

Article

Asymptotic Analysis for One-Stage Stochastic Linear Complementarity Problems and Applications

Shuang Lin ¹, Jie Zhang ^{2,*}  and Chen Qiu ²¹ Department of Basic Courses Teaching, Dalian Polytechnic University, Dalian 116034, China² School of Mathematics, Liaoning Normal University, Dalian 116029, China

* Correspondence: jie_zhang@lnnu.edu.cn

Abstract: One-stage stochastic linear complementarity problem (SLCP) is a special case of a multi-stage stochastic linear complementarity problem, which has important applications in economic engineering and operations management. In this paper, we establish asymptotic analysis results of a sample-average approximation (SAA) estimator for the SLCP. The asymptotic normality analysis results for the stochastic-constrained optimization problem are extended to the SLCP model and then the conditions, which ensure the convergence in distribution of the sample-average approximation estimator for the SLCP to multivariate normal with zero mean vector and a covariance matrix, are obtained. The results obtained are finally applied for estimating the confidence region of a solution for the SLCP.

Keywords: stochastic linear complementarity problem; asymptotic analysis; sample-average approximation; convergence in distribution; confidence regions

MSC: 90C30; 49M37



Citation: Lin, S.; Zhang, J.; Qiu, C. Asymptotic Analysis for One-Stage Stochastic Linear Complementarity Problems and Applications. *Mathematics* **2023**, *11*, 482. <https://doi.org/10.3390/math11020482>

Academic Editor: Alexander Grigoriev

Received: 3 December 2022

Revised: 8 January 2023

Accepted: 13 January 2023

Published: 16 January 2023



Copyright: © 2023 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<https://creativecommons.org/licenses/by/4.0/>).

1. Introduction

The finite-dimensional complementarity problems form a relatively perfect and fruitful topic in mathematical programming. In order to reflect the uncertain factors in practice, stochastic complementarity problems have attracted extensive attention in the recent literature [1–4]. We investigate the following one-stage stochastic linear complementarity problem (SLCP): find $x \in \mathbb{R}^n$, such that

$$\begin{aligned} \mathbb{E}[M(\xi(w))]x + \mathbb{E}[q(\xi(w))] &\geq 0, x \geq 0, \\ (\mathbb{E}[M(\xi(w))]x + \mathbb{E}[q(\xi(w))])^T x &= 0, \end{aligned} \quad (1)$$

where $\mathbb{E}$ denotes the mathematical expectation, $\xi : \Omega \rightarrow \Xi \subset \mathbb{R}^k$ is a random vector defined on a probability space $(\Omega, \mathcal{F}, P)$, and $M : \mathbb{R}^k \rightarrow \mathbb{R}^{n \times n}$ and $q : \mathbb{R}^k \rightarrow \mathbb{R}^n$ are functions. Throughout the paper, $M(\xi(\omega))$ and $q(\xi(\omega))$ are measurable functions of ω , and the condition

$$\mathbb{E}[\|M(\xi(\omega))\|^2 + \|q(\xi(\omega))\|^2] < \infty$$

holds. To ease the notation, $\xi(\omega)$ will be written as ξ .

Problem (1) can be seen as a special case of the stochastic nonlinear complementarity problem (SNCP) in Gürkan et al. [5]. Some examples of stochastic complementarity problems in operational research, finance, economics, and engineering, can be found in [6,7].

Accurately calculating the expected value in (1) is either impossible or costly. The sample-average approximation (SAA) method [8–10] is considered to be an effective method for estimating the expected value. By generating an independent identically distributed (iid) sample $\xi^1, \dots, \xi^N$ of ξ and approximating the expected value with sample average,

the SAA method can effectively estimate the expectations. Throughout the paper, the SLCP (1) will be approximated by

$$\begin{aligned}\widehat{M}_N x + \widehat{q}_N &\geq 0, x \geq 0, \\ \left(\widehat{M}_N x + \widehat{q}_N\right)^T x &= 0,\end{aligned}\quad (2)$$

where

$$\widehat{M}_N x + \widehat{q}_N = \frac{1}{N} \sum_{j=1}^N \left[M(\xi^j) x + q(\xi^j) \right] \quad (3)$$

is the sample-average mapping of $\mathbb{E}[M(\xi(w))]x + \mathbb{E}[q(\xi(w))]$. Equation (1) is called the true problem, and Equation (2) is the SAA problem to Equation (1).

In this paper, for an SAA solution x_N and its almost sure cluster point x^* , we are interested in the asymptotic behavior of the SAA estimator $\sqrt{N}(x_N - x^*)$, that is, establishing conditions on $\mathbb{E}[M(\xi(w))]$ such that for $N \rightarrow \infty$,

$$\sqrt{N}(x_N - x^*) \xrightarrow{\mathcal{D}} Z,$$

where Z in $\mathbb{R}^n$ is a normal random variable, and the symbol $\mathcal{D}$ above the arrow denotes a convergence in the distribution.

The asymptotic behavior of the SAA estimator has been deeply discussed in plenty of the literature [6,8,11,12] and other references. Most of those results are related to convergence in the distribution of the SAA estimators for stochastic-constrained optimizations or normal map formulations of stochastic variational inequalities. In this paper, we first study the asymptotic behavior of the SAA solutions of stochastic quadratic programming, and then obtain the conditions ensuring the asymptotical normality of the SAA estimators of the SLCP. Finally, methods for estimating confidence regions of true solutions to the SLCP are provided.

The paper proceeds as follows: in Section 2, the asymptotic results are obtained under the nonsingularity condition or positive definiteness condition of $\mathbb{E}[M(\xi(w))]$. We then applied the results in Section 3 to obtain the confidence intervals of an SLCP solution.

2. Main Results

Notice that if the solution of an SLCP (1) exists, then the SLCP (1) is equivalent to the following quadratic programming:

$$\begin{aligned}\min_{x \in \mathbb{R}^n} \quad & x^T \mathbb{E}[M(\xi)]x + \mathbb{E}[q(\xi)]^T x \\ \text{s.t.} \quad & \mathbb{E}[M(\xi)]x + \mathbb{E}[q(\xi)] \geq 0, x \geq 0.\end{aligned}\quad (4)$$

In order to provide the asymptotic behavior of the SAA SLCP, we first consider the following stochastic-constrained optimization (SCO) problem:

$$\begin{aligned}\min_x \quad & \mathbb{E}[f(x, \xi(w))] \\ \text{s.t.} \quad & \mathbb{E}[g_i(x, \xi(w))] \leq 0, i = 1, 2, \dots, p, \\ & \mathbb{E}[h_j(x, \xi(w))] = 0, j = 1, 2, \dots, q,\end{aligned}\quad (5)$$

where ξ is defined as in Equation (1), $f : \mathbb{R}^n \times \mathbb{R}^k \rightarrow \mathbb{R}$, $g_i : \mathbb{R}^n \times \mathbb{R}^k \rightarrow \mathbb{R}$, $i = 1, 2, \dots, p$, $h_j : \mathbb{R}^n \times \mathbb{R}^k \rightarrow \mathbb{R}$, $j = 1, 2, \dots, q$ are random functions.

The definition of the Karash–Kuhn–Tucker (KKT) point of the problem (5) is as follows:

Definition 1. Suppose the expectation value functions $\mathbb{E}[f(\cdot, \xi(w))], \mathbb{E}[g(\cdot, \xi(w))], \mathbb{E}[h(\cdot, \xi(w))]$ are continuously differentiable. Then $(x^*, \lambda^*, \mu^*) \in \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^q$ is called the Karash–Kuhn–Tucker (KKT) point of the problem (5) if (x^*, λ^*, μ^*) satisfies

$$0 \in \nabla_x \mathbb{E}[f(x^*, \xi(w))] + \mathcal{J}_x \mathbb{E}[g(x^*, \xi(w))]^T \lambda^* + \mathcal{J}_x \mathbb{E}[h(x^*, \xi(w))]^T \mu^*,$$

$$\lambda_i^* \geq 0, \lambda_i^* \mathbb{E}[g_i(x^*, \xi(w))] = 0, i = 1, 2, \dots, p$$

$$\mathbb{E}[g_i(x^*, \xi(w))] \leq 0, i = 1, 2, \dots, p$$

$$\mathbb{E}[h_j(x^*, \xi(w))] = 0, j = 1, 2, \dots, q,$$

where

$$g(x, \xi(w)) = (g_1(x, \xi(w)), \dots, g_p(x, \xi(w)))^T$$

and

$$h(x, \xi(w)) = (h_1(x, \xi(w)), \dots, h_q(x, \xi(w)))^T.$$

Let $\xi^1, \xi^2, \dots, \xi^N$ be an iid sample of ξ , then the SAA problem of SCO is

$$\begin{aligned} \min_x \quad & \hat{f}_N(x) \\ \text{s.t.} \quad & \hat{g}_i^N(x) \leq 0, i = 1, 2, \dots, p, \\ & \hat{h}_j^N(x) = 0, j = 1, 2, \dots, q. \end{aligned} \quad (6)$$

where

$$\hat{f}_N(x) = \frac{1}{N} \sum_{k=1}^N f(x, \xi^k), \quad \hat{g}_i^N(x) = \frac{1}{N} \sum_{k=1}^N g_i(x, \xi^k), i = 1, 2, \dots, p,$$

$$\hat{h}_j^N(x) = \frac{1}{N} \sum_{k=1}^N h_j(x, \xi^k), j = 1, 2, \dots, q.$$

We make the following assumptions for the future asymptotic analysis. Let X be a nonempty compact subset of $\mathbb{R}^n$, and let $\sigma(x, \xi(w))$ be one of the elements in the following:

$$\{f(\cdot, \xi(w)), \nabla f(\cdot, \xi(w)), \nabla^2 f(\cdot, \xi(w)), \nabla g_i(\cdot, \xi(w)), \nabla^2 g_i(\cdot, \xi(w)), i = 1, 2, \dots, p, \nabla h_j(\cdot, \xi(w)), \nabla^2 h_j(\cdot, \xi(w)), j = 1, 2, \dots, q\}.$$

Consider the following conditions.

- (A1) For each $x \in X$, $\mathbb{E}[\|\sigma(x, \xi(w))\|]$ is finite valued, and $\mathbb{E}[\sigma(x, \xi(w))]$ is well-defined.
 (A2) There exists a positive valued random variable $C(w)$ such that

$$\mathbb{E}[C(w)] < +\infty,$$

and for all $x_1, x_2 \in X$, and almost every $w \in \Omega$, the following inequality holds:

$$\|\sigma(x_1, \xi(w)) - \sigma(x_2, \xi(w))\| \leq C(w) \|x_1 - x_2\|.$$

- (A3) For any fixed $x \in X$,

$$f(\cdot, \xi(w)), g_i(\cdot, \xi(w)), i = 1, 2, \dots, p,$$

$$h_j(\cdot, \xi(w)), j = 1, 2, \dots, q,$$

is twice continuously differentiable at x for almost every $w \in \Omega$.

The above assumptions are commonly used in stochastic optimization. Using Theorem 6.3.2 and Theorem 6.3.6 in [8],

$$\mathbb{E}[f(\cdot, \xi(w))], \mathbb{E}[g_i(\cdot, \xi(w))], \mathbb{E}[h_j(\cdot, \xi(w))], i = 1, 2, \dots, p, j = 1, 2, \dots, q$$

is twice continuously differentiable on X , and

$$\limsup_{N \rightarrow \infty, x \in X} \left\| \frac{1}{N} \sum_{k=1}^N \sigma(x, \xi_k) - \mathbb{E}[\sigma(\cdot, \xi(w))] \right\| = 0 \quad \text{w.p.1.}$$

Let $(x_0, \lambda_0, \mu_0) \in \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^q$. We need the following conditions:

(A4) The second order sufficient condition (SOSC) holds at (x_0, λ_0, μ_0) , i.e.,

$$d^T \nabla_x^2 \mathbb{E}[L(x_0, \lambda_0, \mu_0, \xi(w))] d > 0,$$

for every nonzero vector d satisfying

$$\nabla_x \mathbb{E}[g_i(x_0, \xi(w))]^T d \leq 0, i \in I(x_0), \nabla_x \mathbb{E}[h_j(x_0, \xi(w))]^T d = 0, j = 1, 2, \dots, q,$$

where

$$I(x_0) = \{i = 1, 2, \dots, p : \mathbb{E}[g_i(x_0, \xi)] = 0\}$$

$$L(x, \lambda, \mu, \xi(w)) = f(x, \xi) + g(x, \xi)^T \lambda + h(x, \xi)^T \mu.$$

(A5) The Mangasarian–Fromovitz constraint qualification (MFCQ) holds at x_0 , i.e.,

$$\nabla_x \mathbb{E}[h_j(x_0, \xi(w))], j = 1, 2, \dots, q$$

is linear independent and there exists d satisfying

$$\nabla_x \mathbb{E}[g_i(x_0, \xi(w))]^T d \leq 0, i \in I(x_0), \nabla_x \mathbb{E}[h_j(x_0, \xi(w))]^T d = 0, j = 1, 2, \dots, q.$$

(A6) The linear independent constraint qualification (LICQ) holds at x_0 , i.e.,

$$\nabla_x \mathbb{E}[g_i(x_0, \xi(w))]^T d, i \in I(x_0), \nabla_x \mathbb{E}[h_j(x_0, \xi(w))]^T d, j = 1, 2, \dots, q$$

are linearly independent.

(A7) The strict complementarity condition (SCC) holds, i.e.,

$$(\lambda_0)_i > 0, i \in I(x_0) := J.$$

Suppose that $\hat{L}_N(x, \lambda, \mu)$ is the SAA function of $L(x, \lambda, \mu, \xi)$. Then, the following propositions are directly from Theorems 3.1 and 3.2 in [6].

Proposition 1. Let $(x_0, \lambda_0, \mu_0) \in \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^q$ and there exists a compact neighborhood X of x_0 such that condition (A1)–(A3) holds. If (x_0, λ_0, μ_0) is a KKT point for (5) and condition (A4) and (A5) holds, then there exists (x_N, λ_N, μ_N) satisfying the KKT condition of (6) and $(x_N, \lambda_N, \mu_N) \rightarrow (x_0, \lambda_0, \mu_0)$ w.p.1 as $N \rightarrow \infty$.

Proposition 2. Suppose condition (A1)–(A3) hold on X , where X is a compact neighborhood of x_0 . If a sequence of KKT points (x_N, λ_N, μ_N) for (6) converges to (x_0, λ_0, μ_0) almost surely, and (A4), (A6) hold at (x_0, λ_0, μ_0) , then

$$\sqrt{N}[(x_N, (\lambda_N)_J, \mu_N) - (x_0, (\lambda_0)_J, \mu_0)] \xrightarrow{\mathcal{D}} (u, v, w),$$

where $(u, v, w) \in \mathbb{R}^n \times \mathbb{R}^{|J|} \times \mathbb{R}^q$ is the KKT point for the random quadratic programming problem

$$\begin{aligned} \min_{d \in \mathbb{R}^n} \quad & c_1^* d + \frac{1}{2} d^T \nabla_x^2 \mathbb{E}[L(x_0, \lambda_0, \mu_0, \xi)] d \\ \text{s.t.} \quad & [\mathcal{J}_x \mathbb{E}[g(x_0, \xi)] d + c_2^*]_{J_+} = 0, \\ & [\mathcal{J}_x \mathbb{E}[g(x_0, \xi)] d + c_2^*]_{J_0} \leq 0, \\ & \mathcal{J}_x \mathbb{E}[h(x_0, \xi)] d + c_3^* = 0, \end{aligned}$$

where $J_+ = \{i \in J : (\lambda_0)_i > 0\}$, $J_0 = \{i \in J : (\lambda_0)_i = 0\}$, c_1^*, c_2^*, c_3^* satisfy

$$\begin{aligned} \sqrt{N} [\nabla_x \hat{L}_N(x_0, \lambda_0, \mu_0) - \nabla_x \mathbb{E}[L(x_0, \lambda_0, \mu_0, \xi)]] &\xrightarrow{\mathcal{D}} c_1^*, \\ \sqrt{N} [\hat{g}_N(x_0) - \mathbb{E}[g(x_0, \xi)]] &\xrightarrow{\mathcal{D}} c_2^*, \\ \sqrt{N} [\hat{h}_N(x_0) - \mathbb{E}[h(x_0, \xi)]] &\xrightarrow{\mathcal{D}} c_3^*. \end{aligned}$$

Theorem 1. Suppose conditions in Proposition 2 hold. If a sequence of KKT points (x_N, λ_N, μ_N) for (6) converges to (x_0, λ_0, μ_0) almost surely and (A7) hold, then

$$\sqrt{N} [(x_N, (\lambda_N)_J, \mu_N) - (x_0, (\lambda_0)_J, \mu_0)]$$

converges in distribution to a normal with mean 0 and the covariance matrix

$$T = \begin{pmatrix} \Psi & G^T & H^T \\ G & 0 & 0 \\ H & 0 & 0 \end{pmatrix}^{-1} \begin{pmatrix} \Sigma_{11} & \Sigma_{12} & \Sigma_{13} \\ \Sigma_{21} & \Sigma_{22} & \Sigma_{23} \\ \Sigma_{31} & \Sigma_{32} & \Sigma_{33} \end{pmatrix} \begin{pmatrix} \Psi & G^T & H^T \\ G & 0 & 0 \\ H & 0 & 0 \end{pmatrix}^{-1},$$

where

$$\begin{aligned} \Psi &= \nabla_x^2 \mathbb{E}[L(x_0, \lambda_0, \mu_0, \xi)], G = \mathcal{J}_x \mathbb{E}[g(x_0, \xi)]_J, H = \mathcal{J}_x \mathbb{E}[h(x_0, \xi)], \\ \alpha_1(\xi) &= \nabla_x L(x_0, \lambda_0, \mu_0, \xi) - \nabla_x \mathbb{E}[L(x_0, \lambda_0, \mu_0, \xi)], \alpha_2(\xi) = [g(x_0, \xi) - \mathbb{E}[g(x_0, \xi)]]_J, \\ \alpha_3(\xi) &= h(x_0, \xi) - \mathbb{E}[h(x_0, \xi)], \Sigma_{ij} = \mathbb{E}[\alpha_i(\xi) \alpha_j(\xi)^T], i = 1, 2, 3; j = 1, 2, 3. \end{aligned}$$

Proof. By Proposition 2, under condition (A7), we know that

$$\sqrt{N} [(x_N, (\lambda_N)_J, \mu_N) - (x_0, (\lambda_0)_J, \mu_0)] \xrightarrow{\mathcal{D}} (u, v, w),$$

where (u, v, w) is the KKT point for the stochastic quadratic programming

$$\begin{aligned} \min_{d \in \mathbb{R}^n} \quad & c_1^* d + \frac{1}{2} d^T \nabla_x^2 \mathbb{E}[L(x_0, \lambda_0, \mu_0, \xi)] d \\ \text{s.t.} \quad & [\mathcal{J}_x \mathbb{E}[g(x_0, \xi)] d + c_2^*]_J = 0, \mathcal{J}_x \mathbb{E}[h(x_0, \xi)] d + c_3^* = 0. \end{aligned}$$

That is, (u, v, w) satisfies

$$\begin{bmatrix} \nabla_x^2 \mathbb{E}[L(x_0, \lambda_0, \mu_0, \xi)] & \mathcal{J}_x \mathbb{E}[g(x_0, \xi)]_J^T & \mathcal{J}_x \mathbb{E}[h(x_0, \xi)]^T \\ \mathcal{J}_x \mathbb{E}[g(x_0, \xi)]_J & 0 & 0 \\ \mathcal{J}_x \mathbb{E}[h(x_0, \xi)] & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} -c_1^* \\ -(c_2^*)_J \\ -c_{3*} \end{bmatrix}.$$

Notice that under (A4),

$$\begin{bmatrix} \nabla_x^2 \mathbb{E}[L(x_0, \lambda_0, \mu_0, \xi)] & \mathcal{J}_x \mathbb{E}[g(x_0, \xi)]_J^T & \mathcal{J}_x \mathbb{E}[h(x_0, \xi)]^T \\ \mathcal{J}_x \mathbb{E}[g(x_0, \xi)]_J & 0 & 0 \\ \mathcal{J}_x \mathbb{E}[h(x_0, \xi)] & 0 & 0 \end{bmatrix}$$

is nonsingular and

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} \nabla^2 \mathbb{E}[L(x_0, \lambda_0, \mu_0, \xi)] & \mathcal{J}_x \mathbb{E}[g(x_0, \xi)]_J^T & \mathcal{J}_x \mathbb{E}[h(x_0, \xi)]^T \\ \mathcal{J}_x \mathbb{E}[g(x_0, \xi)]_J & 0 & 0 \\ \mathcal{J}_x \mathbb{E}[h(x_0, \xi)] & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} -c_1^* \\ -(c_2^*)_J \\ -c_3^* \end{bmatrix}.$$

Then the conclusion holds. $\square$

Remark 1. Notice that in [8,11] (Section 5.2.2), the asymptotic analysis of the optimal solutions of SAA stochastic-constrained optimization is established when the constraints are independent of random vectors. The conclusions of Theorem 1 extend the results in [8,11] (Section 5.2.2) to stochastic-constrained optimization in which the constraints contain random vectors.

We next apply the results above to a stochastic linear complementarity problem (1). We first provide some conditions below:

(A8) $\mathbb{E}[M(\xi)]$ is a symmetric matrix, and

$$\forall d \in \mathbb{R}^n, d^T \mathbb{E}[M(\xi)]d = 0 \Rightarrow d = 0.$$

(A9) Let x_0 be a solution of SLCP (1); the nondegenerate condition holds at x_0 , i.e.,

$$(x_0)_i + (\mathbb{E}[M(\xi)]x_0 + \mathbb{E}[q(\xi)])_i > 0, \text{ for } i = 1, 2, \dots, n.$$

(A10) $\mathbb{E}[M(\xi)]$ is positive definite.

Notice that if (A10) holds, then (A8) holds. (A9) is a typical condition in the study of complementarity problems.

Let the $\xi^1, \xi^2, \dots, \xi^N$ be the iid samples of ξ . Thus, the SAA problem of (4) is as follows:

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & x^T \hat{M}_N x + x^T \hat{q}_N \\ \text{s.t.} \quad & \hat{M}_N x + \hat{q}_N \geq 0, x \geq 0. \end{aligned} \quad (7)$$

Next, we provide our results.

Theorem 2. Suppose that $\mathbb{E}[M(\xi)]$ and $\mathbb{E}[q(\xi)]$ are well-defined and finite. Let x_0 be a solution of SLCP (1) and $(x_0, \lambda_0, \mu_0) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$ be a KKT point for (4), (A9) holds at x_0 , and (A8) holds. Then

- (i) for N large enough, there exists (x_N, λ_N, μ_N) satisfying the KKT condition of (7), and $(x_N, \lambda_N, \mu_N) \rightarrow (x_0, \lambda_0, \mu_0)$ w.p.1 as $N \rightarrow \infty$.
- (ii) for the (x_N, λ_N, μ_N) in (i),

$$\sqrt{N}[(x_N, (\lambda_N)_J, (\mu_N)_I) - (x_0, (\lambda_0)_J, (\mu_0)_I)]$$

converges in distribution to a normal with 0 and covariance matrix $D^{-1}YD^{-1}$, where

$$D = \begin{pmatrix} 2\mathbb{E}[M(\xi)] & \mathbb{E}[M(\xi)]_J & E_I \\ \mathbb{E}[M(\xi)]_J^T & 0 & 0 \\ E_I^T & 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} Y_{11} & Y_{12} & 0 \\ Y_{21} & Y_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

with

$$J = \{i \in \{1, 2, \dots, n\} : (\mathbb{E}[M(\xi)]x_0 + \mathbb{E}[q(\xi)])_i = 0\}, I = \{i \in \{1, 2, \dots, n\} : (x_0)_i = 0\},$$

E the identity matrix,

$$\beta_1(\xi) = (M(\xi) - \mathbb{E}[M(\xi)])x_0 + q(\xi) - \mathbb{E}[q(\xi)],$$

$$\beta_2(\xi) = (M(\xi)x_0 - \mathbb{E}[M(\xi)]x_0 + q(\xi) - \mathbb{E}[q(\xi)])_J,$$

$$Y_{ij} = \mathbb{E}[\beta_i(\xi)\beta_j(\xi)^T], i = 1, 2; j = 1, 2.$$

Proof. We will only need to verify conditions in Theorem 1. We advance the proof with the following three steps:

Step 1. Notice that x_0 satisfies

$$\mathbb{E}[M(\xi)]x_0 + \mathbb{E}[q(\xi)] \geq 0, x_0 \geq 0,$$

$$(\mathbb{E}[M(\xi)]x_0 + \mathbb{E}[q(\xi)])^T x_0 = 0$$

and (x_0, μ_0, λ_0) satisfies the KKT condition of (4), that is,

$$0 = 2\mathbb{E}[M(\xi)]x_0 + \mathbb{E}[q(\xi)] - \mathbb{E}[M(\xi)]\lambda_0 - \mu_0.$$

$$\mathbb{E}[M(\xi)]x_0 + \mathbb{E}[q(\xi)] \geq 0, \lambda_0 \geq 0.$$

$$(\mathbb{E}[M(\xi)]x_0 + \mathbb{E}[q(\xi)])^T \lambda_0 = 0. \quad (8)$$

$$x_0 \geq 0, \mu_0 \geq 0, x_0^T \mu_0 = 0. \quad (9)$$

Then we have

$$0 = x_0^T (2\mathbb{E}[M(\xi)]x_0 + \mathbb{E}[q(\xi)]^T x_0 - x_0^T \mathbb{E}[M(\xi)]\lambda_0 - x_0^T \mu_0) \quad (10)$$

and

$$0 = 2\lambda_0^T \mathbb{E}[M(\xi)]x_0 + \lambda_0^T \mathbb{E}[q(\xi)] - \lambda_0^T \mathbb{E}[M(\xi)]\lambda_0 - \lambda_0^T \mu_0, \quad (11)$$

which, by (8) and (9), means that

$$0 = x_0^T \mathbb{E}[M(\xi)]x_0 - x_0^T \mathbb{E}[M(\xi)]\lambda_0$$

and

$$0 = \lambda_0^T \mathbb{E}[M(\xi)]x_0 - \lambda_0^T \mathbb{E}[M(\xi)]\lambda_0 - \lambda_0^T \mu_0.$$

Since

$$\lambda_0 \in N_{\mathbb{R}_+^n}(\mathbb{E}[M(\xi)]x_0 + \mathbb{E}[q(\xi)]),$$

$$\mu_0 \in N_{\mathbb{R}_+^n}(x_0)$$

and condition (A9) hold, we have

$$\lambda_0^T \mu_0 = 0.$$

Then combining (10) and (11), we have

$$(x_0 - \lambda_0)^T \mathbb{E}[M(\xi)](x_0 - \lambda_0) = 0,$$

which by condition (A8) means that

$$x_0 = \lambda_0.$$

Consequently, by (8), we obtain

$$\mu_0 = \mathbb{E}[M(\xi)]x_0 + \mathbb{E}[q(\xi)].$$

Then the SCC in Theorem 1 holds for (4).

Step 2. Let

$$J = \{i \in \{1, \dots, n\} : (\mathbb{E}[M(\xi)]x_0 + \mathbb{E}[q(\xi)])_i = 0\}$$

and

$$I = \{i \in \{1, \dots, n\} : (x_0)_i = 0\}.$$

Under condition (A8), we have

$$J \cap I = \emptyset, \quad J \cup I = \{1, \dots, n\}.$$

Let

$$\mathbb{E}[M(\xi)] = \begin{pmatrix} \mathbb{E}[m_1(\xi)]^T \\ \vdots \\ \mathbb{E}[m_n(\xi)]^T \end{pmatrix},$$

where $m_i(\cdot) : \mathbb{R}^k \rightarrow \mathbb{R}^n, i = 1, 2, \dots, n$ are random vectors. If

$$|J| = k,$$

without loss of generality, we assume that

$$J = \{1, \dots, k\}, I = \{k+1, \dots, n\}.$$

Next, we show

$$\mathbb{E}[m_1(\xi)], \dots, \mathbb{E}[m_k(\xi)], e_{k+1}, \dots, e_n$$

are linearly independent. Indeed, let

$$\tilde{M} = \begin{pmatrix} \mathbb{E}[m_1(\xi)]^T \\ \vdots \\ \mathbb{E}[m_k(\xi)]^T \\ e_{k+1}^T \\ \vdots \\ e_n^T \end{pmatrix},$$

we only need to show that $\tilde{M}d = 0 \implies d = 0$. If there exists $d \in \mathbb{R}^n$ such that

$$\tilde{M}d = 0,$$

then

$$\tilde{M}d = \begin{pmatrix} \mathbb{E}[m_1(\xi)]^T d \\ \vdots \\ \mathbb{E}[m_k(\xi)]^T d \\ e_{k+1}^T d \\ \vdots \\ e_n^T d \end{pmatrix} = 0.$$

Therefore, we have

$$\sum_{i=1}^n (e_i^T d) (\mathbb{E}[m_i(\xi)]^T d) = d^T \mathbb{E}[M(\xi)] d = 0,$$

which by condition (A8), means that

$$d = 0.$$

So $\tilde{M}$ is nonsingular. Consequently, the LICQ for (4) holds at x_0 .

Step 3. Next we show the SOSC for (4) holds at (x_0, λ_0, μ_0) . Notice that under the LICQ, the SOCC is as follows:

$$d^T \mathbb{E}[M(\xi)] d > 0$$

for every nonzero vector d satisfying $d \in C$, where

$$C = \left\{ d \in \mathbb{R}^n : \begin{array}{ll} \mathbb{E}[m_i(\xi)]^T d = 0, & \text{for } \lambda_i > 0, i \in J, \\ e_j^T d = 0, & \text{for } \mu_j > 0, j \in I \end{array} \right\}.$$

Since

$$\lambda_i > 0 \iff i \in J,$$

$$\mu_j > 0 \iff j \in I,$$

then

$$C = \left\{ d \in \mathbb{R}^n : \begin{array}{ll} \mathbb{E}[m_i(\xi)]^T d = 0, & \text{for } i \in J, \\ e_j^T d = 0, & \text{for } j \in I \end{array} \right\}.$$

Similarly to Step 2 above, we have

$$d^T \mathbb{E}[M(\xi)] d = 0, \text{ for } d \in C.$$

Then $d = 0$ follows from condition (A8). Therefore the SOSC holds for (4). As a result, we verify the conditions in Theorem 1. $\square$

Notice that under (A10), the SLCP (1) is equivalent to the following problem.

$$\begin{array}{ll} \min_{x \in \mathbb{R}^n} & \frac{1}{2} x^T \mathbb{E}[M(\xi)] x + q(\xi)^T x \\ \text{s.t.} & x \geq 0 \end{array} \quad (12)$$

By Theorem 2, we obtain the following results.

Corollary 1. Suppose $\mathbb{E}[M(\xi)]$ and $\mathbb{E}[q(\xi)]$ are well-defined and finite. Let x_0 be a solution of SLCP (1), $(x_0, \mu_0) \in \mathbb{R}^n \times \mathbb{R}^n$ be a KKT point for (12), (A9) holds at x_0 , and (A10) holds. Then

(i) for N large enough, there exists (x_N, μ_N) satisfying the KKT condition of the SAA problem of (12) and

$$(x_N, \mu_N) \rightarrow (x_0, \mu_0) \text{ w.p.1 as } N \rightarrow \infty$$

(ii) for (x_N, μ_N) in (i),

$$\sqrt{N}[(x_N, (\mu_N)_I) - (x_0, (\mu_0)_I)]$$

converges in distribution to a normal with 0 and covariance matrix

$$\begin{pmatrix} \mathbb{E}[M(\xi)] & E_I \\ E_I^T & 0 \end{pmatrix}^{-1} \begin{pmatrix} \mathbb{E}[\gamma(\xi)\gamma(\xi)^T] & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbb{E}[M(\xi)] & E_I \\ E_I^T & 0 \end{pmatrix}^{-1}$$

with

$$I = \{i \in \{1, 2, \dots, n\} : (x_0)_i = 0\},$$

$$\gamma(\xi) = (M(\xi) - \mathbb{E}[M(\xi)])x_0 + q(\xi) - \mathbb{E}[q(\xi)].$$

3. Applications

In this section, we apply the results above to estimate the confidence regions of the SLCP solutions (1). Inspired by Theorem 4.2 in [12], we have the following theorem:

Theorem 3. Suppose conditions in Theorem 2.2 hold. Let Σ_N be a sequence converging to $\mathbb{E}[\gamma(\xi)\gamma(\xi)^T]$ with probability one and $\Sigma := \mathbb{E}[\gamma(\xi)\gamma(\xi)^T]$ be nonsingular, where $\gamma(\xi) = (M(\xi) - \mathbb{E}[M(\xi)])x_0 + q(\xi) - \mathbb{E}[q(\xi)]$. Let

$$\tilde{\Sigma}_N = \begin{pmatrix} 2\hat{M}_N \\ \hat{M}_N \\ E \end{pmatrix} \Sigma_N^{-1} \begin{pmatrix} 2\hat{M}_N, \hat{M}_N, E \end{pmatrix}, \quad \tilde{\Sigma} = \begin{pmatrix} 2\mathbb{E}[M(\xi)] \\ \mathbb{E}[M(\xi)] \\ E \end{pmatrix} \Sigma^{-1} (2\mathbb{E}[M(\xi)], \mathbb{E}[M(\xi)], E).$$

Assume the decomposition of $\tilde{\Sigma}_N$ is

$$\tilde{\Sigma}_N = U_N^T C_N U_N = ((U_N)_1^T, (U_N)_2^T) \begin{pmatrix} \Lambda_N & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} (U_N)_1 \\ (U_N)_2 \end{pmatrix}, \quad (13)$$

where $U_N \in \mathbb{R}^{3n \times 3n}$ is an orthogonal matrix, $(U_N)_1 \in \mathbb{R}^{r_N \times 3n}$, $(U_N)_2 \in \mathbb{R}^{(3n-r_N) \times 3n}$, and $\Lambda_N \in \mathbb{R}^{r_N \times r_N}$ are diagonal matrices with monotonically decreasing positive elements, and r_N is the rank of $\tilde{\Sigma}_N$. Thus, for $\alpha \in (0, 1)$ and any $\varepsilon > 0$,

$$\lim_{N \rightarrow \infty} \text{Prob} \left\{ (x_0, \lambda_0, \mu_0) \in \hat{Q}_{N,\varepsilon} \right\} = 1 - \alpha, \quad (14)$$

where

$$\hat{Q}_{N,\varepsilon} = \left\{ z \in \mathbb{R}^{3n} \left| \begin{array}{l} N(z - z_N)^T (U_N)_1^T \Lambda_N^{-1} (U_N)_1 (z - z_N) \leq \chi_{r_N}^2(\alpha), \\ \|\sqrt{N}(U_N)_2(z - z_N)\|_\infty \leq \varepsilon \end{array} \right. \right\}$$

with $z = (x, \lambda, \mu)$, $z_N = (x_N, \lambda_N, \mu_N)$.

Proof. We know from Theorem 2.2 that

$$\begin{pmatrix} 2\mathbb{E}[M(\xi)] & \mathbb{E}[M(\xi)]_J & E_I \\ \mathbb{E}[M(\xi)]_J^T & 0 & 0 \\ E_I^T & 0 & 0 \end{pmatrix} \sqrt{N}((x_N, (\lambda_N)_J, (\mu_N)_I) - (x_0, (\lambda_0)_J, (\mu_0)_I))$$

converges in distribution to a normal with 0 and covariance matrix

$$\begin{pmatrix} Y_{11} & Y_{12} & 0 \\ Y_{21} & Y_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

which, by Theorem 5.1 in [13], means that

$$\sqrt{N} \left[2\hat{M}_N(x_N - x_0) + \hat{M}_N((\lambda_N)_J - (\lambda_0)_J) + E_I((\mu_N)_I - (\mu_0)_I) \right]$$

converges in distribution to a normal with 0 and covariance matrix $Y_{11} = \mathbb{E}[\gamma(\xi)\gamma(\xi)^T]$. Notice that under condition (A9), for a large enough N , for $i \in \{1, 2, \dots, n\} \setminus I$ and $j \in \{1, 2, \dots, n\} \setminus J$, $(\mu_N)_i - (\mu_0)_i = 0$, and $(\lambda_N)_i - (\lambda_0)_i = 0$. We obtain the following:

$$E(\mu_N - \mu_0) = E_I((\mu_N)_I - (\mu_0)_I) \quad \text{and} \quad \hat{M}_N(\lambda_N - \lambda_0) = (\hat{M}_N)_J((\lambda_N)_J - (\lambda_0)_J)$$

Therefore,

$$\sqrt{N} \left[2\hat{M}_N(x_N - x_0) + \hat{M}_N(\lambda_N - \lambda_0) + (\mu_N - \mu_0) \right]$$

converges in distribution to a normal with 0 and covariance matrix $\mathbb{E}[\gamma(\xi)\gamma(\xi)^T]$. Since $\mathbb{E}[\gamma(\xi)\gamma(\xi)^T]$ is nonsingular, for a large enough N , Σ_N is almost surely nonsingular. Thus, we have

$$N \left[2\widehat{M}_N(x_N - x_0) + \widehat{M}_N(\lambda_N - \lambda_0) + \mu_N - \mu_0 \right] \Sigma_N^{-1} \left[2\widehat{M}_N(x_N - x_0) + \widehat{M}_N(\lambda_N - \lambda_0) + \mu_N - \mu_0 \right]$$

weakly converging to an χ_n^2 random variable. Consequently, the result is directly from the proof of Theorem 4.2 in [12]. $\square$

In practice, let

$$\Sigma_N = \frac{1}{N} \sum_{i=1}^N \left[\left(F(x_N, \xi^i) - \frac{1}{N} \sum_{i=1}^N F(x_N, \xi^i) \right) \left(F(x_N, \xi^i) - \frac{1}{N} \sum_{i=1}^N F(x_N, \xi^i) \right)^T \right], \quad (15)$$

where $F(x, \xi) = M(\xi)x + q(\xi)$. Then, Σ_N converges to Σ almost as surely as N tends to infinity.

We next illustrate three examples to show the applications of the results above.

Example 1. Consider an SLCP (1) with

$$M(\xi) = \begin{pmatrix} 2\xi_1 & \xi_2 \\ \xi_2 & \xi_3 \end{pmatrix}, \quad q(\xi) = \begin{pmatrix} \xi_1 \\ -\xi_2 \end{pmatrix},$$

where $\xi = (\xi_1, \xi_2, \xi_3)$, ξ_1, ξ_2, ξ_3 are independent random variables and each one has a normal distribution $\mathcal{N}(1, 0.1)$. Since $\mathbb{E}[M(\xi)]$ is definitely positive, that is, condition (A10) holds, the corresponding true optimization problem (12) is

$$\begin{aligned} \min_{x \in \mathbb{R}^2} \quad & \frac{1}{2} x^T \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} x + \begin{pmatrix} 1 \\ -1 \end{pmatrix}^T x \\ \text{s.t.} \quad & x \geq 0 \end{aligned} \quad (16)$$

Generating an iid sample $\xi^1, \dots, \xi^N$ of ξ , then the SAA problem is

$$\begin{aligned} \min_{x \in \mathbb{R}^2} \quad & \frac{1}{2} x^T \widehat{M}_N x + \widehat{q}_N^T x \\ \text{s.t.} \quad & x \geq 0, \end{aligned} \quad (17)$$

where $\widehat{M}_N$ and $\widehat{q}_N$ are sample-average mappings of $\mathbb{E}[M(\xi)]$ and $\mathbb{E}[q(\xi)]$, respectively.

Next, we verify the conditions in Theorem 3. By simple computing, the optimal solution to the true problem (16) is $x_0 = (0, 1)^T$ and the corresponding multiplier is $\mu_0 = \mathbb{E}[M(\xi)]x_0 + \mathbb{E}[q(\xi)] = (2, 0)^T$. Then we have

$$(x_0)_i + (\mathbb{E}[M(\xi)]x_0 + \mathbb{E}[q(\xi)])_i > 0, \text{ for } i = 1, 2,$$

which means condition (A9) holds. Condition (A8) holds due to the fact that (A10) holds. The matrix

$$\mathbb{E}[\gamma(\xi)\gamma(\xi)^T] = \begin{pmatrix} 0.02 & -0.01 \\ -0.01 & 0.02 \end{pmatrix}$$

is nonsingular. Then all the conditions in Theorem 3 hold. In practice, we can verify such conditions by the corresponding SAA estimators due to the robustness of those conditions.

We denote x_N and μ_N the corresponding SAA optimal solution and multiplier to (17) respectively. By Theorem 3, for fixed ε , the 95% confidence region is

$$\left\{ (x, \mu) \in \mathbb{R}^4 \mid \begin{aligned} & \begin{pmatrix} x - x_N \\ \mu - \mu_N \end{pmatrix}^T (U_N)_1^T \Lambda_N^{-1} (U_N)_1 \begin{pmatrix} x - x_N \\ \mu - \mu_N \end{pmatrix} \leq \chi_{r_N}^2(0.05), \\ & \left\| \sqrt{N} (U_N)_2 \begin{pmatrix} x - x_N \\ \mu - \mu_N \end{pmatrix} \right\|_{\infty} \leq \varepsilon \end{aligned} \right\},$$

where

$$\tilde{\Sigma}_N = \begin{pmatrix} 2\hat{M}_N \\ E \end{pmatrix} \Sigma_N^{-1} \begin{pmatrix} 2\hat{M}_N & E \end{pmatrix} = ((U_N)_1^T, (U_N)_2^T) \begin{pmatrix} \Lambda_N & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} (U_N)_1 \\ (U_N)_2 \end{pmatrix},$$

Σ_N is defined as in (15).

We next examine the performance of the proposed method in Theorem 3 by generating 100 confidence intervals at the 95% level, with different sample sizes N and parameters ε . We show the coverage rates for x_0 . The related quadratic programming is solved through “fmincon” running on MATLAB. The results are illustrated in Table 1.

Table 1. Summary of coverage rates for x_0 with different ε and N .

Types	$N = 10$	$N = 30$	$N = 60$	$N = 100$
$\varepsilon = 0.1$	90%	93%	95%	97%
$\varepsilon = 0.3$	91%	94%	96%	98%
$\varepsilon = 0.5$	93%	94%	98%	99%

We know from Table 1 that when $N = 10$ and $\varepsilon = 0.1$, a reasonable coverage rate for x_0 can be obtained. Furthermore, the coverage rates for x_0 increase with the increase of sample numbers and parameters.

We next apply the results obtained to a two-stage stochastic linear complementarity problem (TSLCP), which is a modified version of Example 2.6 in [14].

Example 2. Consider a TSLCP as follows: finding $x \in \mathbb{R}^2$ such that

$$0 \leq x \perp Ax + \mathbb{E}[B(\xi)y(\xi)] + q_1 \geq 0, \quad (18)$$

$$0 \leq y(\xi) \perp N(\xi)x + M(\xi)y(\xi) + q_2(\xi) \geq 0, \quad \text{for a.e. } \omega \in \Omega, \quad (19)$$

where “a.e.” means “almost everywhere”, “ $\perp$ ” denotes the perpendicularity of two vectors,

$$A = \begin{pmatrix} 4 & 2 \\ 0 & 5 \end{pmatrix}, \quad B(\xi) = \begin{pmatrix} -3+\xi & 0 \\ 2 & 1-\xi \end{pmatrix}, \quad q_1 = \begin{pmatrix} -2 \\ -4 \end{pmatrix},$$

$$N(\xi) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad M(\xi) = \begin{pmatrix} 1 & -3-\xi \\ 0 & 1 \end{pmatrix}, \quad q_2(\xi) = \begin{pmatrix} 0 \\ 1+\xi \end{pmatrix},$$

$y: \mathbb{R} \rightarrow \mathbb{R}^2$, ξ follows a uniform distribution over $[-\frac{1}{2}, \frac{1}{2}]$. For any $x \in \mathbb{R}_+^2$ and a.e. $\xi \in \Xi$, the second-stage TSLCP given by (18) has a unique $y^* = (x_1, 0)^T$. Then (19) can be written as

$$0 \leq \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \perp \mathbb{E} \left[\begin{pmatrix} 1+\xi & 2 \\ 2 & 5 \end{pmatrix} \right] \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} -2 \\ -4 \end{pmatrix} \geq 0.$$

In a manner similar to Example 3.1, for $N = 100$ and $\varepsilon = 0.1$, we obtain the 95% confidence region of x^* is $[-0.0012, 0.0021] \times [-0.0011, 0.0013]$.

The asymptotic analysis results can be applied to a problem in engineering, that is, the refinery production problem, which is illustrated in [15,16].

Example 3. In a refinery, there are two products: gasoline and fuel oil. Their output and demand depend on oil production and weather, respectively, in addition to other daily uncertainties. On the supply side, the problem is to minimize production costs under both technical and demand constraints. In order to balance supply and demand, we need an equilibrium condition, which can be constructed into a stochastic linear complementarity problem. For a detailed description of this

problem, see [15]. According to Section 4 in [15], the expected value formulation of the refinery production problem is as follows: finding $x \in \mathbb{R}^5$ such that

$$0 \leq x \perp \mathbb{E}[M(\xi)x + q(\xi)] \geq 0,$$

where

$$M(\xi) = \begin{pmatrix} 0 & 0 & 1 & -2 - \xi_1 & -3 \\ 0 & 0 & 1 & -6 & \xi_2 - 3.4 \\ -1 & -1 & 0 & 0 & 0 \\ 2 + \xi_1 & 6 & 0 & -\xi_3 & -\xi_3 \\ 3 & 3.4 - \xi_2 & 0 & -\xi_4 & \xi_4 \end{pmatrix}, \quad q_2(\xi) = \begin{pmatrix} -2 \\ -3 \\ -100 \\ 180 + \xi_3 \\ 162 + \xi_4 \end{pmatrix},$$

$2x_1 + 3x_2$ is the initial production cost and $\xi_i, i = 1, 2, 3, 4$ satisfy the distribution $\xi_1 \sim U[-0.8, 0.8]$, $\xi_2 \sim e(\lambda = 2.5)$, $\xi_3 \sim N(0, 12)$, $\xi_4 \sim N(0, 9)$. Similar to Example 3.1, for $N = 100$ and $\varepsilon = 0.1$, we obtain the 95% confidence region of x^* is $[35.9224, 36.0052] \times [17.9634, 18.0031] \times [-0.0012, 0.0065] \times [0.2488, 0.2502] \times [0.4987, 0.5014]$.

4. Conclusions

For a one-stage SLCP, the asymptotic normality results of its SAA estimator were obtained in this paper. Under some typical conditions, we show that the SAA estimator of the true solution of the SLCP converges in distribution to a multivariate normal with a zero mean vector and a covariance matrix. As a result, the methods for estimating the confidence regions of solutions of the SLCP were obtained.

Author Contributions: Conceptualization, S.L. and J.Z.; methodology, S.L.; software, C.Q.; validation, J.Z., S.L. and C.Q.; formal analysis, S.L.; investigation, J.Z.; writing—original draft preparation, S.L.; writing—review and editing, J.Z.; visualization, C.Q.; supervision, J.Z.; project administration, J.Z.; All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by the National Natural Science Foundation of China under Project Grant Nos. 12171219 and 61877032, the Liaoning Revitalization Talents Program No. XLYC2007113, Scientific Research Fund of Liaoning Provincial Education Department under Project No. LJKZ0961.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Rockafellar, R.T.; Wets, J.B. Stochastic variational inequalities: single-stage to multistage. *Math. Program.* **2017**, *165*, 291–330. [\[CrossRef\]](#)
2. Chen, X.J.; Sun, H.; Xu, H. Discrete approximation of two-stage stochastic and distributionally robust linear complementarity problems. *Math. Program.* **2019**, *177*, 255–289. [\[CrossRef\]](#)
3. Rockafellar, R.T.; Sun, J. Solving monotone stochastic variational inequalities and complementarity problems by progressive hedging. *Math. Program.* **2019**, *174*, 453–471. [\[CrossRef\]](#)
4. Zhang, J.; Xu, H.; Zhang, L. Quantitative stability analysis of stochastic quasi-variational inequality problems and applications. *Math. Program.* **2017**, *165*, 433–470. [\[CrossRef\]](#)
5. Gürkan, G.; Özge, A.Y.; Robinson, S.M. Sample-path solution of stochastic variational inequalities. *Math. Program.* **1999**, *84*, 313–333. [\[CrossRef\]](#)
6. King, A. J.; Rockafellar, R.T. Asymptotic Theory for Solutions in Statistical Estimation and Stochastic Programming. *Math. Oper. Res.* **1993**, *18*, 148–162. [\[CrossRef\]](#)
7. Jiang, H.; Xu, H. Stochastic approximation approaches to the stochastic variational inequality problem. *IEEE Trans. Autom. Control* **2008**, *53*, 1462–1475. [\[CrossRef\]](#)
8. Shapiro, A.; Dentcheva, D.; Ruszczyński, A. *Lectures on Stochastic Programming: Modeling and Theory*; SIAM: Philadelphia, PA, USA, 2009.

9. Xu, H. Sample average approximation methods for a class of stochastic variational inequality problems. *Asia-Pac. J. Oper. Res.* **2010**, *27*, 103–119. [[CrossRef](#)]
10. Zhang, J.; Zhang, L.; Pang, L. On the Convergence of Coderivative of SAA Solution Mapping for a Parametric Stochastic Variational Inequality. *Set-Valued Var. Anal.* **2012**, *20*, 75–109. [[CrossRef](#)]
11. Shapiro, A. Simulation-based optimization-convergence analysis and statistical inference. *Commun. Stat. Model.* **1996**, *12*, 425–454. [[CrossRef](#)]
12. Lu, S. Symmetric confidence regions and confidence intervals for normal map formulations of stochastic variational inequalities. *SIAM J. Optim.* **2014**, *24*, 1458–1484. [[CrossRef](#)]
13. Wolfgang, H.; Léopold, S. *Applied Multivariate Statistical Analysis*; Springer: Berlin/Heidelberg, Germany, 2007.
14. Chen, L.; Liu, Y.; Yang, X.; Zhang, J. Stochastic Approximation Methods for the Two-Stage Stochastic Linear Complementarity Problem. *SIAM J. Optim.* **2022**, *32*, 2129–2155. [[CrossRef](#)]
15. Chen, X.; Fukushima, M. Expected residual minimization method for stochastic linear complementarity problems. *Math. Oper. Res.* **2005**, *30*, 1022–1038. [[CrossRef](#)]
16. Wang, Y.; Du, S.; Li, H.; Chen, M. Projected trust region method for stochastic linear complementarity problems. *SCIENCEASIA* **2018**, *44*, 453–460. [[CrossRef](#)]

Disclaimer/Publisher's Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.