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Article

Spectral Conditions, Degree Sequences, and Graphical Properties

Xiao-Min Zhu [1], Weijun Liu [2] and Xu Yang [3,*]

[1] College of Sciences, Shanghai Institute of Technology, Shanghai 201418, China; zxmin@sit.edu.cn
[2] College of General Education, Guangdong University of Science and Technology, Dongguan 523083, China; wjliu6210@126.com
[3] School of Statistics and Mathematics, Shanghai Lixin University of Accounting and Finance, Shanghai 201209, China
* Correspondence: xuyang@lixin.edu.cn

Abstract: Integrity, tenacity, binding number, and toughness are significant parameters with which to evaluate network vulnerability and stability. However, we hardly use the definitions of these parameters to evaluate directly. According to the methods, concerning the spectral radius, we show sufficient conditions for a graph to be k-integral, k-tenacious, k-binding, and k-tough, respectively. In this way, the vulnerability and stability of networks can be easier to characterize in the future.

Keywords: spectral radius; vulnerability; integrity; tenacity; binding number; toughness

MSC: 05C50; 05C12

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1. Introduction

The graphs herein are all finite, undirected, and simple. Let G be a graph. The vertex set of G is denoted by $V(G)$ and the *order* of G is $|V(G)|$; the edge set is $E(G)$ and the *size* is $|E(G)|$. In this paper, we use $e(G)$ to denote the size of G. Let $v \in V(G)$. The *degree* of v in G, denoted by $d_G(v)$, is the number of vertices adjacent to v. If $X \subseteq V(G)$, the *neighbor set of X in G*, denoted by $N_G(X)$, is the set of all vertices in G adjacent to at least one vertex in X. When appropriate, we use $d(v)$ and $N(X)$ for short, respectively. The complement of G is denoted by $\overline{G}$. In particular, K_n is the *complete graph* of order n, and the complement of K_n is the *empty graph* E_n. Given two vertex-disjoint graphs G and H, their joint and disjoint union are denoted as $G \vee H$ and $G + H$, respectively. Furthermore, the complete bipartite graph $K_{s,t}$ is $E_s \vee E_t$ and $K_{1,t}$ is the star graph with t edges.

Computers or communication networks are built in such a way that they are difficult to disrupt under external assault and, if they are, are simple to restore. Toughness, integrity, tenacity, and binding number are just a few of the parameters that can be used to evaluate a network's desirable qualities, such as vulnerability and stability. For instance, a network with a large tenacity generally performs better under external attack. We can refer to [1–3] for more details. Let G be a graph. We utilize the symbols $I(G)$ to represent the integrity of G, $T(G)$ to represent its tenacity, bind (G) to represent its binding number, and $\tau(G)$ to represent its toughness. According to [4–6], they are defined in the following formulas:

$$\begin{aligned} I(G) &= \min\{|X| + m(G-X) \mid X \subset V(G) \text{ and } \omega(G-X) \geq 2\}; \\ T(G) &= \min\left\{\frac{|X| + m(G-X)}{\omega(G-X)} \,\middle|\, X \subset V(G) \text{ and } \omega(G-X) \geq 2\right\}; \\ \text{bind}\,(G) &= \min\left\{\frac{|N(X)|}{|X|} \,\middle|\, \emptyset \neq X \subseteq V(G),\ N(X) \neq V(G)\right\}; \\ \tau(G) &= \min\left\{\frac{|X|}{\omega(G-X)} \,\middle|\, X \subset V(G) \text{ and } \omega(G-X) \geq 2\right\}, \end{aligned} \tag{1}$$

where $m(G-X)$ is the order of the largest component of $G-X$ and $\omega(G-X)$ is the number of components of $G-X$. As in [7], a graph G is *k-integral* if $I(G) \geq k$, *k-tough* if $\tau(G) \geq k$, *k-tenacious* if $T(G) \geq k$, and *k-binding* if bind $(G) \geq k$.

The study and the relationships between certain pairs of vulnerability parameters can be found in [4–6,8–11]. Moreover, it is NP-hard to determine the integrity, toughness, or tenacity of a graph, in terms of computational complexity; we can refer to [12–14] for detail. Moreover, Cunningham [15] has shown that the binding number bind (G) is tractable. In particular, Yatauro [7] recently obtained the stability theorems for the properties of G being k-integral, k-tough, k-tenacious, or k-binding. In general, we hardly determine integrity, tenacity, binding number, or toughness by the definitions of these parameters, i.e., the equations in (1). Therefore, we need some other tools with which to characterize graphs with these properties. In this paper, we connect these vulnerability parameters with the spectral radius. This is the motivation of the paper.

Let G be a graph. The *adjacency matrix* of G is denoted by $A(G)$, where the entries of $A(G)$ is a_{ij} and $a_{ij}=1$ if $v_iv_j \in E(G)$, or $a_{ij}=0$ otherwise. The *characteristic polynomial* of G is $P_G(x)=\det(xI-A(G))$, where I is the identity matrix. The roots of $P_G(x)$ are the *eigenvalues* of G. We call the largest eigenvalue of G the *spectral radius* of G and denote it by $\lambda(G)$. The stability of graphs was first studied by Bondy and Chvátal in [16]. Let P be a property defined on all graphs of order n and let k be a non-negative integer. If whenever $G+uv$ for $uv \notin E(G)$ has property P and

$$d_G(u)+d_G(v) \geq k,$$

then P is *k-stable* and G itself has property P. The k-closure of G, indicated by $\mathrm{cl}_k(G)$, is the smallest one in terms of graph size among all the graphs H of order n such that $G \subseteq H$ and

$$d_H(u)+d_H(v)<k$$

for all $uv \notin E(H)$. By combining two non-adjacent vertices such that their degree sum is at least k, it is implied that $\mathrm{cl}_k(G)$ can be derived recursively from G. We can refer to [17] for a list of problems in structural graph theory where it is crucial. If a graph G exists that has the integer sequence π as its vertex degree sequence, then G is referred to as a *realization* of π. An integer sequence $\pi=(d_1 \leq d_2 \leq \cdots \leq d_n)$ is called *graphical* in this situation. If P is a graph property, such as integrity or toughness, we refer to a graphical sequence as having the property P *forcibly* if it appears in every realization of the sequence. In [18], a survey on degree sequences of graphs may be found.

The investigation of the connection between eigenvalues and graph properties has garnered a lot of interest. This is largely because of the issue that Brualdi and Solheid stated in [19]: *Find an upper bound on the spectral radii of the graphs in the given set S and then describe the graphs for which the maximum spectral radius is reached.* For graphs with a specified number of cut vertices, chromatic number, matching number, etc., this topic is investigated; see [20–26]. For more information, see Stevanović's recent thorough monograph [27]. Initiated by Brouwer and Haemers [28], the study of eigenvalues and the matching number was later advanced by Cioabă and numerous other academics [29,30]. In the past, a graph's vertex degrees have been used to establish the necessary conditions for the graph to have specific properties. Li et al. explored the spectral conditions for various stable aspects of graphs [31,32]. After that, Feng et al. [33] considered some graph properties with the spectral radius, including k-connected, k-edge-connected, k-Hamiltonian, k-edge-Hamiltonian, β-deficient, and k-path-coverable. Recently, the degree sequence is also used in a graph to determine if it is k-integral [34], k-tenacious [11], k-binding [35], or k-tough [36]. In this paper, according to the methods in [33], we shall utilize the degree sequence and the closure concepts to get several sufficient conditions of graphs with certain properties, including k-integral, k-tenacious, k-binding, and k-tough.

2. Preliminaries

In this section, some lemmas will be presented for later use. First of all, we collect several results in the following lemma, each of which contains a sufficient graphical degree sequence condition implying the existence of a certain graph being k-integral, k-tenacious, k-tough, or k-binding.

Lemma 1. *Let $\pi = (d_1 \leq d_2 \leq \cdots \leq d_n)$ be a graphical degree sequence.*

1. *[34] Let $n \geq k$ and $k \geq 1$. If $d_{n-k+2} \geq k-1$, then G is k-integral.*
2. *[36] Let $k \geq 1$ and $n \geq \lceil k \rceil + 2$. If $d_{\lfloor i/k \rfloor} \leq i \Rightarrow d_{n-i} \geq n - \lfloor i/k \rfloor$, for $k \leq i < \frac{kn}{k+1}$, then G is k-tough.*
3. *[11] Let $n \geq k \geq \frac{2}{n-1}$ with $n \geq 2$. If $d_{\lfloor \frac{n-k}{k+1} \rfloor + 2} \geq n - \left\lfloor \frac{n-k}{k+1} \right\rfloor - 1$, then G is k-tenacious.*
4. *[35] Let $n \geq k+1$ and $k \geq 1$. If*
   (a) $d_i \leq n - \left\lfloor \frac{n-i}{k} \right\rfloor - 1 \Rightarrow d_{\lfloor \frac{n-i}{k} \rfloor + 1} \geq n - i$, *for* $1 \leq i \leq \left\lfloor \frac{n}{k+1} \right\rfloor$,
   (b) $d_{\lfloor \frac{n}{k+1} \rfloor + 1} \geq n - \left\lfloor \frac{n}{k+1} \right\rfloor$,
   then G is k-binding.

Next, we consider stability results for the same graph properties, which are obtained by Yatauro in [7].

Lemma 2 ([7])**.** *For graphs of order n, we have the following stability results.*

1. *Given an integer $k \geq 2$, the property of being k-integral is $(2k-3)$-stable.*
2. *Given $k \geq 1$, the property of being k-tough is $\left\lceil \frac{2kn}{k+1} \right\rceil$-stable.*
3. *The property of being k-tenacious is $\left\lceil \frac{2(kn-1)}{k+1} \right\rceil$-stable.*
4. *Let $n \geq k+1$ and $k \geq 1$. The property of being k-binding is $\max\left\{ \left\lceil \frac{2kn}{k+1} \right\rceil, \left\lceil \frac{(2k-1)n+1}{k} \right\rceil - 2 \right\}$-stable.*

We have the following lemma by the definition of the k-closure of a graph.

Lemma 3 ([16])**.** *Let P be a property of a graph G. If P is k-stable and $\mathrm{cl}_k(G)$ has property P, then G itself has property P.*

The following result obtains the bounds on the spectral radius of a graph.

Lemma 4 ([25])**.** *Let G be a graph of order n with m edges and with no isolated vertices. Then, the spectral radius of G satisfies*

$$\lambda(G) \leq \sqrt{2m - n + 1}. \tag{2}$$

Equality holds if and only if G consists of c components for some $c \geq 1$, where $c-1$ components are single edges, and the remaining component is either a complete graph or a star graph.

The next lemma is essential to the proof of our main results later.

Lemma 5 ([33])**.** *Let H be a graph of order n and G be any spanning subgraph of H. If for any pair of non-adjacent vertices $u, v \in V(H)$ we have $d_H(u) + d_H(v) \leq \ell$, then*

$$e(H) \geq \binom{n}{2} - \frac{n\lambda^2(\overline{G})}{2n - 2 - \ell}.$$

At last, we give the spectral radius of the graph $K_s + E_t$.

Lemma 6 ([33])**.** *For $s, t \geq 1$, we have*

$$\lambda(K_s \vee E_t) = \frac{s-1+\sqrt{(s-1)^2+4st}}{2}.$$

In particular, for the star graph $K_{1,t}$, we have $\lambda(K_{1,t}) = \sqrt{t}$.

3. Main Results

3.1. Integral and Tenacious

In this section, we consider the spectral conditions of graphs to be k-integral or k-tenacious.

Theorem 1. *Let $n \geq k \geq 2$ and G be a graph of order n.*

1. *If G has no isolated vertices, and $\lambda(G) \geq \sqrt{2nk-k^2+3k-5n-1}$, then G is k-integral, unless $k=3$ and $G = K_{1,n-1}$.*
2. *If $\lambda(\overline{G}) \leq (n-k+1)\sqrt{\frac{n-k+2}{n}}$, then G is k-integral, unless $k=2$ and $G = E_n$.*

Proof. First of all, we prove the claim as follows.

Claim. Let G be a graph of order $n \geq k \geq 1$. If $e(G) \geq \frac{1}{2}(2n-k+1)(k-2)$, then G is k-integral, unless $e(G) = \frac{1}{2}(2n-k+1)(k-2)$ and $G = K_{k-2} \vee E_{n-k+2}$.

Proof of Claim. Suppose that G is not k-integral. Then, from Lemma 1 (1), we have $d_{n-k+2} \leq k-2$, and thus

$$2e(G) = \sum_{i=1}^{n} d_i \leq (n-k+2)(k-2) + (k-2)(n-1) = (2n-k+1)(k-2). \tag{3}$$

Hence, by the assumption of the claim, $e(G) = \frac{1}{2}(2n-k+1)(k-2)$ and all the inequalities above must be equalities. We have $d_1 = \cdots = d_{n-k+2} = k-2$ and $d_{n-k+3} = \cdots = d_n = n-1$. It follows that $G = K_{k-2} \vee E_{n-k+2}$. However, $I(G) = (k-2)+1 = k-1$, thus G is not k-integral. □

(I). With Lemma 4 and the fact that G has no isolated vertices, we get

$$\sqrt{2nk-5n-k^2+3k-1} \leq \lambda(G) \leq \sqrt{2e(G)-n+1},$$

which yields

$$e(G) \geq \frac{1}{2}(2n-k+1)(k-2).$$

By Lemma 4, $K_{k-2} \vee E_{n-k+2}$ cannot be one of the graphs attaining equality in (2), unless $k=3$. When $k=3$, the inequality in (3) is equality, i.e., $e(G) = n-1$ and $G = K_{1,n-1}$. Then, G is 2-integral, not 3-integral. Therefore, by Lemma 4 and the claim above, we have $e(G) > \frac{1}{2}(2n-k+1)(k-2)$, and G is k-integral.

(II). Assume that G is not k-integral. By Lemma 2, we consider the closure $H := \mathrm{cl}_{2k-3}(G)$. According to Lemma 3, H is not k-integral and $H \neq K_n$. Thus for any two non-adjacent vertices u and v in H, we have $d_H(u) + d_H(v) \leq 2k-4$. Therefore, by Lemma 5 and the assumption, it implies that

$$e(H) \geq \binom{n}{2} - \frac{n\lambda^2(\overline{G})}{2n-2k+2} \geq \frac{1}{2}(2n-k+1)(k-2).$$

Since H is not k-integral, according to the claim, we have $H = K_{k-2} \vee E_{n-k+2}$. Moreover, because $G \subseteq H$, we can see that $\overline{G}$ contains the complete graph K_{n-k+2}. Therefore,

$$\lambda(\overline{G}) \geq \lambda(K_{n-k+2}) = n - k + 1 \geq (n - k + 1)\sqrt{\frac{n-k+2}{n}}.$$

By the assumption, we have equality above. Thus, $k = 2$ and $G = H = E_n$. This completes the proof. □

Remark 1. *Though the bounds of $\lambda(G)$ and $\lambda(\overline{G})$ are sharp in Theorem 3.1, we think it is hard to find the extremal graphs when the equalities occur. For example, we explain the equality about $\lambda(G)$ below. Let G be a graph with no isolated vertices. If $\lambda(G) = \sqrt{2nk - k^2 + 3k - 5n - 1}$ occurs in Theorem 1, according to the inequalities in the proof, that is,*

$$\sqrt{2nk - 5n - k^2 + 3k - 1} \leq \lambda(G) \leq \sqrt{2e(G) - n + 1}, \tag{4}$$

we divide the discussion into two cases.

Case 1. $\lambda(G) = \sqrt{2e(G) - n + 1}$.

In this case, we have $e(G) = \frac{1}{2}(2n - k + 1)(k - 2)$ by (1). Furthermore, if $\lambda(G) = \sqrt{2e(G) - n + 1}$, according to Lemma 4 in Section 2, then G consists of c components for some $c \geq 1$, where $c - 1$ components are single edges, and the remaining component is either a complete graph or a star graph, according to Lemma 4 in Section 2. It follows that $e(G) = (c-1) + \frac{[n-2(c-1)][n-2(c-1)-1]}{2}$ or $e(G) = (c-1) + (n - 2(c-1) - 1) = n - (c-1) - 1 = n - c$. Therefore, if extremal graph G exists, then we have

$$\frac{1}{2}(2n - k + 1)(k - 2) = (c - 1) + \frac{[n - 2(c-1)][n - 2(c-1) - 1]}{2}, \tag{5}$$

or

$$\frac{1}{2}(2n - k + 1)(k - 2) = n - c. \tag{6}$$

We consider Equation (6) first. We can see that $k \geq 3$. If $c = 1$, then $G = K_{1,n-1}$, it is not k-integral. Thus, $c \geq 2$ and $k \geq 4$, otherwise $k = 3$ and $c = 1$, which is a contradiction. Therefore, $\frac{1}{2}(2n - k + 1)(k - 2) \geq 2n - k + 1$. Then, $n - c \geq 2n - k + 1$. Therefore, $k - c - 1 \geq n$. Note that $n \geq k$. Hence, we have $k - c - 1 \geq k$, a contradiction. Therefore, Equation (6) cannot occur.

Now we consider Equation (5). To simplify the equation, we have

$$k^2 - (2n + 3)k + n^2 - 4nc + 7n + 4c^2 - 4c = 0.$$

The discriminant is $\Delta = 16nc - 16n + 16c - 16c^2 + 9 = 16(n - c)(c - 1) + 9 > 0$. It follows that the equation has two real roots, and the roots are

$$k_{1,2} = \frac{2n + 3 \pm \sqrt{16(n-c)(c-1) + 9}}{2}.$$

Since $n \geq k$, we have $k = \frac{2n+3-\sqrt{16(n-c)(c-1)+9}}{2}$. However, it is hard to determine three parameters k, n, r with only one equation. We need some other conditions.

Case 2. $\lambda(G) < \sqrt{2e(G) - n + 1}$

By Lemma 4, this case may occur as well. In this case, the extremal graphs should satisfy two conditions:

1. $\lambda(G) = \sqrt{2nk - k^2 + 3k - 5n - 1}$,
2. $e(G) > \frac{1}{2}(2n - k + 1)(k - 2)$.

We think the graphs are not easy to construct just with the above two conditions.

Above all, we believe that the extremal graphs exist. However, some other conditions are necessary, such as the structure of the graphs. Otherwise, the extremal graphs are not easy to *construct just by several equations. The theorems in the left of the paper have the same difficulty with the extremal graphs when the equality occurs.*

Theorem 2. *Let $n \geq k \geq \frac{2}{n-1}$ with $n \geq 2$ and G be a graph of order n.*

1. *If G has no isolated vertices, and $\lambda(G) \geq \sqrt{(n-1)^2 - \left(\left\lfloor \frac{n-k}{k+1} \right\rfloor + 2\right)\left(\left\lfloor \frac{n-k}{k+1} \right\rfloor + 1\right)}$, then G is k-tenacious.*
2. *If $\lambda(\overline{G}) \leq \sqrt{\frac{\lceil 2(\frac{n+1}{k+1}) \rceil - 1}{n} \binom{\lfloor \frac{n+1}{k+1} \rfloor + 1}{2}}$, then G is k-tenacious.*

Proof. First of all, we prove the claim as follows.

Claim. G be a graph of order n such that $n \geq k \geq \frac{2}{n-1}$ and $n \geq 2$. If $e(G) \geq \binom{n}{2} - \binom{\lfloor \frac{n-k}{k+1} \rfloor + 2}{2}$, then G is k-tenacious, unless $e(G) = \binom{n}{2} - \binom{\lfloor \frac{n-k}{k+1} \rfloor + 2}{2}$ and $G = K_{n - \lfloor \frac{n-k}{k+1} \rfloor - 2} \vee E_{\lfloor \frac{n-k}{k+1} \rfloor + 2}$.

Proof of Claim. Assume that G is not k-tenacious for some k in $n \geq k \geq \frac{2}{n-1}$. By Lemma 1, we know that $d_{\lfloor \frac{n-k}{k+1} \rfloor + 2} \leq n - \left\lfloor \frac{n-k}{k+1} \right\rfloor - 2$. Thus,

$$\begin{aligned} 2e(G) &\leq \left(\left\lfloor \frac{n-k}{k+1} \right\rfloor + 2\right)\left(n - \left\lfloor \frac{n-k}{k+1} \right\rfloor - 2\right) + \left(n - \left\lfloor \frac{n-k}{k+1} \right\rfloor - 2\right)(n-1) \\ &= \left(n - \left\lfloor \frac{n-k}{k+1} \right\rfloor - 2\right)\left(n + \left\lfloor \frac{n-k}{k+1} \right\rfloor + 1\right) \\ &= n(n-1) - \left(\left\lfloor \frac{n-k}{k+1} \right\rfloor + 2\right)\left(\left\lfloor \frac{n-k}{k+1} \right\rfloor + 1\right). \end{aligned}$$

Hence, $e(G) = \binom{n}{2} - \binom{\lfloor \frac{n-k}{k+1} \rfloor + 2}{2}$ and all the inequalities above must be equalities. Therefore, $d_1 = d_2 = \cdots = d_{\lfloor \frac{n-k}{k+1} \rfloor + 2} = n - \left\lfloor \frac{n-k}{k+1} \right\rfloor - 2$ and $d_{\lfloor \frac{n-k}{k+1} \rfloor + 3} = \cdots = d_n = n - 1$. It follows that $G = K_{n - \lfloor \frac{n-k}{k+1} \rfloor - 2} \vee E_{\lfloor \frac{n-k}{k+1} \rfloor + 2}$. However,

$$T(G) = \frac{(n - \left\lfloor \frac{n+1}{k+1} \right\rfloor - 1) + 1}{\left\lfloor \frac{n+1}{k+1} \right\rfloor + 1} = \frac{n - (\left\lfloor \frac{n+1}{k+1} \right\rfloor + 1) + 1}{\left\lfloor \frac{n+1}{k+1} \right\rfloor + 1} < \frac{n - \frac{n+1}{k+1} + 1}{\frac{n+1}{k+1}} = k,$$

i.e., G is not k-tenacious. □

(I). With Lemma 4 and the fact that G has no isolated vertices, we get

$$\sqrt{(n-1)^2 - \left(\left\lfloor \frac{n-k}{k+1} \right\rfloor + 2\right)\left(\left\lfloor \frac{n-k}{k+1} \right\rfloor + 1\right)} \leq \lambda(G) \leq \sqrt{2e(G) - n + 1},$$

implying that

$$e(G) \geq \binom{n}{2} - \binom{\left\lfloor \frac{n-k}{k+1} \right\rfloor + 2}{2}.$$

Because $K_{n - \lfloor \frac{n-k}{k+1} \rfloor - 2} \vee E_{\lfloor \frac{n-k}{k+1} \rfloor + 2}$ cannot achieve equality in inequality (2), the aforementioned claim and Lemma 4 lead to the conclusion that $e(G) > \binom{n}{2} - \binom{\lfloor \frac{n-k}{k+1} \rfloor + 2}{2}$, and G is k-tenacious.

(II). Assume that G is not k-tenacious. By Lemma 2, we consider the closure $H := \mathrm{cl}_{\lceil \frac{2(kn-1)}{k+1} \rceil}(G)$. According to Lemma 3, H is not k-tenacious and $H \neq K_n$.

Thus, for any two non-adjacent vertices u and v in H, we have $d_H(u) + d_H(v) \leq \left\lceil \frac{2(kn-1)}{k+1} \right\rceil - 1$. Therefore, by Lemma 5 and the assumption, it follows that

$$e(H) \geq \binom{n}{2} - \frac{n\lambda^2(\overline{G})}{2n - \left\lceil \frac{2(kn-1)}{k+1} \right\rceil - 1} \geq \binom{n}{2} - \binom{\left\lfloor \frac{n-k}{k+1} \right\rfloor + 2}{2}.$$

Since H is not k-tenacious, according to the claim, we have $H = K_{n - \lfloor \frac{n-k}{k+1} \rfloor - 2} \vee E_{\lfloor \frac{n-k}{k+1} \rfloor + 2}$. Moreover, because $G \subseteq H$, we can see that $\overline{G}$ contains $K_{\lfloor \frac{n-k}{k+1} \rfloor + 2}$. We have

$$\lambda(\overline{G}) \geq \lambda(K_{\lfloor \frac{n-k}{k+1} \rfloor + 2}) = \left\lfloor \frac{n+1}{k+1} \right\rfloor > \sqrt{\frac{\left\lfloor \frac{n+1}{k+1} \right\rfloor \left(\left\lfloor \frac{n+1}{k+1} \right\rfloor + 1 \right) \left(\left\lfloor \frac{2(n+1)}{k+1} \right\rfloor - 1 \right)}{2n}},$$

because of $k \geq \frac{2}{n-1}$ and

$$\begin{aligned} \frac{\left(\left\lfloor \frac{n+1}{k+1} \right\rfloor + 1 \right) \left(\left\lfloor \frac{2(n+1)}{k+1} \right\rfloor - 1 \right)}{2n} &\leq \frac{\left\lfloor \frac{n+1}{\frac{2}{n-1}+1} \right\rfloor + 1}{2} \cdot \frac{\left\lfloor \frac{2(n+1)}{k+1} \right\rfloor - 1}{n} \\ &= \frac{n}{2} \cdot \frac{\left\lfloor \frac{2(n+1)}{k+1} \right\rfloor - 1}{n} \\ &\leq \frac{n}{2} \cdot \frac{2\left\lfloor \frac{n+1}{k+1} \right\rfloor}{n} = \left\lfloor \frac{n+1}{k+1} \right\rfloor, \end{aligned} \tag{7}$$

and all the inequalities in (7) cannot be equalities simultaneously. □

3.2. Binding Number

In this section, we consider the spectral conditions of graphs to be k-binding.

Theorem 3. *Let $G \neq K_n$ be a graph on n vertices. Let $n \geq 2(k+1)$ if $k \geq 2$ or $n \geq 9$ if $1 \leq k < 2$.*

1. *If G has no isolated vertices, and*

$$\lambda(G) \geq \max\left\{ \sqrt{(n-1)^2 - 2\left\lfloor \frac{n-1}{k} \right\rfloor}, \sqrt{(n-1)^2 - \left\lfloor \frac{n}{k+1} \right\rfloor \left(\left\lfloor \frac{n}{k+1} \right\rfloor + 1 \right)} \right\},$$

then G is k-binding;

2. *If*

$$\lambda(\overline{G}) \leq \min\left\{ \sqrt{\frac{\left\lfloor \frac{n}{k+1} \right\rfloor \left(\left\lfloor \frac{n}{k+1} \right\rfloor + 1 \right) \left(\left\lceil \frac{2n}{k+1} \right\rceil - 1 \right)}{2n}}, \sqrt{\frac{\left\lfloor \frac{n-1}{k} \right\rfloor \left(2n - \left\lceil \frac{(2k-1)n+1}{k} \right\rceil + 1 \right)}{n}} \right\},$$

then G is k-binding.

Proof. First of all, we prove the claim as follows.

Claim. Let G be a graph of order n such that $n \geq 2(k+1)$ if $k \geq 2$ or $n \geq 9$ if $1 \leq k < 2$. If

$$e(G) \geq \max\left\{ \binom{n}{2} - \left\lfloor \frac{n-1}{k} \right\rfloor, \binom{n}{2} - \binom{\left\lfloor \frac{n}{k+1} \right\rfloor + 1}{2} \right\},$$

then G is k-binding, unless $e(G) = \binom{n}{2} - \left\lfloor \frac{n-1}{k} \right\rfloor$ and $G = K_{n-\lfloor \frac{n-1}{k} \rfloor -1} \vee \left(K_1 + K_{\lfloor \frac{n-1}{k} \rfloor}\right)$, or $e(G) = \binom{n}{2} - \binom{\lfloor \frac{n}{k+1} \rfloor +1}{2}$ and $G = K_{n-\lfloor \frac{n}{k+1} \rfloor -1} \vee E_{\lfloor \frac{n}{k+1} \rfloor +1}$.

Proof of Claim. Assume that G is not k-binding. Then $\pi(G)$ must fail either (a) of Lemma 1 (4) for some i or (b) of Lemma 1 (4). Thus, the proof will be divided into two cases as follows.

Case 1. First assume that $\pi(G)$ fails (a) of Lemma 1 (4), so

$$d_i \leq n - \left\lfloor \frac{n-i}{k} \right\rfloor - 1 \text{ and } d_{\lfloor \frac{n-i}{k} \rfloor +1} \leq n-i-1 \text{ for } 1 \leq i \leq \left\lfloor \frac{n}{k+1} \right\rfloor.$$

Thus

$$\begin{aligned}
2e(G) &\leq i\left(n - \left\lfloor \frac{n-i}{k} \right\rfloor - 1\right) + \left(\left\lfloor \frac{n-i}{k} \right\rfloor + 1 - i\right)(n-i-1) \\
&\quad + \left(n - \left\lfloor \frac{n-i}{k} \right\rfloor - 1\right)(n-1) \\
&= i^2 - i\left(2\left\lfloor \frac{n-i}{k} \right\rfloor + 1\right) + n(n-1) \\
&< i^2 - i\left(2\left(\frac{n-i}{k} - 1\right) + 1\right) + n(n-1) \\
&= \left(\frac{k+2}{k}\right)i^2 - \left(\frac{2n-k}{k}\right)i + n(n-1).
\end{aligned}$$

Suppose $f(x) = \left(\frac{k+2}{k}\right)x^2 - \left(\frac{2n-k}{k}\right)x$ with $1 \leq x \leq \frac{n}{k+1}$. We find that $f(x)$ is a concave up parabola with vertex at $x = \frac{2n-k}{2(k+2)}$. Since $n \geq 2(k+1)$, we have $\frac{2n-k}{2(k+2)} \geq \frac{4(k+1)-k}{2(k+2)} > 1$. It implies that the maximum value of $f(x)$ occurs at either $x = 1$ or $x = \frac{n}{k+1}$. Then it is easy to check that $f(1) - f(\frac{n}{k+1}) = \frac{(n-(k+1))(kn-2(k+1)^2)}{k(k+1)^2} \geq 0$ if $n \geq \frac{2(k+1)^2}{k}$.

When $n \geq \frac{2(k+1)^2}{k}$, then $f_{\max}(x) = f(1)$ and we have

$$2e(G) < f(1) + n(n-1) = n(n-1) - \frac{2(n-1)}{k} + 2.$$

Therefore,

$$e(G) \leq \binom{n}{2} - \left\lfloor \frac{n-1}{k} \right\rfloor.$$

Hence, $e(G) = \binom{n}{2} - \left\lfloor \frac{n-1}{k} \right\rfloor$, and all the inequalities above must be equalities. Therefore, $d_1 = n - \left\lfloor \frac{n-1}{k} \right\rfloor - 1$, $d_2 = \cdots = d_{\lfloor \frac{n-1}{k} \rfloor +1} = n-2$ and $d_{\lfloor \frac{n-1}{k} \rfloor +2} = \cdots = d_n = n-1$. It follows that $G = K_{n-\lfloor \frac{n-1}{k} \rfloor -1} \vee \left(K_1 + K_{\lfloor \frac{n-1}{k} \rfloor}\right)$. Let $S = V\left(K_1 + K_{\lfloor \frac{n-1}{k} \rfloor}\right)$. Then

$$\text{bind}\,(G) = \frac{|N(S)|}{|S|} = \frac{n-1}{\left\lfloor \frac{n-1}{k} \right\rfloor + 1} < \frac{n-1}{\frac{n-1}{k}} = k,$$

i.e., G is not k-binding.

Thus, if $1 \leq k < 2$ and $n \geq 9 > \frac{2(k+1)^2}{k}$ or $k \geq 2$ and $n \geq \frac{2(k+1)^2}{k}$, we are done. Otherwise, assume $k \geq 2$ and $2(k+1) \leq n < \frac{2(k+1)^2}{k}$. Then $1 \leq i \leq \frac{n}{k+1} < \frac{2(k+1)}{k} \leq 3$, i.e., $i = 1$ or 2. Let

$$P(i) = \binom{n}{2} + \frac{1}{2}\left(i^2 - i\left(2\left\lfloor \frac{n-i}{k} \right\rfloor + 1\right)\right).$$

Therefore, we have $e(G) \leq \max\{P(1), P(2)\}$.

Since $2 \le \frac{n-2}{k} < \frac{n-1}{k} < 4$, we have the following possibilities: (i) $\lfloor \frac{n-2}{k} \rfloor = \left\lfloor \frac{n-1}{k} \right\rfloor = 2$ or 3; (ii) $\lfloor \frac{n-2}{k} \rfloor = 2$ and $\left\lfloor \frac{n-1}{k} \right\rfloor = 3$. Note that $P(2) - P(1) = -2\lfloor \frac{n-2}{k} \rfloor + \left\lfloor \frac{n-1}{k} \right\rfloor + 1$, which is negative for every possibility indicated above. Therefore, $\max\{P(1), P(2)\} = P(1)$, and $e(G) \le P(1) = \binom{n}{2} - \left\lfloor \frac{n-1}{k} \right\rfloor$.

Case 2. Assume that $\pi(G)$ fails (b) of Lemma 1 (4). We have

$$d_{\lfloor \frac{n}{k+1} \rfloor + 1} \le n - \left\lfloor \frac{n}{k+1} \right\rfloor - 1.$$

Then

$$\begin{aligned} 2e(G) &\le \left(\left\lfloor \frac{n}{k+1} \right\rfloor + 1\right)\left(n - \left\lfloor \frac{n}{k+1} \right\rfloor - 1\right) + \left(n - \left\lfloor \frac{n}{k+1} \right\rfloor - 1\right)(n-1) \\ &= n(n-1) - \left\lfloor \frac{n}{k+1} \right\rfloor \left(\left\lfloor \frac{n}{k+1} \right\rfloor + 1\right). \end{aligned}$$

Hence, $e(G) = \binom{n}{2} - \binom{\lfloor \frac{n}{k+1} \rfloor + 1}{2}$ and all the inequalities in this case must be equalities. Therefore, $d_1 = \cdots = d_{\lfloor \frac{n}{k+1} \rfloor + 1} = n - \left\lfloor \frac{n}{k+1} \right\rfloor - 1$ and $d_{\lfloor \frac{n}{k+1} \rfloor + 2} = \cdots = d_n = n - 1$. It follows that $G = K_{n - \lfloor \frac{n}{k+1} \rfloor - 1} \vee E_{\lfloor \frac{n}{k+1} \rfloor + 1}$. Let $S = V\left(E_{\lfloor \frac{n}{k+1} \rfloor + 1}\right)$. Then

$$\operatorname{bind}(G) = \frac{|N(S)|}{|S|} = \frac{n - \left\lfloor \frac{n}{k+1} \right\rfloor - 1}{\left\lfloor \frac{n}{k+1} \right\rfloor + 1} < \frac{n - \frac{n}{k+1}}{\frac{n}{k+1}} = k,$$

i.e., G is not k-binding. □

(I). With Lemma 4 and the fact that G has no isolated vertices, according to the claim, if $e(G) \ge \binom{n}{2} - \left\lfloor \frac{n-1}{k} \right\rfloor$, we get

$$\sqrt{(n-1)^2 - 2\left\lfloor \frac{n-1}{k} \right\rfloor} \le \lambda(G) \le \sqrt{2e(G) - n + 1},$$

which yields

$$e(G) \ge \binom{n}{2} - \left\lfloor \frac{n-1}{k} \right\rfloor.$$

Because $K_{n - \lfloor \frac{n-1}{k} \rfloor - 1} \vee \left(K_1 + K_{\lfloor \frac{n-1}{k} \rfloor}\right)$ cannot achieve equality in (2), the aforementioned claim and Lemma 4 lead to the conclusion that $e(G) > \binom{n}{2} - \left\lfloor \frac{n-1}{k} \right\rfloor$, and G is k-binding.

On the other hand, if $e(G) \ge \binom{n}{2} - \binom{\lfloor \frac{n}{k+1} \rfloor + 1}{2}$, from Lemma 4, we have

$$\sqrt{(n-1)^2 - \left\lfloor \frac{n}{k+1} \right\rfloor \left(\left\lfloor \frac{n}{k+1} \right\rfloor + 1\right)} \le \lambda(G) \le \sqrt{2e(G) - n + 1},$$

which yields

$$e(G) \ge \binom{n}{2} - \binom{\left\lfloor \frac{n}{k+1} \right\rfloor + 1}{2}.$$

Because $K_{n - \lfloor \frac{n}{k+1} \rfloor - 1} \vee E_{\lfloor \frac{n}{k+1} \rfloor + 1}$ cannot achieve equality in the inequality (2), the aforementioned claim and Lemma 4 lead to the conclusion that $e(G) > \binom{n}{2} - \binom{\lfloor \frac{n}{k+1} \rfloor + 1}{2}$, and G is k-binding.

(II). Suppose that G is not k-binding. According to Lemma 2 (4), the proof will be divided into two cases as follows.

Case 1: The property of being k-binding is $\left\lceil \frac{(2k-1)n+1}{k} \right\rceil - 2$-stable.

In this case, (4) (a) of Lemma 1 fails and we consider the closure $H := \mathrm{cl}_{\left\lceil \frac{(2k-1)n+1}{k} \right\rceil - 2}(G)$. According to Lemma 3, H is not k-binding and $H \neq K_n$. Thus, for any two non-adjacent vertices u and v in H, we have $d_H(u) + d_H(v) \leq \left\lceil \frac{(2k-1)n+1}{k} \right\rceil - 3$. Therefore, by Lemma 5 and the assumption, it follows that

$$e(H) \geq \binom{n}{2} - \frac{n\lambda^2(\overline{G})}{2n - \left\lceil \frac{(2k-1)n+1}{k} \right\rceil + 1} \geq \binom{n}{2} - \left\lfloor \frac{n-1}{k} \right\rfloor.$$

Since H is not k-binding, according to the claim, we have

$$H = K_{n - \lfloor \frac{n-1}{k} \rfloor - 1} \vee \left(K_1 + K_{\lfloor \frac{n-1}{k} \rfloor} \right).$$

Moreover, because $G \subseteq H$, we can see that $\overline{G}$ contains the star $K_{1, \lfloor \frac{n-1}{k} \rfloor}$. Using Lemma 6, we have

$$\lambda(\overline{G}) \geq \lambda(K_{1, \lfloor \frac{n-1}{k} \rfloor}) = \sqrt{\left\lfloor \frac{n-1}{k} \right\rfloor} \geq \sqrt{\frac{\left\lfloor \frac{n-1}{k} \right\rfloor \left(2n - \left\lceil \frac{(2k-1)n+1}{k} \right\rceil + 1 \right)}{n}}.$$

By the assumption, we have the equality above. Thus, $k = 1$ and $G = K_1 + K_{n-1}$, which is not binding.

Case 2: The property of being k-binding is $\left\lceil \frac{2kn}{k+1} \right\rceil$-stable.

In this case, (4) (b) of Lemma 1 fails, and we consider the closure $H := \mathrm{cl}_{\left\lceil \frac{2kn}{k+1} \right\rceil}(G)$. According to Lemma 3, H is not k-binding and $H \neq K_n$. Thus for any two non-adjacent vertices u and v in H, we have $d_H(u) + d_H(v) \leq \left\lceil \frac{2kn}{k+1} \right\rceil - 1$. Therefore, by Lemma 5 and the assumption, it follows that

$$e(H) \geq \binom{n}{2} - \frac{n\lambda^2(\overline{G})}{2n - \left\lceil \frac{2kn}{k+1} \right\rceil - 1} \geq \binom{n}{2} - \binom{\left\lfloor \frac{n}{k+1} \right\rfloor + 1}{2}.$$

Since H is not k-binding, according to Claim 1, we have $H = K_{n - \lfloor \frac{n}{k+1} \rfloor - 1} \vee E_{\lfloor \frac{n}{k+1} \rfloor + 1}$. Moreover, because $G \subseteq H$, we can see that $\overline{G}$ contains the complete graph $K_{\lfloor \frac{n}{k+1} \rfloor + 1}$. Furthermore, we have

$$\lambda(\overline{G}) \geq \lambda(K_{\lfloor \frac{n}{k+1} \rfloor + 1}) = \left\lfloor \frac{n}{k+1} \right\rfloor > \sqrt{\frac{\left\lfloor \frac{n}{k+1} \right\rfloor \left(\left\lfloor \frac{n}{k+1} \right\rfloor + 1 \right) \left(\left\lfloor \frac{2n}{k+1} \right\rfloor - 1 \right)}{2n}},$$

because of $k \geq 1$ and $n \geq 2(k+1)$, and

$$\frac{\left(\left\lfloor \frac{n}{k+1} \right\rfloor + 1 \right) \left(\left\lfloor \frac{2n}{k+1} \right\rfloor - 1 \right)}{2n} < \frac{\frac{2n}{k+1} \left(\left\lfloor \frac{n}{k+1} \right\rfloor + 1 \right)}{2n} < \left\lfloor \frac{n}{k+1} \right\rfloor.$$

□

3.3. Toughness

In this section, we consider the property of k-tough. First, we obtain a lemma as follows.

Lemma 7. *Let $k \geq 1$ and $n \geq \lceil k \rceil + 2$. If a graph G of order n is not k-tough, then there exists i such that $k \leq i \leq \frac{kn}{k+1}$ and*

$$e(G) \leq \binom{n}{2} - \frac{T(i)}{2},$$

where $T(i) = \left\lfloor \frac{i}{k} \right\rfloor \left(2n - 2i - \left\lfloor \frac{i}{k} \right\rfloor - 1\right)$.

Proof. Assume that G is not k-tough. Then by Lemma 1, there exists integer i such that $k \leq i \leq \frac{kn}{k+1}$, $d_{\lfloor i/k \rfloor} \leq i$ and $d_{n-i} \leq n - \lfloor i/k \rfloor - 1$. We have

$$\begin{aligned} 2e(G) &\leq \left\lfloor \frac{i}{k} \right\rfloor i + \left(n - i - \left\lfloor \frac{i}{k} \right\rfloor\right)\left(n - \left\lfloor \frac{i}{k} \right\rfloor - 1\right) + i(n-1) \\ &= n(n-1) - \left\lfloor \frac{i}{k} \right\rfloor \left(2n - 2i - \left\lfloor \frac{i}{k} \right\rfloor - 1\right) \\ &= n(n-1) - T(i). \end{aligned} \tag{8}$$

This completes the proof. □

To calculate the minimum of $T(i)$ in Lemma 7, for fixed values of k and $n \geq \lceil k \rceil + 2$, define

$$p(n,k) = \left\lceil \frac{kn}{k+1} \right\rceil - 1,\ q(n,k) = \left\lfloor \frac{p}{k} \right\rfloor.$$

It implies that $p(n,k) = \left\lceil \frac{n(k+1)-n}{k+1} \right\rceil - 1 = n - \left\lfloor \frac{n}{k+1} \right\rfloor - 1$. Since $k \leq i \leq \frac{kn}{k+1}$, we have Table 1 as follows.

Table 1. $T_{\min}(i)$ on each interval.

Interval	$T(i)$	$T_{\min}(i)$
$i \in [k, 2k)$	$2n - 2i - 2$	$\lceil 2k \rceil - 1$
$i \in [2k, 3k)$	$2(2n - 2i - 3)$	$\lceil 3k \rceil - 1$
$\vdots$	$\vdots$	$\vdots$
$i \in [qk, \frac{kn}{k+1}]$	$q(2n - 2i - q - 1)$	p

In the rest of this section, we will divide into three cases of $q(n,k)$. For convenience, let $p = p(n,k)$ and $q = q(n,k)$. Therefore, $T_{\min}(i) = \min\{T(i_1), \cdots, T(i_{q-1}), T(p)\}$, where $i_s = \lceil (s+1)k \rceil - 1$ for $1 \leq s \leq q-1$.

Situation 1. $q = 1$

If $q = 1$, then $p \in [k, 2k)$. Furthermore, $T_{\min}(i) = T(p) = 2n - 2p - 2$.

Situation 2. $q = 2$

When $q = 2$, we have

$$T_{\min}(i) = \min\{T(i_1), T(p)\} = \min\{2n - 2\lceil 2k \rceil, T(p)\},$$

where $T(p) = 2(2n - 2p - 3)$.

By the definition of toughness, k is rational. It follows that integers a and b exist, in the lowest terms, such that $k = \frac{a}{b}$. Let r be the remainder when bn is divided by $a+b$, so that $0 \leq r \leq a + b - 1$ and $bn = (a+b)\left\lfloor \frac{n}{k+1} \right\rfloor + r$. Thus,

$$n = (k+1)\left\lfloor \frac{n}{k+1} \right\rfloor + \frac{r}{b}. \tag{9}$$

In the next, we will discuss the situation of r.

Case 1. $0 \leq r < b$.

In this case, we have

$$p = \left\lceil \frac{kn}{k+1} \right\rceil - 1 > \frac{kn}{k+1} - 1 \Rightarrow \frac{p}{k} > \frac{n}{k+1} - \frac{1}{k} \geq \frac{n}{k+1} - 1.$$

By Equation (9), we have $n < (t+1)\lfloor \frac{n}{t+1} \rfloor + 1$, it follows that $p = n - \left\lfloor \frac{n}{k+1} \right\rfloor - 1 < k\left\lfloor \frac{n}{k+1} \right\rfloor$. Thus, $\frac{p}{k} < \left\lfloor \frac{n}{k+1} \right\rfloor$. Therefore, $\left\lfloor \frac{n}{k+1} \right\rfloor - 1 \leq \lfloor \frac{p}{k} \rfloor < \left\lfloor \frac{n}{k+1} \right\rfloor$, and $q = \lfloor \frac{p}{k} \rfloor = \left\lfloor \frac{n}{k+1} \right\rfloor - 1$.

If $q = \left\lfloor \frac{n}{k+1} \right\rfloor - 1 = 2$, then we have $p = n - 4$ and $T(p) = 10$. Furthermore, $\left\lfloor \frac{n}{k+1} \right\rfloor = 3$, it follows that $4(k+1) > n \geq 3(k+1)$. If $k \geq 3$, then

$$2n - 2\lceil 2k \rceil > 2n - 2(2k+1) = 2n - 4k - 2 \geq 2(3k+3) - 4k - 2 = 2k + 4 \geq 10,$$

and $T_{\min}(i) = T(p) = 10 = \left(\left\lfloor \frac{n}{k+1} \right\rfloor - 1\right)\left(\left\lfloor \frac{n}{k+1} \right\rfloor + 2\right)$.

If $k \leq 2$ and $n \leq 2k+5$, then $2n - 2\lceil 2k \rceil \leq 2n - 4k \leq 10$, and $T_{\min}(i) = T(i_1) = 2n - 2\lceil 2k \rceil$.

Case 2. $b \leq r \leq a + b - 1$.

If $b = r$, we have $n = (k+1)\left\lfloor \frac{n}{k+1} \right\rfloor + 1$, then $p = n - \left\lfloor \frac{n}{k+1} \right\rfloor - 1 = k\left\lfloor \frac{n}{k+1} \right\rfloor$ and $q = \lfloor \frac{p}{k} \rfloor = \left\lfloor \frac{n}{k+1} \right\rfloor$.

If $b < r$, we have $n > (k+1)\left\lfloor \frac{n}{k+1} \right\rfloor + 1$ and $p > k\left\lfloor \frac{n}{k+1} \right\rfloor$, it implies that $\frac{p}{k} > \left\lfloor \frac{n}{k+1} \right\rfloor$. Note that $p = \left\lceil \frac{kn}{k+1} \right\rceil - 1 < \frac{kn}{k+1}$ and $\frac{p}{k} < \frac{n}{k+1}$. Therefore, $q = \lfloor \frac{p}{k} \rfloor = \left\lfloor \frac{n}{k+1} \right\rfloor$.

When $q = 2$, we have $p = n - 3$ and $T(p) = 6$. Furthermore, $\left\lfloor \frac{n}{k+1} \right\rfloor = 2$, it follows that $3(k+1) > n \geq 2(k+1)$. If $n \geq 3k+2$ and $k \geq 2$, then

$$2n - 2\lceil 2k \rceil > 2n - 2(2k+1) = 2n - 4k - 2 \geq 2(3k+2) - 4k - 2 = 2k + 2 \geq 6,$$

and $T_{\min}(i) = T(p) = 6 = \left\lfloor \frac{n}{k+1} \right\rfloor \left(\left\lfloor \frac{n}{k+1} \right\rfloor + 1\right)$.

If $n \leq 2k+3$, then $2n - 2\lceil 2k \rceil \leq 2n - 4k \leq 6$, and $T_{\min}(i) = T(i_1) = 2n - 2\lceil 2k \rceil$.

Situation 3. $q \geq 3$

Note that

$$T(i_s) = s(2n - 2(\lceil (s+1)k \rceil - 1) - s - 1) = s(2n - 2\lceil (s+1)k \rceil - s + 1),$$

where $s \in [1, q-1]$ is an integer. Thus, $T_{\min}(i) = \min_{1 \leq s \leq q-1}\{T(i_s), T(p)\}$. Let $f(s) = s(2n - 2((s+1)k + 1) - s + 1)$ and $h(s) = s(2n - 2(s+1)k - s + 1)$. Then, $f(s) < T(i_s) \leq h(s)$. Moreover, $f(s) = -(2k+1)s^2 + (2n - 2k - 1)s$ and $h(s) = -(2k+1)s^2 + (2n - 2k + 1)s$. Since $q \geq 3$ and $q = \lfloor \frac{p}{k} \rfloor$, we have $3k \leq p = \left\lceil \frac{kn}{k+1} \right\rceil - 1 < \frac{kn}{k+1}$. Thus, $n \geq 3k + 4$. Note that $f(s)$ and $h(s)$ are concave down parabolas with vertex at

$$\frac{2n - 2k - 1}{2(2k+1)} \geq \frac{4k+7}{2(2k+1)} > 1 \text{ and } \frac{2n - 2k + 1}{2(2k+1)} \geq \frac{4k+9}{2(2k+1)} > 1,$$

respectively. Thus, $f(s)$ and $h(s)$ achieve the minimum value at $s = 1$ or $s = q - 1$. Therefore, $T(i_s)$ also achieves its minimum value at $s = 1$ or $s = q - 1$, where $T(i_1) = 2n - 2\lceil 2k \rceil$ and $T(i_{q-1}) = (q-1)(2n - 2\lceil kq \rceil - q + 2)$. Therefore, if $q \geq 3$,

$$\begin{aligned} T_{\min}(i) &= \min\{T(i_1), T(i_{q-1}), T(p)\} \\ &= \min\{2n - 2\lceil 2k \rceil, (q-1)(2n - 2\lceil kq \rceil - q + 2), T(p)\}. \end{aligned}$$

In the rest, we divide into two cases as follows.

Case 1. $0 \leq r < b$.

In this case, $q = \left\lfloor \frac{n}{k+1} \right\rfloor - 1$ and $p = n - q - 2$, it implies that

$$\begin{aligned} T(i_{q-1}) - T(i_1) &= (q-1)(2n - 2\lceil kq \rceil - q + 2) - (2n - 2\lceil 2k \rceil) \\ &> (q-1)(2n - 2(kq+1) - q + 2) - 2n + 4k \\ &= (q-2)[2n - (2k+1)(q+1)] - 2 \geq 2n - \left\lfloor \frac{n}{k+1} \right\rfloor (2k+1) - 2 \\ &\geq 2n - \frac{n}{k+1}(2k+1) - 2 = \frac{n}{k+1} - 2 \geq 0 \end{aligned}$$

if $n \geq 2(k+1)$, and

$$\begin{aligned} T(i_1) - T(p) &= 2n - 2\lceil 2k \rceil - q(2n - 2p - q - 1) \\ &= 2n - 2\lceil 2k \rceil - q(q+3) \\ &= 2n - 2\lceil 2k \rceil - \left(\left\lfloor \frac{n}{k+1} \right\rfloor - 1 \right) \left(\left\lfloor \frac{n}{k+1} \right\rfloor + 2 \right) \\ &> 2n - 2(2k+1) - \left(\frac{n}{k+1} - 1 \right) \left(\frac{n}{k+1} + 2 \right) \\ &= \frac{-n^2 + (2k^2 + 3k + 1)n - 4k^3 - 8k^2 - 4k}{(k+1)^2} \geq 0 \end{aligned}$$

if $n \leq \frac{(2k^2+3k+1)+(k+1)\sqrt{4k^2-12k+1}}{2}$. Note that $4k^2 - 12k + 1 - (2k-5)^2 = 8k - 24 \geq 0$ if $k \geq 3$, it implies that if $k \geq 3$ and $3k + 4 \leq n \leq \frac{(2k^2+3k+1)+(k+1)(2k-5)}{2}$, i.e., $3k + 4 \leq n \leq 2k^2 - 2$, then $T(i_1) - T(p) > 0$. Thus, $T_{\min}(i) = T(p)$.

Case 2. $b \leq r \leq a + b - 1$.

In this case, $q = \left\lfloor \frac{n}{k+1} \right\rfloor$ and $p = n - q - 1$. It implies that

$$\begin{aligned} T(i_{q-1}) - T(p) &= (q-1)(2n - 2\lceil kq \rceil - q + 2) - q(2n - 2p - q - 1) \\ &= (q-1)(2n - 2\lceil kq \rceil - q + 2) - q(q+1) \\ &= 2(q-1)(n - \lceil kq \rceil + 1) \\ &> 4(n - (kq+1) + 1) \\ &= 4(n - kq) > 0 \end{aligned}$$

as $\frac{n}{k} > \frac{n+1}{k+1} > \frac{n}{k+1} \geq \left\lfloor \frac{n}{k+1} \right\rfloor$. Furthermore,

$$\begin{aligned} T(i_1) - T(p) &= 2n - 2\lceil 2k \rceil - q(2n - 2p - q - 1) \\ &= 2n - 2\lceil 2k \rceil - q(q+1) \\ &> 2n - 2(2k+1) - \frac{n}{k+1}\left(\frac{n}{k+1} + 1 \right) \\ &= \frac{-n^2 + (2k^2 + 3k + 1)n - 4k^3 - 10k^2 - 8k - 2}{(k+1)^2} \geq 0 \end{aligned}$$

if $n \leq \frac{2k^2+3k+1+(k+1)\sqrt{(2k-7)(2k+1)}}{2}$. Meanwhile, by simple computation, if $k \geq 4$, then $(2k-7)(2k+1) \geq (2k-5)^2$. Therefore, if $n \leq 2k^2 - 2$, then $n \leq \frac{2k^2+3k+1+(k+1)\sqrt{(2k-7)(2k+1)}}{2}$ and $T(i_1) - T(p) \geq 0$. Thus, $T_{\min}(i) = T(p)$.

As a conclusion, we have the following result.

Theorem 4. *Let $k = \frac{a}{b} \geq 1$ and $n \geq \lceil k \rceil + 2$, where a and b are integers such that $a \geq b \geq 1$. Let G be a graph of order n. Let $p = \left\lceil \frac{kn}{k+1} \right\rceil - 1$, $q = \lfloor \frac{p}{k} \rfloor$, and r be the remainder when bn is divided by $a + b$.*

1. *If $q = 1$, then*
   (a) *If G has no isolated vertices, and $\lambda(G) \geq \sqrt{n^2 - 4n + 2\lceil 2k \rceil + 1}$, then G is k-tough.*
   (b) *If $\lambda(\overline{G}) < \sqrt{\frac{(n-\lceil 2k \rceil)(2n-\lceil \frac{2kn}{k+1} \rceil -1)}{n}}$, then G is k-tough.*
2. *If $q = 2$, $0 \leq r \leq b$, and $k \geq 3$; or $q \geq 3$, $0 \leq r < b$, $k \geq 3$ and $3k + 4 \leq n \leq 2k^2 - 2$, then*
   (a) *If G has no isolated vertices, and $\lambda(G) \geq \sqrt{(n-1)^2 - \left(\left\lfloor \frac{n}{k+1} \right\rfloor - 1\right)\left(\left\lfloor \frac{n}{k+1} \right\rfloor + 2\right)}$, then G is k-tough.*
   (b) *If $\lambda(\overline{G}) \leq \sqrt{\frac{(\lfloor \frac{n}{k+1} \rfloor -1)(\lfloor \frac{n}{k+1} \rfloor +2)(2n-\lceil \frac{2kn}{k+1} \rceil -1)}{2n}}$, then G is k-tough.*
3. *If $q = 2$, $b \leq r \leq a + b - 1$, $n \geq 3k + 2$, and $k \geq 2$; or $q \geq 3$, $b \leq r \leq a + b - 1$, $k \geq 4$, and $n \leq 2k^2 - 2$, then*
   (a) *If G has no isolated vertices, and $\lambda(G) \geq \sqrt{(n-1)^2 - \left\lfloor \frac{n}{k+1} \right\rfloor \left(\left\lfloor \frac{n}{k+1} \right\rfloor + 1\right)}$, then G is k-tough.*
   (b) *If $\lambda(\overline{G}) \leq \sqrt{\frac{\lfloor \frac{n}{k+1} \rfloor (\lfloor \frac{n}{k+1} \rfloor +1)(2n-\lceil \frac{2kn}{k+1} \rceil -1)}{2n}}$, then G is k-tough.*

Proof.

(I). First of all, we prove the claim as follows.

Claim 1. Let G be a graph of order $n \geq \lceil k \rceil + 2$. If $e(G) \geq \frac{n(n-3)}{2} + \lceil 2k \rceil$, then G is k-tough, unless $e(G) = \frac{n(n-3)}{2} + \lceil 2k \rceil$ and $G = K_{\lceil 2k \rceil - 1} \vee \left(K_1 + K_{n - \lceil 2k \rceil}\right)$.

Proof of Claim 1. Suppose that G is not k-tough, according to the discussion in Situation 1, $T_{\min} = 2n - 2p - 2$. Since $q = 1$, we have $p < 2k$. According to Lemma 7, it follows that $e(G) \leq \frac{n(n-1)}{2} - (n - p - 1) = \frac{n(n-3)}{2} + p + 1$. Therefore,

$$e(G) \leq \frac{n(n-3)}{2} + \lceil 2k \rceil.$$

Since $e(G) = \frac{n(n-3)}{2} + \lceil 2k \rceil$, we have $i = p = \lceil 2k \rceil - 1$ and all the inequalities in Equation (8) must be equalities. Thus, $d_1 = \lceil 2k \rceil - 1$, $d_2 = \cdots = d_{n-\lceil 2k \rceil + 1} = n - 2$ and $d_{n-\lceil 2k \rceil + 2} = \cdots = d_n = n - 1$. It follows that $G = K_{\lceil 2k \rceil - 1} \vee \left(K_1 + K_{n-\lceil 2k \rceil}\right)$. Let $S = V(K_{\lceil 2k \rceil - 1})$. Then

$$\tau(G) = \frac{|S|}{\omega(G-S)} = \frac{\lceil 2k \rceil - 1}{2} < \frac{2k}{2} = k,$$

i.e., G is not k-tough. □

(a). With Lemma 4 and the fact that G has no isolated vertices, we get

$$\sqrt{n^2 - 4n + 2\lceil 2k \rceil + 1} \leq \lambda(G) \leq \sqrt{2e(G) - n + 1},$$

which yields

$$e(G) \geq \frac{n(n-3)}{2} + \lceil 2k \rceil.$$

Because $K_{\lceil 2k \rceil - 1} \vee \left(K_1 + K_{n-\lceil 2k \rceil}\right)$ cannot achieve equality in the inequality (2), the aforementioned claim and Lemma 4 lead to the conclusion that $e(G) > \frac{n(n-3)}{2} + \lceil 2k \rceil$, and G is k-tough.

(b). Assume that G is not k-tough. By Lemma 2, we consider the closure $H := \mathrm{cl}_{\lceil \frac{2kn}{k+1} \rceil}(G)$. According to Lemma 3, H is not k-tough and $H \neq K_n$. Thus for any two non-adjacent vertices u and v in H, we have $d_H(u) + d_H(v) \leq \left\lceil \frac{2kn}{k+1} \right\rceil - 1$. Therefore, by Lemma 5 and the assumption, it follows that

$$e(H) \geq \binom{n}{2} - \frac{n\lambda^2(\overline{G})}{2n - \left\lceil \frac{2kn}{k+1} \right\rceil - 1} \geq \frac{n(n-3)}{2} + \lceil 2k \rceil.$$

Since H is not k-tough, according to the claim, we have $H = K_{\lceil 2k \rceil - 1} \vee \left(K_1 + K_{n-\lceil 2k \rceil}\right)$. Moreover, because $G \subseteq H$, we can see that $\overline{G}$ contains the star $K_{1,n-\lceil 2k \rceil}$. With Lemma 6, we have

$$\lambda(\overline{G}) \geq \lambda(K_{1,n-\lceil 2k \rceil}) = \sqrt{n - \lceil 2k \rceil} > \sqrt{\frac{(n - \lceil 2k \rceil)(2n - \left\lceil \frac{2kn}{k+1} \right\rceil - 1)}{n}},$$

because of $k \geq 1$ and $2n - \left\lceil \frac{2kn}{k+1} \right\rceil = \left\lfloor \frac{2n}{k+1} \right\rfloor < \frac{2n}{k+1} \leq n$.

(II). When $q = 2$ and $0 \leq r \leq b$, we first prove the claim as follows.

Claim 2. Let $q = 2$, $0 \leq r \leq b$, and $k \geq 3$. If $e(G) \geq \binom{n}{2} - \frac{1}{2}\left(\left\lfloor \frac{n}{k+1} \right\rfloor - 1\right)\left(\left\lfloor \frac{n}{k+1} \right\rfloor + 2\right)$, then G is k-tough, unless $e(G) = \binom{n}{2} - \frac{1}{2}\left(\left\lfloor \frac{n}{k+1} \right\rfloor - 1\right)\left(\left\lfloor \frac{n}{k+1} \right\rfloor + 2\right)$ and $G = K_{n-\lfloor \frac{n}{k+1} \rfloor - 1} \vee \left(K_2 + E_{\lfloor \frac{n}{k+1} \rfloor - 1}\right)$.

Proof of Claim 2. Suppose that G is not k-tough, according to the discussion in Situation 2, $T_{\min}(i) = T(p)$. Since $q = 2$ and $0 \leq r < b$, then we have $T(p) = \left(\left\lfloor \frac{n}{k+1} \right\rfloor - 1\right)\left(\left\lfloor \frac{n}{k+1} \right\rfloor + 2\right)$. It follows that $e(G) \leq \binom{n}{2} - \frac{1}{2}\left(\left\lfloor \frac{n}{k+1} \right\rfloor - 1\right)\left(\left\lfloor \frac{n}{k+1} \right\rfloor + 2\right)$. Hence, $e(G) = \binom{n}{2} - \frac{1}{2}\left(\left\lfloor \frac{n}{k+1} \right\rfloor - 1\right)\left(\left\lfloor \frac{n}{k+1} \right\rfloor + 2\right)$, and from the proof of Lemma 7, we have $i = p = n - \left\lfloor \frac{n}{k+1} \right\rfloor - 1$, $q = \left\lfloor \frac{i}{k} \right\rfloor = \left\lfloor \frac{n}{k+1} \right\rfloor - 1$. Thus, $d_1 = \cdots = d_{\lfloor \frac{n}{k+1} \rfloor - 1} = n - \left\lfloor \frac{n}{k+1} \right\rfloor - 1$, $d_{\lfloor \frac{n}{k+1} \rfloor} = d_{\lfloor \frac{n}{k+1} \rfloor + 1} = n - \left\lfloor \frac{n}{k+1} \right\rfloor$, $d_{\lfloor \frac{n}{k+1} \rfloor + 2} = \cdots = d_n = n - 1$. Therefore, $G = K_{n-\lfloor \frac{n}{k+1} \rfloor - 1} \vee \left(K_2 + E_{\lfloor \frac{n}{k+1} \rfloor - 1}\right)$. Let $S = V(K_{n-\lfloor \frac{n}{k+1} \rfloor - 1})$. Then,

$$\tau(G) = \frac{|S|}{\omega(G-S)} = \frac{n - \left\lfloor \frac{n}{k+1} \right\rfloor - 1}{\left\lfloor \frac{n}{k+1} \right\rfloor + 1} < \frac{n - \frac{n}{k+1}}{\frac{n}{k+1}} = k,$$

i.e., G is not k-tough. $\square$

(a). With Lemma 4 and the fact that G has no isolated vertices, we get

$$\sqrt{(n-1)^2 - \left(\left\lfloor \frac{n}{k+1} \right\rfloor - 1\right)\left(\left\lfloor \frac{n}{k+1} \right\rfloor + 2\right)} \leq \lambda(G) \leq \sqrt{2e(G) - n + 1},$$

which yields

$$e(G) \geq \binom{n}{2} - \frac{1}{2}\left(\left\lfloor \frac{n}{k+1} \right\rfloor - 1\right)\left(\left\lfloor \frac{n}{k+1} \right\rfloor + 2\right).$$

Because $K_{n-\lfloor \frac{n}{k+1} \rfloor - 1} \vee \left(K_2 + E_{\lfloor \frac{n}{k+1} \rfloor - 1}\right)$ cannot achieve equality in the inequality (2), the aforementioned claim and Lemma 4 lead to $e(G) > \binom{n}{2} - \frac{1}{2}\left(\left\lfloor \frac{n}{k+1} \right\rfloor - 1\right)\left(\left\lfloor \frac{n}{k+1} \right\rfloor + 2\right)$, and G is k-tough.

(b). Assume that G is not k-tough. By Lemma 2, we consider the closure $H := \mathrm{cl}_{\lceil \frac{2kn}{k+1} \rceil}(G)$. According to Lemma 3, H is not k-tough and $H \neq K_n$. Thus, for any two non-adjacent vertices u and v in H, we have $d_H(u) + d_H(v) \leq \left\lceil \frac{2kn}{k+1} \right\rceil - 1$. Therefore, by Lemma 5 and the assumption, it implies

$$e(H) \geq \binom{n}{2} - \frac{n\lambda^2(\overline{G})}{2n - \left\lceil \frac{2kn}{k+1} \right\rceil - 1} \geq \binom{n}{2} - \frac{1}{2}\left(\left\lfloor \frac{n}{k+1} \right\rfloor - 1\right)\left(\left\lfloor \frac{n}{k+1} \right\rfloor + 2\right).$$

Since H is not k-tough, according to the claim, we have $H = K_{n-\lfloor \frac{n}{k+1} \rfloor - 1} \vee \left(K_2 + E_{\lfloor \frac{n}{k+1} \rfloor - 1}\right)$. Moreover, because $G \subseteq H$, we can see that $\overline{G}$ contains $E_2 \vee K_{\lfloor \frac{n}{k+1} \rfloor - 1}$. With Lemma 6, we have

$$\begin{aligned}\lambda(\overline{G}) \geq \lambda(\overline{K}_2 + K_{\lfloor \frac{n}{k+1} \rfloor - 1}) &= \frac{\left\lfloor \frac{n}{k+1} \right\rfloor - 2 + \sqrt{\left\lfloor \frac{n}{k+1} \right\rfloor^2 + 4\left\lfloor \frac{n}{k+1} \right\rfloor - 4}}{2} \\ &> \sqrt{\frac{\left(\left\lfloor \frac{n}{k+1} \right\rfloor - 1\right)\left(\left\lfloor \frac{n}{k+1} \right\rfloor + 2\right)\left(2n - \left\lceil \frac{2kn}{k+1} \right\rceil - 1\right)}{2n}}\end{aligned}$$

as $\frac{2n - \lceil \frac{2kn}{k+1} \rceil - 1}{2n} < \frac{2n - \frac{2kn}{k+1}}{2n} = \frac{1}{k+1}$ and $k \geq 3$.

If $q \geq 3$, $0 \leq r < b$, $k \geq 3$, and $3k + 4 \leq n \leq 2k^2 - 2$, then we can see that $T_{\min}(i) = T(p)$ according to Situation 3, Case 1. Thus, the proof is the same as the case of $q = 2$ above in (II).

(III). When $q = 2$ and $b \leq r \leq a + b - 1$. We first prove the claim as follows.

Claim 3. Let $q = 2$, $b \leq r \leq a + b - 1$, $n \geq 3k + 2$ and $k \geq 2$. If $e(G) \geq \binom{n}{2} - \frac{1}{2}\left\lfloor \frac{n}{k+1} \right\rfloor\left(\left\lfloor \frac{n}{k+1} \right\rfloor + 1\right)$, then G is k-tough, unless $e(G) = \binom{n}{2} - \frac{1}{2}\left\lfloor \frac{n}{k+1} \right\rfloor\left(\left\lfloor \frac{n}{k+1} \right\rfloor + 1\right)$ and $G = K_{n-\lfloor \frac{n}{k+1} \rfloor - 1} \vee E_{\lfloor \frac{n}{k+1} \rfloor + 1}$.

Proof of Claim 3. Suppose that G is not k-tough, according to the discussion in Situation 2, $T_{\min}(i) = T(p)$. Since $q = 2$ and $b \leq r \leq a + b - 1$, then we have $T_{\min}(i) = \left\lfloor \frac{n}{k+1} \right\rfloor\left(\left\lfloor \frac{n}{k+1} \right\rfloor + 1\right)$. It follows that $e(G) \leq \binom{n}{2} - \frac{1}{2}\left\lfloor \frac{n}{k+1} \right\rfloor\left(\left\lfloor \frac{n}{k+1} \right\rfloor + 1\right)$. Hence $e(G) = \binom{n}{2} - \frac{1}{2}\left\lfloor \frac{n}{k+1} \right\rfloor\left(\left\lfloor \frac{n}{k+1} \right\rfloor + 1\right)$, and from the proof of Lemma 7, we have $i = p = n - \left\lfloor \frac{n}{k+1} \right\rfloor - 1$, $q = \left\lfloor \frac{i}{k} \right\rfloor = \left\lfloor \frac{n}{k+1} \right\rfloor$. Thus, $d_1 = \cdots = d_{\lfloor \frac{n}{k+1} \rfloor + 1} = n - \left\lfloor \frac{n}{k+1} \right\rfloor - 1$, $d_{\lfloor \frac{n}{k+1} \rfloor + 2} = \cdots = d_n = n - 1$. Therefore, $G = K_{n-\lfloor \frac{n}{k+1} \rfloor - 1} \vee E_{\lfloor \frac{n}{k+1} \rfloor + 1}$. Let $S = V(K_{n-\lfloor \frac{n}{k+1} \rfloor - 1})$. Then,

$$\tau(G) = \frac{|S|}{\omega(G - S)} = \frac{n - \left\lfloor \frac{n}{k+1} \right\rfloor - 1}{\left\lfloor \frac{n}{k+1} \right\rfloor + 1} < \frac{n - \frac{n}{k+1}}{\frac{n}{k+1}} = k,$$

i.e., G is not k-tough. □

(a). With Lemma 4 and the fact that G has no isolated vertices, we get

$$\sqrt{(n-1)^2 - \left\lfloor \frac{n}{k+1} \right\rfloor(\left\lfloor \frac{n}{k+1} \right\rfloor + 1)} \leq \lambda(G) \leq \sqrt{2e(G) - n + 1},$$

which yields

$$e(G) \geq \binom{n}{2} - \frac{1}{2}\left\lfloor \frac{n}{k+1} \right\rfloor\left(\left\lfloor \frac{n}{k+1} \right\rfloor + 1\right).$$

Because $K_{n-\lfloor \frac{n}{k+1} \rfloor -1} \vee E_{\lfloor \frac{n}{k+1} \rfloor +1}$ cannot achieve equality in the inequality (2), the aforementioned claim and Lemma 4 lead to the conclusion that $e(G) > \binom{n}{2} - \frac{1}{2}\left\lfloor \frac{n}{k+1} \right\rfloor \left(\left\lfloor \frac{n}{k+1} \right\rfloor + 1 \right)$, and G is k-tough.

(b). Assume that G is not k-tough. By Lemma 2, we consider the closure $H := \mathrm{cl}_{\lceil \frac{2kn}{k+1} \rceil}(G)$. According to Lemma 3, H is not k-tough and $H \neq K_n$. Thus for any two non-adjacent vertices u and v in H, we have $d_H(u) + d_H(v) \leq \left\lceil \frac{2kn}{k+1} \right\rceil - 1$. Therefore, by Lemma 5 and the assumption, it implies

$$e(H) \geq \binom{n}{2} - \frac{n\lambda^2(\overline{G})}{2n - \left\lceil \frac{2kn}{k+1} \right\rceil - 1} \geq \binom{n}{2} - \frac{1}{2}\left\lfloor \frac{n}{k+1} \right\rfloor \left(\left\lfloor \frac{n}{k+1} \right\rfloor + 1 \right).$$

Since H is not k-tough, according to the claim, we have $H = K_{n-\lfloor \frac{n}{k+1} \rfloor -1} \vee E_{\lfloor \frac{n}{k+1} \rfloor +1}$. Moreover, because $G \subseteq H$, we can see that $\overline{G}$ contains $K_{\lfloor \frac{n}{k+1} \rfloor +1}$. With Lemma 6, we have

$$\lambda(\overline{G}) \geq \lambda(K_{\lfloor \frac{n}{k+1} \rfloor +1}) = \left\lfloor \frac{n}{k+1} \right\rfloor > \sqrt{\frac{\left\lfloor \frac{n}{k+1} \right\rfloor \left(\left\lfloor \frac{n}{k+1} \right\rfloor + 1 \right) \left(2n - \left\lceil \frac{2kn}{k+1} \right\rceil - 1 \right)}{2n}}$$

as $\frac{2n - \lceil \frac{2kn}{k+1} \rceil - 1}{2n} < \frac{2n - \frac{2kn}{k+1}}{2n} = \frac{1}{k+1}$ and $k \geq 2$.

If $q \geq 3$, $b \leq r \leq a + b - 1$, $k \geq 4$, and $n \leq 2k^2 - 2$, then we can see that $T_{\min}(i) = T(p)$ according to Situation 3, Case 2. Thus, the proof is the same as that for the case of $q = 2$ above in (III).

□

4. Conclusions

In this paper, concerning the spectral radius, for some positive k, we show sufficient conditions for a graph to be k-integral, k-tenacious, k-binding, and k-tough, respectively. These properties are important for network vulnerability and stability. In the future, concerning the spectral radius, we may investigate some other parameters that have close relationships with network vulnerability and stability.

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