mathematics

MDPI

Article

Tangent Bundles Endowed with Quarter-Symmetric Non-Metric Connection (QSNMC) in a Lorentzian Para-Sasakian Manifold

Rajesh Kumar [1], Lalnunenga Colney [1], Samesh Shenawy [2] and Nasser Bin Turki [3,*]

[1] Department of Mathematics, Pachhunga University College, Mizoram University, Aizawl 796001, India; rajesh_mzu@yahoo.com (R.K.); lalnunengacolney_official@yahoo.com (L.C.)
[2] Basic Science Department, Modern Academy for Engineering and Technology, Maadi 11585, Egypt; drssshenawy@eng.modern-academy.edu.eg
[3] Department of Mathematics, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia
* Correspondence: nassert@ksu.edu.sa

Abstract: The purpose of the present paper is to study the complete lifts of a QSNMC from an LP-Sasakian manifold to its tangent bundle. The lifts of the curvature tensor, Ricci tensor, projective Ricci tensor, and lifts of Einstein manifold endowed with QSNMC in an LP-Sasakian manifold to its tangent bundle are investigated. Necessary and sufficient conditions for the lifts of the Ricci tensor to be symmetric and skew-symmetric and the lifts of the projective Ricci tensor to be skew-symmetric in the tangent bundle are given. An example of complete lifts of four-dimensional LP-Sasakian manifolds in the tangent bundle is shown.

Keywords: Lorentzian para-Sasakian manifolds; complete lifts; tangent bundle; quarter-symmetric non-metric connection; partial differential equations; mathematical operators; curvature tensor; projective Ricci tensor; Einstein manifold

MSC: 53C05; 53C07; 53C25; 58A30

Citation: Kumar, R.; Colney, L.; Shenawy, S.; Bin Turki, N. Tangent Bundles Endowed with Quarter-Symmetric Non-Metric Connection (QSNMC) in a Lorentzian Para-Sasakian Manifold. *Mathematics* **2023**, *11*, 4163. https://doi.org/10.3390/math11194163

Academic Editor: Marian Ioan Munteanu

Received: 19 August 2023
Revised: 24 September 2023
Accepted: 2 October 2023
Published: 4 October 2023

1. Introduction

Tangent bundle geometry has long been a source of interest in differential geometry. Tangent bundle investigation introduces several novel challenges to the study of modern differential geometry. Using the lift function, it is convenient to generalize differentiable structures on any manifold M to its tangent bundle. The theory of vertical, complete, and horizontal lifts of geometrical structures and connections from a manifold to its tangent bundle was developed by Yano and Ishihara [1]. Numerous researchers have examined various connections and geometric structures on the tangent bundle like Yano and Kobayashi [2], Tani [3], Pandey and Chaturvedi [4], and Khan [5,6]. Different lifts of metallic structures to tangent bundles have been studied in [7–9]. Tangent bundles immersed with quarter-symmetric non-metric connections, semi-symmetric P-connections, and semi-symmetric non-metric connections on almost Hermitian manifolds, Kähler manifolds, Kenmotsu manifolds, Sasakian manifolds, para-Sasakian manifolds, Riemannian manifolds and their submanifolds, and statistical manifolds and their submanifolds have been studied in [5,10–18]. Recently, Khan et al. [19] studied the tangent bundle of P-Sasakian manifolds endowed with a quarter-symmetric metric connection (QSMC).

On the other hand, the notion of quarter-symmetric connection in a Riemannian manifold with affine connection was introduced by Golab in 1975 [20]. This was further developed by many geometers like Yano and Imai [21], Rastogi [22,23], Mishra and Pandey [24], Mukhopadhyay et al. [25], Biswas and De [26], Sengupta and Biswas [27], Singh and Pandey [28], and others.

Let ∇ be a linear connection on an n-dimensional differentiable manifold M^n of class C^∞. If the torsion tensor T of ∇ defined by

$$T(X_0, Y_0) = \nabla_{X_0} Y_0 - \nabla_{Y_0} X_0 - [X_0, Y_0], \tag{1}$$

satisfies

$$T(X_0, Y_0) = \lambda_0(Y_0)\phi_0 X_0 - \lambda_0(X_0)\phi_0 Y_0, \tag{2}$$

where λ_0 is a 1-form and ϕ_0 is a $(1,1)$ tensor field, then the connection ∇ is called a quarter-symmetric connection [21,29,30]. Also, if ∇ satisfies

$$(\nabla_{X_0} g)(Y_0, Z_0) \neq 0, \tag{3}$$

for all $X_0, Y_0, Z_0 \in \mathfrak{X}(M^n)$, the set of all vector fields on M^n, then ∇ is called a quarter-symmetric non-metric connection (QSNMC).

We start this paper with Section 1. Section 2 is devoted to preliminaries. In Section 3, a QSNMC in an LP-Sasakian manifold is studied. The complete lifts of LP-Sasakian manifolds and QSNMC in an LP-Sasakian manifold to its tangent bundle are investigated in Sections 4 and 5. In Sections 6 and 7, the complete lifts of the curvature tensor and symmetric and skew-symmetric condition of the Ricci tensor in an LP-Sasakian manifold endowed with QSNMC to its tangent bundle are investigated. The skew-symmetric properties of the projective Ricci tensor and Einstein manifold endowed with QSNMC in an LP-Sasakian manifold to its tangent bundle are studied in Sections 8 and 9. Lastly, an example of the lift of four-dimensional LP-Sasakian manifolds to its tangent bundle is shown in Section 9, followed by a conclusion section.

2. Preliminaries

Let M^n be a differentiable manifold and $T_0M^n = \bigcup_{p \in M^n} T_{0p}M^n$ be the tangent bundle, where $T_{0p}M^n$ is the tangent space at a point $p \in M^n$ and $\pi : T_0M^n \to M^n$ is the natural bundle structure of T_0M^n over M^n. For any co-ordinate system (Q, x^h) in M^n, where (x^h) is a local co-ordinate system in the neighborhood Q, then $(\pi^{-1}(Q), x^h, y^h)$ is a co-ordinate system in T_0M^n, where (x^h, y^h) is an induced co-ordinate system in $\pi^{-1}(Q)$ from (x^h) [1].

2.1. Vertical and Complete Lifts

Let us define a vector field X_0, a tensor field F_0 of type $(1,1)$, a function f_0, a 1-form ω_0, and an affine connection ∇ in M^n; their vertical and complete lifts are denoted by f_0^v, X_0^v, ω_0^v, F_0^v, ∇^v, and f_0^c, X_0^c, ω_0^c, F_0^c, ∇^c, respectively. The following formulas of complete and vertical lifts are defined by [1,5]

$$(f_0 X_0)^v = f_0^v X_0^v, \ \ (f_0 X_0)^c = f_0^c X_0^v + f_0^v X_0^c, \tag{4}$$

$$X_0^v f_0^v = 0, \ \ X_0^v f_0^c = X_0^c f_0^v = (X_0 f_0)^v, \ \ X_0^c f_0^c = (X_0 f_0)^c, \tag{5}$$

$$\omega_0(f_0^v) = 0, \ \ \omega_0^v(X_0^c) = \omega_0^c(X_0^v) = \omega_0(X_0)^v, \ \ \omega_0^c(X_0^c) = \omega_0(X_0)^c, \tag{6}$$

$$F_0^v X_0^c = (F_0 X_0)^v, \ \ F_0^c X_0^c = (F_0 X_0)^c, \tag{7}$$

$$[X_0, Y_0]^v = [X_0^c, Y_0^v] = [X_0^v, Y_0^c], \ \ [X_0, Y_0]^c = [X_0^c, Y_0^c], \tag{8}$$

$$\nabla^c{}_{X_0^c} Y_0^c = (\nabla_{X_0} Y_0)^c, \ \ \nabla^c{}_{X_0^c} Y_0^v = (\nabla_{X_0} Y_0)^v. \tag{9}$$

Suppose T_0M is the tangent bundle and let $X_0 = X_0^i \frac{\partial}{\partial x^i}$ be a local vector field on M, then its vertical and complete lifts in the term of partial differential equations are

$$X_0^v = X_0^i \frac{\partial}{\partial y^i} \qquad \text{and} \qquad X_0^c = X_0^i \frac{\partial}{\partial x^i} + \frac{\partial X_0^i}{\partial x^j} y^j \frac{\partial}{\partial y^i}.$$

2.2. LP-Sasakian Manifolds

An n-dimensional differentiable manifold M^n is called a Lorentzian para-Sasakian (briefly LP-Sasakian) [31] of dimension n if it admits a $(1,1)$- tensor field ϕ_0, a contravariant vector field ξ_0, a 1-form η_0, and a Lorentzian metric g which satisfy

$$\phi_0{}^2(X_0) = X_0 + \eta_0(X_0)\xi_0, \tag{10}$$
$$\eta_0(\xi_0) = -1, \tag{11}$$
$$g(\phi_0 X_0, \phi_0 Y_0) = g(X_0, Y_0) + \eta_0(X_0)\eta_0(Y_0), \tag{12}$$
$$g(X_0, \xi_0) = \eta_0(X_0), \tag{13}$$
$$(\nabla_{X_0}\phi_0)(Y_0) = g(X_0, Y_0)\xi_0 + \eta_0(Y_0)X_0 + 2\eta_0(X_0)\eta_0(Y_0)\xi_0, \tag{14}$$
$$\nabla_{X_0}\xi_0 = \phi_0 X_0. \tag{15}$$

In an LP-Sasakian manifold, the following relations also hold:

$$\phi_0\xi_0 = 0, \quad \eta_0 \circ \phi_0 = 0, \tag{16}$$
$$rank\ \phi_0 = n - 1. \tag{17}$$

If we take a tensor field $\Phi_0(X_0, Y_0)$ as

$$\Phi_0(X_0, Y_0) = g(X_0, \phi_0 Y_0), \tag{18}$$

for any vector fields X_0 and Y_0, then the tensor field $\Phi_0(X_0, Y_0)$ is a symmetric $(0,2)$ tensor field [31]. Since the 1-form η_0 is closed in an LP-Sasakian manifold, we have [31,32]

$$(\nabla_{X_0}\eta_0)(Y_0) = \Phi_0(X_0, Y_0), \quad \Phi_0(X_0, \xi_0) = 0, \tag{19}$$

for all $X_0, Y_0 \in M^n$. In an LP-Sasakian manifold, the following relations hold [32,33]:

$$g(R_0(X_0, Y_0)Z_0, \xi_0) = g(Y_0, Z_0)\eta_0(X_0) - g(X_0, Z_0)\eta_0(Y_0), \tag{20}$$
$$R_0(\xi_0, X_0)Y_0 = g(X_0, Y_0)\xi_0 - \eta_0(Y_0)X_0, \tag{21}$$
$$R_0(X_0, Y_0)\xi_0 = \eta_0(Y_0)X_0 - \eta_0(X_0)Y_0, \tag{22}$$
$$R_0(\xi_0, X_0)\xi_0 = X_0 + \eta_0(X_0)\xi_0, \tag{23}$$
$$S_0(X_0, \xi_0) = (n-1)\eta_0(X_0), \tag{24}$$
$$S_0(\phi_0 X_0, \phi_0 Y_0) = S_0(X_0, Y_0) + (n-1)\eta_0(X_0)\eta_0(Y_0), \tag{25}$$

where R_0 is the Riemannian curvature tensor and S_0 is the Ricci tensor of the manifold.

3. QSNMC

In an LP-Sasakian manifold (M^n, g), the linear connection $\ddot{\nabla}$ on M^n is given by [29]

$$\ddot{\nabla}_{X_0} Y_0 = \nabla_{X_0} Y_0 + \eta_0(Y_0)\phi_0 X_0 + a_0(X_0)\phi_0 Y_0, \tag{26}$$

where η_0 and a_0 are 1-form associated with vector field ξ_0 and A_0 on M^n is given by

$$\eta_0(X_0) = g(X_0, \xi_0), \tag{27}$$
$$a_0(X_0) = g(X_0, A_0), \tag{28}$$

for all vector fields $X_0 \in \mathfrak{X}_0(M^n)$, where $\mathfrak{X}_0(M^n)$ is the set of all differentiable vector fields on M^n and the torsion tensor is given by

$$\ddot{T}(X_0, Y_0) = \eta_0(Y_0)\phi_0 X_0 - \eta_0(X_0)\phi_0 Y_0 + a_0(X_0)\phi_0 Y_0 - a_0(Y_0)\phi_0 X_0. \tag{29}$$

A linear connection satisfying (29) is called a quarter-symmetric connection. Further, by using (26), we have

$$(\ddot{\nabla}_{X_0}g)(Y_0,Z_0) = -\eta_0(Y_0)g(\phi_0X_0,Z_0) - \eta_0(Z_0)g(\phi_0X_0,Y_0) - 2a_0(X_0)g(\phi_0Y_0,Z_0). \quad (30)$$

A linear connection $\ddot{\nabla}$ defined by (26) which satisfies (29) and (30) is called QSNMC.

4. Complete Lifts from an LP-Sasakian Manifold to Its Tangent Bundle

Let the tangent bundle be denoted by T_0M^n in an LP-Sasakian manifold (M^n,g). Taking complete lifts by mathematical operators on (10)–(16) and (18)–(25), we obtain

$$(\phi_0^2(X_0))^c = X_0^c + \eta_0^c(X_0^c)\xi_0^v + \eta_0^v(X_0^c)\xi_0^c, \quad (31)$$

$$\eta_0^c(\xi_0^c) = \eta_0^v(\xi_0^v) = 0,\ \ \eta_0^c(\xi_0^v) = \eta_0^v(\xi_0^c) = -1, \quad (32)$$

$$g^c\Big((\phi_0X_0)^c,(\phi_0Y_0)^c\Big) = g^c(X_0^c,Y_0^c) + \eta_0^c(X_0^c)\eta_0^v(Y_0^c) + \eta_0^v(X^c)\eta_0^c(Y_0^c), \quad (33)$$

$$g^c(X_0^c,\xi_0^c) = \eta_0^c(X_0^c), \quad (34)$$

$$\begin{aligned}(\nabla^c_{X_0^c}\phi_0^c)Y_0^c &= g^c(X_0^c,Y_0^c)\xi_0^v + g^c(X_0^v,Y_0^c)\xi_0^c + \eta_0^c(Y_0^c)X_0^v + \eta_0^v(Y_0^c)X_0^c \\ &\quad + 2\Big\{\eta_0^c(X_0^c)\eta_0^c(Y_0^c)\xi_0^v + \eta_0^c(X_0^c)\eta_0^v(Y_0^c)\xi_0^c + \eta_0^v(X_0^c)\eta_0^c(Y_0^c)\xi_0^c\Big\},\end{aligned} \quad (35)$$

$$\nabla^c_{X_0^c}\xi_0^c = (\phi_0X_0)^c, \quad (36)$$

$$\phi_0^c\xi_0^c = \phi_0^v\xi_0^v = \phi_0^c\xi_0^v = \phi_0^v\xi_0^c = 0, \quad (37)$$

$$\eta_0^c\circ\phi_0^c = \eta_0^v\circ\phi_0^v = \eta_0^c\circ\phi_0^v = \eta_0^v\circ\phi_0^c = 0, \quad (38)$$

$$\Phi_0^c(X_0^c,Y_0^c) = g^c(X_0^c,\phi_0^cY_0^c), \quad (39)$$

$$(\nabla^c_{X_0^c}\eta_0^c)Y_0^c = \Phi_0^c(X_0^c,Y_0^c), \quad (40)$$

$$\Phi_0^c(X_0^c,\xi_0^c) = 0, \quad (41)$$

$$\begin{aligned}g^c(R^c(X_0^c,Y_0^c)Z_0^c,\xi_0^c) &= g^c(Y_0^c,Z_0^c)\eta_0^v(X_0^c) + g^c(Y_0^v,Z_0^c)\eta_0^c(X_0^c) \\ &\quad - g^c(X_0^c,Z_0^c)\eta_0^v(Y_0^c) - g^c(X_0^v,Z_0^c)\eta_0^c(Y_0^c),\end{aligned} \quad (42)$$

$$R^c(\xi^c,X_0^c)Y_0^c = g^c(X_0^c,Y_0^c)\xi_0^v + g^c(X_0^v,Y_0^c)\xi_0^c - \eta_0^c(Y_0^c)X_0^v - \eta_0^v(Y_0^c)X_0^c, \quad (43)$$

$$R^c(X_0^c,Y_0^c)\xi_0^c = \eta_0^c(Y_0^c)X_0^v + \eta_0^v(Y_0^c)X_0^c - \eta_0^c(X_0^c)Y_0^v - \eta_0^v(X_0^c)Y_0^c, \quad (44)$$

$$R^c(\xi_0^c,X_0^c)\xi_0^c = X_0^c + \eta_0^c(X_0^c)\xi_0^v + \eta_0^v(X_0^c)\xi_0^c, \quad (45)$$

$$S^c(X_0^c,\xi_0^c) = (n-1)\eta_0^c(X_0^c), \quad (46)$$

$$S^c(\phi_0^cX_0^c,\phi_0^cY_0^c) = S^c(X_0^c,Y_0^c) + (n-1)\Big\{\eta_0^c(X_0^c)\eta_0^v(Y_0^c) + \eta_0^v(X_0^c)\eta_0^c(Y_0^c)\Big\}. \quad (47)$$

5. Complete Lifts of QSNMC of an LP-Sasakian Manifold in the Tangent Bundle

In an LP-Sasakian manifold (M^n,g) and its tangent bundle T_0M^n, let us take complete lifts by mathematical operators on Equations (26)–(30), and we have

$$\begin{aligned}\ddot{\nabla}^c_{X_0^c}Y_0^c &= \nabla^c_{X_0^c}Y_0^c + \eta_0^c(Y_0^c)(\phi_0X_0)^v + \eta_0^v(Y_0^c)(\phi_0X_0)^c + a^c(X_0^c)(\phi_0Y_0)^v \\ &\quad + a^v(X_0^c)(\phi_0Y_0)^c,\end{aligned} \quad (48)$$

$$\begin{aligned}\ddot{T}^c(X_0^c, Y_0^c) = {} & \eta_0^c(Y_0^c)(\phi_0 X_0)^v + \eta_0^v(Y_0^c)(\phi_0 X_0)^c - \eta_0^c(X_0^c)(\phi_0 Y_0)^v \\ & - \eta_0^v(X_0^c)(\phi_0 Y_0)^c + a_0^c(X_0^c)(\phi_0 Y_0)^v + a_0^v(X_0^c)(\phi_0 Y_0)^c \\ & - a_0^c(Y_0^c)(\phi_0 X_0)^v - a_0^v(Y_0^c)(\phi_0 X_0)^c,\end{aligned} \tag{49}$$

$$\eta_0^c(X_0^c) = g^c(X_0^c, \xi_0^c), \tag{50}$$

$$a^c(X_0^c) = g^c(X_0^c, A_0^c), \tag{51}$$

$$\begin{aligned}(\ddot{\nabla}_{X_0^c}^c g^c)(Y_0^c, Z_0^c) = {} & -\eta_0^c(Y_0^c) g^c\Big((\phi_0 X_0)^v, Z_0^c\Big) - \eta_0^v(Y_0^c) g^c\Big((\phi_0 X_0)^c, Z_0^c\Big) \\ & - \eta_0^c(Z_0^c) g^c\Big((\phi_0 X_0)^v, Y_0^c\Big) - \eta_0^v(Z_0^c) g^c\Big((\phi_0 X_0)^c, Y_0^c\Big) \\ & - 2a^c(X_0^c) g^c\Big((\phi_0 Y_0)^v, Z_0^c\Big) - 2a^v(X_0^c) g^c\Big((\phi_0 Y_0)^c, Z_0^c\Big).\end{aligned} \tag{52}$$

The connection given by Equation (48) is said to be a QSNMC on an LP-Sasakian manifold in its tangent bundle if the torsion tensor $\ddot{T}^c$ of $T_0 M^n$ endowed with $\ddot{\nabla}^c$ satisfies Equation (49) and the complete lifts of Lorentzian metric g^c fulfill the relation (52).

Theorem 1. *If an LP-Sasakian manifold (M^n, g) with an almost Lorentzian para-contact metric structure $(\phi_0, \xi_0, \eta_0, g)$ admitting a QSNMC $\ddot{\nabla}$ which satisfies (49) and (52), then the QSNMC in the tangent bundle is given by*

$$\ddot{\nabla}_{X_0^c}^c Y_0^c = \nabla_{X_0^c}^c Y_0 + \eta_0^c(Y_0^c)(\phi_0 X_0)^v + \eta_0^v(Y_0^c)(\phi_0 X_0)^c + a^c(X_0^c)(\phi_0 Y_0)^v + a^v(X_0^c)(\phi_0 Y_0)^c.$$

Proof. Let $\ddot{\nabla}^c$ be the complete lifts of a linear connection in M^n given by

$$\ddot{\nabla}_{X_0^c}^c Y_0^c = \nabla_{X_0^c}^c Y_0^c + H_0^c(X_0^c, Y_0^c). \tag{53}$$

Now, we shall determine the complete lifts of the tensor field H_0^c such that $\ddot{\nabla}^c$ satisfies (49) and (52). From (53), we have

$$\ddot{T}^c(X_0^c, Y_0^c) = H_0^c(X_0^c, Y_0^c) - H_0^c(Y_0^c, X_0^c). \tag{54}$$

We denote

$$G_0^c(X_0^c, Y_0^c, Z_0^c) = (\ddot{\nabla}_{X_0^c}^c g^c)(Y_0^c, Z_0^c). \tag{55}$$

From (53) and (55), we have

$$g^c\Big(H_0^c(X_0^c, Y_0^c), Z_0^c\Big) + g^c\Big(H_0^c(X_0^c, Z_0^c), Y_0^c\Big) = -G_0^c(X_0^c, Y_0^c, Z_0^c). \tag{56}$$

Using (52), (53), (55), and (56) we have

$$\begin{aligned}
&g^c\left(\ddot{T}^c(X_0^c, Y_0^c), Z_0^c\right) + g^c\left(\ddot{T}^c(Z_0^c, X_0^c), Y_0^c\right) + g^c\left(\ddot{T}^c(Z_0^c, Y_0^c), X_0^c\right) \\
&= g^c\left(H_0^c(X_0^c, Y_0^c), Z_0^c\right) - g^c\left(H_0^c(Y_0^c, X_0^c), Z_0^c\right) + g^c\left(H_0^c(Z_0^c, X_0^c), Y_0^c\right) \\
&- g^c\left(H_0^c(X_0^c, Z_0^c), Y_0^c\right) + g^c\left(H_0^c(Z_0^c, Y_0^c), X_0^c\right) - g^c\left(H_0^c(Y_0^c, Z_0^c), X_0^c\right) \\
&= g^c\left(H_0^c(X_0^c, Y_0^c), Z_0^c\right) - g^c\left(H_0^c(X_0^c, Z_0^c), Y_0^c\right) - G_0^c(Z_0^c, X_0^c, Y_0^c) + G_0^c(Y_0^c, X_0^c, Z_0^c) \\
&= 2g^c\left(H_0^c(X_0^c, Y_0^c), Z_0^c\right) + G_0^c(X_0^c, Y_0^c, Z_0^c) + G_0^c(Y_0^c, X_0^c, Z_0^c) - G_0^c(Z_0^c, X_0^c, Y_0^c) \\
&= 2g^c\left(H_0^c(X_0^c, Y_0^c), Z_0^c\right) - 2\left\{\eta_0^c(Z_0^c) g^c\left((\phi_0 X_0)^v, Y_0^c\right) + \eta_0^v(Z_0^c) g^c\left((\phi_0 X_0)^c, Y_0^c\right)\right\} \\
&- 2\left\{a_0^c(X_0^c) g^c\left((\phi_0 Y_0)^v, Z_0^c\right) + a_0^v(X_0^c) g^c\left((\phi_0 Y_0)^c, Z_0^c\right)\right\} - 2\left\{a_0^c(Y_0^c) g^c\left((\phi_0 X_0)^v, Z_0^c\right)\right. \\
&\left. + a_0^v(Y_0^c) g^c\left((\phi_0 X_0)^c, Z_0^c\right)\right\} + 2\left\{a_0^c(Z_0^c) g^c\left((\phi_0 X_0)^v, Y_0^c\right) + a_0^v(Z_0^c) g^c\left((\phi_0 X_0)^c, Y_0^c\right)\right\},
\end{aligned}$$

or,

$$\begin{aligned}
H_0^c(X_0^c, Y_0^c) = \frac{1}{2}&\left\{\ddot{T}^c(X_0^c, Y_0^c) + {}'\ddot{T}^c(X_0^c, Y_0^c) + {}'\ddot{T}^c(Y_0^c, X_0^c)\right\} + a_0^c(X_0^c)(\phi_0 Y_0)^v \\
&+ a_0^v(X_0^c)(\phi_0 Y_0)^c + a_0^c(Y_0^c)(\phi_0 X_0)^v + a_0^v(Y_0^c)(\phi_0 X_0)^c \\
&+ g^c\left(\phi_0 X_0)^c, Y_0^c\right)\xi_0^v + g^c\left(\phi_0 X_0)^v, Y_0^c\right)\xi_0^c \\
&- g^c\left(\phi_0 X_0)^c, Y_0^c\right)A_0^v - g^c\left(\phi_0 X_0)^v, Y_0^c\right)A_0^c,
\end{aligned}$$

where ${}'\ddot{T}^c$ is a tensor field of type $(1,2)$ defined by

$$g^c\left({}'\ddot{T}^c(X_0^c, Y_0^c), Z_0^c\right) = g^c\left(\ddot{T}^c(Z_0^c, X_0^c), Y_0^c\right),$$

or,

$$H_0^c(X_0^c, Y_0^c) = \eta_0^c(Y_0^c)(\phi_0 X_0)^v + \eta_0^v(Y_0^c)(\phi_0 X_0)^c + a^c(X_0^c)(\phi_0 Y_0)^v + a^v(X_0^c)(\phi_0 Y_0)^c,$$

which gives

$$\ddot{\nabla}^c_{X_0^c} Y_0^c = \nabla^c_{X_0^c} Y_0 + \eta_0^c(Y_0^c)(\phi_0 X_0)^v + \eta_0^v(Y_0^c)(\phi_0 X_0)^c + a^c(X_0^c)(\phi_0 Y_0)^v + a^v(X_0^c)(\phi_0 Y_0)^c.$$

□

6. Curvature Tensor of LP-Sasakian Manifolds Endowed with QSNMC to Tangent Bundle

Let $\ddot{R}_0^c$ and R_0^c be the curvature tensors of the connections $\ddot{\nabla}^c$ and ∇^c to tangent bundle T_0M^n, respectively.

$$\ddot{R}_0^c(X_0^c, Y_0^c)Z_0^c = \ddot{\nabla}^c_{X_0^c}\ddot{\nabla}^c_{Y_0^c}Z_0^c - \ddot{\nabla}^c_{Y_0^c}\ddot{\nabla}^c_{X_0^c}Z_0^c - \ddot{\nabla}^c_{[X_0^c, Y_0^c]}Z_0^c. \tag{57}$$

Using (48) in (57), we have

$$\begin{aligned}
\ddot{R}_0^c(X_0^c,Y_0^c)Z_0^c = {} & R_0^c(X_0^c,Y_0^c)Z_0^c + g^c\Big((\phi_0X_0)^c,Z_0^c\Big)(\phi_0Y_0)^v \\
& + g^c\Big((\phi_0X_0)^v,Z_0^c\Big)(\phi_0Y_0)^c - g^c\Big((\phi_0Y_0)^c,Z_0^c\Big)(\phi_0X_0)^v \\
& - g^c\Big((\phi_0Y_0)^v,Z_0^c\Big)(\phi_0X_0)^c + \eta_0^c(Y_0^c)\eta_0^c(Z_0^c)X_0^v \\
& + \eta_0^c(Y_0^c)\eta_0^v(Z_0^c)X_0^c + \eta_0^v(Y_0^c)\eta_0^c(Z_0^c)X_0^c \\
& - \eta_0^c(X_0^c)\eta_0^c(Z_0^c)Y_0^v - \eta_0^c(X_0^c)\eta_0^v(Z_0^c)Y_0^c \\
& - \eta_0^v(X_0^c)\eta_0^c(Z_0^c)Y_0^c + a_0^c(Y_0^c)g^c(X_0^c,Z_0^c)\xi_0^v \\
& + a_0^c(Y_0^c)g^c(X_0^v,Z_0^c)\xi_0^c + a_0^v(Y_0^c)g^c(X_0^c,Z_0^c)\xi_0^c \\
& - a_0^c(X_0^c)g^c(Y_0^c,Z_0^c)\xi_0^v - a_0^c(X_0^c)g^c(Y_0^v,Z_0^c)\xi_0^c \\
& - a_0^v(X_0^c)g^c(Y_0^c,Z_0^c)\xi_0^c + a_0^c(Y_0^c)\eta_0^c(X_0^c)\eta_0^c(Z_0^c)\xi_0^v \\
& + a_0^c(Y_0^c)\eta_0^c(X_0^c)\eta_0^v(Z_0^c)\xi_0^c + a_0^c(Y_0^c)\eta_0^v(X_0^c)\eta_0^c(Z_0^c)\xi_0^c \\
& + a_0^v(Y_0^c)\eta_0^c(X_0^c)\eta_0^c(Z_0^c)\xi_0^c - a_0^c(X_0^c)\eta_0^c(Y_0^c)\eta_0^c(Z_0^c)\xi_0^v \\
& - a_0^c(X_0^c)\eta_0^c(Y_0^c)\eta_0^v(Z_0^c)\xi_0^c - a_0^c(X_0^c)\eta_0^v(Y_0^c)\eta_0^c(Z_0^c)\xi_0^c \\
& - a_0^v(X_0^c)\eta_0^c(Y_0^c)\eta_0^c(Z_0^c)\xi_0^c + da_0^c(X_0^c,Y_0^c)(\phi_0Z_0)^v \\
& + da_0^v(X_0^c,Y_0^c)(\phi_0Z_0)^c,
\end{aligned} \tag{58}$$

where

$$R_0^c(X_0^c,Y_0^c)Z_0^c = \nabla^c_{X_0^c}\nabla^c_{Y_0^c}Z_0 - \nabla^c_{Y_0^c}\nabla^c_{X_0^c}Z_0 - \nabla^c_{[X_0^c,Y_0^c]}Z_0^c, \tag{59}$$

is the curvature tensor of ∇^c with respect to the Riemannian connection. Contracting (58), we obtain

$$\begin{aligned}
\ddot{S}_0^c(Y_0^c,Z_0^c) = {} & S_0^c(Y_0^c,Z_0^c) - \gamma^c g^c\Big((\phi_0Y_0)^c,Z_0^c\Big) + \Big[1 - a_0^c(\xi_0^c)\Big]g^c(Y_0^c,Z_0^c) \\
& + \Big[\eta_0^c - a_0^c(\xi_0^c)\Big]\Big[\eta_0^c(Y_0^c)\eta_0^v(Z_0^c) + \eta_0^v(Y_0^c)\eta_0^c(Z_0^c)\Big] \\
& + da_0^c\Big((\phi_0Z_0)^c,Y_0^c\Big),
\end{aligned} \tag{60}$$

and

$$\ddot{r_0}^c = r_0^c - (n-1)a_0^c(\xi_0^c) + \lambda_0^c - \gamma^{c^2}, \tag{61}$$

where $\ddot{S}_0^c$ and $\ddot{r_0}^c$ are the Ricci tensor and scalar curvature with respect to $\ddot{\nabla}^c$.

$$\lambda_0^c = trace\ da_0^c\Big((\phi_0Z_0)^c,Y_0^c\Big) \text{ and } \gamma^c = trace\ \phi_0^c. \tag{62}$$

Theorem 2. *In an LP-Sasakian manifold (M^n,g) with tangent bundle T_0M^n admitting QSNMC, we have the following:*

1. *The complete lifts of curvature tensor $\ddot{R}_0^c$ are given by Equation (58).*
2. *The complete lifts of Ricci tensor $\ddot{S}_0^c$ are given by Equation (60).*
3. *The complete lifts of scalar curvature $\ddot{r}_0$ are given by Equation (61).*

Let us consider that $\ddot{R}_0^c(X_0^c,Y_0^c) = 0$ in (58), and by contracting it we also obtain

$$\begin{aligned}
S_0^c(Y_0^c,Z_0^c) = {} & \gamma^c g^c\Big((\phi_0Y_0)^c,Z_0^c\Big) - \Big[1 - a_0^c(\xi_0^c)\Big]g^c(Y_0^c,Z_0^c) \\
& - \Big[\eta_0^c - a_0^c(\xi_0^c)\Big]\Big[\eta_0^c(Y_0^c)\eta_0^v(Z_0^c) + \eta_0^v(Y_0^c)\eta_0^c(Z_0^c)\Big] \\
& - da_0^c\Big((\phi_0Z_0)^c,Y_0^c\Big),
\end{aligned} \tag{63}$$

which gives

$$r_0^c = (n-1)a_0^c(\xi_0^c) - \lambda_0^c + \gamma^{c^2}. \tag{64}$$

Theorem 3. *In an LP-Sasakian manifold, (M^n, g), with tangent bundle T_0M^n endowed with QSNMC whose curvature tensor vanishes, then the complete lift of r_0^c is given by (64).*

From (58), it follows that

$$'\ddot{R}_0^c(X_0^c, Y_0^c, Z_0^c, W_0^c) + {}'\ddot{R}_0^c(Y_0^c, X_0^c, Z_0^c, W_0^c) = 0, \tag{65}$$

$$\begin{aligned}
&'\ddot{R}_0^c(X_0^c, Y_0^c, Z_0^c, W_0^c) + {}'\ddot{R}_0^c(X_0^c, Y_0^c, W_0^c, Z_0^c) \\
&= \eta_0^c(Y_0^c)\eta_0^v(Z_0^c)g^c(X_0^c, W_0^c) + \eta_0^v(Y_0^c)\eta_0^c(Z_0^c)g^c(X_0^c, W_0^c) \\
&- \eta_0^c(X_0^c)\eta_0^v(Z_0^c)g^c(Y_0^c, W_0^c) - \eta_0^v(X_0^c)\eta_0^c(Z_0^c)g^c(Y_0^c, W_0^c) \\
&+ \eta_0^c(Y_0^c)\eta_0^v(W_0^c)g^c(X_0^c, Z_0^c) + \eta_0^v(Y_0^c)\eta_0^c(W_0^c)g^c(X_0^c, Z_0^c) \\
&- \eta_0^c(X_0^c)\eta_0^v(W_0^c)g^c(Y_0^c, Z_0^c) - \eta_0^v(X_0^c)\eta_0^c(W_0^c)g^c(Y_0^c, Z_0^c) \\
&+ a_0^c(Y_0^c)\eta_0^v(W_0^c)g^c(X_0^c, Z_0^c) + a_0^v(Y_0^c)\eta_0^c(W_0^c)g^c(X_0^c, Z_0^c) \\
&- a_0^c(X_0^c)\eta_0^v(W_0^c)g^c(Y_0^c, Z_0^c) - a_0^v(X_0^c)\eta_0^c(W_0^c)g^c(Y_0^c, Z_0^c) \\
&+ a_0^c(Y_0^c)\eta_0^v(Z_0^c)g^c(X_0^c, W_0^c) + a_0^v(Y_0^c)\eta_0^c(Z_0^c)g^c(X_0^c, W_0^c) \\
&- a_0^c(X_0^c)\eta_0^v(Z_0^c)g^c(Y_0^c, W_0^c) - a_0^v(X_0^c)\eta_0^c(Z_0^c)g^c(Y_0^c, W_0^c) \\
&+ 2\Big[a_0^c(Y_0^c)\eta_0^c(X_0^c)\eta_0^c(Z_0^c)\eta_0^v(W_0^c) + a_0^c(Y_0^c)\eta_0^c(X_0^c)\eta_0^v(Z_0^c)\eta_0^c(W_0^c) \\
&+ a_0^c(Y_0^c)\eta_0^v(X_0^c)\eta_0^c(Z_0^c)\eta_0^c(W_0^c) + a_0^v(Y_0^c)\eta_0^c(X_0^c)\eta_0^c(Z_0^c)\eta_0^c(W_0^c)\Big] \\
&- 2\Big[a_0^c(X_0^c)\eta_0^c(Y_0^c)\eta_0^c(Z_0^c)\eta_0^v(W_0^c) + a_0^c(X_0^c)\eta_0^c(Y_0^c)\eta_0^v(Z_0^c)\eta_0^c(W_0^c) \\
&+ a_0^c(X_0^c)\eta_0^v(Y_0^c)\eta_0^c(Z_0^c)\eta_0^c(W_0^c) + a_0^v(X_0^c)\eta_0^c(Y_0^c)\eta_0^c(Z_0^c)\eta_0^c(W_0^c)\Big] \\
&+ 2da_0^c(X_0^c, Y_0^c)g^c\Big((\phi_0 Z_0)^c, W_0^c\Big).
\end{aligned} \tag{66}$$

$$\begin{aligned}
&'\ddot{R}_0^c(X_0^c, Y_0^c, Z_0^c, W_0^c) - {}'\ddot{R}_0^c(Z_0^c, W_0^c, X_0^c, Y_0^c) \\
&= \eta_0^c(Y_0^c)\eta_0^v(Z_0^c)g^c(X_0^c, W_0^c) + \eta_0^v(Y_0^c)\eta_0^c(Z_0^c)g^c(X_0^c, W_0^c) \\
&- \eta_0^c(X_0^c)\eta_0^v(W_0^c)g^c(Y_0^c, Z_0^c) - \eta_0^v(X_0^c)\eta_0^c(W_0^c)g^c(Y_0^c, Z_0^c) \\
&+ a_0^c(Y_0^c)\eta_0^v(W_0^c)g^c(X_0^c, Z_0^c) + a_0^v(Y_0^c)\eta_0^c(W_0^c)g^c(X_0^c, Z_0^c) \\
&- a_0^c(X_0^c)\eta_0^v(W_0^c)g^c(Y_0^c, Z_0^c) - a_0^v(X_0^c)\eta_0^c(W_0^c)g^c(Y_0^c, Z_0^c) \\
&+ a_0^c(Z_0^c)\eta_0^v(Y_0^c)g^c(X_0^c, W_0^c) + a_0^v(Z_0^c)\eta_0^c(Y_0^c)g^c(X_0^c, W_0^c) \\
&- a_0^c(W_0^c)\eta_0^v(Y_0^c)g^c(X_0^c, Z_0^c) - a_0^v(W_0^c)\eta_0^c(Y_0^c)g^c(X_0^c, Z_0^c) \\
&+ a_0^c(Y_0^c)\eta_0^c(X_0^c)\eta_0^c(Z_0^c)\eta_0^v(W_0^c) + a_0^c(Y_0^c)\eta_0^c(X_0^c)\eta_0^v(Z_0^c)\eta_0^c(W_0^c) \\
&+ a_0^c(Y_0^c)\eta_0^v(X_0^c)\eta_0^c(Z_0^c)\eta_0^c(W_0^c) + a_0^v(Y_0^c)\eta_0^c(X_0^c)\eta_0^c(Z_0^c)\eta_0^c(W_0^c) \\
&- a_0^c(X_0^c)\eta_0^c(Y_0^c)\eta_0^c(Z_0^c)\eta_0^v(W_0^c) - a_0^c(X_0^c)\eta_0^c(Y_0^c)\eta_0^v(Z_0^c)\eta_0^c(W_0^c) \\
&- a_0^c(X_0^c)\eta_0^v(Y_0^c)\eta_0^c(Z_0^c)\eta_0^c(W_0^c) - a_0^v(X_0^c)\eta_0^c(Y_0^c)\eta_0^c(Z_0^c)\eta_0^c(W_0^c) \\
&+ a_0^c(Z_0^c)\eta_0^c(X_0^c)\eta_0^c(Y_0^c)\eta_0^v(W_0^c) + a_0^c(Z_0^c)\eta_0^c(X_0^c)\eta_0^v(Y_0^c)\eta_0^c(W_0^c) \\
&+ a_0^c(Z_0^c)\eta_0^v(X_0^c)\eta_0^c(Y_0^c)\eta_0^c(W_0^c) + a_0^v(Z_0^c)\eta_0^c(X_0^c)\eta_0^c(Y_0^c)\eta_0^c(W_0^c) \\
&- a_0^c(W_0^c)\eta_0^c(X_0^c)\eta_0^c(Y_0^c)\eta_0^v(Z_0^c) - a_0^c(W_0^c)\eta_0^c(X_0^c)\eta_0^v(Y_0^c)\eta_0^c(Z_0^c) \\
&- a_0^c(W_0^c)\eta_0^v(X_0^c)\eta_0^c(Y_0^c)\eta_0^c(Z_0^c) - a_0^v(W_0^c)\eta_0^c(X_0^c)\eta_0^c(Y_0^c)\eta_0^c(Z_0^c) \\
&+ da_0^c(X_0^c, Y_0^c)g^c\Big((\phi_0 Z_0)^c, W_0^c\Big) - da_0^c(Z_0^c, W_0^c)g^c\Big((\phi_0 X_0)^c, Y_0^c\Big),
\end{aligned} \tag{67}$$

and

$$\begin{aligned}
&{}'\ddot{R}_0^c(X_0^c, Y_0^c, Z_0^c, W_0^c) + {}'\ddot{R}_0^c(Y_0^c, Z_0^c, X_0^c, W_0^c) + {}'\ddot{R}_0^c(Z_0^c, X_0^c, Y_0^c, W_0^c) \\
&\quad = da_0^c(X_0^c, Y_0^c) g^c\Big((\phi_0 Z_0)^c, W_0^c\Big) + da_0^c(Y_0^c, Z_0^c) g^c\Big((\phi_0 X_0)^c, W_0^c\Big) \\
&\quad + da_0^c(Z_0^c, X_0^c) g^c\Big((\phi_0 Y_0)^c, W_0^c\Big).
\end{aligned} \tag{68}$$

If the 1-form a_0^c is closed, then from (68) we have

$$\begin{aligned}
&{}'\ddot{R}_0^c(X_0^c, Y_0^c, Z_0^c, W_0^c) + {}'\ddot{R}_0^c(Y_0^c, Z_0^c, X_0^c, W_0^c) \\
&\quad + {}'\ddot{R}_0^c(Z_0^c, X_0^c, Y_0^c, W_0^c) = 0,
\end{aligned} \tag{69}$$

where

$$\begin{aligned}
&{}'\ddot{R}_0^c(X_0^c, Y_0^c, Z_0^c, W_0^c) = g^c\Big(\ddot{R}_0^c(X_0^c, Y_0^c) Z_0^c, W_0^c\Big) \\
&\text{and } {}'R_0^c(X_0^c, Y_0^c, Z_0^c, W_0^c) = g^c\Big(R_0^c(X_0^c, Y_0^c) Z_0^c, W_0^c\Big).
\end{aligned}$$

Theorem 4. *In an LP-Sasakian manifold, (M^n, g) with tangent bundle T_0M^n endowed with a QSNMC, the curvature tensor satisfies relations (65)–(68). In particular, if the complete lift of 1-form a_0^c is closed, then*

$$ {}'\ddot{R}_0^c(X_0^c, Y_0^c, Z_0^c, W_0^c) + {}'\ddot{R}_0^c(Y_0^c, Z_0^c, X_0^c, W_0^c) + {}'\ddot{R}_0^c(Z_0^c, X_0^c, Y_0^c, W_0^c) = 0.$$

7. Symmetric and Skew-Symmetric Condition of the Ricci Tensor of $\ddot{\nabla}^c$ in an LP-Sasakian Manifold Endowed with a QSNMC to Tangent Bundle

From Equation (60), we have

$$\begin{aligned}
\ddot{S}_0^c(Z_0^c, Y_0^c) &= S_0^c(Z_0^c, Y_0^c) - \gamma^c g^c\Big((\phi_0 Z_0)^c, Y_0^c\Big) \\
&\quad + \Big[1 - a_0^c(\xi_0^c)\Big] g^c(Y_0^c, Z_0^c) + \Big[\eta_0^c - a_0^c(\xi_0^c)\Big]\Big[\eta_0^c(Y_0^c)\eta_0^v(Z_0^c) \\
&\quad + \eta_0^v(Y_0^c)\eta_0^c(Z_0^c)\Big] + da_0^c\Big((\phi_0 Y_0)^c, Z_0^c\Big).
\end{aligned} \tag{70}$$

From (60) and (70), we have

$$\ddot{S}_0^c(Y_0^c, Z_0^c) - \ddot{S}_0^c(Z_0^c, Y_0^c) = da_0^c\Big((\phi_0 Z_0)^c, Y_0^c\Big) - da_0^c\Big((\phi_0 Y_0)^c, Z_0^c\Big). \tag{71}$$

If $\ddot{S}_0^c(Y_0^c, Z_0^c)$ is symmetric, then the left-hand side of (71) vanishes, and then

$$da_0^c\Big((\phi_0 Z_0)^c, Y_0^c\Big) = da_0^c\Big((\phi_0 Y_0)^c, Z_0^c\Big). \tag{72}$$

Moreover, if Equation (72) holds, then from (71), $\ddot{S}_0^c(Y_0^c, Z_0^c)$ is symmetric.

Theorem 5. *In an LP-Sasakian manifold (M^n, g) with tangent bundle T_0M^n endowed with QSNMC $\ddot{\nabla}^c$, the Ricci tensor $\ddot{S}_0^c(Y_0^c, Z_0^c)$ with respect to QSNMC is symmetric if and only if relation (72) holds.*

From (60) and (70), we have

$$
\begin{aligned}
\ddot{S}_0^c(Y_0^c, Z_0^c) + \ddot{S}_0^c(Z_0^c, Y_0^c) = {} & 2S_0^c(Y_0^c, Z_0^c) - 2\gamma^c g^c\Big((\phi_0 Y_0)^c, Z_0^c\Big) \\
& + 2\Big[1 - a_0^c(\xi_0^c)\Big] g^c(Y_0^c, Z_0^c) \\
& + 2\Big[n - a_0^c(\xi_0^c)\Big]\Big[\eta_0^c(Y_0^c)\eta_0^v(Z_0^c) \\
& + \eta_0^v(Y_0^c)\eta_0^c(Z_0^c)\Big] + da_0^c\Big((\phi_0 Y_0)^c, Z_0^c\Big) \\
& + da_0^c\Big((\phi_0 Z_0)^c, Y_0^c\Big).
\end{aligned} \tag{73}
$$

By taking the skew-symmetry of $\ddot{S}_0^c(Y_0^c, Z_0^c)$, the left-hand side of (73) will vanish and we have

$$
\begin{aligned}
S_0^c(Y_0^c, Z_0^c) = {} & \gamma^c g^c\Big((\phi_0 Y_0)^c, Z_0^c\Big) \\
& - \Big[1 - a_0^c(\xi_0^c)\Big] g^c(Y_0^c, Z_0^c) - \Big[n - a_0^c(\xi_0^c)\Big] \\
& \Big[\eta_0^c(Y_0^c)\eta_0^v(Z_0^c) - \eta_0^v(Y_0^c)\eta_0^c(Z_0^c)\Big] \\
& - \frac{1}{2}\Big[da_0^c\Big((\phi_0 Y_0)^c, Z_0^c\Big) + da_0^c\Big((\phi_0 Z_0)^c, Y_0^c\Big)\Big].
\end{aligned} \tag{74}
$$

Moreover if $S_0^c(Y_0^c, Z_0^c)$ is given by (74), then from (73), we have

$$
S_0^c(Y_0^c, Z_0^c) + S_0^c(Z_0^c, Y_0^c) = 0.
$$

Theorem 6. *The necessary and sufficient condition for the Ricci tensor of $\ddot{\nabla}^c$ in an LP-Sasakian manifold (M^n, g) endowed with QSNMC $\ddot{\nabla}^c$ in the tangent bundle T_0M^n to be skew-symmetric is that the Ricci tensor of the Levi-Civita connection ∇^c is given by (74).*

8. Skew-Symmetric Properties of the Projective Ricci Tensor in an LP-Sasakian Manifold Endowed with QSNMC $\ddot{\nabla}^c$ in the Tangent Bundle

Chaki and Saha defined the projective Ricci tensor in a Riemannian manifold as [34]

$$
P_0(X_0, Y_0) = \frac{n}{n-1}\Big[S_0(X_0, Y_0) - \frac{r_0}{n} g(X_0, Y_0)\Big]. \tag{75}
$$

So, the projective Ricci tensor with respect to QSNMC $\ddot{\nabla}$ is defined as

$$
\ddot{P}_0(X_0, Y_0) = \frac{n}{n-1}\Big[\ddot{S}_0(X_0, Y_0) - \frac{\ddot{r}_0}{n} g(X_0, Y_0)\Big]. \tag{76}
$$

Taking a complete lift by mathematical operators on (76), we have

$$
\ddot{P}_0^c(X_0^c, Y_0^c) = \frac{n}{n-1}\Big[\ddot{S}_0^c(X_0^c, Y_0^c) - \frac{\ddot{r}_0^c}{n} g^c(X_0^c, Y_0^c)\Big]. \tag{77}
$$

Using (60) and (61) in (77), we have

$$
\begin{aligned}
\ddot{P}_0^c(X_0^c, Y_0^c) = {} & \frac{n}{n-1}\Big[S_0^c(X_0^c, Y_0^c) - \gamma^c g^c\Big((\phi_0 X_0)^c, Y_0^c\Big) \\
& + \Big(1 - a_0^c(\xi_0^c)\Big) g^c(X_0^c, Y_0^c) + \Big(n - a_0^c(\xi_0^c)\Big)\Big(\eta_0^c(X_0^c)\eta_0^v(Y_0^c) \\
& + \eta_0^v(X_0^c)\eta_0^c(Y_0^c)\Big) + da_0^c\Big((\phi_0 Y_0)^c, X_0^c\Big) \\
& - \frac{1}{n}\Big(r_0^c - (n-1)a_0^c(\xi_0^c) + \lambda_0^c - \gamma^{c^2}\Big) g^c(X_0^c, Y_0^c)\Big].
\end{aligned} \tag{78}
$$

Similarly, we have

$$\ddot{P}_0^c(Y_0^c, X_0^c) = \frac{n}{n-1}\Big[S_0^c(Y_0^c, X_0^c) - \gamma^c g^c\Big((\phi_0 Y_0)^c, X_0^c\Big) \\ + \Big(1 - a_0^c(\xi_0^c)\Big)g^c(Y_0^c, X_0^c) + \Big(n - a_0^c(\xi_0^c)\Big)\Big(\eta_0^c(X_0^c)\eta_0^v(Y_0^c) \\ + \eta_0^v(X_0^c)\eta_0^c(Y_0^c)\Big) + da_0^c\Big((\phi_0 X_0)^c, Y_0^c\Big) \\ - \frac{1}{n}\Big(r_0^c - (n-1)a_0^c(\xi_0^c) + \lambda_0^c - \gamma^{c^2}\Big)g^c(Y_0^c, X_0^c)\Big]. \tag{79}$$

From (78) and (79), we have

$$\ddot{P}_0^c(X_0^c, Y_0^c) + \ddot{P}_0^c(Y_0^c, X_0^c) \\ = \frac{n}{n-1}\Big[2S_0^c(X_0^c, Y_0^c) - 2\gamma^c g^c\Big((\phi_0 X_0)^c, Y_0^c\Big) \\ + 2\Big(1 - a_0^c(\xi_0^c)\Big)g^c(X_0^c, Y_0^c) + 2\Big(n - a_0^c(\xi_0^c)\Big) \\ \Big(\eta_0^c(X_0^c)\eta_0^v(Y_0^c) + \eta_0^v(X_0^c)\eta_0^c(Y_0^c)\Big) \\ - \frac{2}{n}\Big(r_0^c - (n-1)a_0^c(\xi_0^c) + \lambda_0^c - \gamma^{c^2}\Big)g^c(X_0^c, Y_0^c) \\ + da_0^c\Big((\phi_0 X_0)^c, Y_0^c\Big) + da_0^c\Big((\phi_0 Y_0)^c, X_0^c\Big)\Big]. \tag{80}$$

If $\ddot{P}_0^c(X_0^c, Y_0^c)$ is skew-symmetric, then the left-hand side of (80) vanishes and we have

$$S_0^c(X_0^c, Y_0^c) = \Big[\gamma^c g^c\Big((\phi_0 X_0)^c, Y_0^c\Big) - \Big(1 - a_0^c(\xi_0^c)\Big)g^c(X_0^c, Y_0^c) \\ - \Big(n - a_0^c(\xi_0^c)\Big)\Big(\eta_0^c(X_0^c)\eta_0^v(Y_0^c) + \eta_0^v(X_0^c)\eta_0^c(Y_0^c)\Big) \\ + \frac{1}{n}\Big(r_0^c - (n-1)a_0^c(\xi_0^c) + \lambda_0^c - \gamma^{c^2}\Big)g^c(X_0^c, Y_0^c) \\ - \frac{1}{2}\Big(da_0^c\Big((\phi_0 X_0)^c, Y_0^c\Big) + da_0^c\Big((\phi_0 Y_0)^c, X_0^c\Big)\Big)\Big]. \tag{81}$$

Moreover, if $S_0^c(X_0^c, Y_0^c)$ is given by (81), then from (80) we obtain

$$\ddot{P}_0^c(X_0^c, Y_0^c) + \ddot{P}_0^c(Y_0^c, X_0^c) = 0 \ \text{ s.t } \ \ddot{P}_0^c(X_0^c, Y_0^c) = -\ddot{P}_0^c(Y_0^c, X_0^c). \tag{82}$$

which gives a skew-symmetric condition of the projective Ricci tensor of $\ddot{\nabla}^c$.

Theorem 7. *The necessary and sufficient condition for the projective Ricci tensor of $\ddot{\nabla}^c$ in an LP-Sasakian manifold (M^n, g) endowed with QSNMC $\ddot{\nabla}^c$ in the tangent bundle T_0M^n to be skew-symmetric is that the Ricci tensor of the Levi-Civita connection $\ddot{\nabla}^c$ is given by (81).*

9. Lifts of Einstein Manifold Endowed with QSNMC $\ddot{\nabla}^c$ in an LP-Sasakian Manifold to the Tangent Bundle

A Riemannian manifold (M^n, g) is called an Einstein manifold with respect to Riemannian connection if

$$S_0^c(X_0^c, Y_0^c) = \frac{r_0^c}{n} g^c(X_0^c, Y_0^c). \tag{83}$$

Then, the Einstein manifold with respect to QSNMC $\ddot{\nabla}^c$ is given by

$$\ddot{S}_0^c(X_0^c, Y_0^c) = \frac{\ddot{r}_0^c}{n} g^c(X_0^c, Y_0^c). \tag{84}$$

Using (60) and (61) in (84), we have

$$\begin{aligned}
&\ddot{S}_0^c(X_0^c, Y_0^c) - \frac{\ddot{r}_0^c}{n} g^c(X_0^c, Y_0^c) \\
&= S_0^c(X_0^c, Y_0^c) - \frac{r_0^c}{n} g^c(X_0^c, Y_0^c) - \gamma^c g^c\Big((\phi_0 X_0)^c, Y_0^c\Big) \\
&+ da_0^c\Big((\phi_0 Y_0)^c, X_0^c\Big) + \frac{1}{n}\Big[n + \gamma^{c^2} - \lambda_0^c - a_0^c(\xi_0^c)\Big] g^c(X_0^c, Y_0^c) \\
&+ \Big(n - a_0^c(\xi_0^c)\Big)\Big[\eta_0^c(X_0^c)\eta_0^v(Y_0^c) + \eta_0^v(X_0^c)\eta_0^c(Y_0^c)\Big].
\end{aligned} \tag{85}$$

If

$$\begin{aligned}
&\gamma^c g^c\Big((\phi_0 X_0)^c, Y_0^c\Big) + da_0^c\Big(X_0^c, (\phi_0 Y_0)^c\Big) \\
&= \frac{1}{n}\Big[n + \gamma^{c^2} - \lambda_0^c - a_0^c(\xi_0^c)\Big] g^c(X_0^c, Y_0^c) \\
&+ \Big(n - a_0^c(\xi_0^c)\Big)\Big[\eta_0^c(X_0^c)\eta_0^v(Y_0^c) + \eta_0^v(X_0^c)\eta_0^c(Y_0^c)\Big],
\end{aligned} \tag{86}$$

then from (85), we have

$$\ddot{S}_0^c(X_0^c, Y_0^c) - \frac{\ddot{r}_0^c}{n} g^c(X_0^c, Y_0^c) = S_0^c(X_0^c, Y_0^c) - \frac{r_0^c}{n} g^c(X_0^c, Y_0^c). \tag{87}$$

Theorem 8. *In an LP-Sasakian manifold (M^n, g) with tangent bundle $T_0 M^n$ admitting QSNMC if Equation (86) holds, then the manifold reduces to an Einstein manifold for the Riemannian connection if and only if it is an Einstein manifold for the connection $\ddot{\nabla}^c$.*

10. Example

Let M be a four-dimensional manifold defined as

$$M = \Big\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4; x_4 \neq 0\Big\}, \tag{88}$$

where $\mathbb{R}$ is the set of real numbers. Let x_1, x_2, x_3, x_4 be given by

$$e_1 = \frac{x_1}{x_4}\frac{\partial}{\partial x_1}, \qquad e_2 = \frac{x_2}{x_4}\frac{\partial}{\partial x_2}, \qquad e_3 = \frac{x_3}{x_4}\frac{\partial}{\partial x_3}, \qquad e_4 = x_4\frac{\partial}{\partial x_4},$$

where $\{e_1, e_2, e_3, e_4\}$ are a linearly independent global frame on M. Let the 1-form η_0 be given by

$$\eta_0(X_0) = g(X_0, e_4).$$

The Lorentzian metric g is defined by

$$g(e_i, e_j) = \begin{cases} -1, & i = j = 4 \\ 1, & i = j = 1,2,3 \\ 0, & otherwise. \end{cases}$$

Let ϕ_0 be the tensor field defined by

$$\phi_0 e_i = \begin{cases} 0, & i = 4 \\ e_i, & i = 1,2,3. \end{cases}$$

Using the linearity of ϕ_0 and g, we acquire $\eta_0(e_4) = -1$, $\phi_0^2 X_0 = -X_0 + \eta_0(X_0)e_4$ and $g(\phi_0 X_0, \phi_0 Y_0) = g(X_0, Y_0) + \eta_0(X_0)\eta_0(Y_0)$. Thus, for $e_4 = \xi_0$, then the structure (ϕ, ξ_0, η_0, g) is an almost para-contact metric structure on M and M is called an almost para-contact metric manifold. In addition, M satisfies

$$(\nabla_{X_0}\phi_0)Y_0 = g(X_0,Y_0)e_4 + \eta_0(Y_0)X_0 + 2\eta_0(X_0)\eta_0(Y_0)e_4.$$

Here, for $e_4 = \xi_0$, M is an LP-Sasakian manifold. In tangent bundle T_0M, let the complete and vertical lifts of e_1, e_2, e_3, e_4 be $e_1^c, e_2^c, e_3^c, e_4^c$ and $e_1^v, e_2^v, e_3^v, e_4^v$ on M and let g^c be the complete lift of the Lorentzian metric g on T_0M such that

$$g^c(X_0^v, e_4^c) = \Big(g^c(X_0, e_4)\Big)^v = \Big(\eta_0(X_0)\Big)^v \tag{89}$$

$$g^c(X_0^c, e_4^c) = \Big(g^c(X_0, e_4)\Big)^c = \Big(\eta_0(X_0)\Big)^c \tag{90}$$

$$g^c(e_4^c, e_4^c) = -1, \quad g^v(X_0^v, e_4^c) = 0, \quad g^v(e_4^v, e_4^c) = 0, \tag{91}$$

and so on. Let ϕ_0^c and ϕ_0^v be the complete and vertical lifts of the $(1,1)$ tensor field ϕ_0 defined by

$$\phi_0^v(e_4^v) = \phi_0^c(e_4^c) = 0, \tag{92}$$

$$\phi_0^v(e_1^v) = e_1^v, \quad \phi_0^c(e_1^c) = e_1^c, \tag{93}$$

$$\phi_0^v(e_2^v) = e_2^v, \quad \phi_0^c(e_2^c) = e_2^c, \tag{94}$$

$$\phi_0^v(e_3^v) = e_3^v, \quad \phi_0^c(e_3^c) = e_3^c. \tag{95}$$

Using the linearity of ϕ_0 and g, we infer that

$$(\phi_0^2 X_0)^c = X_0^c + \eta_0^c(X_0)e_4^v + \eta_0^v(X_0)e_4^c, \tag{96}$$

$$g^c\Big((\phi_0 e_4)^c, (\phi_0 e_3)^c\Big) = g^c(e_4^c, e_3^c) + \eta_0^c(e_4^c)\eta_0^v(e_3^c) + \eta_0^v(e_4^c)\eta_0^c(e_3^c). \tag{97}$$

Thus, for $e_4 = \xi_0$ in (89)–(91) and (96), the structure $(\phi_0^c, \xi_0^c, \eta_0^c, g^c)$ is an almost para-contact metric structure on T_0M and satisfies the relation

$$\begin{aligned}(\nabla_{e_4^c}^c\phi_0^c)e_3^c = {} & g^c(e_4^c, e_3^c)\xi_0^v + g^c(e_4^v, e_3^c)\xi_0^c + \eta_0^c(e_3^c)e_4^v + \eta_0^v(e_3^c)e_4^c \\ & + 2\Big\{\eta_0^c(e_4^c)\eta_0^c(e_3^c)\xi_0^v + \eta_0^c(e_4^c)\eta_0^v(e_3^c)\xi_0^c + \eta_0^v(e_4^c)\eta_0^c(e_3^c)\xi_0^c\Big\}.\end{aligned}$$

Thus, $(\phi_0^c, \xi_0^c, \eta_0^c, g^c, T_0M)$ is an LP-Sasakian manifold.

11. Conclusions

The current work investigates the lifts of a QSNMC and LP-Sasakian manifold to the tangent bundle. First, the LP-Sasakian manifold lifts to the tangent bundle are presented. The relationship between the Riemannian connection and the QSNMC from an LP-Sasakian manifold to the tangent bundle is established. An expression of the curvature tensor of the lifts of an LP-Sasakian manifold associated with QSNMC to its tangent bundle is given. The Ricci tensor and the scalar curvature lifts to the tangent bundle are provided. Some theorems regarding the properties of the lifts of the curvature tensor of an LP-Sasakian manifold endowed with QSNMC in an LP-Sasakian manifold to the tangent bundle are given.

Necessary and sufficient conditions for the symmetric and skew-symmetric properties of the lifts of the Ricci tensor are investigated. Sufficient conditions for the skew-symmetric property of the lifts of the projective Ricci tensor in the tangent bundle are provided. The lifts of the Einstein manifold associated with QSNMC on an LP-Sasakian manifold to the tangent bundle are also established. An example of the lifts of LP-Sasakian manifolds in the tangent bundle is constructed.

Author Contributions: Conceptualization, R.K., L.C., S.S., and N.B.T.; methodology, R.K., L.C., S.S., and N.B.T.; investigation, R.K., L.C., S.S., and N.B.T.; writing—original draft preparation, R.K., L.C., S.S., and N.B.T.; writing—review and editing, R.K., L.C., S.S., and N.B.T. All authors have read and agreed to the published version of the manuscript.

Funding: This project was supported by the Researchers Supporting Project number (RSP2023R413), King Saud University, Riyadh, Saudi Arabia.

Data Availability Statement: Not applicable.

Acknowledgments: The authors would like to thank the reviewers and the editor for their thorough review of the article and for their insightful suggestions that helped to improve the overall quality of the paper. This project was supported by the Researchers Supporting Project number (RSP2023R413), King Saud University, Riyadh, Saudi Arabia. The second author is thankful to the Government of India, Ministry of Tribal Affairs (Scholarship Division), New Delhi, India, for financial support in the form of the National Fellowship and Scholarship for Higher Education of ST Students (202122-NFST-MIZ-00686).

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Yano, K.; Ishihara, S. *Tangent and Cotangent Bundles: Differential Geometry*; Marcel Dekker, Inc.: New York, NY, USA, 1973.
2. Yano, K.; Kobayashi, S. Prolongations of tensor fields and connections to tangent bundles I general theory. *J. Math. Soc. Jpn.* **1966**, *18*, 194–210. [CrossRef]
3. Tani, M. Prolongations of hypersurfaces to tangent bundles. *Kodai Math. Semin. Rep.* **1969**, *21*, 85–96. [CrossRef]
4. Pandey, P.; Chaturvedi, B.B. On a Käehler manifold equipped with lift of quarter symmetric non-metric connection. *Facta Univ. Ser. Math. Inform.* **2019**, *33*, 539–546.
5. Khan, M.N.I. Tangent bundle endowed with quarter-symmetric non-metric connection on an almost Hermitian manifold. *Facta Univ. Ser. Math. Inform.* **2020**, *35*, 167–178. [CrossRef]
6. Khan, M.N.I. Novel theorems for the frame bundle endowed with metallic structures on an almost contact metric manifold. *Chaos Solitons Fractals* **2021**, *146*, 110872. [CrossRef]
7. Khan, M.N.I.; Choudhary, M.A.; Chaubey, S.K. Alternative equations for horizontal lifts of the metallic structures from manifold onto tangent bundle. *J. Math.* **2022**, *2022*, 5037620. [CrossRef]
8. Khan, M.N.I.; De, U.C. Liftings of metallic structures to tangent bundles of order r. *AIMS Math.* **2022**, *7*, 7888–7897. [CrossRef]
9. Khan, M.N.I.; De, U.C. Lifts of metallic structure on a cross-section. *Filomat* **2022**, *36*, 6369–6373. [CrossRef]
10. De, U.C.; Khan, M.N.I. Kenmotsu manifolds endowed with the semi-symmetric non-metric φ-connection to its tangent bundle. *Novi Sad J. Math.* **2022**. [CrossRef]
11. Khan, M.N.I. Lift of semi-symmetric non-metric connection on a Kähler manifold. *Afr. Mat.* **2016**, *27*, 345–352. [CrossRef]
12. Khan, M.N.I. Tangent bundles endowed with semi-symmetric non-metric connection on a Riemannian manifold. *Facta Univ. Ser. Math. Inform.* **2021**, *1*, 855–878.
13. Khan, M.N.I. Complete lifts of a semi-symmetric metric p-connection on a Riemannian manifold to its tangent bundle. *Int. J. Math. Comput. Sci.* **2022**, *17*, 909–916.
14. Khan, M.N.I. Submanifolds of a Riemannian manifold endowed with a new type of semi-symmetric non-metric connection in the tangent bundle. *Int. J. Math. Comput. Sci.* **2022**, *17*, 265–275.
15. Khan, M.N.I. Liftings from a para-sasakian manifold to its tangent bundles. *Filomat* **2023**, *37*, 6727–6740.
16. Khan, M.N.I.; De, U.C.; Velimirovic, L.S. Lifts of a quarter-symmetric metric connection from a Sasakian manifold to its tangent bundle. *Mathematics* **2023**, *11*, 53. [CrossRef]
17. Khan, M.N.I.; Jun, J. Quarter-symmetric metric connection on tangent bundles. *Far East J. Math. Sci.* **2017**, *101*, 2219–2229. [CrossRef]
18. Kumar, R.; Colney, L.; Khan, M.N.I. Lifts of a semi-symmetric non-metric connection (SSNMC) from Statistical Manifolds to the tangent bundle. *Results Nonlinear Anal.* **2023**, *6*, 50–65. [CrossRef]
19. Khan, M.N.I.; Mofarreh, F.; Haseeb, A. Tangent bundles of P-Sasakian manifolds endowed with a quarter-symmetric metric connection. *Symmetry* **2023**, *15*, 753. [CrossRef]
20. Golab, S. On semi-symmetric and quarter-symmetric linear connections. *Tensor N. S.* **1975**, *29*, 249–254.
21. Yano, K.; Imai, T. Quarter-symmetric metric connections and their curvature tensors. *Tensor N. S.* **1982**, *38*, 13–18.
22. Rastogi, S.C. On quarter-symmetric metric connection. *C. R. Acad. Sci. Bulg.* **1978**, *31*, 811–814.
23. Rastogi, S.C. On quarter-symmetric metric connection. *Tensor N. S.* **1987**, *44*, 133–141.
24. Mishra, R.S.; Pandey, S.N. On quarter symmetric metric F-connections. *Tensor N. S.* **1980**, *34*, 1–7.
25. Mukhopadhyay, S.; Roy, A.K.; Barua, B. Some properties of a quarter-symmetric metric connection on a Riemannian manifold. *Soochow J. Math.* **1991**, *17*, 205–211.

26. Biswas, S.C.; De, U.C. Quarter-symmetric metric connection in an SP-Sasakian manifold. *Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.* **1997**, *46*, 49–56.
27. Sengupta, J.; Biswas, B. Quarter-symmetric non-metric connection on a Sasakian manifold. *Bull. Cal. Math. Soc.* **2003**, *95*, 169–176.
28. Singh, R.N.; Pandey, M.K. On a type of quarter-symmetric non-metric connection in a Kenmotsu manifold. *Bull. Cal. Math. Soc.* **2007**, *99*, 433–444.
29. Prakash, A.; Narain, D. On a quarter symmetric non-metric connection in an Lorentzian para-Sasakian manifolds. *Int. Electron. J. Geom.* **2011**, *4*, 129–137.
30. Friedman, A.; Schouten, J.A. Uber die Geometrie der halbsymmetrischen Ubertragungen. *Math. Zeitschr.* **1924**, *21*, 211–223. [CrossRef]
31. Matsumoto, K. On Lorentzian paracontact manifolds. *Bull. Yamagata Univ. Natur. Sci* **1989**, *12*, 151–156.
32. De, U.C.; Matsumoto, K.; Shaikh, A.A. On Lorentzian para-Sasakian manifolds. *Rend. Del Semin. Mat. Messina* **1999**, *3*, 149–158.
33. Matsumoto, K.; Mihai, I. On a certain transformation in an Lorentzian para-Sasakian manifold. *Tensor N. S.* **1988**, *47*, 189–197.
34. Chaki, M.C.; Saha, S.K. On pseudo projective Ricci-symmetric manifolds. *Bulg. Phys.* **1994**, *24*, 1–7.