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Option Pricing Using LSTM: A Perspective of Realized Skewness

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Abstract: Deep learning has drawn great attention in the financial field due to its powerful ability in nonlinear fitting, especially in the studies of asset pricing. In this paper, we proposed a long short-term memory option pricing model with realized skewness by fully considering the asymmetry of asset return in emerging markets. It was applied to price the ETF50 options of China. In order to emphasize the improvement of this model, a comparison with a parametric method, such as Black-Scholes (BS), and machine learning methods, such as support vector machine (SVM), random forests and recurrent neural network (RNN), was conducted. Moreover, we also took the characteristic of heavy tail into consideration and studied the effect of realized kurtosis on pricing to prove the robustness of the skewness. The empirical results indicate that realized skewness significantly improves the pricing performance of LSTM among moneyness states except for in-the-money call options. Specifically, the LSTM model with realized skewness outperforms the classical method and other machine learning methods in all metrics.

Keywords: deep learning; Option pricing; LSTM; realized skewness

MSC: 91G20; 68T07

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1. Introduction

As a momentous derivative financial instrument, the option plays a crucial role in risk management and price discovery. The rapid development of the modern options market is attributed to the great breakthrough published by Black, Scholes, and Merton [1–3]. Thereafter, option pricing has become a hot topic in the derivative financial market.

As the most famous parametric method for option pricing, the Black-Scholes (BS) formula is put forward based on five assumptions, among which the most controversial ones are the constant volatility and log normality of the underlying asset return. To overcome the drawbacks of the BS formula, many improvements have been worked out to contribute to this field. For example, against the assumption of constant volatility, Heston introduced a stochastic volatility option pricing model [4]. Duan [5] assumed that the volatility of the underlying asset return followed a GARCH process. With the development of data processing capability, realized vpolatility, which is a non-parametric volatility measurement based on high-frequency data, has also been incorporated into the BS pricing framework [6–8].

For the normal assumption, the evidence has been established that the distribution of asset returns in financial market is asymmetric, especially for emerging financial market [9–11]. The characteristic of heavy tail of financial asset return cannot be ignored either [12]. Therefore, some researchers applied skewness for option pricing to get an analytic option pricing formula, achieving more accurate pricing performance [13–15]. Since the Edgeworth expansion has a significant advantage in characterizing the non-normality distribution of underlying assets [16], Duan et al. [17,18] developed a general analytical approximation method for pricing European options based on an Edgeworth series expansion for adjusting skewness and kurtosis of the cumulative asset return in the framework of GARCH.

Another branch of option pricing studies focuses on non-parametric methods, such as machine learning. Neural network, Support vector machine (SVM), Random forests, Extreme gradient boosting (XGBoost), and Light Gradient Boosting Machine (LightGMB) have achieved empirical success in predicting financial asset price [19–26]. However, financial data presents feature of high frequency and serial correlation. Due to the great ability in non-linear fitting and time-series information extraction, a variety of deep-learning methods have already been applied to predict financial asset price, such as long short-term memory (LSTM) model [27–30]. Zhang and Huang [31] applied the long short-term memory recurrent neural network (LSTM-RNN) for hedging. The deep-learning neural network has also been proven efficient for option pricing [32–36]. For example, attempts have been made to use deep-learning models on American options [37,38]. Nevertheless, these methods only focus on refining the structure of the neural network itself, ignoring the stylized facts of financial asset return, especially the skewness.

In light of the above information, an LSTM neural network [39] with realized skewness was proposed. The feasibility of implementing this model in option pricing is tested with the ETF50 options of China. Realized volatility, strike price, maturity, risk-free interest rate, and underlying asset price, which are usually deployed for derivatives pricing, and realized skewness, the one considered in our study, have been adopted as our input features. Furthermore, realized kurtosis has also been considered as one of the input features when testing the robustness of realized skewness. In comparison to the model without realized skewness and the model that contains realized skewness and realized kurtosis, our algorithm containing only realized skewness was applicable to acquire the intraday price and shows better pricing performance. Moreover, to verify the performance of the LSTM model when considering realized skewness, benchmark models were exercised for pricing options [40–45]. Several evaluating metrics, such as mean square error (MSE), root mean square error (RMSE), mean absolute error (MAE) and mean absolute percentage error (MAPE), were used to test the pricing accuracy of our proposed model.

The remainder of this paper is organized as follows. Section 2 provides a brief background of the LSTM neural network and recurrent neural network (RNN) and gives the model structure adopted in this paper. Section 3 conducts an empirical study for pricing ETF50 options and model comparison. Section 4 provides some concluding remarks.

2. Methodology

2.1. Recurrent Neural Network

The recurrent neural network, which is one of the most popular deep-learning algorithms, is a variant structure of an artificial neural network. It is usually used for processing time-sequential data due to its ability to capture the relationship between previous period data and current data. The output features of the previous period are retained during the training process and are computed as input variables for the next period. The result of the computation of the $t-1$ period of the hidden state will be considered as the input variable for period t and multiplied by a weight. Then, the activation function maps the hidden state to the output state. The neurons of each layer are illustrated in Figure 1, where x_t is an input vector at time-step t, h_t is the hidden state vector at time-step t, o_t is the output at time-step t, W is the weight from the hidden layer at time-step $t-1$ to the hidden layer at time-step t, U is the weight from the input layer to the hidden layer, and V is the weight from the hidden layer to the output. The mathematical form of an RNN is given by the following formulas:

$$h_t = f_h(Ux_t + Wh_{t-1}) \tag{1}$$

$$o_t = f_o(Vh_t) \tag{2}$$

where f_h and f_o are the activation function. In this paper, 'ReLU' function was applied to describe nonlinear transformation. The use of 'ReLU' allows the network to introduce sparsity on its own, which can greatly increase the training speed.

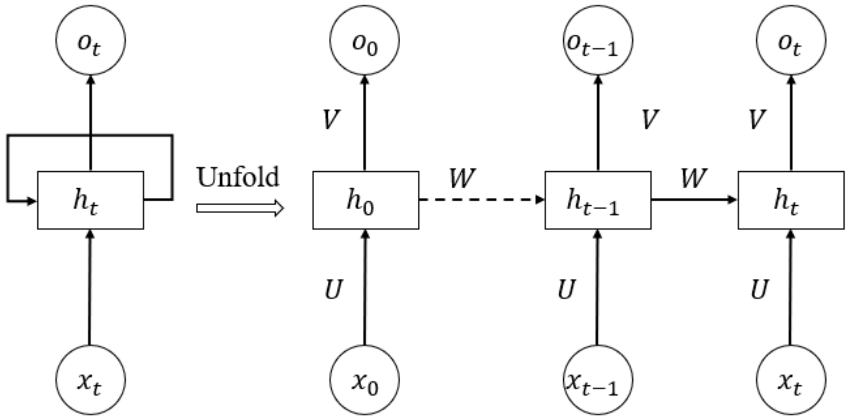


Figure 1. The structure of a neuron of a recurrent neural network.

For this study, an RNN is proposed that is composed of five simple RNN-layers and one dense layer. Additionally, there are some dropout-layers included.

2.2. Long Short-Term Memory Neural Network

The LSTM introduced by Hochreiter and Schmidhuber uses a structure called an LSTM cell to obtain better memory. Meanwhile, in order to avoid gradient explosion and gradient extinction performed in an RNN, the LSTM uses a gate control mechanism and memory cells to control information transmission, which greatly enhances the long-term memory performance [46]. The gate control mechanism consists of an input gate i_t, forget gate f_t, and output gate o_t, as shown in Figure 2. The input gate determines the input variable during the current period and the hidden neuron of the previous period to the memory status of the current period. The forget gate determines the portion to be forgotten from the input memory cell of the previous period to the memory status of the current period. The output gate determines the output information from the memory status. Therefore, the LSTM is specifically designed to learn long-term dependencies.

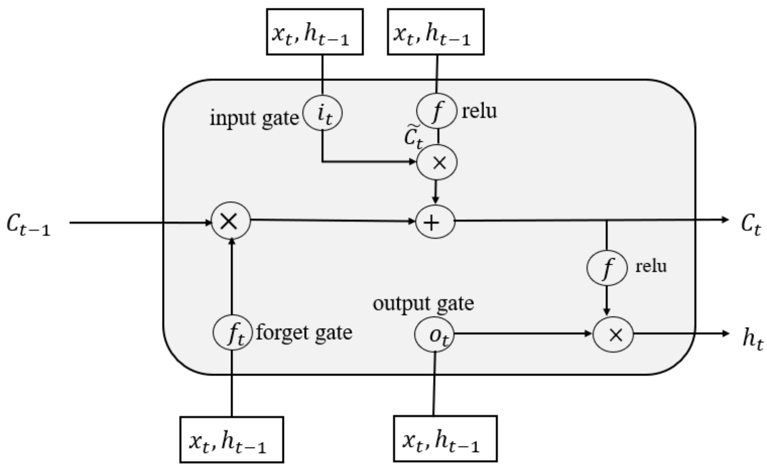


Figure 2. The structure of neuron of LSTM.

The Equations (3)–(8) portray the update of the memory cells of the LSTM. The notations of each equation are presented below:

x_t is the input vector.

W_f, W_i, W_C, and W_o are the weight matrices of gate and cell states.

b_f, b_i, b_C, and b_o are bias vectors.
h_t is the hidden state of the LSTM.
f_t, i_t, and o_t are values of forget gate, input gate, and output gate, respectively.
C_t is the vector for the cell states and $\widetilde{C}_t$ is the temporary vector for the cell states.
$f(\cdot)$ is the activation function.

$$f_t = f\left(W_f \cdot [h_{t-1},\ x_t] + b_f\right) \tag{3}$$

$$i_t = f(W_i \cdot [h_{t-1},\ x_t] + b_i) \tag{4}$$

$$\widetilde{C}_t = f(W_C \cdot [h_{t-1},\ x_t] + b_C) \tag{5}$$

$$C_t = C_{t-1} \cdot f_t + i_t \cdot \widetilde{C}_t \tag{6}$$

$$o_t = f(W_o \cdot [h_{t-1},\ x_t] + b_o) \tag{7}$$

$$h_t = o_t \cdot f(C_t) \tag{8}$$

The structure of the LSTM is composed of an input layer, one or more hidden layers, and an output layer. In addition, the number of neurons in each layer is variant when considering different problems. Figure 3 shows the network architecture. When processing an input sequence, the cell states and hidden states are passed through the neurons to obtain the output. During the training process, the network updates the weights and bias terms in such a way that it minimizes the loss of the objective function over the entire training data set, whose MSE is used in this paper. In Figure 3, I_n are the input features of our model. h_n represents the hidden neurons of hidden layers. It only shows one hidden layer in Figure 3. O is the output of the network, and in our study, it represents the option price. The weights of each connection are adjusted according to the feedback error. In order to construct a pricing model, each layer of the LSTM is stacked with a set of neurons, as shown in Figure 2. With different amounts of neurons and different number of layers, the LSTM presents different structures and adapts to different problems. The specified topology of the LSTM model used in our study is shown in Section 2.3.

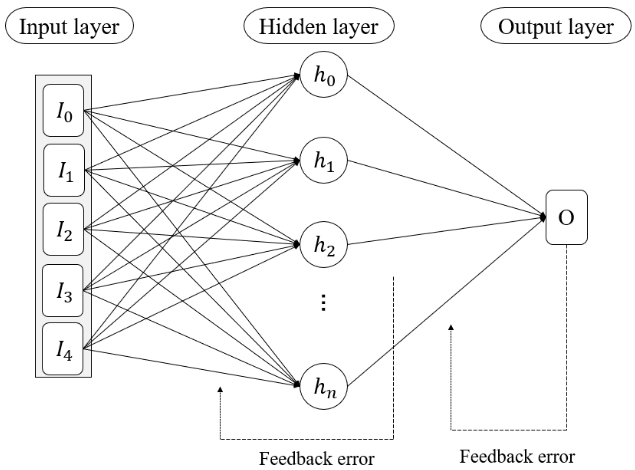


Figure 3. Architecture of the LSTM model used in our paper.

2.3. Model Structure

We construct LSTM networks containing five LSTM layers and four Dropout layers, which are used in our study for pricing the ETF50 options. The Dropout layer can prevent overfitting. With respect to the numerous factors that affect the prices of options, we consider the traditional factors that have been approved using the BS model and the realized skewness and realized kurtosis features proposed in this paper. Then, features that include spot price, strike price, risk-free rate, time-to-maturity, realized volatility, realized skewness, and realized kurtosis are selected as input variables. The structure of the LSTM used in our study is shown in Figure 4. The neural network is trained to retain the prices of the ETF50 options. Then, the predicted option prices acquired from the network and the real option prices should be compared to measure the accuracy of the network.

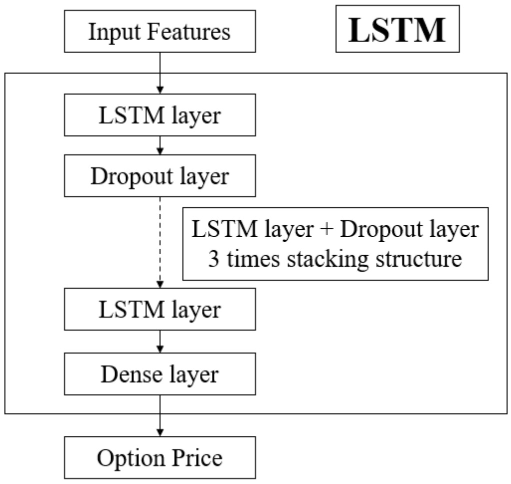


Figure 4. The layers of LSTM model.

In the benchmarks, the structure of the RNN is shown in Figure 5. The overall number of layers of the LSTM and RNN models are equal, and the hyperparameters of each model are shown in Table 1.

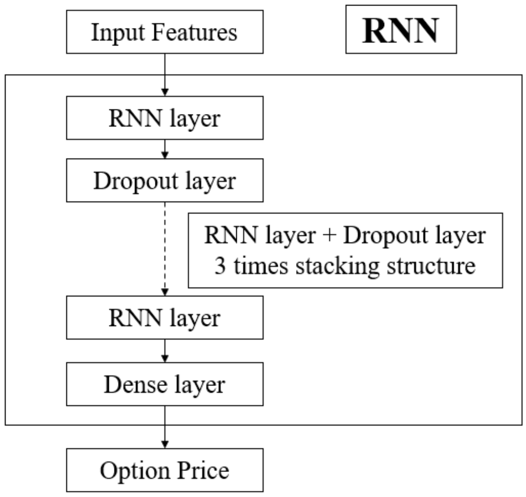


Figure 5. The layers of the RNN model.

Table 1. Hyperparameter of each model.

	LSTM	**RNN**
Activation function	RELU	RELU
Loss function	MSE	MSE
Neurons	[200, 200, 200, 200, 200, 1]	[200, 200, 200, 200, 200, 1]
Learning rate	0.001	0.001
Optimizer	Adam	Adam

Herein, [200, 200, 200, 200, 200, 1] represents the number of neurons from the first network layer to the last network layer.

2.4. Computing Realized Higher Moments

We define the intraday log-returns for each interval, and the definition of the ith log return on day t is given using

$$r_{t,\ i} = 100 * (\log p_{t,i} - \log p_{t,i-1})\ i = 1,\ 2 \ldots N \tag{9}$$

where $p_{t,i}$ is the closing price of ETF50 on day t in ith interval. We use one-minute intraday price so that we have N = 241 in a trading day.

The realized volatility put forward by Andersen and Bollerslev [47] is computed by summing squares of intraday log-returns

$$RV_t = \sum_{i=1}^{N} {r_{t,\ i}}^2 \tag{10}$$

Meanwhile, the intraday realized volatility, such as the interval [a, b], is a consistent estimator for the volatility of the instantaneous log-returns process [48]. In this paper, we consider $N = 5$.

Because the asymmetry of the distribution of log-returns is prominent in financial markets, we are interested in computing realized high moments which may have an impact on the pricing of ETF50 options. We construct a measure of realized skewness and realized kurtosis constructed by Amaya et al. [49], and the definition of each equation is given by

$$RDSkew_t = \frac{\sqrt{N}\sum_{i=1}^{N} {r_{t,i}}^3}{{RV_t}^{3/2}} \tag{11}$$

$$RDKurt_t = \frac{N\sum_{i=1}^{N} {r_{t,i}}^4}{{RV_t}^2} \tag{12}$$

It indicates that the return distribution of stock has a left tail that is fatter than the right tail when there is a negative value of realized skewness and positive values indicate the opposite.

2.5. Measures of Model Performance

In order to evaluate the option pricing accuracy of realized skewness and different models, we consider four widely used metrics: mean squared error, root mean squared error, mean absolute error, and mean absolute percentage error. The smaller the value of these metrics, the more accurate the option price. The definitions of these metrics are provided below:

$$MSE = \frac{1}{n}\sum_{i=1}^{n}(p_i - \hat{p}_i)^2 \tag{13}$$

$$RMSE = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(p_i - \hat{p}_i)^2} \tag{14}$$

$$MAE = \frac{1}{n}\sum_{i=1}^{n}|p_i - \hat{p}_i| \tag{15}$$

$$MAPE = 100\%\frac{1}{n}\sum_{i=1}^{n}\left|\frac{p_i - \hat{p}_i}{p_i}\right| \tag{16}$$

where n is the data number of prediction set, p_i is the actual option price, and $\hat{p}_i$ is the predicted price of option.

Of all these metrics, the MAPE reflects the process of comparison with the original data, which could be more objective. In contrast, the MSE, RMSE, and MAE only measure the deviation between actual values and estimated values, which are susceptible to outliers. Thus, we adopted MAPE as the main estimator to assess the accuracy of our proposed models.

3. Results and Discussion

In this section, we first discuss the details of the China ETF50 options used in our study, and then use these options as a training set to calibrate our hyperparameters. Our aim is to show that realized skewness is reliable for option pricing. This is achieved via a comprehensive testing of the options' performance on the LSTM model with realized skewness and the LSTM model without realized skewness. Furthermore, the LSTM model containing skewness and kurtosis is considered a contrasting model to confirm the ability of realized skewness on pricing options. In this process, we discuss the respective performance of different moneyness of call and put options. Then, we also evaluate the pricing validity between the LSTM model and benchmark models including the RNN, Black-Scholes model, Support vector machine, and Random forests.

3.1. Summary of the Data

The target we used in this study are the ETF50 options traded in the Shanghai Stock Exchange, whose underlying asset is the ETF50. They were first traded on 9 February 2015 with a European-style exercise. Meanwhile, the market share of ETF50 options has rapidly expanded, becoming one of the most important financial derivatives in China. Because quantitative trading has become prevalent with the development of computer technology, studying high frequency option pricing is essential for hedging. We use the one-minute data for the ETF50 options traded from October 2020 to June 2021 for our study, which were obtained from the Wind database. To get a more accurate pricing performance, we take away the data whose volume is zero to calibrate our model and eliminate the option data whose maturity is less than 7 days.

There are one hundred options introduced in our study, among which fifty are call options and the others are put options. Before passing the data to deep-learning networks, the options are sorted by moneyness for the sake of contrasting the pricing accuracy of the different moneyness for call and put options. Table 2 describe the moneyness for each option. A call option should be an in-the-money (ITM) option if moneyness > 1.03, an at-the-money (ATM) option when $0.97 \leq$ moneyness ≤ 1.03, and an out-of-the-money (OTM) option otherwise. We consider a put option to be an OTM option when moneyness > 1.03, an ATM option when $0.97 \leq$ moneyness ≤ 1.03 , and an ITM option when moneyness < 0.97.

Table 2. Description of moneyness.

	State	Moneyness
Call option	ITM	>1.03
	ATM	0.97~1.03
	OTM	<0.97
Put option	ITM	<0.97
	ATM	0.97~1.03
	OTM	>1.03

As presented in Tables 3 and 4, we have ITM, ATM, OTM for options that are also divided by call and put. The statistics of all the features of the ETF50 options are shown in Tables 3 and 4. Herein, "maturity" means the time left to expiration and the moneyness denotes S/K. Among other features, "r" is the SHIBOR rate and "S" represents the prices of underlying asset. "K" stands for the strike prices of options.

Table 3. Statistics of ETF50 Call options sorted by moneyness.

Moneyness		r	S	K	Maturity	Realized Volatility	Realized Skewness	Realized Kurtosis
ITM	count	70205	70205	70205	70205	70205	70205	70205
	mean	0.0139	3.6119	3.2317	56.4780	0.0411	0.0049	2.2384
	std	0.0059	0.1477	0.1499	0.1401	0.0948	1.1394	0.7893
	min	0.0061	3.304	2.908	8	0.0006	−2.2361	1
	max	0.0328	4.096	3.4	237	3.9760	2.2361	5
ATM	count	83637	83637	83637	83637	83637	83637	83637
	mean	0.01442	3.6181	3.4932	58.0846	0.0345	0.0199	2.2216
	std	0.0063	0.1529	0.0949	0.1374	0.0771	1.1112	0.777
	min	0.0061	3.304	3.253	8	0.0006	−2.2361	1
	max	0.0328	4.096	3.6	237	3.9760	2.2361	5
OTM	count	222532	222532	222532	222532	222532	222532	222532
	mean	0.0163	3.6242	3.7427	69.1648	0.0338	0.0151	2.2193
	std	0.0063	0.1569	0.122	0.1594	0.0778	1.1095	0.7772
	min	0.0061	3.304	3.45	8	0.0006	−2.2361	1
	max	0.0328	4.096	3.9	237	3.976	2.2361	5

Table 4. Statistics of ETF50 Put options sorted by moneyness.

Moneyness		r	S	K	Maturity	Realized Volatility	Realized Skewness	Realized Kurtosis
ITM	count	153988	153988	153988	153988	153988	153988	153988
	mean	0.0174	3.6841	3.7128	58.6633	0.0397	0.0071	2.2123
	std	0.0062	0.1529	0.1344	0.1389	0.0939	1.1093	0.7664
	min	0.0061	3.304	3.45	8	0.0006	−2.2361	1
	max	0.0328	4.096	3.9	237	3.976	2.2361	5
ATM	count	103716	103716	103716	103716	103716	103716	103716
	mean	0.0154	3.6317	3.4596	64.8511	0.0346	0.0192	2.2207
	std	0.0064	0.1576	0.1134	0.1463	0.0813	1.1099	0.7756
	min	0.0061	3.304	3.253	8	0.0006	−2.2361	1
	max	0.0328	4.096	3.6	236	3.976	2.2361	5
OTM	count	145085	145085	145085	145085	145085	145085	145085
	mean	0.015	3.5979	3.1774	77.7259	0.0354	0.0164	2.2283
	std	0.0062	0.1502	0.1575	0.1671	0.0883	1.1194	0.7853
	min	0.0061	3.304	2.908	8	0.0007	−2.2361	1
	max	0.0328	4.096	3.4	237	3.976	2.2361	5

Another problem arises because the time-to-maturity has a different magnitude. As a result, the weight adjustment of the network will be overwhelmed by the larger values of an input variable. To prevent larger input variables from becoming dominant, we divide the maturity by 365.

3.2. Pricing Performance of Long Short-Term Memory Model

3.2.1. The Effective Analysis of Realized Skewness

We investigate whether deep-learning models can be applied to estimate given option data well, which is key to the pricing performance of a deep-learning model given market information. We use the intraday option prices to calibrate each model and get the optimal hyperparameters by minimizing the loss function. We then use ETF50 options to compare the performance of the LSTM that possesses realized skewness as an input feature with the LSTM that consists only of five normal features (r, S, K, maturity, and Realized volatility). Additionally, we use benchmark models to emphasize the pricing ability of LSTM. The LSTM model is compared with BS, SVM, Random forests, and an RNN. The ETF50 options data are established in chronological order from 2020 to 2021.

Table 5 presents the pricing errors of the LSTM without realized skewness, and Table 6 presents the pricing errors of the LSTM containing realized skewness on ETF50 options. They indicate that the errors of call options and put options are smaller for the LSTM with realized skewness except for in-the-money call options. When moneyness is not considered, we find that the ETF50 option pricing model that includes realized skewness is decreased in MSA by 15.22% and 29.03% for a call option and put option, respectively. In terms of root mean squared error, the ETF50 option pricing model that includes realized skewness is reduced by 15.26% for a call option and 16.12% for a put option. Additionally, there is a decrease in the mean absolute error of 14.37% and 16.46% for call options and put options, respectively. Further, MAPE decreases by 9.91% and 30.21% for the call option and put option, respectively, which indicates the excellent ability of the realized skewness for modeling pricing options. The empirical results indicate that, from the perspective of realized skewness, the LSTM with realized skewness as one of the input features performs more excellently than the LSTM without realized skewness across all metrics. This implies that our proposed model has a higher accuracy in option pricing.

Table 5. Pricing error of LSTM without realized skewness.

Metrics	**Call Options**			**Put Options**		
	ITM	**ATM**	**OTM**	**ITM**	**ATM**	**OTM**
MSE	0.0067	0.0289	0.0261	0.0442	0.0047	0.0019
RMSE	0.8197	1.7003	1.6149	2.1027	0.6876	0.4307
MAE	0.6437	1.5225	1.3905	1.8732	0.4064	0.2506
MAPE	1.1611	5.2866	28.0592	25.1675	20.1217	46.2594

Note: All values are multiplied by a factor of 100.

Table 6. Pricing error of LSTM with realized skewness.

Metrics	**Call Options**			**Put Options**		
	ITM	**ATM**	**OTM**	**ITM**	**ATM**	**OTM**
MSE	0.0118 −76.12%	0.006 79.24%	0.015 42.53%	0.0207 53.17%	0.0036 23.40%	0.0017 10.53%
RMSE	1.0875 −32.67%	0.7761 54.36%	1.2257 24.10%	1.438 31.61%	0.6035 12.23%	0.4113 4.50%
MAE	0.9053 −40.64%	0.6253 58.96%	1.0454 24.82%	1.3198 29.54%	0.3542 12.84%	0.2331 6.98%
MAPE	1.791 −54.25%	2.3868 54.85%	19.8898 29.11%	16.0634 36.17%	16.7579 16.72%	28.8013 37.74%

Note: All values are multiplied by a factor of 100. The second line of each item is the percentage reduction per metric compared with LSTM without realized skewness. A negative value means the metric is higher than the previous value.

In the process of option pricing, our empirical results demonstrate that the realized skewness has a significant impact on the accuracy of the option pricing. Considering the performance of the LSTM with realized skewness, put options have a higher improvement in accuracy than call options in each metric. The skewness has an effect on the return rate of the underlying asset, which leads to differences in accuracy.

On the moneyness side, compared with the LSTM without realized skewness, the LSTM with realized skewness demonstrates the best accuracy for ITM options, while call options and put options are accurately priced in a majority of the moneyness states.

The effectiveness of the LSTM model with realized skewness is robust in option pricing except for the ITM call option. As we tested based on different maturity and moneyness of ETF50 options, the performance of the LSTM shows that call options have smaller errors than put options in ITM, ATM, and OTM, respectively. This is consistent with the general pattern that the accuracy of the put option pricing model is lower than that of the call option pricing model. The results of the ETF50 option pricing test indicate that the realized skewness can effectively improve the accuracy of deep learning for option pricing.

3.2.2. The Effective Analysis of Realized Kurtosis

To verify the effectiveness of realized kurtosis on pricing options, we consider the LSTM model that contains realized skewness and realized kurtosis compared to the model that contains only normal features and the model that includes realized skewness. Table 7 presents the errors of realized kurtosis when applied to option pricing after training the model using ETF50 option data.

Table 7. Pricing error of LSTM with realized skewness and realized kurtosis.

Metrics	Call Options			Put Options		
	ITM	ATM	OTM	ITM	ATM	OTM
MSE	0.0138	0.0054	0.0282	0.0305	0.0045	0.0015
	−16.64%	9.31%	−87.94%	−47.46%	−24.81%	10.70%
RMSE	1.1732	0.7377	1.6790	1.7471	0.6703	0.3896
	−7.88%	4.95%	−36.99%	−21.50%	−11.07%	5.27%
MAE	0.9242	0.5917	1.4537	1.5390	0.3751	0.2196
	−2.09%	5.37%	−39.06%	−16.61%	−5.90%	5.81%
MAPE	1.6019	2.3151	29.0479	16.6420	16.7752	29.6484
	10.56%	3.01%	−46.04%	−3.60%	−0.10%	−2.94%

Note: All values are multiplied by a factor of 100. The second line of each item is the percentage reduction per metric compared with the LSTM containing realized skewness. A negative value means the metric is higher than the previous value.

As shown in Table 7, the values of the MAPE metric increase in the majority of moneyness states. When considering the MAPE of put options, the value of the ITM option increases 3.6%. For the ATM option, the value is 0.1% and 2.94% on OTM. Especially, the pricing error increases 46.04% in the OTM option. The results of the LSTM option pricing model that uses realized skewness and realized kurtosis show that it performs poorly in each moneyness category of put options when compared with the model containing only realized skewness. Call options also perform poorly in OTM. When considering ITM and ATM call options, the model improves the pricing accuracy of ETF50 options slightly. The results are consistent with the fact that kurtosis is relatively weak on pricing options compared with skewness.

Figure 6 shows the comparison of MAPE among the normal model with of five normal features (the "Original"), the LSTM with realized skewness (the "RDSkew"), and the LSTM with realized skewness and realized kurtosis (the "RDSkew&RDKurt"). As shown in Figure 6, the pricing model with realized skewness has a superior performance in the majority of moneyness states. As for the ITM call option, the model that does not consider realized skewness and realized kurtosis, the Original, performs with the best accuracy.

As shown in Figures 7–12, the option pricing performance of the LSTM with realized skewness (RDSkew) (a) presents more accuracy than that of the LSTM without realized skewness (Original) (b). It also outperforms the model that contains realized skewness and realized kurtosis (RDSkew&RDKurt) (c) in most moneyness states.

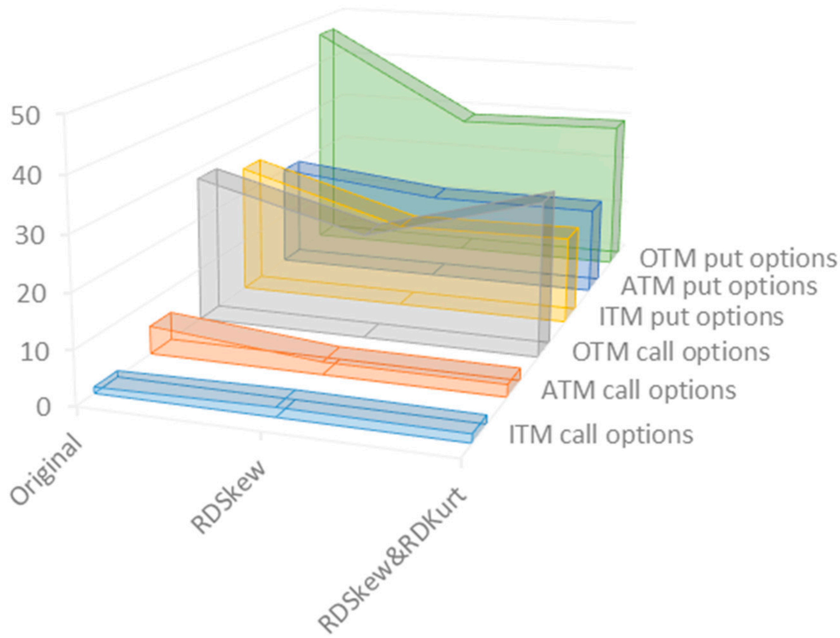


Figure 6. The comparison of the MAPE of different models considering realized skewness and realized kurtosis.

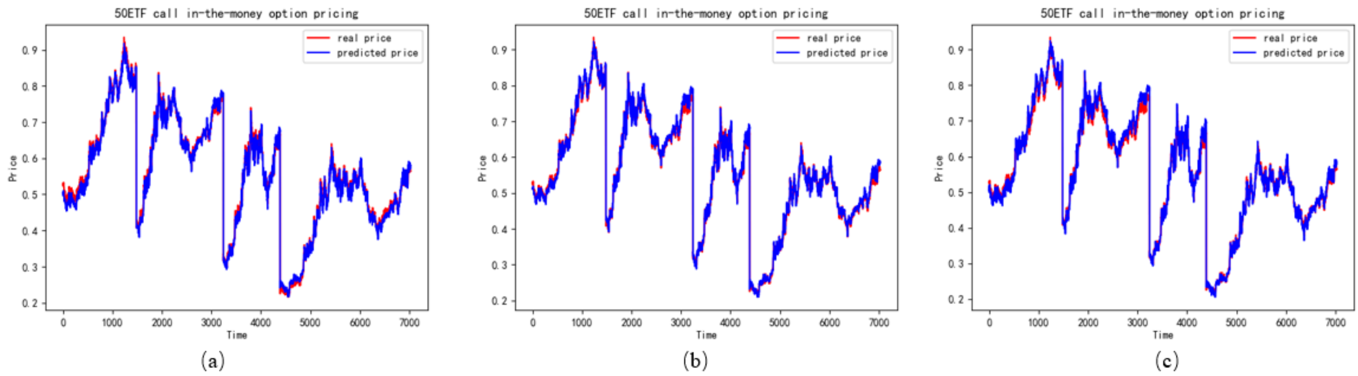


Figure 7. The performance of ITM call options in different features (**a**) represents the LSTM with realized skewness, (**b**) represents the LSTM without realized skewness, (**c**) represents the LSTM with realized skewness and realized kurtosis.

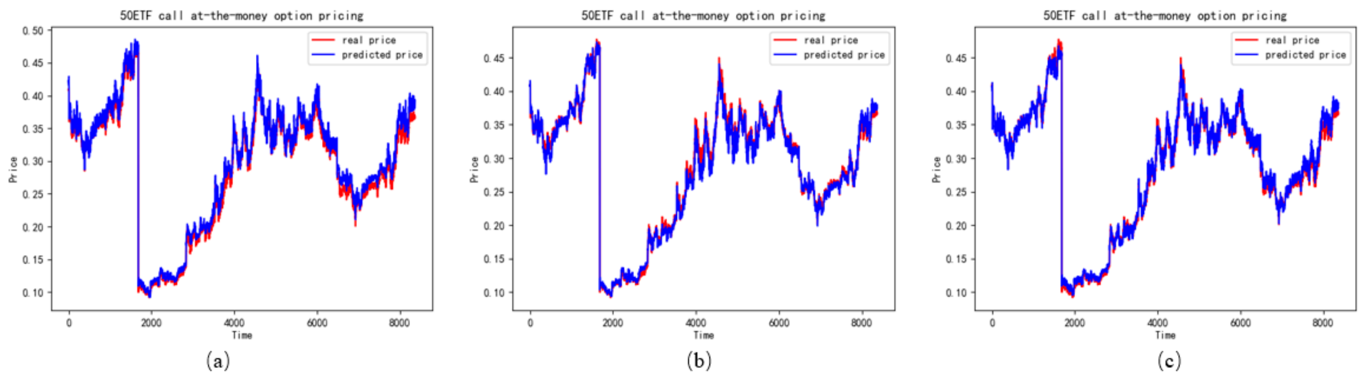
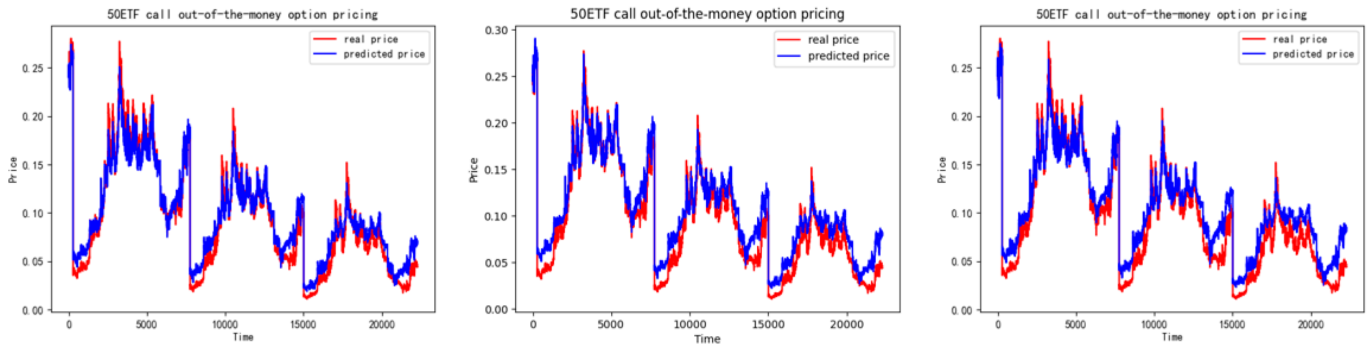
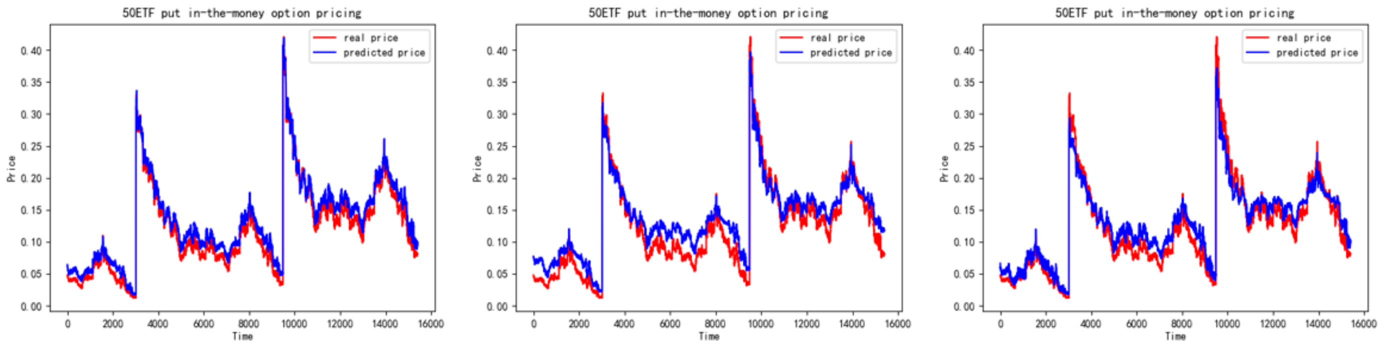


Figure 8. The performance of ATM call options in different features (**a**) represents the LSTM with realized skewness, (**b**) represents the LSTM without realized skewness, (**c**) represents the LSTM with realized skewness and realized kurtosis.



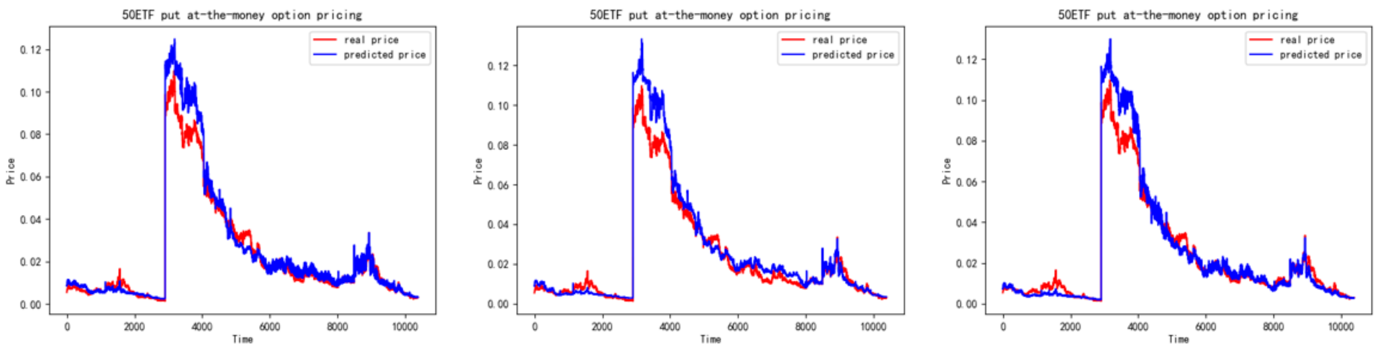
(a) (b) (c)

Figure 9. The performance of OTM call options in different features (**a**) represents the LSTM with realized skewness, (**b**) represents the LSTM without realized skewness, (**c**) represents the LSTM with realized skewness and realized kurtosis.



(a) (b) (c)

Figure 10. The performance of ITM put options in different features (**a**) represents the LSTM with realized skewness, (**b**) represents the LSTM without realized skewness, (**c**) represents the LSTM with realized skewness and realized kurtosis.



(a) (b) (c)

Figure 11. The performance of ATM put options in different features (**a**) represents the LSTM with realized skewness, (**b**) represents the LSTM without realized skewness, (**c**) represents the LSTM with realized skewness and realized kurtosis.

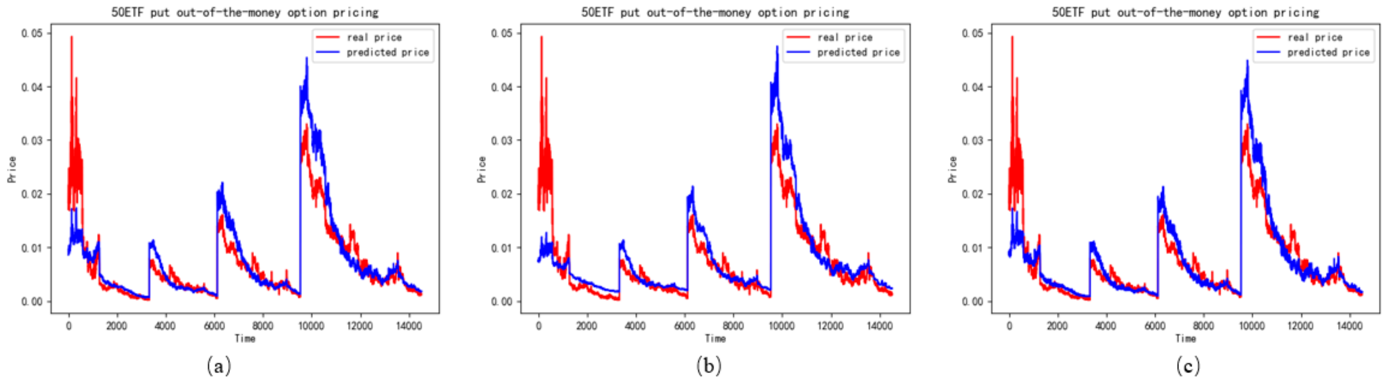


Figure 12. The performance of OTM put options in different features (**a**) represents the LSTM with realized skewness, (**b**) represents the LSTM without realized skewness, (**c**) represents the LSTM with realized skewness and realized kurtosis.

3.3. Pricing Performance of Benchmark Models

The results of benchmark models are measured with the metrics obtained on the dataset with realized skewness, which is an important variable for option pricing. The BS option pricing model, SVM, and Random forests, including 200 decision trees, have been used for comparison with the LSTM model, and, thus, their results are summarized to confirm the pricing performance of the LSTM. Nevertheless, the results obtained with the benchmark machine learning models are used as an indicator of the possible error range.

The first conclusion is that the quality of pricing with the benchmark models varies considerably across the different states of moneyness for options. ITM and ATM options are much more accurate compared with OTM options when it concerns the call option in all of the benchmarks. In the terms of the put option, ITM options have a smaller deviation than ATM and OTM options. Both call and put OTM options are priced with a maximum percentage deviation. For call options and put options, the LSTM model with realized skewness performs the best, followed by the Random forests model.

When it comes to the pricing accuracy of the LSTM model with realized skewness, Tables 8–10 reveal that the LSTM model presented the most excellent pricing performance except for ITM put options, with remarkable nonlinear fitting ability. The BS model provides the least reliable pricing due to the maximum values of metrics. The ITM call options are priced using the LSTM model with an MAPE close to 0.01791, 0.023868 for ATM call options, and around 0.198898 for OTM. For call options priced using LSTM, the MAPE decreases by 98.21%, 97.55%, and 92.68% compared with the BS model with the ITM, ATM, and OTM moneyness states, respectively. Those decreases are 67.80%, 84.23%, and 73.75% when compared with the SVM model and 0.65%, 49.16%, and 13.44% when compared with the Random forests model. Compared with the RNN model, which is most similar to the LSTM model, the MAPE decreases by 83.26% for ITM call options, 82.74% for ATM call options and 72.57% for OTM call options. Similarly, the metric is smallest with put options priced using the LSTM model. When compared with the BS model, the LSTM optimization has 75.01%, 83.93%, and 73.31% in terms of the ITM, ATM, and OTM options. They are 78.87%, 98.18%, and 99.13% compared with SVM model, −37.07%, 44.88%, and 16.44% compared with the Random forests model, and 71.47%, 90.35%, and 84.33% compared with the RNN model. In terms of the moneyness of call and put options, ITM options have the most accurate pricing quality regardless of the pricing model.

Table 8. Pricing error of ITM options estimated using different models when using realized skewness.

Models	ITM Call Options				ITM Put Options			
	MSE	RMSE	MAE	MAPE	MSE	RMSE	MAE	MAPE
BS	33.6885	58.0418	55.9973	99.9041	0.4037	6.3540	5.7960	64.2859
SVM	0.1221	3.4946	2.8661	5.5627	0.3562	5.9681	5.7673	76.0187
RF	0.0209	1.4474	0.9829	1.8028	0.0113	1.0631	0.8555	11.7190
RNN	0.5544	7.4460	6.2965	10.6978	0.2093	4.5747	4.3433	56.2957
LSTM	0.0118	1.0875	0.9053	1.7910	0.0207	1.4380	1.3198	16.0634

Note: 'RF' stands for Random forests model and all values are multiplied by a factor of 100.

Table 9. Pricing error of ATM options estimated using different models when using realized skewness.

Models	ATM Call Options				ATM Put Options			
	MSE	RMSE	MAE	MAPE	MSE	RMSE	MAE	MAPE
BS	0.3303	5.7471	4.1122	97.2872	0.1095	3.3095	2.2106	104.2794
SVM	0.1257	3.5457	3.0895	15.1358	0.5718	7.5617	7.4347	921.8113
RF	0.0257	1.6023	1.2072	4.6944	0.0064	0.7989	0.5151	30.4049
RNN	0.1764	4.1998	3.6286	13.8309	0.0458	2.1405	2.0417	173.6328
LSTM	0.0060	0.7761	0.6253	2.3868	0.0036	0.6035	0.3542	16.7579

Note: 'RF' stands for Random forests model and all values are multiplied by a factor of 100.

Table 10. Pricing error of OTM options estimated using different models when using realized skewness.

Models	OTM Call Options				OTM Put Options			
	MSE	RMSE	MAE	MAPE	MSE	RMSE	MAE	MAPE
BS	2.2385	14.9615	12.1928	271.6490	0.0154	1.2413	0.7371	107.8957
SVM	0.2772	5.2650	4.8966	75.7476	0.8353	9.1395	9.1124	3293.3315
RF	0.0627	2.5038	1.8247	22.9774	0.0023	0.4832	0.2845	34.4681
RNN	0.1661	4.0753	3.8056	72.5007	0.0048	0.6939	0.6531	183.8504
LSTM	0.0150	1.2257	1.0454	19.8898	0.0017	0.4113	0.2331	28.8013

Note: 'RF' stands for Random forests model and all values are multiplied by a factor of 100.

For the sake of robustness, a review of a set of different models has been conducted. We train the model with realized skewness and realized kurtosis for comparison. When containing realized kurtosis, LSTM also has the most accurate pricing capability. Tables 11–13 show that LSTM presents the lowest metrics except for ITM put options.

Table 11. Pricing error of ITM options estimated using different models when using realized skewness and realized kurtosis.

Models	ITM Call Options				ITM Put Options			
	MSE	RMSE	MAE	MAPE	MSE	RMSE	MAE	MAPE
BS	33.6885	58.0418	55.9973	99.9041	0.4037	6.3540	5.7960	64.2859
SVM	0.1226	3.5016	2.8688	5.6109	0.3513	5.9267	5.7186	75.6203
RF	0.0208	1.4406	0.9812	1.7970	0.0112	1.0576	0.8497	11.6567
RNN	0.5987	7.7373	6.5175	11.1190	0.3030	5.5043	5.3030	69.5781
LSTM	0.0138	1.1732	0.9242	1.6019	0.0305	1.7471	1.5390	16.6420

Note: 'RF' stands for Random forests model and all values are multiplied by a factor of 100.

Table 12. Pricing error of ATM options estimated using different models when using realized skewness and realized kurtosis.

Models	ATM Call Options				ATM Put Options			
	MSE	RMSE	MAE	MAPE	MSE	RMSE	MAE	MAPE
BS	0.3303	5.7471	4.1122	97.2872	0.1095	3.3095	2.2106	104.2794
SVM	0.1139	3.3744	2.9022	14.3229	0.5596	7.4806	7.3462	912.8136
RF	0.0253	1.5915	1.2008	4.6725	0.0058	0.7584	0.4925	29.7101
RNN	0.1815	4.2605	3.6802	14.1544	0.0472	2.1731	2.1188	200.7604
LSTM	0.0054	0.7377	0.5917	2.3151	0.0045	0.6703	0.3751	16.7752

Note: 'RF' stands for Random forests model and all values are multiplied by a factor of 100.

Table 13. Pricing error of OTM options estimated using different models when using realized skewness and realized kurtosis.

Models	OTM Call Options				OTM Put Options			
	MSE	RMSE	MAE	MAPE	MSE	RMSE	MAE	MAPE
BS	2.2385	14.9615	12.1928	271.6490	0.0154	1.2413	0.7371	107.8957
SVM	0.2856	5.3446	4.9778	77.0127	0.8353	9.1395	9.1124	3293.3115
RF	0.0628	2.5053	1.8241	23.0268	0.0024	0.4849	0.2853	34.4997
RNN	0.2163	4.6503	4.3962	83.6558	0.0091	0.9544	0.8765	238.5260
LSTM	0.0282	1.6790	1.4537	29.0479	0.0015	0.3896	0.2196	29.6484

Note: 'RF' stands for Random forests model and all values are multiplied by a factor of 100.

Figures 13–18 illustrate the different pricing performance of LSTM (a) and benchmark models (b) stands for the BS model, (c) stands for the SVM model, (d) stands for the Random forests model, and (e) stands for the RNN model when realized skewness is considered. The pictures demonstrate that the prices using the LSTM model with realized skewness are closest to the real prices.

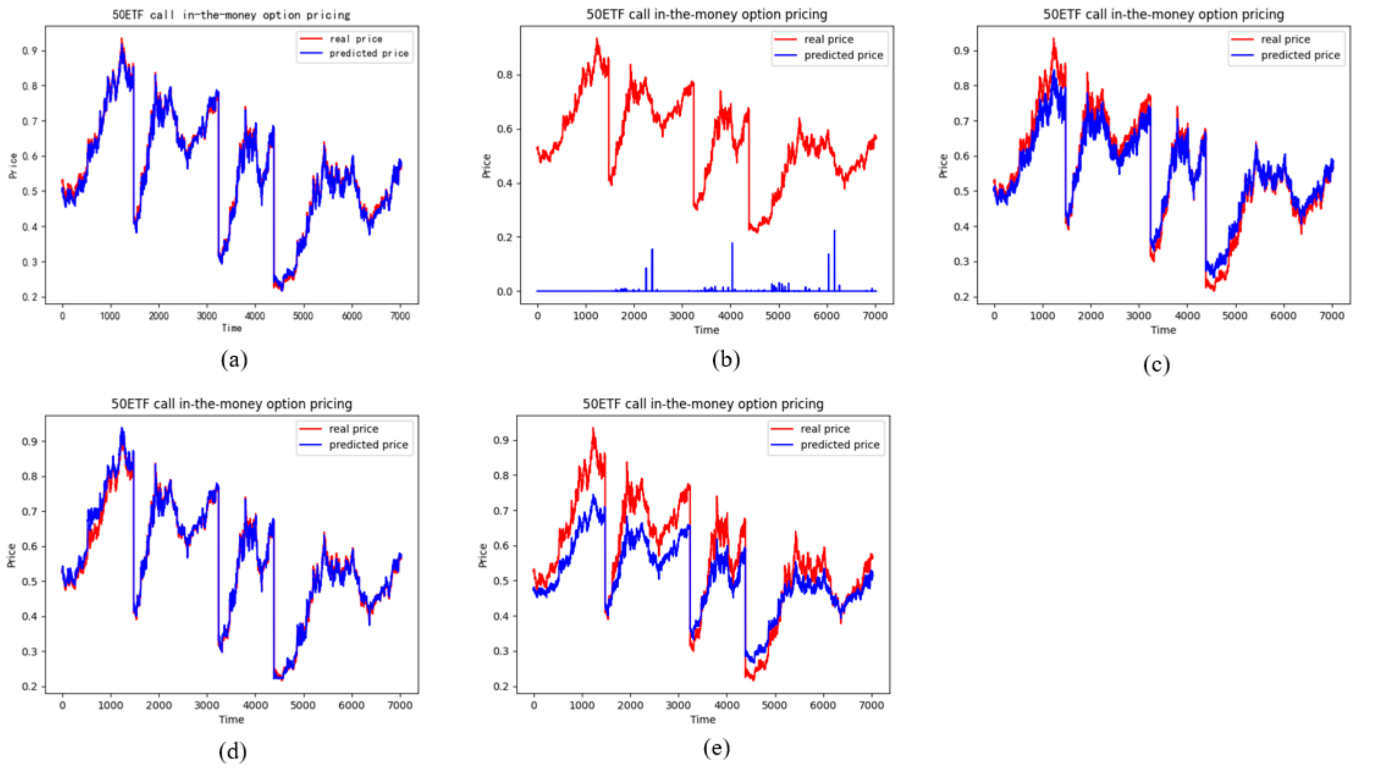


Figure 13. The pricing performance of ITM call options on each model, (**a**) stands for the LSTM model, (**b**) stands for the BS model, (**c**) stands for the SVM model, (**d**) stands for the Ran-dom forests model, (**e**) stands for the RNN model.

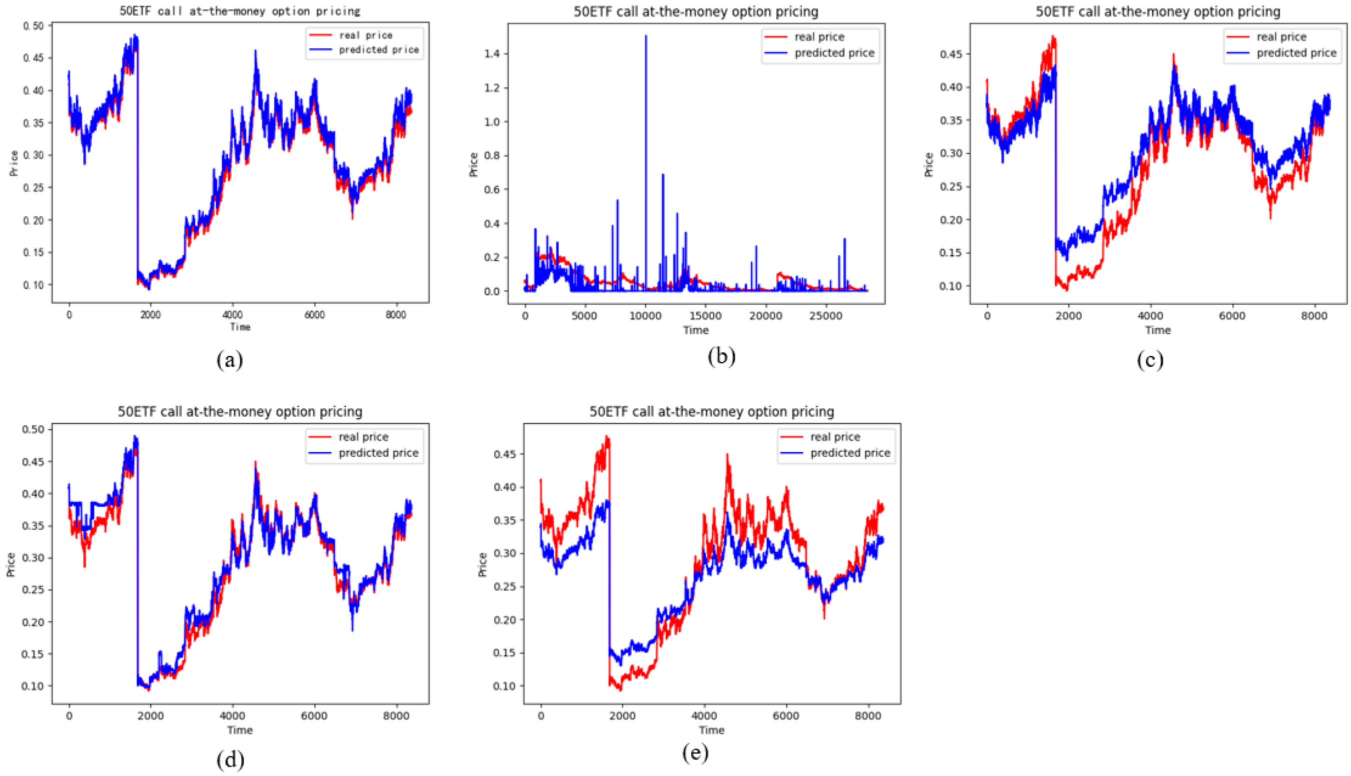


Figure 14. The pricing performance of ATM call options on each model (**a**) stands for the LSTM model, (**b**) stands for the BS model, (**c**) stands for the SVM model, (**d**) stands for the Random forests model, (**e**) stands for the RNN model.

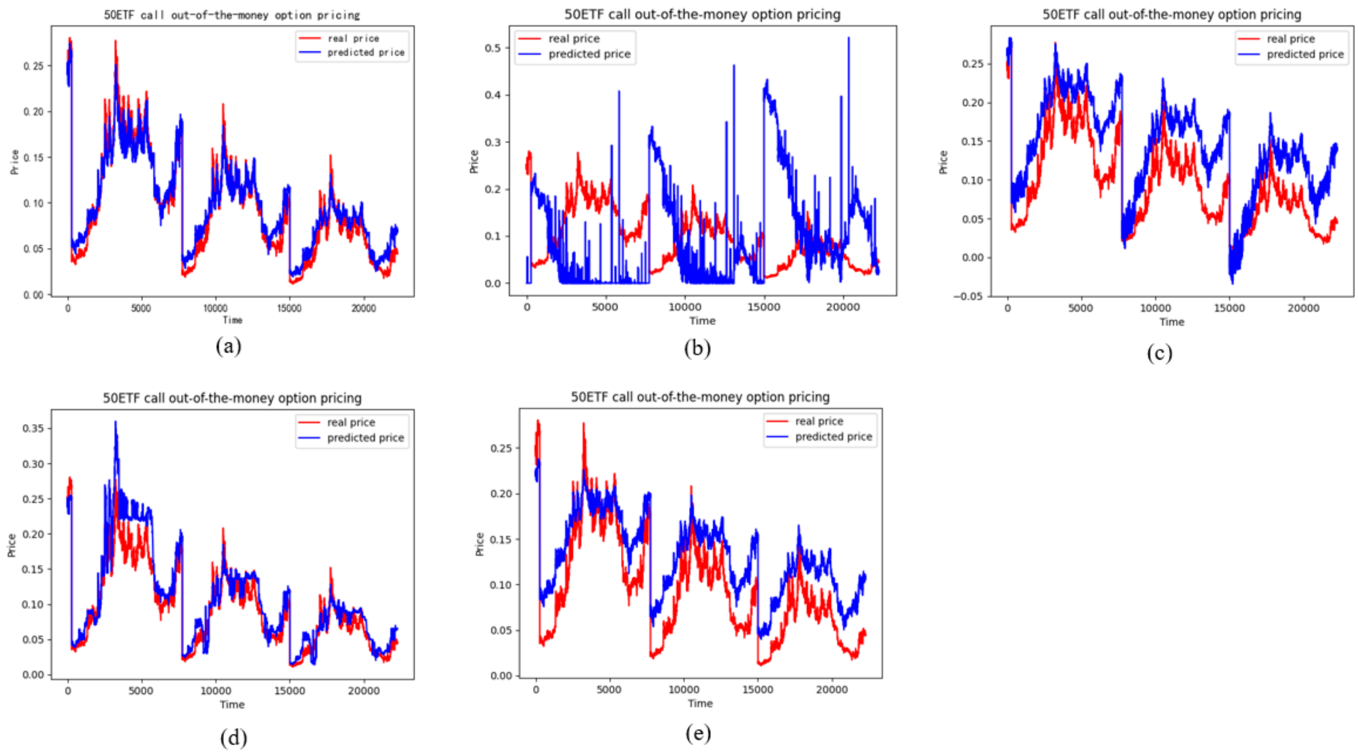


Figure 15. The pricing performance of OTM call options on each model (**a**) stands for the LSTM model, (**b**) stands for the BS model, (**c**) stands for the SVM model, (**d**) stands for the Random forests model, (**e**) stands for the RNN model.

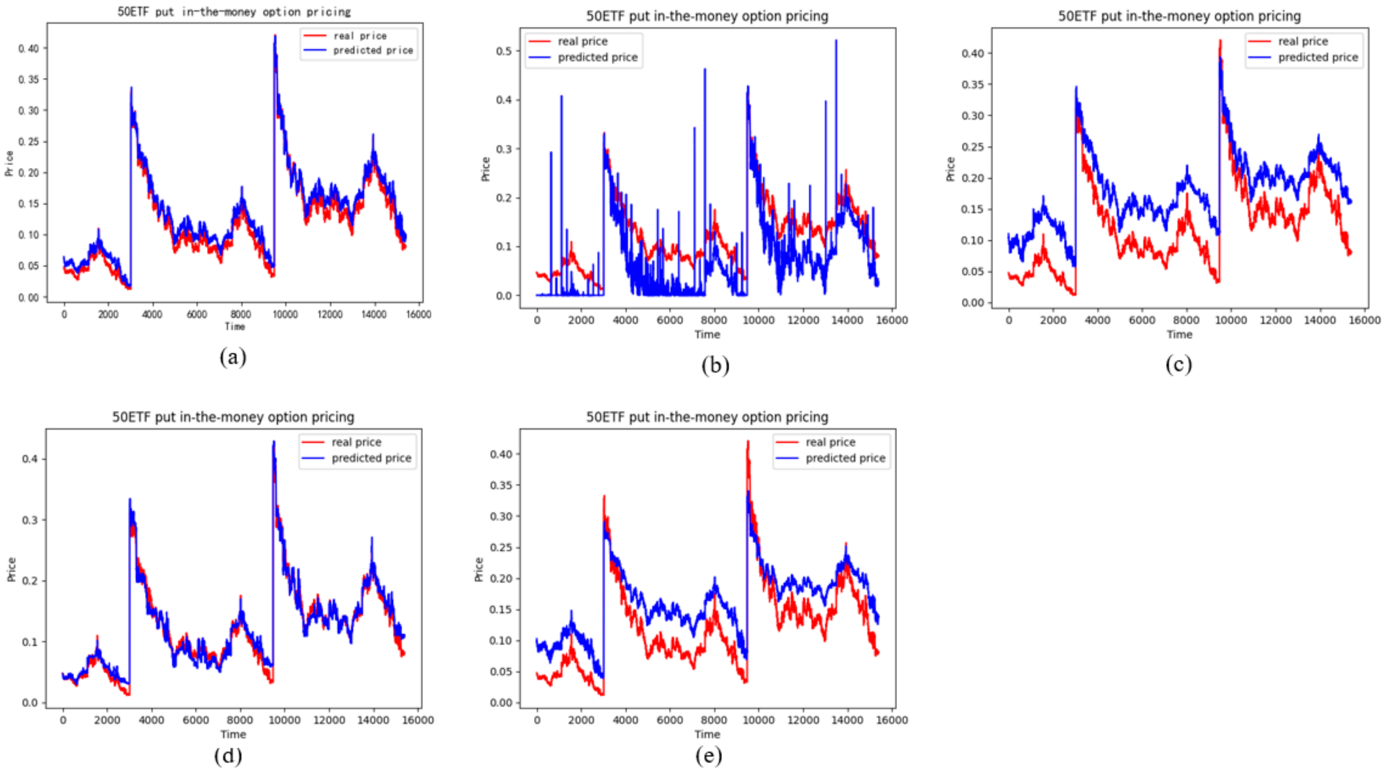


Figure 16. The pricing performance of ITM put options on each model (**a**) stands for the LSTM model, (**b**) stands for the BS model, (**c**) stands for the SVM model, (**d**) stands for the Random forests model, (**e**) stands for the RNN model.

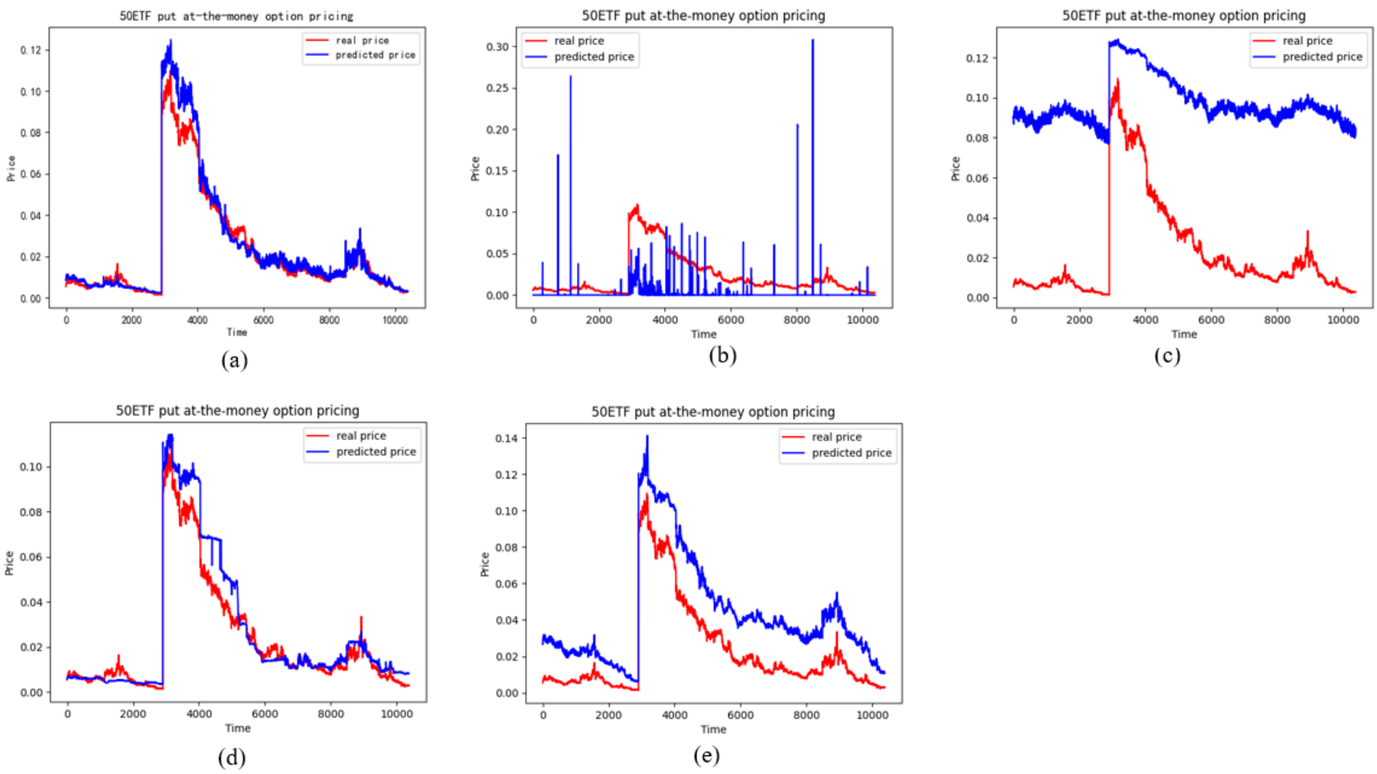


Figure 17. The pricing performance of ATM put options on each model (**a**) stands for the LSTM model, (**b**) stands for the BS model, (**c**) stands for the SVM model, (**d**) stands for the Random forests model, (**e**) stands for the RNN model.

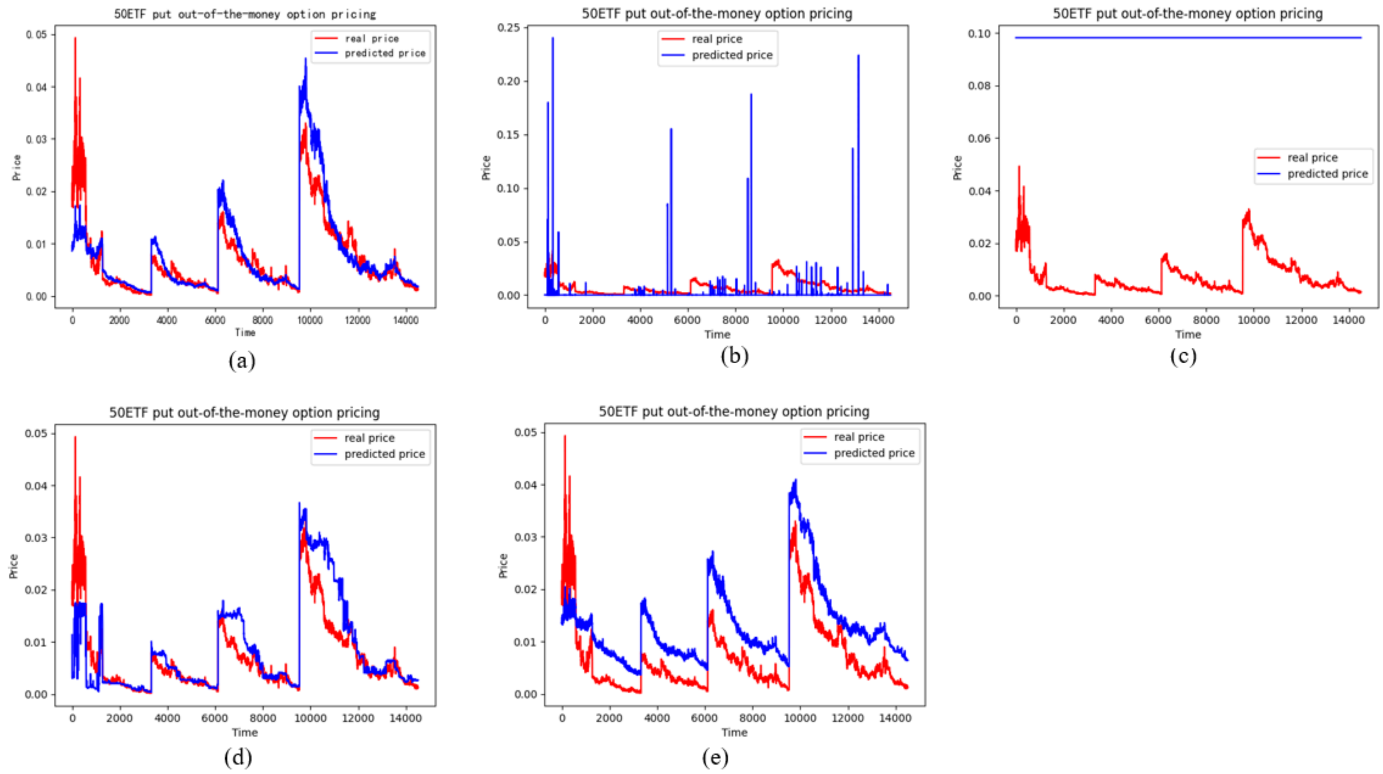


Figure 18. The pricing performance of OTM put options on each model (**a**) stands for the LSTM model, (**b**) stands for the BS model, (**c**) stands for the SVM model, (**d**) stands for the Random forests model, (**e**) stands for the RNN model.

The obtained results suggest that the LSTM model, which consists of realized skewness, has outstanding pricing performance for ETF50 options. So far, the price provided using the LSTM model with realized skewness is more reliable than that from BS, SVM, Random forest, and RNN models, as the resulting errors are lowest for the LSTM model with realized skewness. Thus, it confirms the ability of realized skewness in option pricing with a deep-learning model.

4. Conclusions

In this study, an LSTM model with realized skewness has been proposed for option pricing. We validated the efficiency and accuracy of the proposed model by pricing the ETF50 options of China from 2020 to 2021. The deep-learning models were calibrated in a data-driven approach, which means that accuracy metrics can be used to tune the LSTM model. To check the pricing ability, the data is split into a training sample, a validation sample for adjusting hyperparameters, and a test sample for verifying the accuracy of option prices. Then, in order to confirm the pricing performance of the LSTM model with realized skewness, BS, Support vector machine, Random forest, and RNN models with realized skewness were considered as benchmarks. To confirm the robustness of LSTM, we also constructed the model containing realized skewness and realized kurtosis for comparison. The results are presented for call and put options and are separated by three different moneyness states.

The results obtained from the empirical analysis demonstrate that the pricing accuracy can be improved when realized skewness serves as an input feature, except for ITM call options. One of the possibilities for dealing with the difficulties met by ITM call options would be to develop different models for this characteristic. However, it performs poorly in the model that contains realized skewness and realized kurtosis, compared with the model only containing realized skewness, which confirms the fact that skewness has more significant ability on pricing options when compared with kurtosis. In terms of all metrics, the LSTM model with realized skewness reduces the pricing error by 4.50–79.24%, which confirms the fact that the distribution of real market option data performs better with skewness. As to how we approach the problem, the constructed model improves option pricing significantly compared with benchmarks. Especially, an LSTM model with realized skewness shows the best overall performance.

From the perspective of each model, the LSTM model has certain improvement in option pricing task compared with the other types of models in two aspects of comparison: realized skewness, and realized skewness and realized kurtosis. For the realized skewness dimension, by comparing LSTM, RNN, BS, SVM, and Random forests that contain realized skewness, the LSTM model shows a strong extracting ability. As for the realized skewness and realized kurtosis aspects, the LSTM model improves the accuracy of option pricing in most moneyness states.

For future research, some new features could be applied to a deep-learning neural network. For example, new market contingencies, such as Covid-19, which may influence investors' behavior, could be quantified to price options. Furthermore, one could apply the proposed model to price other financial derivatives, such as Asian options. In addition, more effort could be committed to explore the interpretability of a long short-term neural network when applying it to solve financial problems.

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Conflicts of Interest: The authors declare no conflict of interest.

Abbreviation

LSTM	Long short-term memory
BS	Black-Scholes
ETF	Exchange Traded Funds
SVM	Support vector machine
RNN	Recurrent neural network
GARCH	Generalized Auto Regressive Conditional Heteroskedasticity
XGBoost	Extreme gradient boosting
LightGBM	Light gradient boosting machine
LSTM-RNN	Long short-term memory recurrent neural network
ReLU	Rectified linear unit
MSE	Mean squared error
RMSE	Root mean squared error
MAE	Mean absolute error
MAPE	Mean absolute percentage error
RV	Realized volatility
RDSkew	Realized skewness
RDKurt	Realized kurtosis
ITM	In-the-money
ATM	At-the-money
OTM	Out-of-the-money
RF	Random forests

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