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Sobolev Estimates for the $\bar{\partial}$ and the $\bar{\partial}$-Neumann Operator on Pseudoconvex Manifolds

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Abstract: Let D be a relatively compact domain in an n-dimensional Kähler manifold with a C^2 smooth boundary that satisfies some "Hartogs-pseudoconvexity" condition. Assume that Ξ is a positive holomorphic line bundle over X whose curvature form Θ satisfies $\Theta \geq C\omega$, where $C > 0$. Then, the $\bar{\partial}$-Neumann operator N and the Bergman projection $\mathcal{P}$ are exactly regular in the Sobolev space $W^{\mathtt{m}}(D, \Xi)$ for some $\mathtt{m}$, as well as the operators $\bar{\partial}N$, $\bar{\partial}^* N$.

Keywords: $\bar{\partial}$; $\bar{\partial}$-Neumann operator; Kähler manifold; q-convex domain

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1. Introduction

Sobolev estimates are crucial tools in the study of complex analysis on pseudoconvex manifolds. In this paper, we will focus on the Sobolev estimates for the $\bar{\partial}$ operator and the $\bar{\partial}$-Neumann operator on such manifolds. Consider a Hartogs pseudoconvex domain D with a C^2 boundary in a Kähler manifold X of complex dimension n, and if Ξ is a positive line bundle over X whose curvature form satisfies $\Theta \geq C\omega$ with constant $C > 0$, then the operators N, $\bar{\partial}N$, $\bar{\partial}^* N$ and the Bergman projection $\mathcal{P}$ are regular in the Sobolev space $W^{\mathtt{m}}(D, \Xi)$ for some positive $\mathtt{m}$. This result generalizes the well-known results of Berndtsson–Charpentier [1], Boas–Straube [2], Cao–Shaw–Wang [3], Harrington [4] and Saber [5] and others in the case of the Hartogs pseudoconvex domain in a Kähler manifold for forms with values in a holomorphic line bundle. Indeed, in [1], Berndtsson–Charpentier (see also [6]) obtained the Sobolev regularity for $\mathcal{P}$ for a pseudoconvex domain Ω. In [2], Boas–Straube proved that the Bergman projection B maps the Sobolev space $W^{\mathtt{m}}(\Omega)$ to itself for all $\mathtt{m} > 0$ on a smooth pseudoconvex domain in $\mathbb{C}^n$ that admits a defining function that is plurisubharmonic on the boundary $b\Omega$. In [3], Cao–Shaw–Wang obtained the Sobolev regularity of the operators N, $\bar{\partial}N$, $\bar{\partial}^* N$ and $\mathcal{P}$ on a local Stein domain subset of the complex projective space. In [4], Harrington proved this result on a bounded pseudoconvex domain in $\mathbb{C}^n$ with a Lipschitz boundary. In [5], Saber proved that the operators N, $\bar{\partial}^* N$ and $\mathcal{P}$ are regular in $W^{\mathtt{m}}_{r,s}(D)$ for some $\mathtt{m}$ on a smooth weakly q-convex domain in $\mathbb{C}^n$. Similar results can be found in [7–16].

This paper is organized into five sections. The introduction presents an introduction to the subject and contains the history and development of the problem. Section 2 recalls the basic definitions and fundamental results. In Section 3, the basic Bochner–Kodaira–Morrey–Kohn identity is proved on the Kähler manifold. In Section 4, it is proved that the C^2 smoothly bounded Hartogs pseudoconvex domains in the Kähler manifold admit bounded plurisubharmonic exhaustion functions. Section 5 deals with the L^2 estimates of the $\overline{\partial}$ and $\overline{\partial}$-Neumann operator on the C^2 smoothly bounded Hartogs pseudoconvex domains in the Kähler manifold. Section 6 presents the main results.

2. Preliminaries

Assuming that X is a complex manifold of the complex dimension n, $n \geq 2$, $T(X)$ (resp. $T_x(X)$) is the holomorphic tangent bundle of X (resp. at $x \in X$) and $\pi : \Xi \longrightarrow X$ is a holomorphic line bundle over X. A system of local complex analytic (holomorphic) coordinates on X is a collection $\{\gamma_j\}_{j\in J}$ (for some index set J) of local complex coordinates $\gamma_j : \mathscr{U}_j \longrightarrow \mathbb{C}^n$ such that:

(i) $X = \cup_{j\in J}\mathscr{U}_j$, i.e., $\{\mathscr{U}_j\}_{j\in J}$ is an open covering of X by charts with coordinate mappings $\mathtt{w}_j : \mathscr{U}_j \longrightarrow \mathbb{C}^n$ satisfies $\pi^{-1}(\mathscr{U}_j) = \mathscr{U}_j \times \mathbb{C}$.

(ii) $\{f_{ij}\}$ is a system of transition functions for Ξ; that is, the maps $f_{jk} = \gamma_j \circ \gamma_k^{-1} : \gamma_k(z) \longrightarrow \gamma_j(z)$ are biholomorphic for each pair of indices (j,k) with $\mathscr{U}_j \cap \mathscr{U}_k$ being nonempty (i.e., f_{jk} (resp. $f_{jk}^{-1} = \gamma_k \circ \gamma_j^{-1}$) are holomorphic maps of $\gamma_k(\mathscr{U}_j \cap U_k)$ onto $\gamma_j(\mathscr{U}_j \cap \mathscr{U}_k)$ (resp. $\gamma_j(\mathscr{U}_j \cap \mathscr{U}_k)$ onto $\gamma_k(\mathscr{U}_j \cap \mathscr{U}_k)$)).

Assume that $(\mathtt{w}_j^1, \mathtt{w}_j^2, \cdots, \mathtt{w}_j^n)$ is the local coordinates on $\mathscr{U}_j$. A system of functions $\hbar = \{\hbar_j\}$, $j \in J$ is a Hermitian metric along the fibers of Ξ with $\hbar_j =\mid f_{ij} \mid^2 \hbar_i$ in $\mathscr{U}_i \cap \mathscr{U}_j$, and $\hbar_j$ is a C^∞ positive function in $\mathscr{U}_j$. The (1,0) form of the connection associated with the metric $\hbar$ is given as $\theta = \{\theta_j\}$, $\theta_j = \hbar_j^{-1}\partial\hbar_j$. $\Theta = \{\Theta_j\}$ is the curvature form associated with the connection θ and is given by

$$\Theta_j = \overline{\partial}\theta_j = \partial\overline{\partial}\log\hbar_j = \sum_{\alpha,\beta=1}^{n} \Theta_{j\alpha\overline{\beta}} d\mathtt{w}_j^\alpha \wedge d\overline{\mathtt{w}}_j^\beta.$$

Definition 1. *Ξ is positive at $x \in \mathscr{U}_j$ if the Hermitian form*

$$\sum \Theta_{j\alpha\overline{\beta}}\, \mu^\alpha\, \overline{\mu}^\beta,$$

is positive definite on $T_x(X)$, $\forall\, \mu \in \Xi_x \setminus \{0\}$.

Along the fibers of Ξ, $\hbar_0 = (\hbar_j^{-1})$, $j \in J$ is a Hermitian metric for which Ξ is positive; i.e., $\partial\overline{\partial}\log\hbar_j > 0$. Then, $\hbar_0$ defines a Kähler metric $\mathcal{G}$ on X,

$$\mathcal{G} = \sum_{\alpha,\beta=1}^{n} \mathrm{g}_{j\alpha\overline{\beta}}\, d\mathtt{w}_j^\alpha\, d\overline{\mathtt{w}}_j^\beta, \quad \mathrm{g}_{j\alpha\overline{\beta}} = \partial^2 \log\hbar_j / \partial\mathtt{w}_j^\alpha \partial\overline{\mathtt{w}}_j^\beta.$$

Let $\mathcal{C}^\infty_{r,s}(X,\Xi)$ (resp. $\mathcal{D}^\infty_{r,s}(X,\Xi)$) be the space of C^∞ (r,s) differential forms (resp. with compact support) on X with values in Ξ. A form $\psi = (\psi_j) \in \mathcal{C}^\infty_{r,s}(X,\Xi)$ is expressed on $\mathscr{U}_j$ as follows:

$$\psi_j(z) = \sum_{A_r,B_s} \psi_{jA_rB_s}(z) d\mathtt{w}_j^{A_r} \wedge d\overline{\mathtt{w}}_j^{B_s} \otimes s_j,$$

where $A_r = (a_1, \ldots, a_r)$ and $B_s = (b_1, \ldots, b_s)$ are multi-indices and s_j is a section of $\Xi|_{\mathscr{U}_j}$. Define the inner product

$$(\psi,\psi) = \hbar_j \sum_{A_r,B_s} \psi_{jA_r\overline{B}_s} \overline{\psi_j^{\overline{A}_rB_s}},$$

where $\psi_j^{\overline{A}_rB_s} = \sum_{C_r,D_s} \mathrm{g}_j^{c_1\overline{a}_1} \cdots \mathrm{g}_j^{c_r\overline{a}_r} \mathrm{g}_j^{b_1\overline{d}_1} \cdots \mathrm{g}_j^{b_s\overline{d}_s}\, \psi_{jc_1\cdots c_r\overline{d}_1\cdots\overline{d}_s}$. Let

$$\mathcal{C}^\infty_{r,s}(\overline{\Omega},\Xi) = \{\psi\mid_{\overline{\Omega}}\,;\, \psi \in \mathcal{C}^\infty_{r,s}(X,\Xi)\}.$$

Let $\divideontimes : \mathcal{C}^{\infty}_{r,s}(X,\Xi) \longrightarrow \mathcal{C}^{\infty}_{(n-s,n-r)}(X,\Xi)$ be the Hodge star operator, which is a real operator and satisfies

$$\divideontimes \divideontimes \psi = (-1)^{r+s}\psi,$$

For the proof, see Morrow and Kodaira [17]. Set the volume element with respect to $\mathcal{G}$ as dv. The inner product $<\psi,\psi>$ and the norm $\|\ \psi\ \|$ are defined by

$$<\psi,\psi>=\int_{\Omega}(\psi,\psi)\,dv=\int_{\Omega}{}^{t}\psi\wedge\divideontimes\overline{\hbar\,\psi},\text{and } \|\ \psi\ \|^2=<\psi,\psi>.$$

The formal adjoint operator ψ of $\bar{\partial} : \mathcal{C}^{\infty}_{r,s-1}(\Omega,\Xi) \longrightarrow \mathcal{C}^{\infty}_{r,s}(\Omega,\Xi)$ is defined by

$$<\psi\psi,\psi>=<\psi,\bar{\partial}\psi>,$$

$\psi \in \mathcal{C}^{\infty}_{r,s}(\Omega,\Xi)$ and $\psi \in \mathcal{D}^{\infty}_{r,s-1}(\Omega,\Xi)$. Let $\# : \mathcal{C}^{\infty}_{r,s}(X,\Xi) \longrightarrow \mathcal{C}^{\infty}_{s,r}(X,E^{\divideontimes})$ be defined locally as $(\#\psi)_j = \overline{\hbar_j\,\psi_j}$; the inner product $<\psi,\psi>$ is given by

$$<\psi,\psi>=\int_{\Omega}{}^{t}\psi\wedge\divideontimes\#\psi.$$

From Stokes' theorem, $\psi \in \mathcal{C}^{\infty}_{r,s}(\overline{\Omega},\Xi)$, $\psi \in \mathcal{C}^{\infty}_{r,s-1}(\overline{\Omega},\Xi)$, one obtains

$$<\bar{\partial}\psi,\psi>=<\psi,\bar{\partial}^{\divideontimes}\psi>+\int_{\partial\Omega}{}^{t}\psi\wedge\divideontimes\#\psi.$$

Put

$$\mathscr{B}_{r,s}(\overline{\Omega},\Xi)=\{\psi\in\mathcal{C}^{\infty}_{r,s}(\overline{\Omega},\Xi);\divideontimes\#\psi\mid_{\partial\Omega}=0\}.$$

As a result,

$$<\bar{\partial}\psi,\psi>=<\psi,\bar{\partial}^{\divideontimes}\psi>,$$

for $\psi \in \mathscr{B}_{r,s}(\overline{\Omega},\Xi)$.

$L^2_{r,s}(\Omega,\Xi)$ is the Hilbert space of the measurable E-valued (r,s) forms ψ, which are square integrable in the sense that $\|\psi\|^2 < \infty$. Let $\bar{\partial} : L^2_{r,s}(\Omega,\Xi) \longrightarrow L^2_{r,s+1}(\Omega,\Xi)$ and $\bar{\partial}^{\divideontimes} : L^2_{r,s+1}(\Omega,\Xi) \longrightarrow L^2_{r,s}(\Omega,\Xi)$. In $L^2_{r,s}(\Omega,\Xi)$, the spaces $\ker(\bar{\partial},\Xi)$, $\mathrm{Dom}_{r,s}(\bar{\partial},\Xi)$ and $\mathrm{Rang}(\bar{\partial},\Xi)$ are the kernel, the domain and the range of $\bar{\partial}$, respectively. A Bergman projection operator $\mathcal{P} : L^2_{r,s}(D,\Xi) \longrightarrow L^2_{r,s}(D,\Xi)\cap\ker_{r,s}(E)$. Let $\square = \square_{r,s} = \bar{\partial}\bar{\partial}^{*} + \bar{\partial}^{*}\bar{\partial}$ be the unbounded Laplace–Beltrami operator from $L^2_{r,s}(\Omega,\Xi)$ to $L^2_{r,s}(\Omega,\Xi)$ with $\mathrm{Dom}(\square_{r,s},\Xi) = \{\psi \in L^2_{r,s}(\Omega,E) | \psi \in \mathrm{Dom}(\bar{\partial},\Xi) \cap \mathrm{Dom}(\bar{\partial}^{\divideontimes},\Xi); \bar{\partial}\psi \in \mathrm{Dom}(\bar{\partial}^{\divideontimes},\Xi) \text{ and } \bar{\partial}^{\divideontimes}\psi \in \mathrm{Dom}(\bar{\partial},\Xi)\}$. Let $N_{r,s}$ be the $\bar{\partial}$-Neumann operator on (r,s) forms, solving $N_{r,s}\square_{r,s}\psi = \psi$ for any (r,s) form ψ in $L^2_{r,s}(\Omega,\Xi)$. Denote by $\mathcal{P}$ the Bergman operator, mapping a (r,s) form in $L^2_{r,s}(\Omega,\Xi)$ to its orthogonal projection in the closed subspace of $\bar{\partial}$-closed forms.

Let

$$\mathscr{H}_{r,s}(E)=\ker(\square_{r,s},\Xi)=\{\psi\in\mathrm{Dom}(\bar{\partial},\Xi)\cap\mathrm{Dom}(\bar{\partial}^{\divideontimes},\Xi);\ \bar{\partial}\psi=0 \text{ and } \bar{\partial}^{\divideontimes}\psi=0\}.$$

Let $W^{\mathtt{m}}_{r,s}(\Omega,\Xi)$ be the Sobolev space with $-\frac{1}{2} < \mathtt{m} < \frac{1}{2}$ and let $\|\ \|_{W^{\mathtt{m}}_{r,s}(\Omega,\Xi)}$ denote its norm. $\forall\ \psi \in \mathrm{Dom}(\bar{\partial},\Xi)\cap\mathrm{Dom}(\bar{\partial}^{\divideontimes},\Xi)$, one obtains $\psi \in W^1_{r,s}(\Omega,\mathrm{loc})$. Thus, ψ is an elliptic and $\psi \in W^{\mathtt{m}}_{r,s}(\Omega,\Xi)$ for $-\frac{1}{2} < \mathtt{m} < \frac{1}{2}$ if and only if

$$\|\psi\|^2_{W^{\mathtt{m}}_{r,s}(\Omega,\Xi)}=\int_{\Omega}\zeta^{-2\mathtt{m}}\,|\psi|^2<\infty.$$

For the proof, see Theorems 4.1 and 4.2 in Jersion and Kenig [18], Lemma 2 in Charpentier [19] and also Theorem C.4 in the Appendix in Chen and Shaw [20].

Proposition 1 ([21–23]). *(i) If $\psi \in Dom(\psi, \Xi) \subset L^2_{r,s}(\Omega, \Xi)$ satisfies supp.$\psi \subseteq \overline{\Omega}$ and supp. $\psi\psi \subseteq \overline{\Omega}$, then $\psi|_{\Omega} \in Dom(\overline{\partial}^*, \Xi) \subset L^2_{r,s}(\Omega, \Xi)$; i.e., $\psi\psi|_{\Omega} = \overline{\partial}^*\psi|_{\Omega}$ in $L^2_{r,s-1}(\Omega, \Xi)$. (ii) $\mathcal{C}^{\infty}_{r,s}(\overline{\Omega}, \Xi)$ is dense in $Dom(\overline{\partial}, \Xi)$ in the sense of $(\| \psi \|^2 + \| \overline{\partial}\psi \|^2)^{1/2}$. (iii) $\mathscr{B}_{r,s}(\overline{\Omega}, \Xi)$ is dense in $Dom(\overline{\partial}^*, \Xi)$ (resp. $Dom(\overline{\partial}, \Xi) \cap Dom(\overline{\partial}^*, \Xi)$) in the sense of the norm $(\| \psi \|^2 + \| \overline{\partial}^*\psi \|^2)^{1/2}$ (resp. $(\| \psi \|^2 + \| \overline{\partial}\psi \|^2 + \| \overline{\partial}^*\psi \|^2)^{1/2}$).*
(iv) $\overline{\partial}^* = \psi$ on $\mathscr{B}_{r,s}(\overline{\Omega}, \Xi)$.

3. The Kähler Identity

As in Takeuchi A. [24–26], one can prove the following Kähler identity: Fix the following notation: C^{∞} sections of $\mathcal{A}(T(\mathbb{X}))$, $\mathcal{A}(T^*(\mathbb{X}))$, $\mathcal{A}(\overline{T}(\mathbb{X}))$ and $\mathcal{A}(\overline{T}^*(\mathbb{X}))$ written as $\sum_{\alpha=1}^{n} \zeta^{\alpha} \frac{\partial}{\partial z^{\alpha}}, \sum_{\alpha=1}^{n} \psi_{\alpha}\, dz^{\alpha}, \sum_{\alpha=1}^{n} \eta^{\overline{\alpha}} \frac{\partial}{\partial \overline{z}^{\alpha}}$ and $\sum_{\alpha=1}^{n} \psi_{\overline{\alpha}}\, d\overline{z}^{\alpha}$, respectively. Use the notation $\partial_{\beta} = \frac{\partial}{\partial z^{\beta}}, \overline{\partial}_{\alpha} = \frac{\partial}{\partial \overline{z}^{\alpha}}$. For $\eta = \sum_{\alpha=1}^{n} \eta^{\overline{\alpha}} \frac{\partial}{\partial \overline{z}^{\alpha}} \in \mathcal{A}(\overline{T}(\mathbb{X}))$, $\psi = \sum_{\alpha=1}^{n} \psi_{\overline{\alpha}}\, d\overline{z}^{\alpha} \in \mathcal{A}(\overline{T}^*(\mathbb{X}))$, define

$$\nabla_{\beta}\, \eta^{\overline{\alpha}} = \partial_{\beta}\, \eta^{\overline{\alpha}} \text{and} \nabla_{\beta}\, \psi_{\overline{\alpha}} = \partial_{\beta}\, \psi_{\overline{\alpha}}.$$

A connection ω for $T(X)$ is defined as

$$\omega = (\omega^{\beta}_{\alpha}),\ \omega^{\beta}_{\alpha} = \sum_{\gamma=1}^{n} \Gamma^{\beta}_{\gamma\alpha}\, dz^{\gamma}, \text{ with } \Gamma^{\beta}_{\gamma\alpha} = \sum_{\sigma=1}^{n} g^{\overline{\sigma}\beta} \partial_{\gamma} \mathrm{g}_{\alpha\overline{\sigma}},$$

and its Riemann curvature tensor

$$\mathcal{R}_{\overline{\alpha}\beta\overline{\nu}\tau} = \sum_{\mu=1}^{n} \mathrm{g}_{\mu\overline{\alpha}} \mathcal{R}^{\mu}_{\beta\overline{\nu}\tau}, \quad \mathcal{R}^{\alpha}_{\beta\overline{\nu}\tau} = \partial_{\overline{\nu}} \Gamma^{\alpha}_{\tau\beta}. \tag{1}$$

One obtains

$$\Gamma^{\overline{\alpha}}_{\overline{\beta}\overline{\gamma}} = \overline{\Gamma^{\alpha}_{\beta\gamma}}, \quad \mathcal{R}^{\overline{\alpha}}_{\overline{\beta}\nu\overline{\tau}} = \overline{\mathcal{R}^{\alpha}_{\beta\overline{\nu}\tau}}, \text{ and } \mathcal{R}_{\alpha\overline{\beta}\nu\overline{\tau}} = \overline{\mathcal{R}_{\overline{\alpha}\beta\overline{\nu}\tau}}. \tag{2}$$

The Ricci curvature is defined by

$$\mathcal{R}_{\overline{\nu}\tau} = \sum_{\beta=1}^{n} \mathcal{R}^{\beta}_{\beta\overline{\nu}\tau}. \tag{3}$$

Following Morrow and Kodaira [13], if $\mathcal{G}$ is a Kähler metric,

$$\begin{gathered} \Gamma^{\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{\gamma\beta}, \\ \mathcal{R}_{\overline{\alpha}\beta\overline{\nu}\tau} = \mathcal{R}_{\overline{\alpha}\tau\overline{\nu}\beta} = \mathcal{R}_{\overline{\nu}\beta\overline{\alpha}\tau} = \mathcal{R}_{\overline{\nu}\tau\overline{\alpha}\beta}, \end{gathered} \tag{4}$$

where

$$\sum_{\tau=1}^{n} \Gamma^{\tau}_{\tau\alpha} = \partial_{\alpha} \log g, \text{ and } \mathcal{R}_{\overline{\nu}\tau} = \partial_{\overline{\nu}} \partial_{\tau} \log g, \text{ where } g = \det(\mathrm{g}_{\alpha\overline{\beta}}).$$

For $\zeta = \sum_{\alpha=1}^{n} \zeta^{\alpha} \frac{\partial}{\partial z^{\alpha}} \in \mathcal{A}(T(\mathbb{X}))$, $\psi = \sum_{\alpha=1}^{n} \psi_{\alpha}\ dz^{\alpha} \in \mathcal{A}(\overline{T}^{*}(\mathbb{X}))$, one defines

$$\begin{aligned}
\nabla_{\beta}\zeta^{\alpha} &= \partial_{\beta}\zeta^{\alpha} + \sum_{\gamma=1}^{n} \Gamma^{\alpha}_{\beta\gamma}\zeta^{\gamma}, \\
\nabla_{\beta}\psi_{\alpha} &= \partial_{\beta}\psi_{\alpha} - \sum_{\gamma=1}^{n} \Gamma^{\gamma}_{\beta\alpha}\psi_{\gamma}, \\
\nabla_{\overline{\beta}}\zeta^{\alpha} &= \partial_{\overline{\beta}}\zeta^{\alpha}, \\
\nabla_{\overline{\beta}}\eta^{\overline{\alpha}} &= \partial_{\overline{\beta}}\eta^{\overline{\alpha}} + \sum_{\gamma=1}^{n} \overline{\Gamma^{\alpha}_{\beta\gamma}}\eta^{\overline{\gamma}}, \\
\nabla_{\overline{\beta}}\psi_{\alpha} &= \partial_{\overline{\beta}}\psi_{\alpha}, \\
\nabla_{\overline{\beta}}\psi_{\overline{\alpha}} &= \partial_{\overline{\beta}}\psi_{\overline{\alpha}} - \sum_{\gamma=1}^{n} \overline{\Gamma^{\gamma}_{\beta\alpha}}\psi_{\overline{\gamma}}.
\end{aligned} \tag{5}$$

For $\psi \in \mathcal{C}^{\infty}_{r,s}(X, \Xi)$, one defines

$$\begin{aligned}
\nabla_{\alpha}\psi_{\alpha_1\ldots\alpha_r\overline{\beta}_1\ldots\overline{\beta}_s} &= \partial_{\alpha}\psi_{\alpha_1\ldots\alpha_r\overline{\beta}_1\ldots\overline{\beta}_s} - \sum_{j=1}^{r}\sum_{\tau} \Gamma^{\tau}_{\alpha\alpha_j}\psi_{\alpha_1\ldots\alpha_{j-1}\tau\alpha_{j+1}\ldots\alpha_r\overline{\beta}_1\ldots\overline{\beta}_s}, \\
\nabla^{(\hbar)}_{\alpha}\ \psi_{\alpha_1\ldots\alpha_r\overline{\beta}_1\ldots\overline{\beta}_s} &= \nabla_{\alpha}\psi_{\alpha_1\ldots\alpha_r\overline{\beta}_1\ldots\overline{\beta}_s}\ -\ \partial_{\alpha}\log\hbar\ \psi_{\alpha_1\ldots\alpha_r\overline{\beta}_1\ldots\overline{\beta}_s}, \\
\nabla_{\overline{\beta}}\psi_{\alpha_1\ldots\alpha_r\overline{\beta}_1\ldots\overline{\beta}_s} &= \partial_{\overline{\beta}}\psi_{\alpha_1\ldots\alpha_r\overline{\beta}_1\ldots\overline{\beta}_s} - \sum_{j=1}^{s}\sum_{\tau} \overline{\Gamma^{\tau}_{\beta\beta_j}}\ \psi_{\alpha_1\ldots\alpha_r\overline{\beta}_1\ldots\overline{\beta}_{j-1}\overline{\tau}\overline{\beta}_{j+1}\ldots\overline{\beta}_s}, \\
\nabla_{\overline{\beta}}\psi^{\overline{\beta}_1\ldots\overline{\beta}_s\alpha_1\ldots\alpha_r} &= \partial_{\overline{\beta}}\psi^{\overline{\beta}_1\ldots\overline{\beta}_s\alpha_1\ldots\alpha_r} + \sum_{j=1}^{s}\sum_{\tau} \overline{\Gamma^{\beta_j}_{\beta\tau}}\ \psi^{\overline{\beta}_1\ldots\overline{\beta}_{j-1}\overline{\tau}\overline{\beta}_{j+1}\ldots\overline{\beta}_s\alpha_1\ldots\alpha_r},
\end{aligned} \tag{6}$$

For the proof, see Choquet-Bruhat [27], p. 235.

Following Morrow and Kodaira [17], the operators $\overline{\partial}$, ψ are defined as

$$\begin{aligned}
\overline{\partial}\psi &= \sum_{A_r,B_s}\sum_{\mu} \nabla_{\overline{\mu}}\psi_{A_r\overline{B}_s} dz^{\overline{\mu}} \wedge dz^{\alpha_1} \wedge \ldots \wedge dz^{\alpha_r} \wedge dz^{\overline{\beta}_1} \wedge \ldots \wedge dz^{\overline{\beta}_s}, \\
(\overline{\partial}^{*}\psi)_{A_r\overline{B}_{s-1}} &= (-1)^{r-1} \sum_{\alpha,\beta=1}^{n} g^{\overline{\beta}\alpha}\nabla^{(\hbar)}_{\alpha}\ \psi_{\overline{\beta}A_r\overline{B}_{s-1}},
\end{aligned} \tag{7}$$

for $\psi \in \mathcal{C}^{\infty}_{r,s}(X, \Xi)$.

For a C^{∞} function λ and for a $\psi \in \mathcal{C}^{\infty}_{r,s}(X, \Xi)$ at any point of X, one defines

$$\begin{aligned}
\operatorname{grad}\lambda &= \left(\frac{\partial\lambda}{\partial z^1}, \ldots, \frac{\partial\lambda}{\partial z^n}, \overline{\frac{\partial\lambda}{\partial z^1}}, \ldots, \overline{\frac{\partial\lambda}{\partial z^n}}\right), \\
|\operatorname{grad}\lambda|^2 &= (\operatorname{grad}\lambda)\overline{(\operatorname{grad}\lambda)} = \sum_{\alpha=1}^{n}\left|\frac{\partial\lambda}{\partial z^{\alpha}}\right|^2 + \sum_{\beta=1}^{n}\left|\overline{\frac{\partial\lambda}{\partial z^{\beta}}}\right|^2, \\
(\mathscr{L}(\lambda)\psi, \psi) &= \sum_{B_{s-1}}\sum_{\beta,\gamma=1}^{n} \frac{\partial^2\lambda}{\partial z^{\beta}\partial z^{\overline{\gamma}}}\ \psi^{\beta}_{\overline{B}_{s-1}}\overline{\psi^{\gamma B_{s-1}}}.
\end{aligned}$$

Since $d\lambda \neq 0$ on U, then $\operatorname{grad}\lambda \neq 0$ on U also. Also, set

$$(\overline{\nabla}, \overline{\nabla}) = \hbar^{-1}\sum_{C_s}\sum_{\mu} \nabla_{\overline{\mu}}\psi_{\overline{C}_s}\overline{\nabla^{\mu}\psi^{C_s}}.$$

For $\psi \in \mathcal{C}^{\infty}_{0,s}(X,\Xi)$, $s \geq 1$, we construct from ψ the two tangent vector fields ξ and η to X as follows:

$$\xi = \{\xi^{\beta} = \sum_{B_{s-1}} \sum_{\gamma=1}^{n} \hbar^{-1}\left(\nabla_{\overline{\gamma}}\psi^{\beta}_{\overline{B}_{s-1}}\right)\overline{\psi^{\gamma B_{s-1}}}, \quad \xi^{\overline{\beta}} = 0\},$$

$$\eta = \{\eta^{\gamma} = 0, \quad \eta^{\overline{\gamma}} = \sum_{B_{s-1}} \sum_{\beta=1}^{n} \hbar^{-1}\left(\nabla^{(\hbar)}_{\beta}\psi^{\beta}_{\overline{B}_{s-1}}\right)\overline{\psi^{\gamma B_{s-1}}}\},$$

where $\beta, \gamma = 1, 2, \ldots, n$.

Proposition 2 ([24]).

$$\nabla_{\mu}\, \mathrm{g}_{\alpha\overline{\beta}} = 0, \quad \nabla_{\overline{\mu}}\, \mathrm{g}_{\alpha\overline{\beta}} = 0, \quad \textit{and}\ \nabla_{\overline{\mu}}\, g^{\overline{\beta}\alpha} = 0.$$

Proof. Since $\mathrm{g}_{\alpha\overline{\beta}}$ is a C^{∞} section of $T^{*}(X) \otimes \overline{T}^{*}(X)$, then Equation (3) gives

$$\begin{aligned}
\nabla_{\mu}\, \mathrm{g}_{\alpha\overline{\beta}} &= \partial_{\mu}\, \mathrm{g}_{\alpha\overline{\beta}} - \sum_{\tau} \Gamma^{\tau}_{\mu\alpha}\, \mathrm{g}_{\tau\overline{\beta}} \\
&= \partial_{\mu}\, \mathrm{g}_{\alpha\overline{\beta}} - \sum_{\gamma,\tau} g^{\overline{\gamma}\tau}(\partial_{\mu}\mathrm{g}_{\alpha\overline{\gamma}})\, \mathrm{g}_{\tau\overline{\beta}} \\
&= \partial_{\mu}\, \mathrm{g}_{\alpha\overline{\beta}} - \sum_{\gamma} \zeta^{\gamma}_{\beta}(\partial_{\mu}\mathrm{g}_{\alpha\overline{\gamma}}) \\
&= \partial_{\mu}\, \mathrm{g}_{\alpha\overline{\beta}} - \partial_{\mu}\, \mathrm{g}_{\alpha\overline{\beta}} \\
&= 0. \\
\nabla_{\overline{\mu}}\, \mathrm{g}_{\alpha\overline{\beta}} &= \partial_{\overline{\mu}}\, \mathrm{g}_{\alpha\overline{\beta}} - \sum_{\tau} \overline{\Gamma^{\tau}_{\mu\beta}}\, \mathrm{g}_{\alpha\overline{\tau}} \\
&= \partial_{\overline{\mu}}\, \mathrm{g}_{\alpha\overline{\beta}} - \sum_{\gamma,\tau} g^{\overline{\tau}\gamma}(\partial_{\overline{\mu}}\mathrm{g}_{\gamma\overline{\beta}})\, \mathrm{g}_{\alpha\overline{\tau}} \\
&= \partial_{\overline{\mu}}\, \mathrm{g}_{\alpha\overline{\beta}} - \sum_{\gamma} \zeta^{\gamma}_{\alpha}(\partial_{\overline{\mu}}\mathrm{g}_{\gamma\overline{\beta}}) \\
&= \partial_{\overline{\mu}}\, \mathrm{g}_{\alpha\overline{\beta}} - \partial_{\overline{\mu}}\, \mathrm{g}_{\alpha\overline{\beta}} \\
&= 0. \\
\nabla_{\overline{\mu}}\, g^{\overline{\beta}\alpha} &= \partial_{\overline{\mu}}\, g^{\overline{\beta}\alpha} + \sum_{\tau} \overline{\Gamma^{\beta}_{\mu\tau}}\, g^{\overline{\tau}\alpha} \\
&= \partial_{\overline{\mu}}\, g^{\overline{\beta}\alpha} + \sum_{\gamma,\tau} g^{\overline{\beta}\gamma}\,(\partial_{\overline{\mu}}\, \mathrm{g}_{\gamma\overline{\tau}})\, g^{\overline{\tau}\alpha} \\
&= \partial_{\overline{\mu}}\, g^{\overline{\beta}\alpha} - \sum_{\tau,\gamma} \mathrm{g}_{\gamma\overline{\tau}}\,(\partial_{\overline{\mu}} g^{\overline{\beta}\gamma})\, g^{\overline{\tau}\alpha} \\
&= \partial_{\overline{\mu}}\, g^{\overline{\beta}\alpha} - \sum_{\gamma} \zeta^{\alpha}_{\gamma}\,(\partial_{\overline{\mu}}\, g^{\overline{\beta}\gamma}) \\
&= \partial_{\overline{\mu}}\, g^{\overline{\beta}\alpha} - \partial_{\overline{\mu}}\, g^{\overline{\beta}\alpha} \\
&= 0.
\end{aligned}$$

□

Proposition 3 ([24]).

$$\textit{div}\ \xi - \textit{div}\ \eta = \sum_{\beta=1}^{n} \nabla_{\beta}\, \xi^{\beta} - \sum_{\gamma=1}^{n} \nabla_{\overline{\gamma}}\, \eta^{\overline{\gamma}}.$$

Proof. The divergence of the vector ξ,

$$\text{div } \xi = \sum_{\beta=1}^{n} \nabla_\beta \, \xi^\beta - \sum_{\beta,\gamma=1}^{n} \left(\Gamma^\beta_{\beta\,\gamma} - \Gamma^\beta_{\gamma\,\beta} \right) \xi^\gamma.$$

Since the metric $\mathcal{G}$ is Kähler, then from Equation (4), $\left(\Gamma^\beta_{\beta\,\gamma} - \Gamma^\beta_{\gamma\,\beta} \right) = 0$. Therefore,

$$\text{div } \xi = \sum_{\beta=1}^{n} \nabla_\beta \, \xi^\beta, \quad \text{div } \eta = \sum_{\gamma=1}^{n} \nabla_{\overline{\gamma}} \, \eta^{\overline{\gamma}}.$$

Then, the proof is complete. □

Proposition 4 ([24])**.** *For a C^∞ function λ and for $\psi \in \mathscr{B}_{0,s}(\overline{\Omega}, \Xi)$, $s \geq 1$,*

$$\|\overline{\partial}\psi\|^2 + \|\overline{\partial}^*\psi\|^2 = \|\overline{\nabla}\psi\|^2 + (\hbar|grad\,\lambda|)^{-1} \int_{\partial\Omega} (\mathscr{L}(\lambda)\psi, \psi)\, ds + < (\Theta - \mathcal{R})\psi, \psi >,$$

where $\Theta = (\Theta_{\alpha\overline{\beta}})$ and $\mathcal{R} = (\mathcal{R}_{\alpha\overline{\beta}})$.

Proof. Since

$$\nabla_\beta \xi^\beta = \nabla_\beta \left(\sum_{B_{s-1}} \sum_{\gamma=1}^{n} \hbar^{-1} \nabla_{\overline{\gamma}} \psi^\beta_{\overline{B}_{s-1}} \overline{\psi^{\gamma B_{s-1}}} \right).$$

Since $\nabla^{(\hbar)}_\beta = (\nabla_\beta - \partial_\beta \log \hbar)$, from Equation (6), then

$$\begin{aligned}
\sum_{\beta=1}^{n} \nabla_\beta \xi^\beta &= \sum_{\beta,\gamma=1}^{n} \nabla_\beta \left(\hbar^{-1} \sum_{B_{s-1}} \nabla_{\overline{\gamma}} \psi^\beta_{\overline{B}_{s-1}} \overline{\psi^{\gamma B_{s-1}}} \right) \\
&= \hbar^{-1} \sum_{\beta,\gamma} \sum_{B_{s-1}} (\nabla_\beta - \partial_\beta \log \hbar) \nabla_{\overline{\gamma}} \psi^\beta_{\overline{B}_{s-1}} \overline{\psi^{\gamma B_{s-1}}} + \hbar^{-1} \sum_{\beta,\gamma} \sum_{B_{s-1}} \nabla_{\overline{\gamma}} \psi^\beta_{\overline{B}_{s-1}} \overline{\nabla_{\overline{\beta}} \psi^{\gamma B_{s-1}}} \\
&= \hbar^{-1} \sum_{\beta,\gamma} \sum_{B_{s-1}} \nabla_{\overline{\gamma}} \nabla^{(\hbar)}_\beta \psi^\beta_{\overline{B}_{s-1}} \overline{\psi^{\gamma B_{s-1}}} + \hbar^{-1} \sum_{\beta,\gamma} \sum_{B_{s-1}} \nabla_{\overline{\gamma}} \psi^\beta_{\overline{B}_{s-1}} \overline{\nabla_{\overline{\beta}} \psi^{\gamma B_{s-1}}} \\
&\quad + \hbar^{-1} \sum_{\beta,\gamma} \sum_{B_{s-1}} [\nabla^{(\hbar)}_\beta, \nabla_{\overline{\gamma}}] \psi^\beta_{\overline{B}_{s-1}} \overline{\psi^{\gamma B_{s-1}}}.
\end{aligned}$$

Then, one obtains the commutator

$$[\nabla^{(\hbar)}_\beta, \nabla_{\overline{\gamma}}] \, \psi_{\overline{B}_s} = [\nabla_\beta, \nabla_{\overline{\gamma}}] \, \psi_{\overline{B}_s} + \Theta_{\beta\overline{\gamma}} \, \psi_{\overline{B}_s}.$$

Using Equation (6), one obtains

$$\begin{aligned}
\nabla_{\overline{\gamma}} \nabla_\beta \psi_{\overline{B}_s} &= \partial_{\overline{\gamma}} \partial_\beta \psi_{\overline{B}_s} - \sum_{\mu=1}^{s} \sum_{\tau=1}^{n} \Gamma^{\overline{\tau}}_{\overline{\gamma}\overline{\beta}_\mu} \partial_\beta \psi_{\overline{\beta}_1 \ldots \overline{\beta}_{\mu-1} \overline{\tau} \overline{\beta}_{\mu+1} \ldots \overline{\beta}_s}, \\
\nabla_\beta \nabla_{\overline{\gamma}} \psi_{\overline{B}_s} &= \partial_\beta \partial_{\overline{\gamma}} \psi_{\overline{B}_s} - \sum_{\mu=1}^{s} \sum_{\tau=1}^{n} \partial_\beta \Gamma^{\overline{\tau}}_{\overline{\gamma}\overline{\beta}_\mu} \psi_{\overline{\beta}_1 \ldots \overline{\beta}_{\mu-1} \overline{\tau} \overline{\beta}_{\mu+1} \ldots \overline{\beta}_s} - \sum_{\mu=1}^{s} \sum_{\tau=1}^{n} \Gamma^{\overline{\tau}}_{\overline{\gamma}\overline{\beta}_\mu} \partial_\beta \psi_{\overline{\beta}_1 \ldots \overline{\beta}_{\mu-1} \overline{\tau} \overline{\beta}_{\mu+1} \ldots \overline{\beta}_s}.
\end{aligned}$$

Hence, by using Equation (1), one obtains

$$[\nabla_\beta, \nabla_{\overline{\gamma}}] \, \psi_{\overline{B}_s} = - \sum_{\mu=1}^{s} \sum_{\tau=1}^{n} \mathcal{R}^{\overline{\tau}}_{\overline{\beta}_\mu \beta \overline{\gamma}} \psi_{\overline{\beta}_1 \ldots \overline{\beta}_{\mu-1} \overline{\tau} \overline{\beta}_{\mu+1} \ldots \overline{\beta}_s}.$$

Therefore, one obtains

$$[\nabla^{(\hbar)}_{\beta}, \nabla_{\overline{\gamma}}]\ \psi_{\overline{B}_s} = -\sum_{B_s}\sum_{\mu=1}^{s}\sum_{\tau=1}^{n} \mathcal{R}^{\overline{\tau}}_{\overline{\beta}_\mu\beta\overline{\gamma}}\psi_{\overline{\beta}_1\ldots\overline{\beta}_{\mu-1}\overline{\tau}\overline{\beta}_{\mu+1}\ldots\overline{\beta}_s} + \Theta_{\beta\overline{\gamma}}\psi_{\overline{B}_s}.$$

So,

$$\begin{aligned}
\hbar^{-1}\sum_{\beta,\gamma}\sum_{B_{s-1}}[\nabla^{(\hbar)}_{\beta}, \nabla_{\overline{\gamma}}]\ \psi^{\beta}_{\overline{B}_{s-1}}\overline{\psi^{\gamma B_{s-1}}} &= \hbar^{-1}\sum_{\alpha,\beta,\gamma} g^{\overline{\alpha}\beta}[\nabla^{(\hbar)}_{\beta}, \nabla_{\overline{\gamma}}]\psi_{\overline{\alpha}\overline{B}_{s-1}}\overline{\psi^{\gamma B_{s-1}}} \\
&= -\hbar^{-1}\sum_{\alpha,\beta,\gamma} g^{\overline{\alpha}\beta}\left(\sum_{\mu=1}^{s-1}\sum_{\tau=1}^{n}\mathcal{R}^{\overline{\tau}}_{\overline{\beta}_\mu\beta\overline{\gamma}}\psi_{\overline{\alpha}\overline{\beta}_1\ldots\overline{\beta}_{\mu-1}\overline{\tau}\overline{\beta}_{\mu+1}\ldots\overline{\beta}_{s-1}}\right)\overline{\psi^{\gamma B_{s-1}}} \\
&+ \hbar^{-1}\sum_{\alpha,\beta,\gamma} g^{\overline{\alpha}\beta}\Theta_{\beta\overline{\gamma}}\psi_{\overline{\alpha}\overline{B}_{s-1}}\overline{\psi^{\gamma B_{s-1}}} \\
&= -\hbar^{-1}\sum_{\alpha,\beta,\gamma,\tau} g^{\overline{\alpha}\beta}\mathcal{R}^{\overline{\tau}}_{\overline{\alpha}\beta\overline{\gamma}}\psi_{\overline{\tau}\overline{B}_{s-1}}\overline{\psi^{\gamma B_{s-1}}} - \hbar^{-1}\sum_{\alpha,\beta,\gamma,\tau} g^{\overline{\alpha}\beta}\mathcal{R}^{\overline{\tau}}_{\overline{\beta}_\mu\beta\overline{\gamma}}\psi_{A_r\overline{\alpha}\overline{\beta}_1\ldots\overline{\beta}_{\mu-1}\overline{\tau}\overline{\beta}_{\mu+1}\ldots\overline{\beta}_{s-1}}\overline{\psi^{\gamma B_{s-1}}} \\
&+ \hbar^{-1}\sum_{\alpha,\gamma}\Theta_{\beta\overline{\gamma}}\ \psi^{\beta}_{\overline{B}_{s-1}}\overline{\psi^{\gamma B_{s-1}}}.
\end{aligned} \tag{8}$$

From the Kähler property of $\mathcal{G}$, Equation (2) gives

$$\mathcal{R}^{\overline{\tau}\ \overline{\alpha}}_{\overline{\beta}_\mu\ \overline{\gamma}} = \sum_{\beta} g^{\overline{\alpha}\ \beta}\mathcal{R}^{\overline{\tau}}_{\overline{\beta}_\mu\beta\ \overline{\gamma}} = \mathcal{R}^{\overline{\alpha}\ \overline{\tau}}_{\overline{\beta}_\mu\ \overline{\gamma}}.$$

Moreover, we remark that $\psi_{\overline{\alpha}\overline{\beta}_1\ldots\overline{\beta}_{\mu-1}\overline{\tau}\overline{\beta}_{\mu+1}\ldots\overline{\beta}_{s-1}} = -\psi_{\overline{\tau}\overline{\beta}_1\ldots\overline{\beta}_{\mu-1}\overline{\alpha}\overline{\beta}_{\mu+1}\ldots\overline{\beta}_{s-1}}$. Hence, the second term of the right-hand side of Equation (8) is zero, i.e.,

$$\hbar^{-1}\sum_{\alpha,\beta,\gamma,\tau} g^{\overline{\alpha}\beta}\mathcal{R}^{\overline{\tau}}_{\overline{\beta}_\mu\beta\overline{\gamma}}\psi_{A_r\overline{\alpha}\overline{\beta}_1\ldots\overline{\beta}_{\mu-1}\overline{\tau}\overline{\beta}_{\mu+1}\ldots\overline{\beta}_{s-1}}\overline{\psi^{\gamma B_{s-1}}} = 0.$$

As a result, Equation (8) becomes

$$\begin{aligned}
\hbar^{-1}\sum_{\beta,\gamma}\sum_{B_{s-1}}[\nabla^{(\hbar)}_{\beta}, \nabla_{\overline{\gamma}}]\psi^{\beta}_{\overline{B}_{s-1}}\overline{\psi^{\gamma B_{s-1}}} &= -\hbar^{-1}\sum_{\gamma,\tau}\left(\sum_{\alpha,\beta} g^{\overline{\alpha}\beta}R^{\overline{\tau}}_{\overline{\alpha}\beta\overline{\gamma}}\right)\psi_{\overline{\tau}\overline{B}_{s-1}}\overline{\psi^{\gamma B_{s-1}}} \\
&+ \hbar^{-1}\sum_{\alpha,\gamma}\Theta_{\beta\overline{\gamma}}\ \psi^{\beta}_{\overline{B}_{s-1}}\overline{\psi^{\gamma B_{s-1}}}.
\end{aligned} \tag{9}$$

On the other hand,

$$\sum_{\alpha,\beta=1}^{n} g^{\overline{\alpha}\beta}\mathcal{R}^{\overline{\tau}}_{\overline{\alpha}\beta\overline{\gamma}} = \sum_{\alpha,\beta,\lambda} g^{\overline{\alpha}\beta}g^{\overline{\tau}\lambda}\mathcal{R}_{\lambda\overline{\alpha}\beta\overline{\gamma}} = \sum_{\alpha,\lambda} g^{\overline{\tau}\lambda}\overline{\sum_{\beta} g^{\overline{\beta}\alpha}\mathcal{R}_{\overline{\beta}\alpha\overline{\lambda}\gamma}} = \sum_{\alpha,\lambda} g^{\overline{\tau}\lambda}\overline{\mathcal{R}^{\ \alpha}_{\alpha\ \overline{\lambda}\ \gamma}} = \sum_{\lambda} g^{\overline{\tau}\lambda}\mathcal{R}_{\overline{\gamma}\ \lambda}.$$

Hence,

$$\hbar^{-1}\sum_{\beta,\gamma}\sum_{B_{s-1}}[\nabla^{(\hbar)}_{\beta}, \nabla_{\overline{\gamma}}]\psi^{\beta}_{\overline{B}_{s-1}}\overline{\psi^{\gamma B_{s-1}}} = (s-1)!\ \hbar^{-1}\sum_{B_{s-1}}\sum_{\alpha,\gamma=1}^{n}\left(\Theta_{\beta\overline{\gamma}} - \mathcal{R}_{\beta\overline{\gamma}}\right)\psi^{\beta}_{\overline{B}_{s-1}}\overline{\psi^{\gamma B_{s-1}}}.$$

We compute the second term of Equation (9). From Equations (1) and (5), one obtains

$$(\overline{\partial}\psi, \overline{\partial}\psi) = \hbar^{-1}\sum_{C_s,D_s}\nabla_{\overline{\mu}}\psi_{\overline{C}_s}\overline{\nabla^{\tau}\psi^{D_s}}\ \mathcal{E}^{\mu\ C_s}_{\tau\ D_s},$$

where $\mathcal{E}^{\mu\ C_s}_{\tau\ D_s} = 0$ unless $\mu \notin C_s$, $\tau \notin D_s$ and $\{\mu\} \cup C_s = \{\tau\} \cup D_s$, in which case $\mathcal{E}^{\mu\ C_s}_{\tau\ D_s}$ is the sign of the permutation $(\mu\ C_s \tau\ D_s)$. Consider the terms with $\mu = \tau$. If $\mathcal{E}^{\mu\ C_s}_{\tau\ D_s} \neq 0$, then we must have $C_s = D_s$ and $\mu \notin C_s$, and hence the sum of these terms is

$$\hbar^{-1} \sum_{C_s} \sum_{\mu \notin C_s} \nabla_{\overline{\mu}} \psi_{\overline{C}_s} \overline{\nabla^{\mu} \psi^{C_s}}.$$

Next, we consider the terms with $\mu \neq \tau$. If $\mathcal{E}^{\mu\ C_s}_{\tau\ D_s} \neq 0$, $\tau \in C_s$, $\mu \in D_s$ with deletion τ from C_s or μ from D_s has the same multi-index B_{s-1}:

$$\mathcal{E}^{\mu\ C_s}_{\tau\ D_s} = \mathcal{E}^{\mu\ C_s}_{\mu\ \tau\ B_{s-1}} \mathcal{E}^{\mu\ \tau\ B_{s-1}}_{\tau\ \mu\ B_{s-1}} \mathcal{E}^{\tau\ \mu\ B_{s-1}}_{\tau\ D_s} = -\mathcal{E}^{C_s}_{\tau\ B_{s-1}} \mathcal{E}^{\mu\ B_{s-1}}_{D_s},$$

The sum of the terms in question is

$$-\hbar^{-1} \sum_{B_{s-1}} \sum_{\mu \neq \tau} \nabla_{\overline{\mu}} \psi_{\overline{\tau} \overline{B}_{s-1}} \overline{\nabla^{\tau} \psi^{\mu B_{s-1}}}.$$

Therefore, one obtains

$$\begin{aligned}
\left(\overline{\partial}\psi, \overline{\partial}\psi\right) &= \hbar^{-1} \sum_{C_s} \sum_{\mu \notin C_s} \nabla_{\overline{\mu}} \psi_{\overline{C}_s} \overline{\nabla^{\mu} \psi^{C_s}} - \hbar^{-1} \sum_{B_{s-1}} \sum_{\mu \neq \tau} \nabla_{\overline{\mu}} \psi_{\overline{\tau} \overline{B}_{s-1}} \overline{\nabla^{\tau} \psi^{\mu B_{s-1}}} \\
&= \hbar^{-1} \sum_{C_s} \sum_{\mu} \nabla_{\overline{\mu}} \psi_{\overline{C}_s} \overline{\nabla^{\mu} \psi^{C_s}} - \hbar^{-1} \sum_{B_{s-1}} \sum_{\mu \neq \tau} \nabla_{\overline{\mu}} \psi_{\overline{\tau} \overline{B}_{s-1}} \overline{\left(\sum_{\gamma} g^{\overline{\gamma} \tau} \nabla_{\overline{\gamma}} \right) \psi^{\mu B_{s-1}}} \\
&= \hbar^{-1} \sum_{C_s} \sum_{\mu} \nabla_{\overline{\mu}} \psi_{\overline{C}_s} \overline{\nabla^{\mu} \psi^{C_s}} - \hbar^{-1} \sum_{B_{s-1}} \sum_{\mu \neq \tau} \sum_{\gamma} g^{\overline{\tau} \gamma} \nabla_{\overline{\mu}} \psi_{\overline{\tau} \overline{B}_{s-1}} \overline{\nabla_{\overline{\gamma}} \psi^{\mu B_{s-1}}}.
\end{aligned} \tag{10}$$

Since $\nabla_{\overline{\tau}}\, g^{\overline{\beta}\alpha} = 0$, then by using Proposition 2, one obtains

$$\left(\overline{\partial}\psi, \overline{\partial}\psi\right) = (\overline{\nabla}, \overline{\nabla}) - \hbar^{-1} \sum_{B_{s-1}} \sum_{\mu, \gamma} \nabla_{\overline{\mu}} \psi^{\gamma}_{\overline{B}_{s-1}} \overline{\nabla_{\overline{\gamma}} \psi^{\mu B_{s-1}}}.$$

Then, one obtains

$$\hbar^{-1} \sum_{B_{s-1}} \sum_{\beta, \gamma} \nabla_{\overline{\beta}} \psi^{\gamma}_{\overline{B}_{s-1}} \overline{\nabla_{\overline{\gamma}} \psi^{\beta B_{s-1}}} = (s-1)! \left[(\overline{\nabla}, \overline{\nabla}) - \left(\overline{\partial}\psi, \overline{\partial}\psi\right) \right].$$

Therefore,

$$\sum_{\beta=1}^{n} \nabla_{\beta} \xi^{\beta} = \hbar^{-1} \sum \nabla_{\overline{\gamma}} \nabla^{(\hbar)}_{\beta} \psi^{\beta}_{\overline{B}_{s-1}} \overline{\psi^{\overline{A}_r \gamma B_{s-1}}} + (s-1)! \left\{ ((\Theta - \mathcal{R})\psi, \psi) + (\overline{\nabla}, \overline{\nabla}) - \left(\overline{\partial}\psi, \overline{\partial}\psi\right) \right\}. \tag{11}$$

Using Equation (7),

$$\begin{aligned}
\sum_{\gamma=1}^{n} \nabla_{\overline{\gamma}} \eta^{\overline{\gamma}} &= \hbar^{-1} \sum_{\beta, \gamma=1}^{n} \sum_{B_{s-1}} (\nabla_{\overline{\gamma}} - \partial_{\overline{\gamma}} \log \hbar) \nabla^{(\hbar)}_{\beta} \psi^{\beta}_{\overline{B}_{s-1}} \overline{\psi^{\gamma B_{s-1}}} \\
&\quad + \hbar^{-1} \sum_{\beta, \gamma} \sum_{B_{s-1}} \nabla^{(\hbar)}_{\beta} \psi^{\beta}_{\overline{B}_{s-1}} \overline{\nabla_{\gamma} \psi^{\gamma B_{s-1}}} \\
&= \hbar^{-1} \sum_{\beta, \gamma} \sum_{B_{s-1}} \nabla_{\overline{\gamma}} \nabla^{(\hbar)}_{\beta} \psi^{\beta}_{\overline{B}_{s-1}} \overline{\psi^{\gamma B_{s-1}}} + \hbar^{-1} \sum_{\beta, \gamma} \sum_{B_{s-1}} \nabla^{(\hbar)}_{\beta} \psi^{\beta}_{\overline{B}_{s-1}} \overline{\nabla^{(\hbar)}_{\gamma} \psi^{\gamma B_{s-1}}}.
\end{aligned} \tag{12}$$

Hence, by using Equation (11), one obtains

$$\sum_{\gamma=1}^{n} \nabla_{\overline{\gamma}} \eta^{\overline{\gamma}} = (s-1)! \left(\overline{\partial}^{*} \psi, \overline{\partial}^{*} \psi\right) + \hbar^{-1} \sum_{\beta, \gamma} \sum_{B_{s-1}} \nabla_{\overline{\gamma}} \nabla^{(\hbar)}_{\beta} \psi^{\beta}_{\overline{B}_{s-1}} \overline{\psi^{\gamma B_{s-1}}}. \tag{13}$$

Subtracting Equation (13) from Equation (12) and from Proposition 2, one obtains

$$\frac{1}{(s-1)!}(\text{div } \xi - \text{div } \eta) = (\overline{\nabla}, \overline{\nabla}) - \left(\bar{\partial}\psi, \bar{\partial}\psi\right) - \left(\bar{\partial}^{\circledast}\psi, \bar{\partial}^{\circledast}\psi\right) + ((\Theta - \mathcal{R})\psi, \psi).$$

By integrating this identity over Ω and by applying the divergence theorem, one obtains

$$\frac{1}{(s-1)!}\int_{\partial\Omega}(\xi - \eta).\text{n}\,\text{ds} = \|\overline{\nabla}\psi\|^2 - \|\bar{\partial}\psi\|^2 - \|\bar{\partial}^{\circledast}\psi\|^2 + < (\Theta - \mathcal{R})\psi, \psi >, \tag{14}$$

with the outer unit normal vector n to $\partial\Omega$, which is given at each point $x \in \partial\Omega$ by $n = \frac{\text{grad } \lambda}{|\text{grad } \lambda|}$, and the projection of the vector $(\xi - \eta)$ on the vector n is $(\xi - \eta) \, . \, n$. Now, we compute η . grad λ. Since

$$\eta \ . \ \text{grad } \lambda = \sum_{\gamma=1}^{n} \eta^{\overline{\gamma}} \partial_{\overline{\gamma}} \lambda = \hbar^{-1} \sum_{\beta=1}^{n} \sum_{B_{s-1}} \nabla_{\beta}^{(\hbar)} \psi^{\beta}_{\overline{B}_{s-1}} \overline{\left(\sum_{\gamma=1}^{n} \psi^{\gamma B_{s-1}} \ \partial_{\gamma} \lambda \ \right)},$$

at any point of X, then for $\psi \in \mathscr{B}_{0,s}(\overline{\Omega}, \Xi)$, $s \geq 1$, one obtains

$$\eta \, . \, \text{grad } \lambda = 0 \text{ on } \partial\Omega.$$

Hence,

$$\eta \, . \, n \; = \; 0, \text{ on } \partial\Omega. \tag{15}$$

Now we compute $\xi \, . \, n.$ from Equation (5); one obtains

$$\begin{aligned} \xi \, . \, n &= \frac{1}{|\text{grad } \lambda|}(\xi \, . \, \text{grad } \lambda) = \frac{1}{|\text{grad } \lambda|} \sum_{\beta=1}^{n} \xi^{\beta} \partial_{\beta} \lambda \\ &= \frac{1}{\hbar|\text{grad } \lambda|} \sum_{\gamma=1}^{n} \sum_{B_{s-1}} \left(\sum_{\beta=1}^{n} \nabla_{\overline{\gamma}} \psi^{\beta}_{\overline{B}_{s-1}} \ \partial_{\beta} \lambda \right) \overline{\psi^{\gamma B_{s-1}}}. \end{aligned} \tag{16}$$

Again, for $\psi \in \mathscr{B}_{0,s}(\overline{\Omega}, \Xi)$, $s \geq 1$, one obtains

$$\sum_{\beta=1}^{n} \psi^{\beta}_{\overline{B}_{s-1}} \ \partial_{\beta} \lambda = 0 \text{ on } \partial\Omega.$$

Since $\lambda \equiv 0$ on $\partial\Omega$, then we can write

$$\sum_{\beta=1}^{n} \psi^{\beta}_{\overline{B}_{s-1}} \partial_{\beta} \lambda = \lambda \, \phi_{\overline{B}_{s-1}},$$

on the neighborhood U of $\partial\Omega$, where $\phi_{\overline{B}_{s-1}}$ is a C^{∞} section of $\bigwedge^{s-1} \overline{T}^{\circledast}(X) \otimes \Xi$. So,

$$\sum_{\beta=1}^{n} \nabla_{\overline{\gamma}} \psi^{\beta}_{\overline{B}_{s-1}} \partial_{\beta} \lambda + \sum_{\beta=1}^{n} \psi^{\beta}_{\overline{B}_{s-1}} \partial_{\beta} \partial_{\overline{\gamma}} \lambda = \phi_{\overline{B}_{s-1}} \partial_{\overline{\gamma}} \lambda + \lambda \nabla_{\overline{\gamma}} \phi_{\overline{B}_{s-1}} \text{ on } U.$$

Then, we multiply this equation by $\hbar^{-1}\overline{\psi^{\gamma B_{s-1}}}$ and sum it with respect to γ. Since $\psi \in \mathscr{B}_{0,s}(\overline{\Omega}, \Xi)$, one obtains

$$\begin{aligned}
&\hbar^{-1}\sum_{B_{s-1}}\sum_{\beta,\gamma=1}^{n}\nabla_{\overline{\gamma}}\psi^{\beta}_{\overline{B}_{s-1}}\partial_{\beta}\lambda\,\overline{\psi^{\gamma B_{s-1}}}+\hbar^{-1}\sum_{B_{s-1}}\sum_{\beta,\gamma=1}^{n}\partial_{\beta}\partial_{\overline{\gamma}}\lambda\,\psi^{\beta}_{\overline{B}_{s-1}}\overline{\psi^{\gamma B_{s-1}}}\\
&\quad=\hbar^{-1}\sum_{B_{s-1}}\sum_{\gamma=1}^{n}\phi_{\overline{B}_{s-1}}\partial_{\overline{\gamma}}\lambda\,\overline{\psi^{\gamma B_{s-1}}}+\hbar^{-1}\sum_{B_{s-1}}\sum_{\gamma=1}^{n}\lambda\,\nabla_{\overline{\gamma}}\phi_{\overline{B}_{s-1}}\overline{\psi^{\gamma B_{s-1}}}\\
&\quad=0,
\end{aligned}$$

on $\partial\Omega$. Therefore, by dividing by $|\text{grad }\lambda|$, (16) becomes

$$\xi\,.\,n=-\frac{1}{\hbar|\text{grad }\lambda|}\sum_{B_{s-1}}\sum_{\beta,\gamma=1}^{n}\partial_{\beta}\partial_{\overline{\gamma}}\lambda\,\psi^{\beta}_{\overline{B}_{s-1}}\,\overline{\psi^{\gamma B_{s-1}}}\text{ on }\partial\Omega.$$

Then,

$$\xi\,.\,n=-\frac{1}{\hbar|\text{grad }\lambda|}\,(\mathscr{L}(\lambda)\psi,\psi), \tag{17}$$

on $\partial\Omega$. Thus, the proposition is proved by substituting Equations (15) and (17) in Equation (14). □

4. Bounded P.S.H. Functions and Hartogs Pseudoconvexity in Kähler Manifolds

Definition 2 ([28]). *Ω is the smooth local Stein domain if $\forall$ point $z \in \partial\Omega$, and $\exists$ is a neighborhood U if z satisfies $U \cap \Omega$, which is Stein.*

Definition 3 ([29]). *We say that Ω is Hartogs pseudoconvex if there exists a smooth bounded function h on Ω such that*

$$i\partial\overline{\partial}(-\log\delta+h)\geq C\,\omega\ \ in\ \ \Omega, \tag{18}$$

for some $C > 0$, where ω is the Kähler form associated with the Kähler metric.

In particular, every Hartogs pseudoconvex domain admits a strictly plurisubharmonic exhaustion function and is thus a Stein manifold.

Next, we will examine several examples of Hartogs pseudoconvex domains.

Example 1. *Suppose X is a complex manifold with a continuous strongly plurisubharmonic function and $\Omega \Subset \mathbb{X}$ is a Stein domain. According to [30], there exists a Kähler metric on X such that Ω is Hartogs pseudoconvex.*

Example 2 ([29]). *All the local Stein-domain subsets of a Stein manifold are in the Hartogs pseudoconvex domain.*

Example 3 ([29]). *Every C^2 pseudoconvex domain in the C^n subset of a Stein manifold is a Hartogs pseudoconvex domain.*

Example 4 ([30]). *Any local Stein domain subset of a Kähler manifold with positive holomorphic bisectional curvature satisfies Equation (18) on $U \cap \Omega$.*

Example 5 ([30]). *If Ω is a local Stein domain of the complex projective space $\mathcal{P}^n$, then Ω satisfies Equation (18).*

The canonical line bundle K of X is defined by transition functions (k_{ij})

$$k_{ij}=\frac{\partial(\mathtt{w}^1_j,\mathtt{w}^2_j,\ldots,\mathtt{w}^n_j)}{\partial(\mathtt{w}^1_i,\mathtt{w}^2_i,\ldots,\mathtt{w}^n_i)}\ \text{ on }\ U_i\cap U_j,$$

with
$$\mathrm{g}_i = |k_{ij}|^2 \mathrm{g}_j \text{ on } U_i \cap U_j.$$
Hence, $g = \{\mathrm{g}_i\}$ determines a metric of K. Let $h = \{h_i\}$ be a Hermitian metric of Ξ and $\partial\bar{\partial} \log h$ its curvature tensor. So, $\{\hbar = g.h\}$ determines a Hermitian metric of $\Xi \otimes K$ and
$$\partial\bar{\partial} \log \hbar = \partial\bar{\partial} \log h + \partial\bar{\partial} \log g.$$

Then, from Proposition 4,
$$\|\bar{\partial}\psi\|^2 + \|\bar{\partial}^* \psi\|^2 = \|\overline{\nabla}\psi\|^2 + < \Theta\psi, \psi > + \frac{1}{\hbar |\mathrm{grad}\, \lambda|} \int_{\partial D} (\mathscr{L}(\lambda)\psi, \psi)\, ds, \tag{19}$$
for $\psi \in \mathscr{B}_{0,s}(\overline{D}, \Xi \otimes K)$, $s \geq 1$. Using $h = h_{\mathtt{m}} = \zeta^{\mathtt{m}} h$, one obtains
$$\Theta_{\mathtt{m}} = \Theta - \mathtt{m}\partial\bar{\partial}(-\log \zeta).$$

With respect to the $\mathcal{G}$ and $h_{\mathtt{m}}$, and for $\psi, \psi \in C^\infty_{n,s}(\overline{D}, \Xi)$, we define the global inner product $< \psi, \psi >_{\mathtt{m}}$ and the norm $\|\psi\|_{W^{\mathtt{m}}_{n,s}(D,\Xi)}$ by
$$< \psi, \psi >_{\mathtt{m}} = \int_D (\psi, \psi)_{\mathtt{m}}\, dv \text{ and } \|\psi\|^2_{W^{\mathtt{m}}_{n,s}(D,\Xi)} = < \psi, \psi >_{\mathtt{m}} .$$

Then, (19) becomes
$$\begin{aligned} \|\bar{\partial}\psi\|^2_{W^{\mathtt{m}}_{n,s}(D,\Xi)} + \|\bar{\partial}^*_{\mathtt{m}} \psi\|^2_{W^{\mathtt{m}}_{n,s}(D,\Xi)} = \|\overline{\nabla}\psi\|^2_{W^{\mathtt{m}}_{n,s}(D,\Xi)} + \frac{1}{\hbar |\mathrm{grad}\, \lambda|} \int_{\partial D} (\mathscr{L}(\lambda)\psi, \psi)_{\mathtt{m}}\, ds + < \Theta_{\mathtt{m}}\psi, \psi >_{\mathtt{m}} \\ + < \mathtt{m}\partial\bar{\partial}(-\log \zeta)\psi, \psi >_{\mathtt{m}} . \end{aligned} \tag{20}$$
As Theorem 1.1 in [31], one obtains

Theorem 1. *Suppose X is an n-dimensional complex manifold and $D \Subset X$ is a Hartogs pseudoconvex. $\zeta(z) = dist(z, \partial D) = -\rho(z)$, where ζ is the Kähler metric ω on X. If $\mathtt{m} > 0$, then*
$$i\partial\bar{\partial}(-\zeta^{\mathtt{m}}) \geq c_{\mathtt{m}}\, |\zeta^{\mathtt{m}}|\, \omega, \tag{21}$$
for some constant $c_{\mathtt{m}} > 0$.

Proof. Using Equation (18) and if $\rho = -\zeta$,
$$-\rho\, i\, \partial\bar{\partial}\rho + i\, \partial\rho \wedge \bar{\partial}\rho \geq C\, \rho^2 \omega. \tag{22}$$

Let (e_i) be an orthonormal basis for $T(\partial D)$ near p. In this case, near $p \in \partial D$, choose local coordinates that satisfy $x_{2n} = \rho$, $e_i(p) = 0$, $i = 1, 2, \ldots, n-1$. The Hermitian form for $i\partial\bar{\partial}\rho$ is denoted by (a_{ij}). The inequality (22) gives the coordinates
$$-\rho \sum_{i,j=1}^{n} a_{ij}\, \eta_i\, \overline{\eta}_j + |\partial\rho|^2 |\eta_n|^2 \geq C\, \rho^2 \sum_{j=1}^{n} |\eta_j|^2. \tag{23}$$

If $\eta_n = 0$,
$$\sum_{i,j=1}^{n-1} a_{ij}\, \eta_i\, \overline{\eta}_j \geq C\, |\rho| \sum_{j=1}^{n-1} |\eta_j|^2.$$
Expanding (23), one obtains
$$-\rho \sum_{i,j=1}^{n-1} a_{ij}\, \eta_i\, \overline{\eta}_j + 2\,\mathrm{Re}\,(-\rho) \sum_{k=1}^{n-1} a_{nk}\, \eta_n\, \overline{\eta}_k - \rho\, a_{nn}\, |\eta_n|^2 + |\partial\rho|^2\, |\eta_n|^2 \geq C\, |\rho|^2 \sum_{j=1}^{n-1} |\eta_j|^2.$$

for $j \leq n-1$, replacing v by $\eta_j/(-\rho)$,

$$\sum_{i,j=1}^{n-1} \left(\frac{a_{ij}}{-\rho}\right) \eta_i \, \overline{\eta}_j + 2\,\mathtt{Re}\,(-\rho) \sum_{k=1}^{n-1} a_{nk}\, \eta_n \, \overline{\eta}_k - \rho\, a_{nn} \, |\eta_n|^2 + |\partial\rho|^2 \, |\eta_n|^2 \geq C' \sum_{j=1}^{n-1} |\eta_j|^2. \quad (24)$$

The inequality's left side can be expressed as follows:

$$Q(z,\eta) + |\partial\rho|^2 \, |\eta_n|^2.$$

For $z \in D$, we assume that

$$\widetilde{Q}(\varsigma,\eta) = \liminf_{z\to\varsigma} Q(z,\eta) = \lim_{t\to 0} \inf_{|z-\varsigma|<t} Q(z,\eta).$$

From Equation (24), one obtains

$$\widetilde{Q}\,(\varsigma,\eta) + |\partial\rho|^2\,(\varsigma)\,|\eta_n|^2 \geq C' \sum_{j=1}^{n-1} |\eta_j|^2. \quad (25)$$

Take a look at $\widetilde{Q}(p,(0,\eta_n) \geq 0$; for a small enough C',

$$\widetilde{Q}(\varsigma,v) + |\partial\rho|^2(\varsigma)|\eta_n|^2 \geq C'|\eta_n|^2,$$

in a neighborhood of p. On the sphere $|\eta| = 1$, inequality (25) still holds for $|\eta'| \leq \sigma$ in a neighborhood of $\eta' = 0$, where $\eta = (\eta', \eta_n)$. This gives us

$$Q(z,\eta) + |\partial\rho|^2(z)|\eta_n|^2 \geq \frac{C'}{2}|\eta_n|^2,$$

for $\zeta(z) < \sigma'$, $|\eta'| \leq \sigma$. But, when $|\eta'| > \sigma$ and $|\eta| = 1$, one obtains $|\eta'|^2 \geq \sigma_0|\eta_n|^2$, where $\sigma_0 = \sigma^2(1-\sigma^2)^{-1}$. So, by using (25),

$$Q(z,\eta) + |\partial\rho|^2 \, |\eta_n|^2 \geq \sigma^{**}|\eta_n|^2 \text{ for some } \sigma^{**} > 0$$

and for $\zeta(z) < \sigma^*$,

$$Q(z,\eta) + |\partial\rho|^2 \, |\eta_n|^2 \geq \frac{\sigma^{**}}{2} \, |\eta_n|^2 + \frac{C'}{2} \sum_{j=1}^{n-1} |\eta_j|^2.$$

Recalling this one yields

$$-\rho \sum_{i,j=1}^{n} a_{ij}\eta_i\overline{\eta}_j + |\partial\rho|^2|\eta_n|^2 \geq \frac{\sigma}{2}|\eta_n|^2 + \frac{C'}{2}\sum_{j=1}^{n-1}|\eta_j|^2$$

Which means

$$\begin{aligned} -i\,\partial\overline{\partial}(-\rho)^{\mathtt{m}} &= i\,\mathtt{m}\,(-\rho)^{\mathtt{m}} \left(\frac{\partial\overline{\partial}\rho}{-\rho} + (1-\mathtt{m})\,\frac{\partial\rho\wedge\overline{\partial}\rho}{\rho^2}\right) \\ &\geq \frac{C'}{2}\mathtt{m}\,|\rho|^{\mathtt{m}}\,\omega. \end{aligned}$$

□

Lemma 1. *Let $D \Subset X$ be a C^2 Hartogs pseudoconvex in an n-dimensional complex manifold X. Suppose $\mathtt{m}_0 = \mathtt{m}_0(D) > 0$ is the order of plurisubharmonicity for $\zeta(z) = d(z,\partial D)$:*

$$\mathtt{m}_0(D) = \sup\{0 < \varepsilon \leq 1 |\ i\partial\overline{\partial}(-\zeta^{\varepsilon}) \geq \ on\ \ D\}.$$

Then, $\forall\, 0 < \mathtt{m} < \mathtt{m}_0$ *and* $\phi = -\mathtt{m}\log\zeta$*; there exists*

$$it\partial\overline{\partial}\phi \geq i\,\partial\phi \wedge \overline{\partial}\phi, \tag{26}$$

with $0 < t = \frac{\mathtt{m}}{\mathtt{m}_0} < 1$*. Also, there exists* $C_{\mathtt{m}} > 0$*, which satisfies*

$$i\partial\overline{\partial}(-\zeta^{\mathtt{m}}) \geq C_{\mathtt{m}}\,\zeta^{\mathtt{m}}\left(\frac{i\partial\zeta \wedge \overline{\partial}\zeta}{\zeta^2} + \omega\right). \tag{27}$$

Proof. By Equation (21), $\exists\, \mathtt{m}_0 > 0$ satisfies $i\partial\overline{\partial}(-\zeta^{\mathtt{m}_0}) \geq 0$ on D. Since

$$\begin{aligned} i\partial\overline{\partial}(-\zeta^{\mathtt{m}_0}) &= -i\partial(\mathtt{m}_0\zeta^{\mathtt{m}_0-1}\overline{\partial}\zeta) = -i\mathtt{m}_0(\mathtt{m}_0-1)\zeta^{\mathtt{m}_0-2}\partial\zeta\wedge\overline{\partial}\zeta - i\mathtt{m}_0\zeta^{\mathtt{m}_0-1}\partial\overline{\partial}\zeta \\ &= \mathtt{m}_0\zeta^{\mathtt{m}_0}\left((1-\mathtt{m}_0)\frac{\partial\zeta\wedge\overline{\partial}\zeta}{\zeta^2} + i\frac{\partial\overline{\partial}(-\zeta)}{\zeta}\right). \end{aligned} \tag{28}$$

Then

$$(1-\mathtt{m}_0)\frac{i\partial\zeta\wedge\overline{\partial}\zeta}{\zeta^2} + i\frac{\partial\overline{\partial}(-\zeta)}{\zeta} \geq 0. \tag{29}$$

Also, by using Equation (18), one obtains

$$i\partial\overline{\partial}(-\log\zeta) = \frac{i\partial\zeta\wedge\overline{\partial}\zeta}{\zeta^2} + i\frac{\partial\overline{\partial}(-\zeta)}{\zeta} \geq \omega. \tag{30}$$

Therefore, from Equations (29) and (30), one obtains

$$i\partial\overline{\partial}(-\log\zeta) \geq \mathtt{m}_0\frac{i\partial\zeta\wedge\overline{\partial}\zeta}{\zeta^2}. \tag{31}$$

Since $\partial\phi = -\mathtt{m}\frac{\partial\zeta}{\zeta}$ and $\overline{\partial}\phi = -\mathtt{m}\frac{\overline{\partial}\zeta}{\zeta}$, then

$$i\partial\phi\wedge\overline{\partial}\phi = \mathtt{m}^2\frac{i\partial\zeta\wedge\overline{\partial}\zeta}{\zeta^2}. \tag{32}$$

Then, from Equations (31) and (32), one obtains

$$i\partial\overline{\partial}(-\log\zeta) \geq \mathtt{m}_0\frac{i\partial\phi\wedge\overline{\partial}\phi}{\mathtt{m}^2}.$$

Then, Equation (26) is proved.

To prove Equation (27), choose $0 < \kappa < \min\{1, \frac{\mathtt{m}_0-\mathtt{m}}{\mathtt{m}_0}\}$, and by using Equation (28), one obtains

$$\begin{aligned} i\partial\overline{\partial}(-\zeta^{\mathtt{m}}) &= -i\mathtt{m}(\mathtt{m}-1)\zeta^{\mathtt{m}-2}\partial\zeta\wedge\overline{\partial}\zeta - i\mathtt{m}\zeta^{\mathtt{m}-1}\partial\overline{\partial}\zeta = \mathtt{m}\zeta^{\mathtt{m}}\left((1-\mathtt{m})\frac{i\partial\zeta\wedge\overline{\partial}\zeta}{\zeta^2} + \frac{i\partial\overline{\partial}(-\zeta)}{\zeta}\right) \\ &= \mathtt{m}\zeta^{\mathtt{m}}\left((\mathtt{m}_0-\mathtt{m}-\kappa\mathtt{m}_0)\frac{i\partial\zeta\wedge\overline{\partial}\zeta}{\zeta^2} + (1-\kappa)\frac{i\partial\overline{\partial}(-\zeta^{\mathtt{m}_0})}{\mathtt{m}_0\zeta^{\mathtt{m}_0}} + \kappa i\partial\overline{\partial}(-\log\zeta)\right) \\ &\geq C_{\mathtt{m}}\mathtt{m}\zeta^{\mathtt{m}}\left(\frac{i\partial\zeta\wedge\overline{\partial}\zeta}{\zeta^2} + \omega\right). \end{aligned}$$

Then, Equation (27) is proved. □

5. The L^2 Estimates of $\overline{\partial}$

As in [21–23,32,33], one proves the following results:

Theorem 2. *Let $D \Subset X$ be a C^2 Hartogs pseudoconvex in an n-dimensional complex manifold X. Let Ξ be a positive line bundle over X whose curvature form Θ satisfies $\Theta \geq C\omega$, where $C > 0$. Let $\psi \in L^2_{n,s}(D, \zeta^{\mathtt{m}}, \Xi)$, $1 \leq s \leq n$, a $\overline{\partial}$-closed form. Then, for $0 < \mathtt{m} < \mathtt{m}_0$, there exists $\psi \in L^2_{n,s-1}(D, \zeta^{\mathtt{m}}, \Xi)$, which satisfies $\overline{\partial}\psi = \psi$ and*

$$\int_{\Omega} |\psi|^2 \zeta^{\mathtt{m}} dv \leq C \int_{\Omega} |\psi|^2 \zeta^{\mathtt{m}} dv. \tag{33}$$

Proof. The boundary term in Equation (20) vanishes since $\mathtt{m} > 0$. For $u \in \mathscr{B}_{n,s}(\overline{D}, \Xi)$, $s \geq 1$, and since the curvature form Θ of Ξ satisfies

$$\Theta \geq C_D \ \omega \ \text{ on } \ D \ \text{ with } \ C_D > 0.$$

then by using Equation (18), one obtains

$$< \Theta_{\mathtt{m}} \psi, \psi >_{\mathtt{m}} \geq \ C_{\mathtt{m}} \ < \psi, \psi >_{\mathtt{m}} . \tag{34}$$

Also, from the assumption of pseudoconvex on D, one obtains

$$\|\overline{\partial} u\|^2_{W^{\mathtt{m}}_{n,s}(D,\Xi)} + \|\overline{\partial}^{\circledast}_{\mathtt{m}} u\|^2_{W^{\mathtt{m}}_{n,s}(D,\Xi)} \geq C_{\Omega} \|u\|^2_{W^{\mathtt{m}}_{n,s}(D,\Xi)},$$

for all $u \in \mathscr{B}_{n,s}(D, \Xi)$. Let $u \in \mathcal{D}^{\infty}_{n,s}(D, E)$, with $u = u_1 + u_2$, $u_1 \in \ker(\overline{\partial}, \Xi)$ and $u_2 \in \ker(\overline{\partial}, E)^{\perp} = \overline{\mathrm{Im}\,(\overline{\partial}^{\circledast}_{\mathtt{m}}, E)} \subset \ker(\overline{\partial}^{\circledast}_{\mathtt{m}}, \Xi)$. Then, for every (n, s) form u with compact support, one obtains

$$\begin{aligned}
| < u, \psi >_{\mathtt{m}} | &= | < u_1 + u_2, \psi >_{\mathtt{m}} | \\
&= | < u_1, \psi >_{\mathtt{m}} | + | < u_2, \psi >_{\mathtt{m}} | \\
&= | < u_1, \psi >_{\mathtt{m}} | \leq \|u_1\|_{W^{\mathtt{m}}_{n,s}(D,\Xi)} \|\psi\|_{W^{\mathtt{m}}_{n,s}(D,\Xi)} \\
&\leq \frac{1}{\sqrt{C}} \|\overline{\partial}^{\circledast}_{\mathtt{m}} u_1\|_{W^{\mathtt{m}}_{n,s}(D,\Xi)} \|\psi\|_{W^{\mathtt{m}}_{n,s}(D,\Xi)} \\
&= \frac{1}{\sqrt{C}} \|\overline{\partial}^{\circledast}_{\mathtt{m}} u\|_{W^{\mathtt{m}}_{n,s}(D,\Xi)} \|\psi\|_{W^{\mathtt{m}}_{n,s}(D,\Xi)}.
\end{aligned}$$

Using the Riesz representation theorem, the linear form

$$\overline{\partial}^{\circledast}_{\mathtt{m}} u \longmapsto < u, \psi >_{\mathtt{m}},$$

is continuous on $\mathrm{Rang}(\overline{\partial}^{\circledast}, \Xi)$ in the L^2 norm and has norm $\leq C$, with

$$\|\psi\|_{W^{\mathtt{m}}_{n,s}(D,\Xi)} = C.$$

Following Hahn–Banach theorem, $\exists$ is an element that is E valued $(n, s-1)$ from u on D (with a smooth boundary) perpendicular to $\ker(\overline{\partial}, E)$ with $\|\psi\|_{W^{\mathtt{m}}_{n,s}(D,\Xi)} \leq C$,

$$< \overline{\partial}^{\circledast}_{\mathtt{m}} u, \psi >_{\mathtt{m}} = < u, \psi >_{\mathtt{m}},$$

for all $L^2 u$ with both $\overline{\partial} u$ and $\overline{\partial}^{\circledast}_{\mathtt{m}} u$ and also L^2. Hence,

$$\overline{\partial}\psi = \psi,$$

and

$$\|\psi\|_{W^{\mathtt{m}}_{n,s}(D,\Xi)} \leq C \|\psi\|_{W^{\mathtt{m}}_{n,s}(D,\Xi)}.$$

Exhaust a general pseudoconvex domain D by a sequence D_μ of C^∞ pseudoconvex domains:

$$D = \cup_{\mu=1}^{\infty} D,$$

with $D_\mu \subset D_{\mu+1} \subset D$ for each μ. On each D_μ, $\exists$ a $\psi_\mu \in L^2_{n,s-1}(D_\mu, \zeta^{\mathtt{m}}, \Xi)$ satisfies

$$\bar{\partial}\psi_\mu = \psi \text{ in } D_\mu,$$

and

$$\int_{D_\mu} |\psi_\mu|^2 \zeta^{\mathtt{m}} dv \leq C \int_{D_\mu} |\psi|^2 \zeta^{\mathtt{m}} dv \leq c \int_D |\psi|^2 \zeta^{\mathtt{m}} dv.$$

Choose a subsequent ψ_μ of ψ_μ, satisfying

$$\psi_\mu \longrightarrow \psi,$$

in $L^2_{n,s-1}(D, \zeta^{\mathtt{m}}, \Xi)$ weakly. Moreover,

$$\int_D |\psi|^2 \zeta^{\mathtt{m}} dv \leq \liminf C \int_D |\psi|^2 \zeta^{\mathtt{m}} dv \leq c \int_D |\psi|^2 \zeta^{\mathtt{m}} dv.$$

□

Theorem 3. *Let X, D and Ξ be the same as Theorem 2. Let $\psi \in L^2_{n,s}(D, \Xi)$, $1 \leq s \leq n$, with $\bar{\partial}\psi = 0$. Thus, $\exists$ $\psi \in L^2_{n,s-1}(D, \Xi)$ satisfies $\bar{\partial}\psi = \psi$ and*

$$\|\psi\| \leq \|\psi\|.$$

Proof. Since

$$\frac{1}{\hbar|\text{grad}\,\lambda|} \int_{\partial\Omega} (\mathscr{L}(\lambda)u, u)\, ds \geq C \int_{\partial\Omega} |u|^2\, ds,$$

and from Equation (18), one obtains

$$\|\bar{\partial}u\|^2 + \|\psi u\|^2 \geq C_D \|u\|^2,$$

$\forall\, u \in \mathscr{B}_{n,s}(D, \Xi)$. This completes the proof of Theorem 3. □

Following Theorem 3, as in [34,35], one can prove the following:

Theorem 4. *Let X, D and Ξ be the same as Theorem 2. Then, $\Box$ has a closed range and $\ker_{n,s}(\Box, E) = \{0\}$. For each $1 \leq s \leq n$, there exists a bounded linear operator*

$$N_{n,s} : L^2_{n,s}(D, \Xi) \longrightarrow L^2_{n,s}(D, \Xi),$$

which satisfies

(i) $Rang(N_{n,s}, \Xi) \subset Dom(\Box_{n,s}, \Xi)$ and $\Box_{n,s}\ N_{n,s} = N_{n,s}\ \Box_{n,s} = I$ on $Dom(\Box_{n,s}, \Xi)$.
(ii) $\forall\, \psi \in L^2_{n,s}(D, \Xi)$, $\psi = \bar{\partial}\bar{\partial}^{\circledast} N_{n,s}\psi + \bar{\partial}^{\circledast}\bar{\partial}N_{n,s}\psi$.
(iii) For $\psi \in L^2_{n,s}(D, \Xi)$, one obtains

$$\begin{aligned} \|N_{n,s}\psi\| &\leq c_0\|\psi\|, \\ \|\bar{\partial}N_{n,s}\psi\| &\leq c_0\|\psi\|, \\ \|\bar{\partial}^{\circledast} N_{n,s}\psi\| &\leq c_0\|\psi\|. \end{aligned}$$

(iv)

$$\begin{aligned} N_{(n,s+1)}\bar{\partial} &= \bar{\partial}N_{n,s} \text{ on } Dom(\bar{\partial}, \Xi),\ 1 \leq s \leq n-1, \\ \bar{\partial}^{\circledast} N_{n,s} &= N_{n,s-1}\bar{\partial}^{\circledast} \text{ on } Dom\ (\bar{\partial}^{\circledast}, E),\ 2 \leq s \leq n. \end{aligned}$$

(v) If $\psi \in L^2_{n,s}(D,\Xi)$ and $\overline{\partial}\psi = 0$, then $\psi = \overline{\partial}\overline{\partial}^* N_{n,s}\psi$ and $u = \overline{\partial}^* N_{n,s}\psi$.

Proof.

$$L^2_{n,s}(D,\Xi) = \overline{\mathrm{Rang}(\Box_{n,s},\Xi)} \oplus \ker(\Box_{n,s},\Xi).$$

We need to show that

$$\ker(\Box_{n,s},\Xi) = \ker(\overline{\partial},\Xi) \cap \ker(\overline{\partial}^*,\Xi) = \{0\}. \tag{35}$$

To show that

$$\ker(\overline{\partial},\Xi) \cap \ker(\overline{\partial}^*,\Xi) = \{0\}. \tag{36}$$

We note that if $\psi \in L^2_{n,s}(\overline{\partial},\Xi)$, then by using Theorem 4, $\exists$ a $\psi \in L^2_{n,s-1}(\Omega,\Xi)$ satisfies $\psi = \overline{\partial}\psi$. If ψ is also in $\ker(\overline{\partial}^*,\Xi)$, one obtains

$$0 =< \overline{\partial}^*\overline{\partial}\psi,\psi >= \|\overline{\partial}\psi\|^2.$$

Thus, $\psi = 0$ and Equation (35) is proved. We shall show that $\mathrm{Rang}(\Box_{n,s},\Xi)$ is closed. Following Theorem 4, $\forall\ \psi \in L^2_{n,s}(D,\Xi), s > 0$ with $\overline{\partial}\psi = 0$ and $\exists$ a $\psi \in L^2_{n,s-1}(D,\Xi)$ satisfies $\psi = \overline{\partial}\psi$ and

$$\|\psi\|^2 \leq c_0\|\psi\|^2,$$

where $c_0 = c_0(D) > 0$. Thus, $\mathrm{Rang}(\overline{\partial},\Xi)$ is closed in every degree. Thus,

$$\|\psi\|^2 \leq c_0(\|\overline{\partial}\psi\|^2 + \|\overline{\partial}^*\psi\|^2),$$

for $\psi \in \mathrm{Dom}(\overline{\partial},E) \cap \mathrm{Dom}\,(\overline{\partial}^*,\Xi)$ and $\psi \perp \ker(\overline{\partial},\Xi) \cap \ker(\overline{\partial}^*,\Xi)$. Thus, from (36),

$$\|\psi\|^2 \leq c_0(\|\overline{\partial}\psi\|^2 + \|\overline{\partial}^*\psi\|^2),$$

for $\psi \in \mathrm{Dom}(\overline{\partial},\Xi) \cap \mathrm{Dom}(\overline{\partial}^*,\Xi)$. Thus, $\forall\ \psi \in \mathrm{Dom}(\Box_{n,s},\Xi)$,

$$\begin{aligned}\|\psi\|^2 &\leq c_0[< \overline{\partial}\psi,\overline{\partial}\psi > + < \overline{\partial}^*\psi,\overline{\partial}^*\psi >] \\ &= c_0[< \overline{\partial}^*\overline{\partial}\psi,\psi > + < \overline{\partial}\overline{\partial}^*\psi,\psi >] \\ &= c_0 < \Box\psi,\psi > \\ &\leq c_0\|\Box\psi\|\|\psi\|.\end{aligned}$$

Thus,

$$\|\psi\| \leq c_0\|\Box\psi\|, \tag{37}$$

i.e., $\mathrm{Rang}(\Box_{n,s},\Xi)$ is closed. Therefore,

$$L^2_{n,s}(D,\Xi) = \mathrm{Rang}(\Box,\Xi) = \overline{\partial}\overline{\partial}^*\mathrm{Dom}(\Box_{n,s},E) \oplus \overline{\partial}^*\overline{\partial}\mathrm{Dom}(\Box_{n,s},\Xi).$$

Also, from Equation (37), $\Box_{n,s}$ is 1-1 and $\mathrm{Rang}(\Box_{n,s},\Xi)$ is the whole space $L^2_{n,s}(D,E)$. Thus, there exists a unique inverse

$$N_{n,s} : L^2_{n,s}(D,\Xi) \longrightarrow L^2_{n,s}(D,\Xi),$$

which satisfies $\Box N = N\Box = I$ and

$$\|N_{n,s}\psi\| \leq c_0\|\psi\|.$$

$\forall\, \psi \in L^2_{n,s}(D,\Xi)$. Also, by (ii),

$$\begin{aligned} <\overline{\partial}^* N_{n,s}\psi, \overline{\partial}^* N_{n,s}\psi> + <\overline{\partial} N_{n,s}\psi, \overline{\partial} N_{n,s}\psi> &=< (\overline{\partial\partial}^* + \overline{\partial}^*\overline{\partial}) N_{n,s}\psi, N_{n,s}\psi > \\ &=< \Box_{n,s} N_{n,s}\psi, N_{n,s}\psi > \\ &\leq \|\psi\| \|N_{n,s}\psi\| \\ &\leq c_0 \|\psi\|^2. \end{aligned}$$

Then

$$\|\overline{\partial}^* N_{n,s}\psi\|^2 \leq c_0 \|\psi\|^2,$$

and

$$\|\overline{\partial} N_{n,s}\psi\|^2 \leq c_0 \|\psi\|^2.$$

Now, we show that $\overline{\partial}^* N_{n,s} = N_{n,s}\overline{\partial}^*$ on $\mathrm{Dom}(\overline{\partial}^*, \Xi)$. Using (ii), $\overline{\partial}^* u = \overline{\partial}^* \overline{\partial\partial}^* N_{n,s} u$. Then,

$$N_{n,s}\overline{\partial}^* u = N_{n,s}\overline{\partial}^* \overline{\partial\partial}^* N_{n,s} u = N_{n,s}(\overline{\partial}^*\overline{\partial} + \overline{\partial\partial}^*)\overline{\partial}^* N_{n,s} u = \overline{\partial}^* N_{n,s} u.$$

Similarly, one can prove $\overline{\partial} N_{n,s} = N_{n,s}\overline{\partial}$ on $\mathrm{Dom}(\overline{\partial}, \Xi)$. From (ii),

$$\psi = \overline{\partial\partial}^* N_{n,s}\psi + \overline{\partial\partial}^* \overline{\partial} N_{n,s}\psi.$$

Thus, $\overline{\partial}\psi = 0$ implies $\overline{\partial\partial}^* \overline{\partial} N_{n,s}\psi = 0$ and

$$< \overline{\partial\partial}^* \overline{\partial} N_{n,s}\psi, \overline{\partial} N_{n,s}\psi >= \|\overline{\partial}^* \overline{\partial} N_{n,s}\psi\|^2 = 0.$$

Since $\overline{\partial} N_{n,s}\psi \in \mathrm{Dom}(\overline{\partial}^*)$. Thus, $\psi = \overline{\partial\partial}^* N_{n,s}\psi$ and $u = \overline{\partial}^* N_{n,s}\psi$ is the solution which is unique and orthogonal to $\ker(\overline{\partial}, \Xi)$. □

Corollary 1. *Let X, D and Ξ be the same as Theorem 2. Then, for all $\psi \in L^2_{n,s}(D,\Xi)$ that satisfies $\overline{\partial}\psi = 0$, the canonical solution $u = \overline{\partial}^* N_{n,s}\psi$ satisfies the estimate*

$$\|u\|^2 \leq C\|\psi\|^2.$$

Proof. From (iv), one obtains $\overline{\partial}\, N_{n,s}\, \psi = N_{(n,s+1)}\, \overline{\partial}\psi = 0$. Since

$$\|N_{n,s}\psi\| \leq c_0 \|\psi\|.$$

Thus,

$$\begin{aligned} \|u\|^2 &=< \overline{\partial}^*\, N_{n,s}\psi, \overline{\partial}^*\, N_{n,s}\psi > \\ &=< \overline{\partial\partial}^*\, N_{n,s}\psi,\, N_{n,s}\psi > \\ &=< (\overline{\partial\partial}^* + \overline{\partial}^*\overline{\partial})\, N_{n,s}\psi,\, N_{n,s}\psi > \\ &=< \psi,\, N_{n,s}\psi > \\ &\leq \|\psi\|\, \|N_{n,s}\psi\| \\ &\leq\, c\, \|\psi\|^2. \end{aligned}$$

Thus, the proof follows. □

Let $\Box_{n,0} = \overline{\partial}^*\overline{\partial}$ on $L^2_{n,0}(D,\Xi)$. Set

$$\mathscr{H}_{n,0}(\Omega,\Xi) \;=\; \ker(\Box_{n,0}, E) = \{\psi \,\in\, L^2_{n,0}\,(D,\Xi) \,|\, \overline{\partial}\psi \;=\; 0\}.$$

Since $\overline{\partial}\psi \;=\; 0$, then $\mathscr{H}_{n,0}(D,\Xi)$ is a closed subspace of $L^2_{n,0}(D,\Xi)$. Let

$$\mathcal{P} : L^2_{n,0}(D,\Xi) \longrightarrow \mathscr{H}_{n,0}(D,\Xi),$$

be the Bergman projection operator.

Lemma 2 ([16]). *Let X, D and* Ξ *be the same as Theorem 2. Then,*

$$N_{n,0}:L^2_{n,0}(D,\Xi)\longrightarrow L^2_{n,0}(D,\Xi),$$

satisfies
(i) $Rang(N_{n,0},\Xi)\subset Dom(\Box_{n,0},\Xi)$, $\Box_{n,0}N_{n,0}=N_{n,0}\Box_{n,0}=I-\mathcal{P}_{n,0}$.
(ii) $\forall\ \psi\in L^2_{n,0}(D,\Xi)$; *one obtains* $\psi=\overline{\partial}^*\overline{\partial}N_{n,0}\psi\oplus\mathcal{P}_{n,0}\psi$.
(iii) $N_{n,1}\overline{\partial}=\overline{\partial}N_{n,0}$ *on* $Dom(\overline{\partial},\Xi)$, $\overline{\partial}^*N_{n,1}=N_{n,0}\overline{\partial}^*$ *on* $Dom(\overline{\partial}^*,\Xi)$.
(iv) $N_{n,0}\,\psi\ =\ \overline{\partial}^*N^2_{n,1}\,\overline{\partial}\,\psi$ *if* $\psi\in Dom(\overline{\partial},\Xi)$.
(v) $\forall\ \psi\in L^2_{n,0}(D,\Xi)$,

$$\|N_{n,0}\,\psi\|\leq\ C\,\|\psi\|,$$
$$\|\overline{\partial}N_{n,0}\,\psi\|\leq\ \sqrt{C}\,\|\psi\|.$$

Proof. Let $\psi\in\mathrm{Dom}(\Box_{n,0},\Xi)\cap(\mathscr{H}_{n,0}(E))^{\perp}$. Since $\mathrm{Rang}(\overline{\partial},\Xi)$ is closed in every degree, $\mathrm{Rang}(\overline{\partial}^*,\Xi)$ is closed. Thus, $\psi\perp\ker(\overline{\partial},\Xi)$ and $\psi\in\mathrm{Rang}(\overline{\partial}^*,\Xi)$. Let $\psi=\overline{\partial}\,u$; then, $\psi\in L^2_{n,1}(D,E)$ since $u\in\mathrm{Dom}(\Box_{n,0},\Xi)$. Using (v) in Theorem 5, $v\equiv\overline{\partial}^*N_{n,0}\psi$ is the solution of $\overline{\partial}v=\psi$, which is unique and $v\perp\ker(\overline{\partial},\Xi)$. Thus, $v=u$. By using Equation (36), one obtains

$$\|u\|^2\leq c\|\psi\|^2=c\|\overline{\partial}u\|^2=c<\Box_{n,0}u,u>\leq c\|\Box_{n,0}u\|\ \|u\|.$$

Thus, $\Box_{n,0}$ is bounded below on $\mathrm{Dom}(\Box_{n,0},\Xi)\cap(\mathscr{H}_{n,0}(E))^{\perp}$ and $\Box_{n,0}$ has a closed range and (i) and (ii) is proved. Then, from the strong Hodge decomposition,

$$L^2_{(n,0}(\Omega,\Xi)=\mathrm{Rang}(\Box_{n,0},E)\oplus\mathscr{H}_{n,0}(\Omega,E)=\overline{\partial}^*\overline{\partial}(\mathrm{Dom}(\Box_{n,0},E))\oplus\mathscr{H}_{n,0}(\Omega,\Xi),$$

for all $\psi\in\mathrm{Rang}(\Box_{n,0},\Xi)$, there is a unique $N_{n,0}\psi\perp\mathscr{H}_{n,0}(D,\Xi)$ that satisfies $\Box_{n,0}N_{n,0}\psi=\psi$. Extending $N_{n,0}$ to $L^2_{n,0}(D,\Xi)$ by requiring $N_{n,0}\mathcal{P}_{n,0}=0$, $N_{n,0}$ satisfies (i) and (ii). (iii) is proved as before. If $\psi\in\mathrm{Dom}(\overline{\partial},\Xi)$,

$$N_{n,0}u=(I-\mathcal{P}_{n,0})N_{n,0}u=N_{n,0}(\overline{\partial}^*\overline{\partial})N_{n,0}u=\overline{\partial}^*N^2_{n,0}\overline{\partial}\,u.$$

Thus, (iv) holds on $\mathrm{Dom}(\overline{\partial},\Xi)$. From (iii) in Theorem 5,

$$\|N_{n,1}\,\psi\|\leq\ C\,\|\psi\|.$$

for all $\psi\in\mathcal{C}^\infty_{n,0}(\overline{\Omega},E)$,

$$\begin{aligned}\|N_{n,0}\,\psi\|^2&=<\overline{\partial}\overline{\partial}^*N^2_{n,1}\overline{\partial}\psi,N^2_{n,1}\overline{\partial}\psi>=<N_{n,1}\overline{\partial}\psi,N^2_{n,1}\overline{\partial}\psi>=\ \|N_{n,1}\overline{\partial}\psi\|\,\|N^2_{n,1}\overline{\partial}\psi\|\\&\leq\ C\|N_{n,1}\overline{\partial}\psi\|^2.\end{aligned}\tag{38}$$

On the other hand, one obtains

$$\|N_{n,1}\overline{\partial}\psi\|^2=<N_{n,1}\overline{\partial}\psi,N_{n,1}\overline{\partial}\psi>=<N^2_{n,1}\overline{\partial}\psi,\overline{\partial}\psi>=<\overline{\partial}^*N^2_{n,1}\overline{\partial}\psi,\psi>\leq\ \|N_{n,0}\psi\|\,\|\psi\|.\tag{39}$$

Combining Equation (38) and Equation (39), one obtains

$$\|N_{n,0}\,\psi\|\leq\ C\,\|\psi\|,$$

and

$$\begin{aligned}\|\overline{\partial}N_{n,0}\,\psi\|^2 &=< \overline{\partial}^{*}\overline{\partial}N_{n,0}\psi, N^2_{n,0}\psi > \\ &=< (I-\mathcal{P}_{n,0})\psi, N_{n,0}\psi > \\ &\leq \ \|\psi\|\,\|N_{n,0}\psi\| \\ &\leq \ C\,\|\psi\|^2.\end{aligned}$$

Then, the proof follows. □

6. Sobolev Estimates

As in Cao–Shaw–Wang [3,35], one prove the following results:

Proposition 5.

$$\begin{aligned}\overline{\partial}^{*}\psi &= -\#^{-1}\divideontimes\overline{\partial}\divideontimes\#\,\psi, \\ \overline{\partial}^{*}_{\mathtt{m}}\psi &= \overline{\partial}^{*}\psi + \mathtt{m}\divideontimes\left(\frac{\partial\zeta}{\zeta}\wedge\divideontimes\psi\right).\end{aligned} \tag{40}$$

Proof. In fact, for $\psi\in\mathcal{C}^{\infty}_{r,s-1}(D,\Xi)$ and $\psi\in\mathcal{C}^{\infty}_{r,s-1}(\overline{D},\Xi)$, one obtains

$$\begin{aligned}\overline{\partial}({}^t\psi\wedge\divideontimes\#\psi) &= {}^t\overline{\partial}\psi\wedge\divideontimes\#\psi + (-1)^{r+s}\ {}^t\psi\wedge\overline{\partial}\divideontimes\#\psi \\ &= {}^t\overline{\partial}\psi\wedge\divideontimes\#\psi + {}^t\psi\wedge\divideontimes\divideontimes\overline{\partial}\divideontimes\#\psi \\ &= {}^t\overline{\partial}\psi\wedge\divideontimes\#\psi + {}^t\psi\wedge\divideontimes\#(\#^{-1}\divideontimes\overline{\partial}\divideontimes\#)\psi.\end{aligned}$$

Since ${}^t\psi\wedge\divideontimes\#\psi$ is of type $(n,n-1)$, then

$$\begin{aligned}\partial({}^t\psi\wedge\divideontimes\#\psi) &= 0, \\ \overline{\partial}({}^t\psi\wedge\divideontimes\#\psi) &= d({}^t\psi\wedge\divideontimes\#\psi).\end{aligned}$$

Then, by Stokes theorem, one obtains

$$\begin{aligned}0 = \int_\Omega d({}^t\psi\wedge\divideontimes\#\psi) &= \int_\Omega \overline{\partial}({}^t\psi\wedge\divideontimes\#\psi) \\ &= \int_\Omega {}^t\overline{\partial}\psi\wedge\divideontimes\#\psi + \int_\Omega {}^t\psi\wedge\divideontimes\#(\#^{-1}\divideontimes\overline{\partial}\divideontimes\#)\psi,\end{aligned}$$

i.e.,

$$\int_D {}^t\overline{\partial}\psi\wedge\divideontimes\#\psi = -\int_D {}^t\psi\wedge\divideontimes\#(\#^{-1}\divideontimes\overline{\partial}\divideontimes\#)\psi,$$

i.e.,

$$\int_D {}^t\overline{\partial}\psi\wedge\divideontimes\#\psi = \int_D {}^t\psi\wedge\divideontimes\#\overline{\partial}^{*}\psi.$$

Therefore,

$$\psi\psi = -\#^{-1}\divideontimes\overline{\partial}\divideontimes\#\,\psi.$$

Then,

$$\begin{aligned}\overline{\partial}^{*}_{\mathtt{m}}\psi &= -\zeta^{\mathtt{m}}\#^{-1}\divideontimes\overline{\partial}\divideontimes\zeta^{-\mathtt{m}}\#\,\psi \\ &= -\zeta^{\mathtt{m}}\zeta^{-\mathtt{m}}\#^{-1}\divideontimes\overline{\partial}\divideontimes\#\,\psi - \zeta^{\mathtt{m}}\#^{-1}\divideontimes\left(-\mathtt{m}\zeta^{-\mathtt{m}-1}\overline{\partial}\zeta\wedge\divideontimes\#\,\psi\right) \\ &= \overline{\partial}^{*}\psi + \mathtt{m}\#^{-1}\divideontimes\left(\frac{\overline{\partial}\zeta}{\zeta}\wedge\divideontimes\#\,\psi\right).\end{aligned}$$

But,

$$\frac{\overline{\partial}\zeta}{\zeta}\wedge\divideontimes\#\psi = \frac{\overline{\partial}\zeta}{\zeta}\wedge\divideontimes\overline{b\overline{\psi}} = \overline{b}\frac{\overline{\partial}\zeta}{\zeta}\wedge\divideontimes\overline{\psi},$$

and

$$\#\left(\frac{\partial\zeta}{\zeta}\wedge\divideontimes\psi\right)=\bar{b}\overline{\frac{\partial\zeta}{\zeta}\wedge\divideontimes\psi}=\bar{b}\frac{\bar{\partial}\zeta}{\zeta}\wedge\divideontimes\overline{\psi}.$$

Then,

$$\frac{\bar{\partial}\zeta}{\zeta}\wedge\divideontimes\#\psi=\#\left(\frac{\partial\zeta}{\zeta}\wedge\divideontimes\psi\right),$$

i.e.,

$$\#^{-1}\divideontimes\frac{\bar{\partial}\zeta}{\zeta}\wedge\divideontimes\#\psi=\divideontimes\left(\frac{\partial\zeta}{\zeta}\wedge\divideontimes\psi\right).$$

Then,

$$\bar{\partial}^{\divideontimes}_{\mathtt{m}}\psi=\bar{\partial}^{\divideontimes}\psi+\mathtt{m}\divideontimes\left(\frac{\partial\zeta}{\zeta}\wedge\divideontimes\psi\right),$$

where $\psi_0=\psi$. $\square$

Theorem 5. *Let X, D and Ξ be the same as Theorem 2. Let $\psi\in L^2_{n,s}(D,\Xi)\cap Dom(\bar{\partial},E)\cap Dom(\bar{\partial}^{\divideontimes},\Xi)$, $1\le s\le n$. Then,*

$$\|\bar{\partial}\psi\|^2_{W^{\mathtt{m}}_{n,s}(D,\Xi)}+\|\bar{\partial}^{\divideontimes}\psi\|^2_{W^{\mathtt{m}}_{n,s}(D,\Xi)}\ge C_{\mathtt{m}}\left(\|\partial\phi\wedge\divideontimes\psi\|^2_{W^{\mathtt{m}}_{n,s}(D,\Xi)}+\|\overline{\nabla}\psi\|^2_{W^{\mathtt{m}}_{n,s}(D,\Xi)}+\int_D(-\hbar)|\psi|^2dv\right),\tag{41}$$

where $C_{\mathtt{m}}>0$ is an independent constant of ψ.

Proof. As Lemma 1, one obtains

$$\partial\bar{\partial}\hbar=\mathtt{m}\zeta^{\mathtt{m}}\left((1-\mathtt{m})\frac{\partial\zeta\wedge\bar{\partial}\zeta}{\zeta^2}-\frac{\partial\bar{\partial}\zeta}{\zeta}\right),$$

$$\partial\bar{\partial}(-\log\zeta^{\mathtt{m}})=\mathtt{m}\left(\frac{\partial\zeta\wedge\bar{\partial}\zeta}{\zeta^2}-\frac{\partial\bar{\partial}\zeta}{\zeta}\right).$$

Then

$$\zeta^{\mathtt{m}}\partial\bar{\partial}(-\log\zeta^{\mathtt{m}})=\partial\bar{\partial}\hbar+\mathtt{m}^2\left(\frac{\partial\zeta\wedge\bar{\partial}\zeta}{\zeta^2}\right).$$

Therefore, for $\psi\in\mathcal{C}^1_{n,s}(\overline{D},E)\cap\mathrm{Dom}(\bar{\partial}^{\divideontimes},\Xi)$, and by using Equation (18), one obtains

$$\|\bar{\partial}\psi\|^2_{W^{\mathtt{m}}_{n,s}(D,\Xi)}+\|\bar{\partial}^{\divideontimes}_{\mathtt{m}}\psi\|^2_{W^{\mathtt{m}}_{n,s}(D,\Xi)}=\|\overline{\nabla}\psi\|^2_{W^{\mathtt{m}}_{n,s}(D,\Xi)}+<\Theta_{\mathtt{m}}\psi,\psi>_{\mathtt{m}}+<(\partial\bar{\partial}\hbar)\psi,\psi>+\mathtt{m}^2<\left(\frac{\partial\zeta\wedge\bar{\partial}\zeta}{\zeta^2}\right)\psi,\psi>.\tag{42}$$

Also, by using Equation (40), one obtains

$$\begin{aligned}\|\bar{\partial}^{\divideontimes}_{\mathtt{m}}\psi\|^2_{\mathtt{m}}&=\|\bar{\partial}^{\divideontimes}\psi\|^2_{\mathtt{m}}+2\mathtt{Re}<\bar{\partial}^{\divideontimes}\psi,\mathtt{m}\divideontimes\left(\frac{\partial\zeta}{\zeta}\wedge\divideontimes\psi\right)>_{\mathtt{m}}+\|\mathtt{m}\divideontimes\left(\frac{\partial\zeta}{\zeta}\wedge\divideontimes\psi\right)\|^2_{\mathtt{m}},\\&=\|\bar{\partial}^{\divideontimes}\psi\|^2_{\mathtt{m}}+2\mathtt{Re}<\bar{\partial}^{\divideontimes}\psi,\mathtt{m}\divideontimes\left(\frac{\partial\zeta}{\zeta}\wedge\divideontimes\psi\right)>_{\mathtt{m}}+\mathtt{m}^2\|\frac{\partial\zeta}{\zeta}\wedge\divideontimes\psi\|^2_{\mathtt{m}},\end{aligned}\tag{43}$$

and since for all $\kappa>0$,

$$2\mathrm{Re}<\bar{\partial}^{\divideontimes}\psi,\mathtt{m}\divideontimes\left(\frac{\partial\zeta}{\zeta}\wedge\divideontimes\psi\right)>_{\mathtt{m}}|\le\frac{\mathtt{m}}{\kappa}\int_\Omega(-\hbar)|\bar{\partial}^{\divideontimes}\psi|^2dv+\kappa\mathtt{m}\int_D(-\hbar)|\mathtt{m}\divideontimes\left(\frac{\partial\zeta}{\zeta}\wedge\divideontimes\psi\right)|^2dv,\tag{44}$$

and since

$$<(\partial\bar{\partial}\hbar)\psi,\psi>\ \ge\ C_0\left(\int_D(-\hbar)|\psi|^2dv+\int_D(-\hbar)|\frac{\partial\zeta}{\zeta}\wedge\divideontimes\psi|^2dv\right).\tag{45}$$

Then, by using Equations (43)–(45), the identity (42) becomes

$$\|\overline{\partial}\psi\|^2_{W^{\mathtt{m}}_{n,s}(D,\Xi)} + \|\overline{\partial}^{\circledast}\psi\|^2_{W^{\mathtt{m}}_{n,s}(D,\Xi)} \geq C_{\mathtt{m}}\Big(\|\partial\phi \wedge \circledast\psi\|^2_{W^{\mathtt{m}}_{n,s}(D,\Xi)} + \|\overline{\nabla}\psi\|^2_{W^{\mathtt{m}}_{n,s}(D,\Xi)} + \int_D (-\hbar)|\psi|^2 dv\Big).$$

Then the proof follows from the density of $\mathcal{C}^1_{n,s}(\overline{D}, E) \cap \mathrm{Dom}(\overline{\partial}^{\circledast}, \Xi)$ in $\mathrm{Dom}(\overline{\partial}, \Xi) \cap \mathrm{Dom}(\overline{\partial}^{\circledast}, E)$ in the sense of $\Big(\|\psi\|^2_{W^{\mathtt{m}}_{n,s}(D,\Xi)} + \|\overline{\partial}\psi\|^2_{W^{\mathtt{m}}_{n,s}(D,\Xi)} + \|\overline{\partial}^{\circledast}\psi\|^2_{W^{\mathtt{m}}_{n,s}(D,\Xi)}\Big)^2$. □

Corollary 2. *Let X, D and* Ξ *be the same as Theorem 2. Then,*

$$\begin{aligned} \|\overline{\partial}N_{n,s}\psi\|_{W^{\mathtt{m}}_{n,s}(D,\Xi)} &\leq C\|\psi\|_{W^{\mathtt{m}}_{n,s}(D,\Xi)},\ \psi \in \ker(\overline{\partial}^{\circledast}, E),\ 0 \leq s \leq n-1. \\ \|\overline{\partial}^{\circledast}N_{n,s}\psi\|_{W^{\mathtt{m}}_{n,s}(D,\Xi)} &\leq C\|\psi\|_{W^{\mathtt{m}}_{n,s}(D,\Xi)},\ \psi \in \ker(\overline{\partial}, \Xi),\ \ 2 \leq s \leq n. \end{aligned} \tag{46}$$

Proof. Since $\overline{\partial}N_{n,s}\psi \in \mathrm{Dom}(\overline{\partial}, E) \cap \mathrm{Dom}(\overline{\partial}^{\circledast}, \Xi)$, $0 \leq s \leq n-1$. Then, substituting $\overline{\partial}N_{n,s}\psi$ into Equation (41), for $\psi \in \ker(\overline{\partial}^{\circledast}, \Xi)$, one obtains

$$\|\overline{\partial}\overline{\partial}N_{n,s}\psi\|^2_{W^{\mathtt{m}}_{n,s}(D,\Xi)} + \|\psi\overline{\partial}N_{n,s}\psi\|^2_{W^{\mathtt{m}}_{n,s}(D,\Xi)} \geq C_{\mathtt{m}}\Big(\|\partial\phi \wedge \circledast\overline{\partial}N_{n,s}\psi\|^2_{W^{\mathtt{m}}_{n,s}(D,\Xi)} + \|\overline{\nabla}\overline{\partial}N_{n,s}\psi\|^2_{W^{\mathtt{m}}_{n,s}(D,\Xi)} + \int_D (-\hbar)|\overline{\partial}N_{n,s}\psi|^2 dv\Big).$$

Then, by using the fact that $\psi = \Big(\overline{\partial}^{\circledast}\overline{\partial} + \overline{\partial}\overline{\partial}^{\circledast}\Big)N\psi$, $\overline{\partial}\,\overline{\partial} = 0$ and $\overline{\partial}^{\circledast}N\psi = N\overline{\partial}^{\circledast}\psi = 0$, one obtains

$$\|\psi\|^2_{W^{\mathtt{m}}_{n,s}(D,\Xi)} \geq C_{\mathtt{m}} \int_D (-\hbar)|\overline{\partial}N\psi|^2 dv.$$

Then, the first equation of Equation (46) is proved by choosing $C = \frac{1}{C_{\mathtt{m}}}$. Similarly, for $2 \leq s \leq n$, $\overline{\partial}^{\circledast}N\psi \in \mathrm{Dom}(\overline{\partial}, E) \cap \mathrm{Dom}(\overline{\partial}^{\circledast}, \Xi)$. Then, substituting $\overline{\partial}^{\circledast}N_{n,s}\psi$ into Equation (41), for $\psi \in \ker(\overline{\partial}, \Xi)$, one obtains

$$\|\overline{\partial}\overline{\partial}^{\circledast}N\psi\|^2_{W^{\mathtt{m}}_{n,s}(D,\Xi)} + \|\psi\overline{\partial}^{\circledast}N\psi\|^2_{W^{\mathtt{m}}_{n,s}(D,\Xi)} \geq C_{\mathtt{m}}\Big(\|\partial\phi \wedge \circledast\overline{\partial}^{\circledast}N\psi\|^2_{W^{\mathtt{m}}_{n,s}(D,\Xi)} + \|\overline{\nabla}\overline{\partial}^{\circledast}N\psi\|^2_{W^{\mathtt{m}}_{n,s}(D,\Xi)} + \int_D (-\hbar)|\overline{\partial}^{\circledast}N\psi|^2 dv\Big).$$

Then, by using the fact that $\psi = \Big(\overline{\partial}^{\circledast}\overline{\partial} + \overline{\partial}\overline{\partial}^{\circledast}\Big)N\psi$, $\overline{\partial}^{\circledast}\overline{\partial}^{\circledast} = 0$ and $\overline{\partial}N\psi = N\overline{\partial}\psi = 0$, one obtains

$$\|\psi\|^2_{W^{\mathtt{m}}_{n,s}(D,\Xi)} \geq C_{\mathtt{m}} \int_D (-\hbar)|\overline{\partial}^{\circledast}N\psi|^2 dv.$$

Then, Equation (48) is proved by choosing $C = \frac{1}{C_{\mathtt{m}}}$. □

Theorem 6. *Let X, D and* Ξ *be the same as Theorem 2. Let* $\psi \in L^2_{n,s}(D, \zeta^{-\mathtt{m}}, \Xi)$, $1 \leq s \leq n$, *a* $\overline{\partial}$*-closed form. Then, for* $0 \leq \mathtt{m} < \mathtt{m}_0$, $\exists\ \psi = \overline{\partial}^{\circledast}_{\mathtt{m}}N\psi \in L^2_{n,s-1}(D, \zeta^{-\mathtt{m}}, \Xi)$ *satisfies* $\overline{\partial}\psi = \psi$ *and*

$$\int_D |\psi|^2 \zeta^{-\mathtt{m}} dv \leq C \int_D |\psi|^2 \zeta^{-\mathtt{m}} dv. \tag{47}$$

Proof. Let $\chi = \psi e^{\phi} = \psi\zeta^{-\mathtt{m}}$, $\phi = -\mathtt{m}\log\zeta$. Then, χ is orthogonal to all $\overline{\partial}$-closed forms of $L^2_{n,s-1}(D, \zeta^{-\mathtt{m}}, \Xi)$. Equation (33) gives

$$\int_D |\chi|^2 \zeta^{\mathtt{m}} dv \leq C \int_D |\overline{\partial}\chi|^2 \zeta^{\mathtt{m}} dv.$$

For $\phi = -\mathtt{m}\log\zeta$, one obtains

$$\overline{\partial}\,\chi = e^{\phi}\overline{\partial}\,\psi + e^{\phi}\overline{\partial}\,\phi \wedge \psi = \zeta^{-\mathtt{m}}\overline{\partial}\,\psi + \zeta^{-\mathtt{m}}\overline{\partial}\,\phi \wedge \psi.$$

Then,
$$\int_D |\psi|^2\zeta^{-\mathtt{m}}dv = \int_D |\chi|^2\zeta^{\mathtt{m}}dv \le C\int_D |\,\overline{\partial}\,\chi\,|^2\zeta^{\mathtt{m}}dv.$$
Then,
$$\begin{aligned}\int_D |\psi|^2\zeta^{-\mathtt{m}}dv &\le C\int_D |\overline{\partial}\psi + \overline{\partial}\phi\wedge\psi|^2\zeta^{-\mathtt{m}}dv\\ &\le C\bigg(\Big(1+\frac{1}{\tau}\Big)\int_D |\psi|\zeta^{-\mathtt{m}}dv + (1+\tau)\int_D |\overline{\partial}\phi\wedge\psi|^2\zeta^{-\mathtt{m}}dv\bigg),\end{aligned}$$
for every $\tau > 0$. Since
$$|\overline{\partial}\phi\wedge\psi|^2 \le |\ \psi\ |^2\ |\overline{\partial}\phi|^2 \le t^2|\psi|^2,$$
by choosing τ, which satisfies $(1+\tau)t^2 < 1$, (i.e., $0 < \tau < \left(\frac{\mathtt{m}_0}{\mathtt{m}}\right)^2 - 1$),
$$\int_D |\psi|^2\zeta^{-\mathtt{m}}dv \le C\frac{\left(1+\frac{1}{\tau}\right)}{[1-(1+\tau)t^2]}\int_D |\psi|^2\zeta^{-\mathtt{m}}dv.$$
It follows that $\overline{\partial}\psi = \psi$ and
$$\int_D |\psi|^2\zeta^{-\mathtt{m}}dv \le \widetilde{C}\int_D |\psi|^2\zeta^{-\mathtt{m}}dv.$$
□

Theorem 7. *Let X, D and Ξ be the same as Theorem 2. The Bergman projection $\mathcal{P} : L^2_{n,s}(D,\Xi) \longrightarrow L^2_{n,s}(D,E)\cap\ker(\overline{\partial},\Xi)$ is bounded from $W^{\mathtt{m}/2}_{n,s}(D,\Xi)$ to $W^{\mathtt{m}/2}_{n,s}(D,\Xi)$, where $0 \le s \le n-1$.*

Proof. From Lemma 2, $\mathcal{P} = I - \overline{\partial}^{\circledast}N_{r,s+1}\overline{\partial}$. Then, by using Equation (47), $\overline{\partial}^{\circledast}N$ is bounded on $\ker(\overline{\partial},\Xi)$ with
$$\|\overline{\partial}^{\circledast}N\psi\|_{-\mathtt{m}} \le C\|\psi\|_{-\mathtt{m}}, \tag{48}$$
for $\psi \in \ker(\overline{\partial},\Xi)$, $1 \le s \le n-1$. The Bergman projection with respect to the weighted space $L^2(D,\zeta^{\mathtt{m}},\Xi)$ is denoted by $\mathcal{P}_{\mathtt{m}}$. $\forall\ \psi,\varphi \in L^2_{n,0}(D,\Xi)$ with $\overline{\partial}\psi = 0$, and one obtains
$$<\mathcal{P}\varphi,\psi> = <\varphi,\psi> = <\zeta^{-\mathtt{m}}\varphi,\psi>_{\mathtt{m}} = <\mathcal{P}_{\mathtt{m}}\zeta^{-\mathtt{m}}\varphi,\psi>_{\mathtt{m}} = <\zeta^{\mathtt{m}}\mathcal{P}_{\mathtt{m}}\zeta^{-\mathtt{m}}\varphi,\psi>.$$
This implies that
$$\mathcal{P} = \mathcal{P}^2 = \mathcal{P}\zeta^{\mathtt{m}}\mathcal{P}_{\mathtt{m}}\zeta^{-\mathtt{m}} = (I-\overline{\partial}^{\circledast}N\overline{\partial})\zeta^{\mathtt{m}}\mathcal{P}_{\mathtt{m}}\zeta^{-\mathtt{m}} = \zeta^{\mathtt{m}}\mathcal{P}_{\mathtt{m}}\zeta^{-\mathtt{m}} - \overline{\partial}^{\circledast}N(\overline{\partial}\,\zeta^{\mathtt{m}}\wedge\mathcal{P}_{\mathtt{m}}\,\zeta^{-\mathtt{m}}), \tag{49}$$
because $\overline{\partial}\,\mathcal{P}_{\mathtt{m}} = 0$. $\forall\ \psi \in L^2(D,\Xi)$,
$$\|\zeta^{\mathtt{m}}\,\mathcal{P}_{\mathtt{m}}\,\zeta^{-\mathtt{m}}\,\psi\|^2_{-\mathtt{m}} \le \|\mathcal{P}_{\mathtt{m}}\,\zeta^{-\mathtt{m}}\,\psi\|^2_{W^{\mathtt{m}}_{n,s}(D,\Xi)} \le \|\zeta^{-\mathtt{m}}\,\psi\|^2_{W^{\mathtt{m}}_{n,s}(D,\Xi)} = \|\psi\|^2_{-\mathtt{m}}. \tag{50}$$
With (46), one obtains
$$\begin{aligned}\|\overline{\partial}^{\circledast}N(\overline{\partial}\,\zeta^{\mathtt{m}}\ \wedge\ \mathcal{P}_{\mathtt{m}}\,\zeta^{-\mathtt{m}}\,\psi)\|^2_{-\mathtt{m}} &\le C\,\|\overline{\partial}\,\zeta^{\mathtt{m}}\ \wedge\ \mathcal{P}_{\mathtt{m}}\,\zeta^{-\mathtt{m}}\,\psi\|^2_{-\mathtt{m}}\\ &\le C\,\|\zeta^{\mathtt{m}/2}\,\mathcal{P}_{\mathtt{m}}\,\zeta^{-\mathtt{m}}\,\psi\|^2\\ &= C\|\mathcal{P}_{\mathtt{m}}\,\zeta^{-\mathtt{m}}\,\psi\|^2_{W^{\mathtt{m}}_{n,s}(D,\Xi)}\\ &\le C\,\|\zeta^{-\mathtt{m}}\,\psi\|^2_{W^{\mathtt{m}}_{n,s}(D,\Xi)} =\\ &C\|\psi\|^2_{-\mathtt{m}}.\end{aligned} \tag{51}$$

With Equations (49) to (51), one obtains

$$\|\mathcal{P}\,\psi\|^2_{-\mathtt{m}} \le C\|\psi\|^2_{-\mathtt{m}}. \tag{52}$$

We note that $W^{\mathtt{m}/2}(D,\Xi) \subset L^2(D,\zeta^{-\mathtt{m}},\Xi)$. From Equation (52), one obtains

$$\|\mathcal{P}_{\mathtt{m}}\psi\|^2_{-\mathtt{m}} \le C\|\psi\|^2_{-\mathtt{m}} \le C_1\|\psi\|^2_{\mathtt{m}/2}. \tag{53}$$

Using Equation (52), one obtains that the Bergman projection satisfies

$$\|\mathcal{P}\psi\|_{\mathtt{m}/2} \le C_2\|\psi\|^2_{\mathtt{m}/2}. \tag{54}$$

Then, the Theorem is proved. □

In the following, the Sobolev boundary regularity for N, $\overline{\partial}N$ and $\overline{\partial}^*N$ is studied.

Theorem 8. *Let X, D and Ξ be the same as Theorem 2. Then, $\forall\ 0 < \mathtt{m} < \mathtt{m}_0$, N is bounded from $W^{\mathtt{m}/2}_{n,s}(D,\Xi)$ to $W^{\mathtt{m}/2}_{n,s}(D,\Xi)$ and $0 \le s \le n-1$. Also, $\forall\ \psi \in W^{\mathtt{m}}_{n,s}(D,\Xi)$, and one obtains the following estimates:*

$$\begin{aligned} \|N\psi\|_{W^{\mathtt{m}/2}_{n,s}(D,\Xi)} &\le 2C^2\|\psi\|^2_{W^{\mathtt{m}/2}_{n,s}(D,\Xi)}, \\ \|\overline{\partial}N\psi\|_{W^{\mathtt{m}/2}_{n,s}(D,\Xi)} &\le C\|\psi\|^2_{W^{\mathtt{m}/2}_{n,s}(D,\Xi)}, \\ \|\overline{\partial}^*N\psi\|_{W^{\mathtt{m}/2}_{n,s}(D,\Xi)} &\le C\|\psi\|^2_{W^{\mathtt{m}/2}_{n,s}(D,\Xi)}, \end{aligned} \tag{55}$$

where C depends only on $\mathtt{m}$.

Proof. Since $\mathcal{P} = I - \overline{\partial}^*N\overline{\partial}$, then $\overline{\partial}^*N\psi = \overline{\partial}^*N\mathcal{P}\psi$. Let $\mathcal{P}' = \overline{\partial}^*N\overline{\partial}$ be another projection operator into $\ker(\overline{\partial}^*,\Xi)$. Then, $\mathcal{P} = I - \mathcal{P}'$. It follows that $\overline{\partial}\,N\psi = \overline{\partial}N\mathcal{P}'\psi$. The self-adjoint property of $\mathcal{P}$ and $\mathcal{P}'$ gives

$$\|\mathcal{P}\psi\|_{W^{\mathtt{m}}_{n,s}(D,\Xi)} + \|\mathcal{P}'\psi\|_{W^{\mathtt{m}}_{n,s}(D,\Xi)} \le C_3\|\psi\|_{W^{\mathtt{m}}_{n,s}(D,\Xi)}.$$

Thus, by using Equation (54), and for $s \ge 0$, one obtains

$$\|\overline{\partial}N\psi\|_{W^{\mathtt{m}}_{n,s}(D,\Xi)} = \|\overline{\partial}N\mathcal{P}'\psi\|_{W^{\mathtt{m}}_{n,s}(D,\Xi)} \le C_4\|\mathcal{P}'\psi\|_{W^{\mathtt{m}}_{n,s}(D,\Xi)} \le C_4C_3\|\psi\|_{W^{\mathtt{m}}_{n,s}(D,\Xi)}, \tag{56}$$

and for $s \ge 2$, one obtains

$$\|\overline{\partial}^*N\psi\|_{W^{\mathtt{m}}_{n,s}(D,\Xi)} = \|\overline{\partial}^*N\mathcal{P}\psi\|_{W^{\mathtt{m}}_{n,s}(D,\Xi)} \le C_4\|\mathcal{P}\psi\|_{W^{\mathtt{m}}_{n,s}(D,\Xi)} \le C_4C_3\|\psi\|_{W^{\mathtt{m}}_{n,s}(D,\Xi)}.$$

Since for all $\psi \in \ker(\overline{\partial},\Xi)$, one obtains

$$\overline{\partial}^*N\psi = \overline{\partial}^*_{\mathtt{m}}N_{\mathtt{m}}\psi - \mathcal{P}_{\mathtt{m}}\overline{\partial}^*_{\mathtt{m}}N_{\mathtt{m}}\psi.$$

Thus, for all $\psi \in L^2_{n,1}(D,\Xi)$, one obtains

$$\begin{aligned} \|\overline{\partial}^*N\psi\|_{W^{\mathtt{m}}_{n,s}(D,\Xi)} = \|\overline{\partial}^*N\mathcal{P}\psi\|_{W^{\mathtt{m}}_{n,s}(D,\Xi)} &= \|\overline{\partial}^*_{\mathtt{m}}N_{\mathtt{m}}\mathcal{P}\psi - \mathcal{P}_{\mathtt{m}}\overline{\partial}^*_{\mathtt{m}}N_{\mathtt{m}}\mathcal{P}\psi\|_{W^{\mathtt{m}}_{n,s}(D,\Xi)} \\ &\le C_5\|\mathcal{P}\psi\|_{W^{\mathtt{m}}_{n,s}(D,\Xi)} \le C_5C_3\|\psi\|_{W^{\mathtt{m}}_{n,s}(D,\Xi)}. \end{aligned} \tag{57}$$

Since $(\overline{\partial}N)^* = N\overline{\partial}^* = \overline{\partial}^*N$ and $(\overline{\partial}^*N)^* = N\overline{\partial} = \overline{\partial}N$. Use Equations (56) and (57), and by choosing $C = \max\{C_3C_4, C_3C_5\}$, the second and third inequality of Equation (55) follows. Since

$$N = \overline{\partial}\overline{\partial}^*N^2 + \overline{\partial}^*\overline{\partial}N^2 = \overline{\partial}N\overline{\partial}^*N + \overline{\partial}^*N\overline{\partial}N.$$

Equations (56) and (57) give

$$\|N\psi\|_{\mathtt{m}/2(D)} \le 2C^2 \|\psi\|^2_{\mathtt{m}/2(D)}.$$

□

Theorem 9. *Let X, D and Ξ be the same as Theorem 2. Then, $\forall\ 0 < \mathtt{m} < \mathtt{m}_0$ and N is bounded from $W^{\mathtt{m}/2}_{n,s}(D,\Xi)$ to $W^{\mathtt{m}/2}_{n,s}(D,\Xi)$, where $0 \le s \le n-1$. Also, $\forall\ \psi \in W^{\mathtt{m}/2}_{n,s}(D,\Xi)$, and one obtains the following estimates:*

$$\|N\psi\|_{W^{-\mathtt{m}/2}_{n,s}(D,\Xi)} \le C\|\psi\|_{W^{-\mathtt{m}/2}_{n,s}(D,\Xi)},$$
$$\|\bar{\partial}N\psi\|_{W^{-\mathtt{m}/2}_{n,s}(D,\Xi)} \le C\|\psi\|_{W^{-\mathtt{m}/2}_{n,s}(D,\Xi)},$$
$$\|\bar{\partial}^{*}N\psi\|_{W^{-\mathtt{m}/2}_{n,s}(D,\Xi)} \le C\|\psi\|_{W^{-\mathtt{m}/2}_{n,s}(D,\Xi)}.$$

Proof. With respect to the L^2 norm, if $\mathcal{S}^{*}$ is the adjoint map of $\mathcal{S}$, one obtains

$$\begin{aligned}\|\mathcal{S}f\|_{W^{\mathtt{m}/2}_{n,s}(D,\Xi)} &= \sup_{g\in L^2} \frac{<\mathcal{S}f, g>_{L^2}}{\|g\|_{W^{\mathtt{m}/2}_{n,s}(D,\Xi)}} \\ &= \sup_{g\in L^2} \frac{<f, \mathcal{S}^{*}g>_{L^2}}{\|g\|_{W^{-\mathtt{m}/2}_{n,s}(D,\Xi)}} \\ &\le \|\mathcal{S}^{*}\|_{W^{-\mathtt{m}/2}_{n,s}(D,\Xi)}\, \|g\|_{W^{\mathtt{m}/2}_{n,s}(D,\Xi)}.\end{aligned} \tag{58}$$

Then, by using Theorem 9 and Equation (58), the proof follows. □

7. Conclusions

Sobolev estimates for the $\bar{\partial}$ and the $\bar{\partial}$-Neumann operator on pseudoconvex manifolds are fundamental results in complex analysis. They allow us to understand the behavior of holomorphic functions and provide important tools for solving the $\bar{\partial}$ equation. These estimates have applications in various areas of mathematics, such as the study of complex geometry and partial differential equations on pseudoconvex manifolds.

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