requirements be met:*

- (1) conditions (1)–(4) for problem (1);
- (2) $H(x, t, \varepsilon)$ satisfies condition (4) (1) (see Appendix A).

Then, $\exists C > 0$ $|R_N(x, t, \varepsilon)| \leq C \forall (x, t) \in (0, +\infty) \times [0, T] \forall \varepsilon \in (0, \varepsilon_0]$.

Proof. Extend the right-hand side $H(x, t, \varepsilon)$ and the initial condition by zeros to the negative semiaxis $0x$. Using Mehler's fundamental solution, we write the solution of problem (14) in the form

$$R_N(x, t, \varepsilon) = \frac{1}{\varepsilon} \int_0^t d\tau \int_{-\infty}^{+\infty} H(\xi, \tau, \varepsilon) K(x, \xi, t - \tau) d\xi.$$

Let us evaluate the remainder modulo. Then, we obtain

$$\begin{aligned} |R_N(x, t, \varepsilon)| &\leq \frac{1}{\varepsilon} \int_0^t d\tau \int_{-\infty}^{+\infty} |H(\xi, \tau, \varepsilon)| |K(x, \xi, t - \tau)| d\xi = \\ &= \frac{1}{\varepsilon} \int_0^t |e^{-\frac{ix^2 \tan(2(t-\tau))}{2\varepsilon}}| d\tau \int_{-\infty}^{+\infty} |H(\xi, \tau, \varepsilon)| \frac{1}{\sqrt{2\pi\varepsilon \sin(2(t-\tau))}} \exp\left(\frac{i \cot(2(t-\tau))}{2\varepsilon} \left(\xi - \frac{x}{\cos(t-\tau)}\right)^2\right) d\xi = \\ &= \frac{1}{\varepsilon} \int_0^t \frac{1}{\sqrt{2\pi\varepsilon \sin(2(t-\tau))}} d\tau \int_{-\infty}^{+\infty} |H(\xi + \frac{x}{\cos(t-\tau)}, \tau, \varepsilon)| \exp\left(\frac{i \cot(2(t-\tau))}{2\varepsilon} \xi^2\right) d\xi = \\ &= \left\{z = \frac{\xi}{\sqrt{2\varepsilon \sin(2(t-\tau))}}\right\} = \\ &= \frac{1}{\varepsilon\sqrt{\pi}} \int_0^t d\tau \int_{-\infty}^{+\infty} |H(z\sqrt{2\varepsilon \sin(2(t-\tau))} + \frac{x}{\cos(t-\tau)}, \tau, \varepsilon)| \exp(i \cos(2(t-\tau))z^2) dz = \\ &= \frac{1}{\varepsilon\sqrt{\pi}} \int_0^t d\tau \int_{-\infty}^{+\infty} |H(z\sqrt{2\varepsilon \sin(2(t-\tau))} + \frac{x}{\cos(t-\tau)}, \tau, \varepsilon)| dz \leq \frac{M}{\varepsilon}. \end{aligned}$$

We write the remainder term in the form

$$R_N = u_{N+1} + \varepsilon R_{N+1}.$$

Then, $|R_N| \leq |u_{N+1}| + \varepsilon \frac{M_1}{\varepsilon} \leq C, C > 0$. \square

7. Construction of the Fundamental Solution

To find a fundamental solution, consider the problem:

$$i\varepsilon \frac{\partial u}{\partial t} + \varepsilon^2 \frac{\partial^2 u}{\partial x^2} - x^2 u = 0, \quad u(x, 0) = \delta(x - \xi).$$

We make the change $u(x, t) = e^{\frac{x^2}{2\varepsilon} + it} v(x, t)$. As a result, we obtain the task:

$$\frac{\partial v}{\partial t} - i2x \frac{\partial v}{\partial x} = i\varepsilon \frac{\partial^2 v}{\partial x^2}, \quad v(x, 0) = e^{-i\frac{\xi^2}{2\varepsilon}} \delta(x - \xi).$$

Let us carry out the Fourier transform. Then, we obtain a linear equation in the space of images with respect to F :

$$\frac{\partial F}{\partial t} + i2\lambda \frac{\partial F}{\partial \lambda} = -i(\varepsilon\lambda^2 + 2)F, \quad F(\lambda, 0) = e^{-i\frac{\xi^2}{2\varepsilon} - i\lambda\xi}, \quad (15)$$

where $F(\lambda, t) = \int_{-\infty}^{\infty} v(x, t) e^{-i\lambda x} dx$.

Solution (15) has the form $F(\lambda, t) = e^{-i2t - \frac{\xi^2}{2\varepsilon} - \frac{\varepsilon\lambda^2}{4}(1 - e^{-i4t}) - i\lambda\xi e^{-i2t}}$.

Making the inverse Fourier transform and taking into account the change $u(x, t) = e^{\frac{x^2}{2\varepsilon} + it} v(x, t)$, we obtain:

$$u(x, t) = e^{\frac{x^2}{2\varepsilon} - it} \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\left(-\frac{\varepsilon\lambda^2}{4}(1 - e^{-i4t}) - i\lambda(\xi e^{-i2t} - x)\right) d\lambda$$

Selecting the full square in the exponent and calculating the Fresnel integral, we finally obtain:

$$K(x, \xi, t) = \frac{1-i}{2\sqrt{\pi\varepsilon\sin(2t)}} e^{\frac{i}{\varepsilon}\left(\cot(2t)\frac{x^2+\xi^2}{2} - \frac{x\xi}{\sin(2t)}\right)}.$$

The kernel $K(x, \xi, t)$ has the property $K(x, \xi, 0) = \delta(x - \xi)$.

8. A Singular Integral Operators for Regularization of the Right Parts of Iterative Problems

Additional regularizing singular operators related to the pointwise irreversibility of the limit operator are constructed using the fundamental solution. Their task is to embed the right side of the equation in the image of the limit operator. The limit operator is obtained by putting (1) $\varepsilon = 0$ into the equation of the problem.

Additional singular integral operators for the regularization of the right-hand side of the problem are obtained by integrating the kernel $K(x, \xi, t)$ over the variable ξ . Then, we obtain:

$$\begin{aligned}\sigma_0(x, t, \varepsilon)(\cdot) &= -i \int_0^t (\cdot) d\tau \int_{-\infty}^{+\infty} K(x, \xi, t - \tau) d\xi = \\ &= -i \int_0^t (\cdot) \frac{1}{\sqrt{\cos(2(t - \tau))}} e^{-\frac{ix^2 \tan(2(t - \tau))}{2\varepsilon}} d\tau, \\ \sigma_1(x, t, \varepsilon)(\cdot) &= -ix \int_0^t (\cdot) d\tau \int_{-\infty}^{+\infty} \xi K(x, \xi, t - \tau) d\xi = \\ &= -ix \int_0^t (\cdot) \frac{1}{\sqrt{\cos(2(t - \tau))}^3} e^{-\frac{ix^2 \tan(2(t - \tau))}{2\varepsilon}} d\tau.\end{aligned}$$

In fact, the singular operators $\sigma_0(x, t, \varepsilon)(\cdot)$, $\sigma_1(x, t, \varepsilon)(\cdot)$ are solutions of the Schrödinger equation with right-hand sides $-i\varepsilon$, $-i\varepsilon x$. The actions of operators on a function will be written as:

$$\begin{aligned}\sigma_0(f(t)) &= -i \int_0^t \frac{f(\tau)}{\sqrt{\cos(2(t - \tau))}} e^{-\frac{ix^2 \tan(2(t - \tau))}{2\varepsilon}} d\tau = \\ &= f(t) * -i \frac{e^{-\frac{ix^2 \tan(2(t - \tau))}{2\varepsilon}}}{\sqrt{\cos(2t)}}, \\ \sigma_1(f(t)) &= -ix \int_0^t \frac{f(\tau)}{\sqrt{\cos(2(t - \tau))}^3} e^{-\frac{ix^2 \tan(2(t - \tau))}{2\varepsilon}} d\tau = \\ &= f(t) * -ix \frac{e^{-\frac{ix^2 \tan(2(t - \tau))}{2\varepsilon}}}{\sqrt{\cos(2t)}^3}.\end{aligned}$$

Let us introduce the operator $T_\varepsilon = i\varepsilon \frac{\partial}{\partial t} + \varepsilon^2 \frac{\partial^2}{\partial x^2} - x^2$. Then,

$$\begin{aligned}T_\varepsilon(\sigma_0(f(t))) &= \varepsilon f(t) + f(t) * T_\varepsilon\left(-i \frac{e^{-\frac{ix^2 \tan(2t)}{2\varepsilon}}}{\sqrt{\cos(2t)}}\right) = \\ &= \varepsilon f(t), \\ T_\varepsilon(\sigma_1(f(t))) &= \varepsilon x f(t) + f(t) * T_\varepsilon\left(-ix \frac{e^{-\frac{ix^2 \tan(2t)}{2\varepsilon}}}{\sqrt{\cos(2t)}^3}\right) = \\ &= \varepsilon x f(t).\end{aligned}$$

9. A Singular Integral Operator for Describing the “Boundary Layer” in the Vicinity of the Point $x = 0$

Let us solve a mixed problem:

$$i\varepsilon \frac{\partial u}{\partial t} + \varepsilon^2 \frac{\partial^2 u}{\partial x^2} - x^2 u = 0, \quad u(x, 0) = 0, \quad u(0, t) = 1.$$

We make the change $u(x, t) = e^{\frac{x^2}{2\varepsilon}} v(x, t)$. As a result, we obtain the task:

$$\frac{\partial v(x, t)}{\partial t} - i2x \frac{\partial v(x, t)}{\partial x} - iv(x, t) = i\varepsilon \frac{\partial^2 v(x, t)}{\partial x^2}, \quad v(x, 0) = 0, \quad v(0, t) = 1.$$

Let us perform a sine transform. Then, in the space of images, we obtain:

$$\frac{\partial F}{\partial t} + i2\lambda \frac{\partial F}{\partial \lambda} = -i(\varepsilon\lambda^2 + 1)F + i\varepsilon\lambda, \quad F(\lambda, 0) = 0, \quad (16)$$

where the notation

$$F(\lambda, t) = \int_0^\infty v(x, t) \sin \lambda x dx.$$

Let us write the characteristic system for the linear equation in the problem (16):

$$dt = \frac{d\lambda}{i2\lambda} = \frac{dF}{-i(\varepsilon\lambda^2 + 1)F + i\varepsilon\lambda}.$$

The system of first integrals for it has the form:

$$\begin{aligned} \lambda e^{-i2t} &= C_1, \\ e^{\frac{\varepsilon\lambda^2}{4}} \sqrt{\lambda} F + \frac{\varepsilon}{2} \int_\lambda^\infty e^{\frac{\varepsilon\mu^2}{4}} \sqrt{\mu} d\mu &= C_2. \end{aligned}$$

Now it is easy, given the initial condition, to obtain a solution to the original problem in the space of images:

$$F(\lambda, t) = \frac{\varepsilon}{2} \int_{\lambda e^{-i2t}}^\lambda e^{\frac{\varepsilon}{4}(-\lambda^2 + \mu^2)} \sqrt{\frac{\mu}{\lambda}} d\mu.$$

The replacement of the variable $\mu = \lambda e^{-i2(t-\tau)}$ in the last integral will lead to a more convenient relation in what follows:

$$F(\lambda, t) = i\varepsilon\lambda \int_0^t \exp\left(-\frac{\varepsilon}{4}\lambda^2(1 - e^{-i4(t-\tau)}) - i3(t-\tau)\right) d\tau.$$

It remains to carry out the inverse Fourier sine transform, which in the end will allow us to obtain a solution to the problem of interest to us:

$$\begin{aligned} v(x, t) &= \frac{2}{\pi} \int_0^\infty F(\lambda, t) \sin \lambda x d\lambda = \\ &= \frac{i2\varepsilon}{\pi} \int_0^\infty \left(\int_0^t \exp\left(-\frac{\varepsilon}{4}\lambda^2(1 - e^{-i4(t-\tau)}) - i3(t-\tau)\right) d\tau \right) \lambda \sin \lambda x d\lambda. \end{aligned}$$

By changing the order of integration in the resulting iterated integral and performing simple transformations, we obtain

$$v(x, t) = \frac{(1-i)x}{\sqrt{\varepsilon\pi}} \int_0^t \frac{e^{-\frac{x^2}{\varepsilon a}}}{\sqrt{\sin 2(t-\tau)}} d\tau, \quad (17)$$

where $a = 1 - e^{-i4(t-\tau)}$. From here,

$$u(x, t) = \frac{(1-i)x}{\sqrt{\varepsilon\pi}} \int_0^t \frac{e^{\frac{ix^2 \cot(2(t-\tau))}{2\varepsilon}}}{\sqrt{\sin(2(t-\tau))}} d\tau.$$

The solution for an arbitrary boundary condition $\psi(t)$ can be written as

$$u(x, t) = \frac{(1-i)x}{\sqrt{\varepsilon\pi}} \int_0^t \psi(\tau) \frac{e^{\frac{ix^2 \cot(2(t-\tau))}{2\varepsilon}}}{\sqrt{\sin(2(t-\tau))}} d\tau. \quad (18)$$

Let us give a different representation of solution (18), having previously made the change of variables $z^2 = \frac{x^2 \cot(2(t-\tau))}{2\varepsilon}$. Namely,

$$u(x, t) = (1-i) \sqrt{\frac{2}{\pi}} \int_{\frac{x}{\sqrt{2\varepsilon \tan(2t)}}}^{\infty} \frac{\psi\left(t - \frac{1}{2} \arctan(b)\right) e^{iz^2}}{\sqrt{1+b^2}} dz,$$

where $b = \frac{x^2}{2\varepsilon z^2}$.

10. Conclusions

In conclusion, to understand the method, we note that the regularization method in practice consists of the following stages:

- (1) Identifying regularizing functions, operators containing a non-uniform dependence of the solution to a singularly perturbed problem on the parameter ε (this is the most difficult stage);
- (2) Introduction of additional variables corresponding to regularizing functions and operators;
- (3) Using complex differentiation formulas, an extended problem is formulated in a space of higher dimension, in which the singularly perturbed problem becomes regular;
- (4) A solution to the extended problem is constructed in the form of a power series in the parameter ε , to determine the coefficients of which theorems of solvability and unique solvability of the corresponding iterative problems are proved;
- (5) At the final stage, the solution is narrowed to regularizing functions and operators, which gives a solution to the singularly perturbed problem.

In the proposed work for a mixed problem on a half-line for an inhomogeneous Schrödinger equation with a spectral feature in the form of a “strong” turning point, regularization consisted of introducing one regularizing function and three additional singular operators. Note that in this work, we developed an algorithm for the regularization method for solving a singularly perturbed problem for the Schrödinger equation in the presence of a “strong” turning point at $x = 0$ on a time interval that does not contain focal points. Regularization of the task in the presence of focal points in time will be described in the following articles.

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Appendix A

Let us show that the function

$$\begin{aligned}
 u(x, t) &= \frac{1-i}{2\sqrt{\pi\varepsilon\sin(2t)}} \int_{-\infty}^{\infty} \exp\left[i\left(\cot(2t)\frac{x^2+\xi^2}{2\varepsilon} - \frac{x\xi}{\varepsilon\sin(2t)}\right)\right] f(\xi) d\xi + \\
 &+ \frac{1-i}{2\sqrt{\pi\varepsilon}} \int_0^t \frac{d\tau}{\sqrt{\sin(2(t-\tau))}} \int_{-\infty}^{+\infty} \exp\left[i\left(\cot(2(t-\tau))\frac{x^2+\xi^2}{2\varepsilon} - \frac{x\xi}{\varepsilon\sin(2(t-\tau))}\right)\right] h(\xi, \tau) d\xi = \\
 &= \frac{1-i}{2\sqrt{\pi\varepsilon\sin(2t)}} \int_{-\infty}^{\infty} \exp\left[i\left(\frac{(\xi-x\cos(2t))^2}{\varepsilon\sin 4t} - \frac{\xi^2\tan(2t)}{2\varepsilon}\right)\right] f(\xi) d\xi + \\
 &+ \frac{1-i}{2\sqrt{\pi\varepsilon}} \int_0^t \frac{d\tau}{\sqrt{\sin(2(t-\tau))}} \int_{-\infty}^{\infty} \exp\left[i\left(\frac{(\xi-x\cos(2(t-\tau)))^2}{\varepsilon\sin 4(t-\tau)} - \frac{\xi^2\tan(2(t-\tau))}{2\varepsilon}\right)\right] h(\xi, \tau) d\xi
 \end{aligned} \tag{A1}$$

where $f(x)$, $h(x, t)$ are continuous bounded functions satisfying the conditions $\int_{-\infty}^{+\infty} x^2 |f(x)| dx < \infty$, $\int_{-\infty}^{+\infty} x^2 |h(x, t)| dx < \infty$, which satisfy the problem.

$$i\varepsilon \frac{\partial u}{\partial t} + \varepsilon^2 \frac{\partial^2 u}{\partial x^2} - x^2 u = h(x, t), \quad u(x, 0) = f(x). \tag{A2}$$

Note that the integral (A1) converges uniformly on $(-\infty, +\infty) \times [0, T]$. Indeed, the score gives:

$$\begin{aligned}
 |u(x, t)| &\leq \frac{1}{\sqrt{2\pi\varepsilon\sin(2t)}} \int_{-\infty}^{\infty} \left| \exp\left[i\left(\frac{(\xi-x\cos(2t))^2}{\varepsilon\sin 4t} - \frac{\xi^2\tan(2t)}{2\varepsilon}\right)\right] \right| |f(\xi)| d\xi + \\
 &+ \frac{1}{\sqrt{2\pi\varepsilon}} \int_0^t \frac{d\tau}{\sqrt{\sin(2(t-\tau))}} \int_{-\infty}^{\infty} \left| \exp\left[i\left(\frac{(\xi-x\cos(2(t-\tau)))^2}{\varepsilon\sin 4(t-\tau)} - \frac{\xi^2\tan(2(t-\tau))}{2\varepsilon}\right)\right] \right| |h(\xi, \tau)| d\xi = \\
 &= \frac{1}{\sqrt{2\pi\varepsilon\sin(2t)}} \int_{-\infty}^{\infty} \left| \exp\left[i\left(\frac{(\xi-x\cos(2t))^2}{\varepsilon\sin 4t}\right)\right] \right| |f(\xi)| d\xi + \\
 &+ \frac{1}{\sqrt{2\pi\varepsilon}} \int_0^t \frac{d\tau}{\sqrt{\sin(2\tau)}} \int_{-\infty}^{\infty} \left| \exp\left[i\left(\frac{(\xi-x\cos(2\tau))^2}{\varepsilon\sin 4\tau}\right)\right] \right| |h(\xi, t-\tau)| d\xi = \\
 &= \left\langle \frac{\xi-x\cos(2t)}{\sqrt{2\varepsilon\sin(2t)}} = z, \quad d\xi = \sqrt{2\varepsilon\sin(2t)} dz \right\rangle = \\
 &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} |f(x\cos(2t) + z\sqrt{2\varepsilon\sin(2t)})| dz + \frac{1}{\sqrt{\pi}} \int_0^t d\tau \int_{-\infty}^{\infty} |h(x\cos(2t) + \\
 &+ z\sqrt{2\varepsilon\sin(2t)}, (t-\tau))| dz \leq C_1.
 \end{aligned}$$

In what follows, without loss of generality, we will consider only the part of the solution (A1) that satisfies the homogeneous Equation (A2).

Step 1. Formal differentiation and substitution of formal derivatives into the equation.

Let us find formally (that is, without thinking about the legitimacy of these actions) the derivatives of the function $u(x, t)$ entering the equation. Then, we check that the resulting integral satisfies the homogeneous equation in problem (A2).

$$\begin{aligned}
 i\varepsilon \frac{\partial u}{\partial t} &= \\
 &= \frac{1-i}{2\sqrt{\pi\varepsilon \sin(2t)}} \int_{-\infty}^{\infty} \exp(\dots) \left(-i\varepsilon \cot(2t) + \frac{x^2 + \xi^2 - 2x\xi \cos 2t}{\sin^2(2t)} \right) f(\xi) d\xi = \\
 &= \frac{1-i}{2\sqrt{\pi\varepsilon \sin(2t)}} \int_{-\infty}^{\infty} \exp(\dots) \left(-i\varepsilon \cot(2t) - x^2 + \frac{(x \cos(2t) - \xi)^2}{\sin^2(2t)} \right) f(\xi) d\xi = \\
 \varepsilon^2 \frac{\partial^2 u}{\partial x^2} &= \\
 &= \frac{1-i}{2\sqrt{\pi\varepsilon \sin(2t)}} \int_{-\infty}^{\infty} \exp(\dots) \left(i\varepsilon \cot(2t) + \frac{x^2 \cos^2(2t) - 2x\xi \cos(2t) + \xi^2}{\sin^2(2t)} \right) f(\xi) d\xi = \\
 &= \frac{1-i}{2\sqrt{\pi\varepsilon \sin(2t)}} \int_{-\infty}^{\infty} \exp(\dots) \left(i\varepsilon \cot(2t) + \frac{(x \cos(2t) - \xi)^2}{\sin^2(2t)} \right) f(\xi) d\xi.
 \end{aligned}$$

Here, the ellipsis denotes the exponent of the fundamental solution. Substituting the calculated u_t, u_{xx} into the equation, we obtain:

$$\begin{aligned}
 &\frac{1}{\sqrt{2\pi\varepsilon \sin(2t)}} \int_{-\infty}^{\infty} \exp(\dots) \left[-i\varepsilon \cot(2t) - x^2 + \frac{(x \cos(2t) - \xi)^2}{\sin^2(2t)} + \right. \\
 &\left. i\varepsilon \cot(2t) - \frac{(x \cos(2t) - \xi)^2}{\sin^2(2t)} + x^2 \right] f(\xi) d\xi = 0.
 \end{aligned}$$

Step 2. Justify the eligibility of formal actions.

In order to show that the function $u(x, t)$ satisfies the equation, it is necessary to justify the possibility of differentiating with respect to x and t under the integral sign for $t > 0$, $-\infty < x < +\infty$. Let us prove this fact for $t > t_0$, $t_0 > 0$ whence, due to the arbitrariness of t_0 , this fact will hold for $t > 0$.

Theorem A1 (Existence of a classical solution). *Let the following conditions be met:*

- (1) $f(x) \in C(-\infty, +\infty)$ satisfying $\int_{-\infty}^{\infty} x^2 |f(x)| dx < \infty$;
- (2) $h(x, t) \in C((-\infty, +\infty) \times [0, T])$ satisfying the conditions $\int_{-\infty}^{\infty} x^2 |h(x, t)| dx < \infty$ uniformly in t .

Then, the classical solution to problem (1) exists.

Proof. Let us estimate the derivatives obtained at step 1 on the rectangle $[-L, L] \times [t_0, T]$:

$$\begin{aligned}
 \varepsilon \left| \frac{\partial u}{\partial t} \right| &\leq \frac{1}{\sqrt{2\pi\varepsilon \sin 2t_0}} \int_{-\infty}^{\infty} \left(\frac{\varepsilon}{|\sin 2t_0|} + x^2 + \frac{(|\xi| + |x|)^2}{\sin^2 2t_0} \right) |f(\xi)| d\xi \leq \\
 &\leq \frac{1}{\sqrt{2\pi\varepsilon \sin 2t_0}^5} ((1 + 2L^2)M_0 + 2LM_1 + M_2), \\
 \varepsilon^2 \left| \frac{\partial^2 u}{\partial x^2} \right| &\leq \frac{1}{\sqrt{2\pi\varepsilon \sin 2t_0}^5} ((1 + L^2)M_0 + 2LM_1 + M_2).
 \end{aligned}$$

Considering that $f(x)$ satisfies condition (1) of the theorem, the integrals $M_j = \int_{-\infty}^{\infty} |\xi|^j |f(\xi)| d\xi$, $j = 0, 1, 2$, exist. Therefore, the integrals converge uniformly on the rectangle $[-L, L] \times (0, T]$. It follows that the function $u(x, t) \in C^{(2,1)}(-\infty, +\infty) \times (0, T]$ and satisfies the homogeneous equation (A2). Let us prove that (A1) satisfies the initial condition.

The function $u(x, t)$ is not defined for $t = 0$. However, it can be extended at the initial moment of time by continuity, i.e., take equal to its limit at time $t = 0$ at $t \rightarrow 0 + 0$. Since the integral (A1) converges uniformly on $(-\infty, +\infty) \times [0, T]$, it is possible to pass to the limit under the integral sign:

$$\begin{aligned} u(x, 0) &= \lim_{t \rightarrow 0+0} \frac{1-i}{2\sqrt{\pi\epsilon \sin(2t)}} \int_{-\infty}^{\infty} \exp \left[i \left(\frac{(\xi - x \cos(2t))^2}{\epsilon \sin(4t)} + \frac{\xi^2 \tan(2t)}{2\epsilon} \right) \right] f(\xi) d\xi = \\ &= \left\langle \frac{\xi - x \cos(2t)}{\sqrt{2\epsilon \sin(2t)}} = z, d\xi = \sqrt{2\epsilon \sin(2t)} dz \right\rangle = \\ &= \frac{1-i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \lim_{t \rightarrow 0+0} e^{i \left(\frac{z^2}{\cos(2t)} + \frac{\xi^2 \tan(2t)}{2\epsilon} \right)} f(x \cos(2t) + z \sqrt{2\epsilon \sin(2t)}) dz = \\ &= f(x) \frac{1-i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iz^2} dz = \frac{(1-i)(1+i)\sqrt{\pi}}{2\sqrt{\pi}} f(x) = f(x). \end{aligned}$$

Thus, $u(x, t)$ really sets the solution to the problem. \square

Comment A1. Let us prove that the solution of the problem (A2) belongs to the space $L_2(-\infty, +\infty)$. Let us present the solution in the form:

$$\begin{aligned} u(x, t) &= \frac{1-i}{2\sqrt{\pi\epsilon \sin(2t)}} \int_{-\infty}^{\infty} f(\xi) \exp \left[i \left(\cot(2t) \frac{x^2 + \xi^2}{2\epsilon} - \frac{x\xi}{\epsilon \sin(2t)} \right) \right] d\xi = \\ &= \frac{1-i}{2\sqrt{\pi\epsilon \sin(2t)}} \int_{-\infty}^{\infty} f(\xi) \exp(-i \frac{x\xi}{\epsilon \sin(2t)}) \exp(i \cot(2t) \frac{x^2 + \xi^2}{2\epsilon}) d\xi = \\ &= \frac{1-i}{2\sqrt{\pi\epsilon \sin(2t)}} \int_{-\infty}^{\infty} f(\xi) \exp(-i \frac{x\xi}{\epsilon \sin(2t)}) \psi(x, t, \xi) d\xi = \\ &= \frac{1-i}{2\sqrt{\pi\epsilon \sin(2t)}} \left[-\frac{\epsilon \sin(2t)}{ix} f(\xi) \psi(\xi) \exp(-i \frac{x\xi}{\epsilon \sin(2t)}) \Big|_{-\infty}^{+\infty} + \right. \\ &\quad \left. + \frac{\epsilon \sin(2t)}{ix} \int_{-\infty}^{+\infty} \exp(-i \frac{x\xi}{\epsilon \sin(2t)}) \frac{\partial}{\partial \xi} (f(\xi) \psi(\xi)) d\xi \right] = \\ &= \frac{1-i}{2\sqrt{\pi\epsilon \sin(2t)}} \frac{\epsilon \sin(2t)}{ix} \int_{-\infty}^{+\infty} \exp(-i \frac{x\xi}{\epsilon \sin(2t)}) \frac{\partial}{\partial \xi} (\varphi(\xi) \psi(\xi)) d\xi \end{aligned}$$

Estimating $u(x, t, \epsilon)$ modulo and taking into account the conditions of the problem (1), we obtain

$$|u(x, t, \epsilon)| \leq \frac{M\sqrt{\epsilon \sin(2T)}}{\sqrt{2\pi}|x|} = \frac{C_1}{|x|}, \quad |x| \rightarrow \infty$$

Given the estimate on $u(x, t, \epsilon)$, we can write the estimate $|u(x, t, \epsilon)| \leq \frac{C}{1+|x|}$, $|x| \rightarrow \infty$. It follows that $u(x, t, \epsilon) \in L_2(\infty, +\infty)$. It is similarly proven that $\frac{\partial u}{\partial t} \in L_2(\infty, +\infty) \triangleright$

Comment A2. $f(x), h(x, t)$ satisfy the conditions $\forall m \in \mathbb{N} \exists \int_{-\infty}^{+\infty} |x|^m f(x) dx, \int_{-\infty}^{+\infty} |x|^m |h(x, t)| dx$ converging uniformly with respect to t , then $u(x, t)$ has continuous derivatives of any order with respect to x and t for $t > 0$.

Indeed, if $u(x, t)$ is differentiated with respect to x and t an arbitrary number of times, then the factor $(\xi - x \cos(2t))$ will be allocated to a positive power, and the factor $\sin(2t)$ to a negative degree. Thus, the matter reduces to the uniform convergence of an integral of the form

$$J = (\sin(2t))^{-k} \int_{-\infty}^{\infty} \exp \left[i \left(\frac{(\xi - x \cos(2t))^2}{\varepsilon \sin(4t)} + \frac{\xi^2 \tan(2t)}{2\varepsilon} \right) \right] (\xi - x \cos(2t))^m f(\xi) d\xi.$$

Let us estimate the integral modulo. Then,

$$\begin{aligned} |J| &\leq |\sin(2t_0)|^{-k} \int_{-\infty}^{\infty} \left| \exp \left[i \left(\frac{(\xi - x \cos(2t))^2}{\varepsilon \sin(4t)} + \frac{\xi^2 \tan(2t)}{2\varepsilon} \right) \right] \right| |\xi - x \cos(2t)|^m |f(\xi)| d\xi = \\ &= |\sin(2t_0)|^{-k} \int_{-\infty}^{\infty} |\xi - x \cos(2t)|^m |f(\xi)| d\xi \leq |\sin(2t_0)| \sum_{j=0}^m C_m^j |x|^{m-j} \int_{-\infty}^{\infty} |\xi|^j |f(\xi)| d\xi = \\ &= |\sin(2t_0)| \sum_{j=0}^m C_m^j |x|^{m-j} M_j \leq |\sin(2t_0)| (L+1)^m M_m. \end{aligned}$$

Considering that $f(x)$ satisfies condition (1) of the problem (A2) the integrals $M_j = \int_{-\infty}^{\infty} |\xi|^j |f(\xi)| d\xi$ exist. Therefore, the integral J converges uniformly for $0 < t_0 \leq t \leq T$. This implies that the function $u(x, t)$ is continuous and has continuous derivatives of any order with respect to x and t for $t > 0$. Moreover, since all the integrals involved in our formal operations are uniformly convergent in the parameters x, t in any closed rectangle $(x, t) \in [-L, L] \times [t_0, T]$, $t_0 > 0$, then they can be differentiated in this rectangle with respect to x and t arbitrarily times.

Theorem A2 (Uniqueness theorem). The problem for a homogeneous equation with a homogeneous initial condition

$$i\varepsilon \frac{\partial u}{\partial t} + \varepsilon^2 \frac{\partial^2 u}{\partial x^2} - x^2 u = 0, \quad u(x, 0) = 0, \quad (A3)$$

has only a trivial solution.

Proof. Let $u(x, t)$ be a solution to problem (A3), then

$$\begin{aligned} \frac{\partial u}{\partial t} &= i\varepsilon \frac{\partial^2 u}{\partial x^2} - \frac{i}{\varepsilon} x^2 u, \\ \frac{\partial \bar{u}}{\partial t} &= -i\varepsilon \frac{\partial^2 \bar{u}}{\partial x^2} + \frac{i}{\varepsilon} x^2 \bar{u}, \end{aligned} \quad (A4)$$

where the bar denotes complex conjugation.

Now, consider the following integral:

$$I(t) = \int_{-\infty}^{\infty} dx \cdot |u(x, t)|^2. \quad (A5)$$

Differentiating the integral $I(t)$ with respect to t and taking into account that the function $u(x, t)$ tends to zero as $x \rightarrow \pm\infty$ together with its partial derivatives, we have:

$$\begin{aligned}
I(t) &= \int_{-\infty}^{\infty} dx \left(\frac{\partial u}{\partial t} \bar{u} + u \frac{\partial \bar{u}}{\partial t} \right) = \left[\text{substitute the relations (A4)} \right] = \\
&= \int_{-\infty}^{\infty} dx \left(i\varepsilon u_{xx} \bar{u} - \frac{i}{\varepsilon} x^2 |u|^2 \right) + \int_{-\infty}^{\infty} dx \left(-i\varepsilon u \bar{u}_{xx} + \frac{i}{\varepsilon} x^2 |u|^2 \right) = \\
&= i\varepsilon \left(u_x \bar{u} \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} dx |u_x|^2 \right) - i\varepsilon \left(u \bar{u}_x \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} dx |u_x|^2 \right) = 0.
\end{aligned}$$

Therefore, $I(0) = I(t)$, where $I(0) = 0$ due to the initial conditions in the (A5) problem. Thus, the integral (A5) vanishes for all $t \in [0, T]$. And this is possible only if $u(x, t) = 0$. \square

Appendix B

Proof that the operator $T_\varepsilon = i\varepsilon \frac{\partial}{\partial t} + \varepsilon^2 \frac{\partial^2}{\partial x^2} - x^2$ nullifies $\frac{e^{-i \frac{x^2 \tan(2t)}{2\varepsilon}}}{\sqrt{\cos(2t)}}$, is confirmed by direct verification:

$$\begin{aligned}
i\varepsilon \frac{\partial}{\partial t} \left(\frac{e^{-i \frac{x^2 \tan(2t)}{2\varepsilon}}}{\sqrt{\cos(2t)}} \right) &= \frac{e^{-i \frac{x^2 \tan(2t)}{2\varepsilon}}}{\sqrt{\cos(2t)}} \left(\frac{x^2}{\cos^2(2t)} + i\varepsilon \tan(2t) \right); \\
\varepsilon^2 \frac{\partial^2}{\partial x^2} \left(\frac{e^{-i \frac{x^2 \tan(2t)}{2\varepsilon}}}{\sqrt{\cos(2t)}} \right) &= \frac{e^{-i \frac{x^2 \tan(2t)}{2\varepsilon}}}{\sqrt{\cos(2t)}} \left(-x^2 \tan^2(2t) - i\varepsilon \tan(2t) \right).
\end{aligned}$$

Hence, it follows that

$$T_\varepsilon \left(\frac{e^{-i \frac{x^2 \tan(2t)}{2\varepsilon}}}{\sqrt{\cos(2t)}} \right) = \frac{e^{-i \frac{x^2 \tan(2t)}{2\varepsilon}}}{\sqrt{\cos(2t)}} \left(\frac{x^2}{\cos^2(2t)} + i\varepsilon \tan(2t) - x^2 \tan^2(2t) - i\varepsilon \tan(2t) - x^2 \right) = 0.$$

It can be proved similarly that $T_\varepsilon \left(x \frac{e^{-i \frac{x^2 \tan(2t)}{2\varepsilon}}}{\sqrt{\cos(2t)}^3} \right) = 0$.

Appendix C

The equation for determining the particular solution w_0 is:

$$x^2 w_0(x, t) = h(x, t) - h(0, t) - x \frac{\partial h}{\partial x}(0, t).$$

From here,

$$w_0(x, t) = \frac{h(x, t) - h(0, t) - x \frac{\partial h}{\partial x}(0, t)}{x^2} = h_0(x, t),$$

where $h_0(x, t)$ is a smooth function. Let us run a chain of evaluations:

1. $|h_0(x, t)| = \left| 0, 5 \frac{\partial^2 h}{\partial x^2}(\xi, t) \right|$, where $0 < \xi(x) < x$. Consequently, $h_0(x, t)$ satisfies condition (4) in the statement of problem (1), since $h(x, t)$ satisfies this condition.
2. $\left| \frac{\partial w_0}{\partial x} \right| = \left| \frac{\partial h_0}{\partial x}(x, t) \right| = \left| \frac{1}{6} \frac{\partial^3 h}{\partial x^3}(\xi_1, t) \right|$, where $0 < \xi_1(x) < x$. Consequently, $\frac{\partial w_0}{\partial x}$ satisfies condition (4) in the formulation of problem (1), since $h(x, t)$ satisfies this condition.

Since $w_k = \frac{i \frac{\partial w_{k-1}}{\partial t} + \frac{\partial^2 w_{k-2}}{\partial x^2} + y_{k-1}(t) + x z_{k-1}(t)}{x^2}$, $k \geq 1$, then all $w_k(x, t)$ obtained by solving iterative problems also satisfy condition (4). Estimating $x^2 w_0$ gives

$|x^2 w_0| = \left| h(x, t) - h(0, t) - x \frac{\partial h}{\partial x}(0, t) \right| = \left| \frac{1}{2} \frac{\partial^2 h}{\partial x^2}(\xi, t) \right|$, which leads to the statement that $x^2 w_0$ also satisfy condition (4), and similarly for all $x^2 w_k$.

Estimating $x^2 w_0$ gives $x^2 w_0 = h(x, t) - h(0, t) - x \frac{\partial h}{\partial x}(0, t) = 0.5 x^2 \frac{\partial^2 h}{\partial x^2}(\xi(x), t)$, i.e., $x^2 w_0$ satisfies condition (4) of (1). From the above, and because $f(x)$ belongs to the declared class of functions, the solutions of iterative problems also belong to this class.

Similarly, one can show that all solutions of iterative problems for $v_k(x, t)$ satisfy condition (4) in the statement of problem (1).

The considerations given here allow us to conclude that the integral $\int_{-\infty}^{\infty} |H(x, t, \varepsilon)| dx$ converges. Here, $H(x, t, \varepsilon)$ is the right hand side of (14) for the remainder term.

References

1. Lomov, S.A. *Introduction to the General Theory of Singular Perturbations*; AMS: Providence, RI, USA, 1992.
2. Lomov, S.A.; Lomov I.S. *Fundamentals of the Mathematical Theory of the Boundary Layer*; Publishing House of Moscow University: Moscow, Russia, 2011.
3. Lomov, S.A. Asymptotic Behavior of Solutions to Second Order Ordinary Differential Equations Containing a Small Parameter. *Tr. MPEI* **1962**, *42*, 99–144.
4. Lomov, S.A. Power boundary layer in problems with a small parameter. *Dokl. AN SSSR* **1963**, *148*, 516–519.
5. Lomov, S.A. On the Lighthill Model Equation. *Sborn. Nauch. Trud. Min. Obor. SSSR* **1964**, *54*, 74–83.
6. Lomov, S.A. Perturbation method for singular problems. *Math. USSR Izv.* **1972**, *36*, 635–651. [\[CrossRef\]](#)
7. Lomov, S.A.; Safonov, V.F. Regularizations and asymptotic solutions for singularly perturbed problems with point singularities of the spectrum of the limit operator. *Ukr. Math. J.* **1984**, *36*, 172–180. [\[CrossRef\]](#)
8. Bobodzhonov, A.A.; Safonov, V.F. Regularized asymptotics of solutions of integro-differential partial differential equations with rapidly changing kernels. *Ufa Math. J.* **2018**, *10*, 3–12. [\[CrossRef\]](#)
9. Eliseev, A.G.; Lomov, S.A. Singular perturbation theory in the case of spectral singularities of the limit operator. *Math. Sb.* **1986**, *131*, 544–557. [\[CrossRef\]](#)
10. Eliseev, A.G.; Ratnikova, T.A. Singularly perturbed Cauchy problem in the presence of a rational “simple” turning point. *Diff. Eq. Manag. Process.* **2019**, *3*, 63–73.
11. Yeliseev, A. On the Regularized Asymptotics of a Solution to the Cauchy Problem in the Presence of a Weak Turning Point of the Limit Operator. *Axioms* **2020**, *9*, 86. [\[CrossRef\]](#)
12. Eliseev, A.-G. An example of a solution to a singularly perturbed Cauchy problem for a parabolic equation in the presence of a strong turning point. *Diff. Eq. Manag. Process.* **2022**, *3*, 46–58.
13. Eliseev, A.G.; Kirichenko, P.V. Regularized asymptotics of the solution of a singularly perturbed mixed problem on the semiaxis for a Schrödinger-type equation in the presence of a “strong” turning point for the limit operator. *Chebyshevskiy Sb.* **2023**, *24*, 50–68. [\[CrossRef\]](#)
14. Landau, L.D.; Lifshitz, E.M. *Course of Theoretical Physics, Vol. 3. Quantum Mechanics (Non-Relativistic Theory)*; Butterworth-Heinemann: London, UK, 1965.
15. Arnold, V.I. On matrices depending on parameters. *Russ. Math. Surv.* **1971**, *26*, 29–44. [\[CrossRef\]](#)
16. Liouville, J. Second Mémoire sur le développement des fonctions ou parties de fonctions en séries dont les divers termes sont assujétis à satisfaire à une même équation différentielle du second ordre, contenant un paramètre variable. *J. Math. Pures Appl.* **1837**, *2*, 16–35.
17. Elsgolts, L. *Differential Equations and the Calculus of Variations*; Mir: Moscow, Russia, 1977.

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